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Recounting the Odds of an Even Derangement

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We see the potential value of this proof as twofold. First, it appears cleaner and shorter than what is found in most texts. And our constant, $\ln(8)$, is modest compared to Sierpinski's 4 [**7**], Apostol's 6 [**1**], or the $32\ln(2)$ offered in earlier editions of Niven and Zuckerman [**6**]. LeVeque [**5**], Hardy and Wright [**4**], and the latest edition of Niven and Zuckerman [**6**] give no particular constant, merely proving that one exists. Chebychev [3] achieved a much smaller constant than ours, but with considerably more effort. We hope that our short proof will be found to have pedagogical value.

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Recounting the Odds of an Even Derangement

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Odd as it may sound, when n exams are randomly returned to n students, the probability that no student receives his or her own exam is almost exactly $1/e$ (approximately 0.368), for all $n \geq 4$. We call a permutation with no fixed points, a *derangement*, and we let $D(n)$ denote the number of derangements of n elements. For $n \geq 1$, it can be shown that $D(n) = \sum_{k=0}^{n}(-1)^k n!/k!$, and hence the *odds* that a random permutation of n elements has no fixed points is $D(n)/n!$, which is within $1/(n+1)!$ of $1/e$ [**1**].

Permutations come in two varieties: even and odd. A permutation is even if it can be achieved by making an even number of swaps; otherwise it is odd. Thus, one might *even* be interested to know that if we let $E(n)$ and $O(n)$ respectively denote the number of even and odd derangements of n elements, then (oddly enough),

$$E(n) = \frac{D(n) + (n-1)(-1)^{n-1}}{2}$$

and

$$O(n) = \frac{D(n) - (n-1)(-1)^{n-1}}{2}.$$

The above formulas are an immediate consequence of the equation $E(n) + O(n) = D(n)$, which is obvious, and the following theorem, which is the focus of this note.

THEOREM. *For* $n \geq 1$,

$$E(n) - O(n) = (-1)^{n-1}(n-1). \tag{1}$$

Proof 1: Determining a Determinant The fastest way to derive equation (1), as is done in [**3**], is to compute a determinant. Recall that an n-by-n matrix $A = [a_{ij}]_{i,j=1}^{n}$ has determinant

$$\det(A) = \sum_{\pi \in S_n} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \operatorname{sgn}(\pi), \tag{2}$$

where S_n is the set of all permutations of $\{1, \ldots, n\}$, $\operatorname{sgn}(\pi) = 1$ when π is even, and $\operatorname{sgn}(\pi) = -1$ when π is odd. Let A_n denote the n-by-n matrix whose nondiagonal entries are $a_{ij} = 1$ (for $i \neq j$), with zeroes on the diagonal. For example, when $n = 4$,

$$A_4 = J_4 - I_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

By (2), every permutation that is not a derangement will contribute 0 to the sum (since it uses at least one of the diagonal entries), every even derangement will contribute 1 to the sum, and every odd derangement will contribute -1 to the sum. Consequently, $\det(A_n) = E(n) - O(n)$. To see that $\det(A_n) = (-1)^{n-1}(n-1)$, observe that $A_n = J_n - I_n$, where J_n is the matrix of all ones and I_n is the identity matrix. Since J_n has rank one, zero is an eigenvalue of J_n, with multiplicity $n-1$, and its other eigenvalue is n (with an eigenvector of all 1s). Apply $J_n - I_n$ to the eigenvectors of J_n to find the eigenvalues of A_n: -1 with multiplicity $n-1$ and $n-1$ with multiplicity 1. Multiplying the eigenvalues gives us $\det(A_n) = (-1)^{n-1}(n-1)$, as desired. ■

A 1996 Note in the MAGAZINE [**2**] gave *even odder* ways to determine the determinant of A_n.

Although the proof by determinants is quick, the form of (1) suggests that there should also exist an *almost* one-to-one correspondence between the set of even derangements and the set of odd derangements.

Proof 2: Involving an Involution Let D_n denote the set of derangements of $\{1, \ldots, n\}$, and let X_n be a set of $n-1$ *exceptional* derangements (that we specify later), each with sign $(-1)^{n-1}$. We exhibit a *sign reversing involution* on $D_n - X_n$. That is, letting $T_n = D_n - X_n$, we find an invertible function $f : T_n \to T_n$ such that for π in T_n, π and $f(\pi)$ have opposite signs, and $f(f(\pi)) = \pi$. In other words, except for the $n-1$ exceptional derangements, every even derangement "holds hands" with an odd derangement, and vice versa. From this, it immediately follows that $|E_n| - |O_n| = (-1)^{n-1}(n-1)$.

Before describing f, we establish some notation. We express each π in D_n as the product of k disjoint cycles $C_1, \ldots, C_k$ with respective lengths $m_1, \ldots, m_k$ for some

$k \geq 1$. We follow the convention that each cycle begins with its smallest element, and the cycles are listed from left to right in increasing order of the first element. In particular, $C_1 = (1\ a_2\ \cdots\ a_{m_1})$ and, if $k \geq 2$, C_2 begins with the smallest element that does not appear in C_1. Since π is a derangement on n elements, we must have $m_i \geq 2$ for all i, and $\sum_{i=1}^{k} m_i = n$. Finally, since a cycle of length m has sign $(-1)^{m-1}$, it follows that π has sign $(-1)^{\sum_{i=1}^{k}(m_i-1)} = (-1)^{n-k}$.

Let π be a derangement in D_n with first cycle $C_1 = (1\ a_2\ \cdots\ a_m)$ for some $m \geq 2$. We say that π has *extraction point* $e \geq 2$ if e is the smallest number in the set $\{2, \ldots, n\} - \{a_2\}$ for which C_1 does *not* end with the numbers of $\{2, \ldots, e\} - \{a_2\}$ written in decreasing order. Note that π will have extraction point $e = 2$ unless the number 2 appears as the second term or last term of C_1. We illustrate this definition with some pairs of examples from D_9. Notice that in each pair below, the number of cycles of π and π' differ by one, and the extraction point e occurs in the first cycle of π and is the leading element of the second cycle of π'.

$$\begin{array}{llll}
\pi = (1\,9\,7\,2\,8)(3\,6)(4\,5) & \text{and} & \pi' = (1\,9\,7)(2\,8)(3\,6)(4\,5) & \text{have } e = 2. \\
\pi = (1\,2\,9\,7\,3\,8\,5)(4\,6) & \text{and} & \pi' = (1\,2\,9\,7)(3\,8\,5)(4\,6) & \text{have } e = 3. \\
\pi = (1\,9\,7\,3\,8\,5\,2)(4\,6) & \text{and} & \pi' = (1\,9\,7\,2)(3\,8\,5)(4\,6) & \text{have } e = 3. \\
\pi = (1\,9\,4\,8\,5\,3\,2)(6\,7) & \text{and} & \pi' = (1\,9\,3\,2)(4\,8\,5)(6\,7) & \text{have } e = 4. \\
\pi = (1\,4\,9\,5\,8\,3\,2)(6\,7) & \text{and} & \pi' = (1\,4\,9\,3\,2)(5\,8)(6\,7) & \text{have } e = 5. \\
\pi = (1\,3\,8\,6\,9\,7\,5\,4\,2) & \text{and} & \pi' = (1\,3\,8\,5\,4\,2)(6\,9\,7) & \text{have } e = 6.
\end{array}$$

Observe that every derangement π in D_n contains an extraction point unless π consists of a single cycle of the form $\pi = (1\ a_2\ Z)$, where Z is the ordered set $\{2, 3, \ldots, n-1, n\} - \{a_2\}$, written in decreasing order. For example, the 9-element derangement $(1\,5\,9\,8\,7\,6\,4\,3\,2)$ has no extraction point. Since a_2 can be any element of $\{2, \ldots, n\}$, there are exactly $n - 1$ derangements of this type, all of which have sign $(-1)^{n-1}$. We let X_n denote the set of derangements of this form. Our problem reduces to finding a sign reversing involution f over $T_n = D_n - X_n$.

Suppose π in T_n has extraction point e. Then the first cycle C_1 of π ends with the (possibly empty) ordered subset Z consisting of the elements of $\{2, \ldots, e-1\} - \{a_2\}$ written in decreasing order. Our sign reversing involution $f : T_n \to T_n$ can then be succinctly described as follows:

$$(1\ a_2\ X\ e\ Y\ Z)\sigma \overset{f}{\longleftrightarrow} (1\ a_2\ X\ Z)(e\ Y)\sigma, \tag{3}$$

where X and Y are ordered subsets, Y is nonempty, and σ is the rest of the derangement π.

Notice that since the number of cycles of π and $f(\pi)$ differ by one, they must be of opposite signs. The derangements on the left side of (3) are those for which the extraction point e is in the first cycle. In this case, Y must be nonempty, since otherwise "$e\ Z$" would be a longer decreasing sequence and e would not be the extraction point. The derangements on the right side of (3) are those for which the extraction point e is not in the first cycle (and must therefore be the leading element of the second cycle). In this case, Y is nonempty since π is a derangement. Thus for any derangement π, the derangement $f(\pi)$ is also written in standard form, with the same extraction point e and with the same associated ordered subset Z. Another way to see that π and $f(\pi)$ have opposite signs is to notice that $f(\pi) = (xy)\pi$ (multiplying from left to right), where x is the last element of X ($x = a_2$ when X is empty), and y is the last element of Y. Either way, $f(f(\pi)) = \pi$, and f is a well-defined, sign-reversing involution, as desired. ■

In summary, we have shown combinatorially that for all values of n, there are almost as many even derangements as odd derangements of n elements. Or to put it another way, when randomly choosing a derangement with at least five elements, the *odds* of having an even derangement are nearly *even*.

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Volumes of Generalized Unit Balls

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Diamonds, cylinders, squares, stars, and balls. These geometric figures are familiar to undergraduate students, but what could they possibly have in common? One answer is: They are generalized balls. The standard Euclidean ball can be distorted into a variety of strange-shaped balls by linear and nonlinear transformations. The purpose of this note is to give a unified formula for computing the volumes of generalized unit balls in n-dimensional spaces.

A generalized unit ball in $\mathbb{R}^n$ is described by the set

$$\mathbb{B}_{p_1 p_2 \cdots p_n} = \{\mathbf{x} = (x_1, \ldots, x_n) : |x_1|^{p_1} + \cdots + |x_n|^{p_n} \leq 1\}, \tag{1}$$

where $p_1 > 0, p_2 > 0, \ldots, p_n > 0$.

When the numbers $p_1, \ldots, p_n$ are all greater than or equal to 1, the unit ball $\mathbb{B}_{p_1 \ldots p_n}$ is convex. Since $|x|^p$ is not concave on $[-1, 1]$ for $0 < p < 1$, $\mathbb{B}_{p_1 \ldots p_n}$ is not necessarily convex anymore when $n > 1$. When $p_1 = p_2 = \cdots = p_n = p \geq 1$, we obtain the usual l_p ball. The l_2 ball is denoted by $\mathbb{B}$. By choosing different numbers p_i, we can alter the appearance of the generalized balls greatly, as shown in FIGURE 1 with examples in $\mathbb{R}^3$.

Motivated by an article by Folland [**5**], I derived a unified formula for calculating the volume of these balls. Although the volume formulas for the standard Euclidean ball $\mathbb{B}$ and simplex have been known for a long time [**4**, pp. 208, 220], the unified formula is (relatively) new. It is surprising that no matter how strange the balls look, the volume of any ball can be computed by a single formula, as follows:

THEOREM. *Assume $p_1, \ldots, p_n > 0$. The volume of the unit ball $\mathbb{B}_{p_1 p_2 \cdots p_n}$ in $\mathbb{R}^n$ is equal to*