3-1-2000

A Rational Solution to Cootie

Arthur T. Benjamin
Harvey Mudd College

Matthew T. Fluet '99

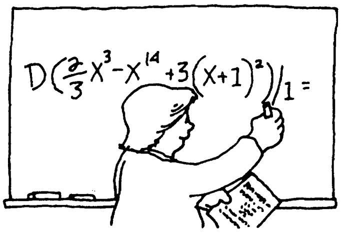
Recommended Citation

Benjamin, Arthur T. and Matthew T. Fluet. "A Rational Solution to Cootie." The College Mathematics Journal, Vol 31, No. 2, pp. 124-125, March, 2000.

CLASSROOM CAPSULES

EDITOR

Thomas A. Farmer
Department of Mathematics and Statistics
Miami University
Oxford, OH 45056-1641



A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

A Rational Solution to *Cootie*

Arthur T. Benjamin (benjamin@hmc.edu), and Matthew T. Fluet (mfluet@hmc.edu), Harvey Mudd College, Claremont, CA 91711

A recent Classroom Capsule **[1]** described the game of *Cootie* and posed a question about the game's playing time. The authors used an infinite sum of four summations with complicated summand terms to answer this question. Here, we describe a simpler way to obtain an exact answer using only a finite number of calculations.

In *Cootie*, players race to construct a "cootie bug" by rolling a die to collect component parts. Players must first roll a 1 in order to acquire the body and then roll a 2 for the head. Once a player has both the body and the head, the remaining parts can be obtained in any order by rolling two 3's for the eyes, one 4 for the nose, two 5's for the antennae, and six 6's for the legs. The previous article asked the question: what is the theoretical expected value of the number of rolls required to make a cootie?

The rules of *Cootie* naturally break an analysis of the playing time T into three parts:

$$T = B + H + T_{2,1,2,6},$$

where B and H denote the number of rolls to obtain the body and head, respectively, and $T_{2,1,2,6}$ is the number of rolls to subsequently obtain two 3's, one 4, two 5's, and six 6's. Since $E[B] = E[H] = 6$, we have, by the linearity of expectation,

$$E[T] = 12 + E[T_{2,1,2,6}]. \tag{1}$$

We calculate $E[T_{2,1,2,6}]$ by a recursive calculation that exploits the law of conditional expectation:

$$E[X] = \sum_{y} E[X \mid Y = y] P[Y = y]. \tag{2}$$

For $a, b, c, d > 0$, we let $T_{a,b,c,d}$ denote the number of rolls to obtain a 3's, b 4's, c 5's, and d 6's. To exploit (2), we condition on Y, the outcome of the first roll. Since

$P[Y=y]=\frac{1}{6}$, we obtain

$$E[T_{a,b,c,d}]=\frac{1}{6}\sum_{y=1}^{6}E[T_{2,1,2,6}|Y=y]. \tag{3}$$

Next, we note that

$$E[T_{a,b,c,d}|Y=1]=1+E[T_{a,b,c,d}],$$

since an initial roll of 1 uses a roll and has not changed our situation. However, we note that

$$E[T_{a,b,c,}|Y=3]=1+E[T_{a-1,b,c,d}],$$

since an initial roll of 3 uses a roll and has changed our goal to rolling $a-1$ 3's, b 4's, c 5's, and d 6's. The other cases follow similarly. Solving (3) for $E[T_{a,b,c,d}]$ yields

$$E[T_{a,b,c,d}]=\frac{6+E[T_{a-1,b,c,d}]+E[T_{a,b-1,c,d}]+E[T_{a,b,c-1,d}]+E[T_{a,b,c,d-1}]}{4}.$$

Now, if a, b, c, or d is 0, then $T_{a,b,c,d}$ reduces to a "smaller" problem. For instance, $T_{a,b,c,0}=T_{a,b,c}$ denotes the number of rolls needed to obtain a rolls of one type, b of another type and c of a third type. Exploiting (2) by again conditioning on the outcome of the first roll, we can derive that

$$E[T_{a,b,c}]=\frac{6+E[T_{a-1,b,c}]+E[T_{a,b-1,c}]+E[T_{a,b,c-1}]}{3}$$

$$E[T_{a,b}]=\frac{6+E[T_{a-1,b}]+E[T_{a,b-1}]}{2},$$

with the appropriate reductions to a "smaller" problem if a, b, or c is 0. Finally, we have the trivial base case

$$E[T_a]=6a.$$

These calculations can either be carried out by hand (an arduous task requiring the calculation of 126 intermediate values) or through a computer program. A quick computer calculation yields

$$E[T_{2,1,2,6}]=\frac{441357301}{11943936}=36.9524167745+.$$

From (1) it follows that $E[T]=48.9524167745+$. We note that this value differs with the number calculated in [**1**]. In the vast majority of *Cootie* games, the legs will be the last body part completed. Thus it is not too surprising that $E[T_{2,1,2,6}]$ is only slightly bigger than 36, the time required to get six 6's. On the other hand, we were very surprised to notice that the expected number of rolls to get all of the 3's, 4's, 5's, and 6's has a denominator equal to 3456 squared!

Acknowledgment. We thank Professor Janet Myhre and the Institute of Decision Sciences for supporting this research.

Reference

1. Min Deng and Mary T. Whalen. The Mathematics of *Cootie*. This JOURNAL, 29 (1998) 222–224.