2-1-1993

Optimal Leapfrogging

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This article arose from trying to determine the fastest way of moving checkers from the lower left-hand corner of a Go board to the upper right-hand corner, with no opponent in the way. The pieces move in "Chinese checkers" fashion by shifting or jumping in a way we soon illustrate and later describe precisely. Our goal is to move our pieces from a prescribed origin position to a prescribed destination position in as few turns as possible.

We regard our Go board as a subset of the integer-point lattice, $\mathbf{Z}^2$. Suppose we have four indistinguishable checkers initially situated at the points $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, and we wish to move them to the points $(17,17)$, $(17,18)$, $(18,17)$, and $(18,18)$. See FIGURE 1. One might begin by maneuvering into the *snake* configuration $\{(0,0),(1,1),(2,2),(3,3)\}$. (This can be done in four moves: for example, a hop, $(0,1) \to (2,1)$; a shift, $(2,1) \to (2,2)$; a two-hop jump $(1,0) \to (1,2) \to (3,2)$; and a shift $(3,2) \to (3,3)$.) Then apply the following three-move procedure: Shift the bottom piece (at $(0,0)$) to the right (to $(1,0)$), then triple-hop that piece (to $(3,4)$), then shift that piece to the right (to $(4,4)$). The pieces end up in the snake configuration (starting at $(1,1)$), and the same three-move procedure can be applied. If we apply this procedure 15 times, we reach $\{(15,15),(16,16),(17,17),(18,18)\}$. Four moves later, we will have reached our destination using $4+(15\times 3)+4=53$ moves altogether. However, a much faster trajectory exists. In one move, hop the piece at $(1,0)$ to $(1,2)$ reaching the *serpent* configuration $\{(0,0),(0,1),(1,1),(1,2)\}$. See FIGURE 2. Then apply the following two-move procedure: Double-hop the bottom piece (from $(0,0)$ to $(2,2)$), then double-hop the left-most piece (from $(0,1)$ to $(2,3)$). Once again, the pieces end up in the "serpent configuration" (starting at $(1,1)$), and the same two-move procedure can be applied. If we apply this procedure 16 times, we reach $\{(16,16),(16,17),(17,17),(17,18)\}$. Two moves later, we will have reached our destination using only 35 moves. The second trajectory is faster because the serpent configuration requires only two moves to translate itself in the direction $(1,1)$, a feat requiring three moves by the snake configuration. In this article, we characterize the speediest configurations for the above game (played in n dimensions) and thereby prove that the second trajectory is in fact optimal.

Precisely, we consider the problem of efficiently moving a collection of p indistinguishable pieces over the integer lattice $\mathbf{Z}^n$. The movement rules are analogous to those of Chinese checkers, and are as follows. At all times, pieces occupy distinct points in $\mathbf{Z}^n$. At each move, exactly one piece is displaced. If a piece is situated at the point $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbf{Z}^n$ and, for some $i\in\{1,\ldots,n\}$, the point $\mathbf{x}+\mathbf{e}_i$ is unoccupied (where $\mathbf{e}_i$ is the i-th unit vector), then the piece may *shift* there; similarly for $\mathbf{x}-\mathbf{e}_i$.

If $\mathbf{x}+\mathbf{e}_i$ is occupied, but $\mathbf{x}+2\mathbf{e}_i$ is not, then the piece can *hop* over the occupant of $\mathbf{x}+\mathbf{e}_i$ to arrive at $\mathbf{x}+2\mathbf{e}_i$, where it may either remain or hop over another adjacent piece, etc. (Similarly for a hop over $\mathbf{x}-\mathbf{e}_i$ to $\mathbf{x}-2\mathbf{e}_i$. A *move* consists of either a shift or a *jump* (a sequence of one or more hops by a single piece).

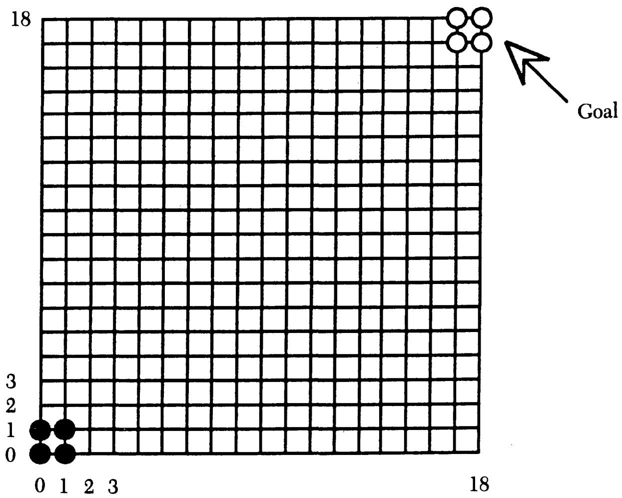

FIGURE 1
Solitaire Chinese checkers on a Go board.

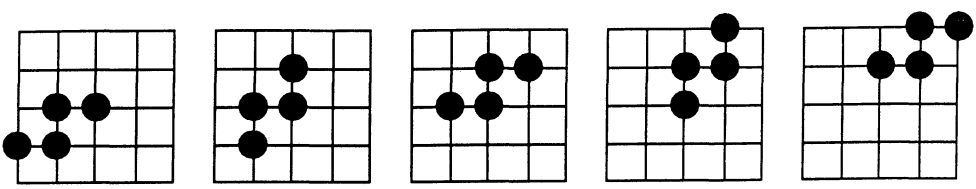

FIGURE 2
Serpent movement.

We begin with some definitions. A *placement* of pieces is a size p subset of $\mathbf{Z}^n$, usually denoted by $X=\{\mathbf{x}_1,\ldots,\mathbf{x}_p\}$. We define the *centroid* of a placement X to be

$$\mathbf{c}(X)=\frac{1}{p}\sum_{u=1}^{p}\mathbf{x}_u,$$

which is a vector in $\frac{1}{p}\mathbf{Z}^n$. For placements X and Y, the *displacement* from X to Y is defined by

$$\mathbf{d}(X,Y)=\sum_{i=1}^{n}\left(c_i(Y)-c_i(X)\right)$$

where $c_i(Y) - c_i(X)$ is the i-th component of $\mathbf{c}(Y) - \mathbf{c}(X)$. Loosely, displacement measures the distance between placements, where the directions $\mathbf{e}_i$ are viewed as positive directions. Note that d can be negative, or nontrivially zero. For $m \geq 1$, an *m-move trajectory* $X_0, X_1, \ldots, X_m$ is a sequence of placements where X_{v+1} is reachable from X_v in a single move. The *speed* of an m-move trajectory from X to Y is defined as $\mathbf{d}(X,Y)/m$.

THEOREM 1. *Any trajectory speed is* $\leq 2 - 2/p$, *where* $p \geq 2$ *is the number of pieces. When* $p = 1$, *the speed is bounded by* 1.

Proof. From the definition, the speed of the trajectory $X_0, X_1, \ldots, X_m$ is the average of the speeds of its m moves. Each of these is of the form $\mathbf{d}(X_v, X_{v+1})/1$. If $p = 1$, the move is a shift and the displacement is 1, provided the shift is positive. If $p \geq 2$, the move is either a shift or a jump of at most $p - 1$ hops. The centroid is displaced by at most $(2p - 2)/p$, the maximum being attained by $p - 1$ positive hops.

Note that this maximum speed is not sustainable: There are various configurations in which a "long-jump" of $p - 1$ hops can be made, but the next move can be another long-jump only in the rather trivial one-dimensional case with $p = 2$. This is a special case of our next theorem, that a "repeatable" trajectory has speed at most 1, which we will call the *speed of light*.

We say that placement Y is a *translate* of X if there exists $\mathbf{a} \in \mathbf{Z}^n$ such that $Y = X + \mathbf{a}$, i.e., $\{\mathbf{y}_1, \ldots, \mathbf{y}_p\} = \{\mathbf{x}_1 + \mathbf{a}, \ldots, \mathbf{x}_p + \mathbf{a}\}$. Such placements X and Y are said to be represented by the same *configuration*. For $\mathbf{x} \in \mathbf{Z}^n$, define $\|\mathbf{x}\|$ to be $\sum_{i=1}^n x_i$, and for all integers M, let the *border* M be $\{\mathbf{x} \in \mathbf{Z}^n\colon \|\mathbf{x}\| = M\}$. Define the *tail* and *head* of a placement X as

$$t(X) = \min_u \|x_u\|, \quad h(X) = \max_u \|x_u\|.$$

THEOREM 2. *Let* Y *be a translate of* X. *Then any trajectory from* X *to* Y *has speed at most* 1.

Proof. Suppose $Y = X + \mathbf{a}$ for some $\mathbf{a} \in \mathbf{Z}^n$. For ease of notation, assume that $\mathbf{y}_i = \mathbf{x}_i + \mathbf{a}$, $i = 1, \ldots, p$, whence

$$\mathbf{d}(X, Y) = \sum_{i=1}^{n} (c_i(Y) - c_i(X)) = \sum_{i=1}^{n} \frac{1}{p} \sum_{u=1}^{p} a_i = \|\mathbf{a}\|.$$

Next we observe that the tail (and the head) cannot increase by more than 1 after each move. Therefore, since $t(Y) = \min_{i=1}^p \|(\mathbf{x}_i + \mathbf{a})\| = \min_{i=1}^p \|\mathbf{x}_i\| + \|a_i\| = t(X) + \|\mathbf{a}\|$, it follows that the number of moves m needed for a trajectory from X to Y is at least $\|\mathbf{a}\| = \mathbf{d}(X, Y)$. If $\mathbf{d}(X, Y) \leq 0$, then since $m \geq 1$, the trajectory has nonpositive speed. Otherwise, since $m \geq \mathbf{d}(X, Y)$, its speed is $\mathbf{d}(X, Y)/m \leq 1$.

A placement X is called a *speed-of-light placement* if there exists a nonzero vector $\mathbf{a} \in \mathbf{Z}^n$ and a speed one trajectory (called a *speed-of-light trajectory*) from X to $X + \mathbf{a}$. In FIGURE 3, we illustrate speed-of-light configurations for the two-dimensional case (where $p = 1$, 2, and 4, respectively). In fact, the next theorem demonstrates that these are the *only* such configurations in two dimensions, and essentially the only ones for higher dimensions, too.

THEOREM 3. *The following are speed-of-light configurations*:

The ***atom*** $\{\mathbf{x}\}$ (*when* $p = 1$),

the ***frog*** $\{\mathbf{x}, \mathbf{x} + \mathbf{e}_i\}$ $1 \leq i \leq n$ (*when* $p = 2$), *and*

the ***serpent*** $\{\mathbf{x}, \mathbf{x} + \mathbf{e}_i, \mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, \mathbf{x} + 2\mathbf{e}_i + \mathbf{e}_j\}$ $1 \leq i \neq j \leq n$ (*when* $p = 4$).

No other speed-of-light configurations exist.

The first part of the theorem is straightforward. The atom can translate itself (in the direction $\mathbf{e}_i$) by shifting itself from $\{\mathbf{x}\}$ to $\{\mathbf{x} + \mathbf{e}_i\}$, a speed one maneuver when

$p = 1$. The frog $\{\mathbf{x}, \mathbf{x} + \mathbf{e}_i\}$ translates itself (in the direction $\mathbf{e}_i$) in a single hop to $\{\mathbf{x} + \mathbf{e}_i, \mathbf{x} + 2\mathbf{e}_i\}$, a speed-one maneuver when $p = 2$. When $p = 4$, the serpent performs two consecutive double-hops to go from $\{\mathbf{x}, \mathbf{x} + \mathbf{e}_i, \mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, \mathbf{x} + 2\mathbf{e}_i + \mathbf{e}_j\}$ to $\{\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, \mathbf{x} + 2\mathbf{e}_i + \mathbf{e}_j, \mathbf{x} + 2\mathbf{e}_i + 2\mathbf{e}_j, \mathbf{x} + 3\mathbf{e}_i + 2\mathbf{e}_j\}$, translating itself in the direction $\mathbf{e}_i + \mathbf{e}_j$ in two moves.

We establish the second part of the theorem by proving a series of necessary conditions that must be satisfied by speed-of-light objects.

LEMMA 1. *Every move in a speed-of-light trajectory must simultaneously increase the values of the back border and the front border. Hence a speed-of-light placement X contains a unique piece on border $t(X)$, and a speed-of-light move must "jump" that piece to a point on border $h(X) + 1$.*

Proof. As argued in proving Theorem 2, an m-move trajectory from X to $X + \mathbf{a}$ has speed $\|\mathbf{a}\|/m$, where $m \geq \|\mathbf{a}\|$. Note that $t(X + \mathbf{a}) = t(X) + \|\mathbf{a}\|, h(X + \mathbf{a}) = h(X) + \|\mathbf{a}\|$. We observe (as in the proof of Theorem 2) that the functions t and h cannot increase by more than 1 each move. Hence, in order for $m = \|\mathbf{a}\|$, we must simultaneously increase the values of both borders each move.

LEMMA 2. *Given a speed-of-light placement X and $t(X) \leq M \leq h(X)$, there is at most one occupied point $\mathbf{x} \in X$ with $\|\mathbf{x}\| = M$.*

Proof. Suppose, to the contrary, that more than one piece is situated on border M. By Lemma 1, the first $M - t(X)$ moves of the trajectory involve moving pieces from borders with values less than M to borders with values greater than M, after which our new back border has value M. But then this border has more than one piece, contradicting Lemma 1.

LEMMA 3. *When $p \geq 2$, every move in a speed-of-light trajectory is a jump.*

Proof. Since $p \geq 2$, we have $h(X) > t(X)$ by Lemma 1. Since a shift can not take a back border piece beyond border $t(X) + 1 \leq h(X)$, it can not expand the front border, as required.

Notice that in a speed-of-light trajectory, for a piece on border M to make "forward progress," it must hop over a piece on border $M + 1$ and land on border $M + 2$. It follows from Lemma 3 that every speed-of-light placement X has at least one piece on every border between $t(X)$ and $h(X)$.

Thus, we have

LEMMA 4. *Every speed-of-light placement X must have exactly one piece on each border between border $t(X)$ and border $h(X)$. Consequently, $h(X) = t(X) + p - 1$.*

LEMMA 5. *If X is a speed-of-light placement with $p \geq 2$ pieces, then p must be even.*

Proof. By Lemmas 1 and 4, the first move must jump a piece from border $t(X)$ to border $t(X) + p$. Since a jump changes the border value by an even number, p must be even.

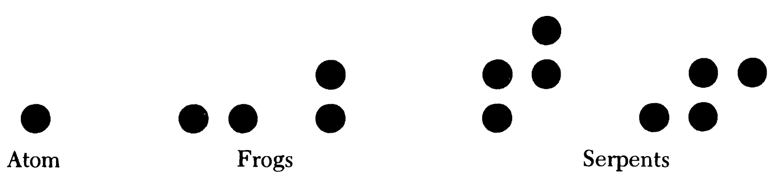

FIGURE 3
Speed-of-light configurations.

Proof of Theorem 3. By Lemma 3, when $p = 2$, all speed-of-light configurations must be of the form $\{\mathbf{x}, \mathbf{x} + \mathbf{e}_i\}$ for some $1 \leq i \leq n$, i.e., the frogs. By Lemma 5, $p \neq 3$. We can restrict our attention to the case where $p \geq 4$. It remains to prove that the only possible remaining speed-of-light configurations are of the serpent variety. Imagine that we have a speed-of-light trajectory that makes one move every second. Suppose the front border piece of our speed-of-light placement X is presently $(t = 0)$ situated at $\mathbf{x} \in \mathbf{Z}^n$, with $\|x\| = M$. We shall focus our attention on only those points in $\mathbf{Z}^n$ that occupy borders of value M or higher. Thus at $t = 0$, all that we see is a single piece, situated at $\mathbf{x}$ (see FIGURE 4).

When $t = 1$, the front border's value has increased to $M + 1$. The new front border piece must have made its final hop over $\mathbf{x}$ to land on the point $\mathbf{x} + \mathbf{e}_i$ for some $1 \leq i \leq n$. See FIGURE 5.

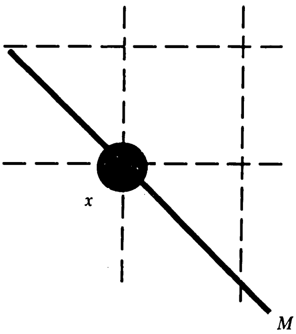

FIGURE 4
What we see when $t = 0$.

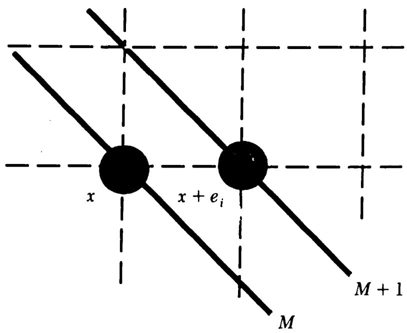

FIGURE 5
What we see when $t = 1$.

When $t = 2$, the front border's value has increased to $M + 2$. The piece that landed there had to make its final hop over $\mathbf{x} + \mathbf{e}_i$. Hence the piece on border $M + 2$ must be situated at $\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j$ for some $1 \leq j \leq n$. Observe that $i \neq j$, for otherwise the piece at $\mathbf{x}$ would have hopped over $\mathbf{x} + \mathbf{e}_i$ to point $\mathbf{x} + 2\mathbf{e}_i$ thus leaving no piece with border value M, contradicting Lemma 4.

Thus, at $t = 2$, we see three pieces, situated at $\mathbf{x}$, $\mathbf{x} + \mathbf{e}_i$, $\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j$, where $1 \leq i \neq j \leq n$. (See FIGURE 6.)

When $t = 3$, a piece is jumped to the new front border $M + 3$ and, since it had to hop over the piece at $\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j$, it must end up at $\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k$ for some $1 \leq k \leq n$. We know that $k \neq j$ by the same argument as $i \neq j$ above. we now show that, in fact, $k = i$. The new front border piece, before it made its final hop over $\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j$ to $\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k$, must have been at the point $\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k$ on border $M + 1$. But how did it get there? It had to hop over the sole piece on border M, situated at $\mathbf{x}$. But this requires $\mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k$ to be a unit vector, which is only possible when $k = i$ or $k = j$. And since $k \neq j$, we have $k = i$. Hence at $t = 3$, we see four pieces, situated at $\mathbf{x}$, $\mathbf{x} + \mathbf{e}_i$, $\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j$, and $\mathbf{x} + 2\mathbf{e}_i + \mathbf{e}_j$, as in FIGURE 7.

When $t = 4$, the back border piece, wherever it is, jumps to border $M + 4$, landing on $\mathbf{x} + 2\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k$ for some $1 \leq k \leq n$. By the argument in the preceding paragraph, the hop over the piece at $\mathbf{x} + 2\mathbf{e}_i + \mathbf{e}_j$ had to come from the point $\mathbf{x} + 2\mathbf{e}_i$ reached by a hop over $\mathbf{x} + \mathbf{e}_i$, and so that hop came from $\mathbf{x}$. Hence the jump originated at $\mathbf{x}$. Thus $p = 4$, and our current configuration (as well as our original one) must have been a serpent.

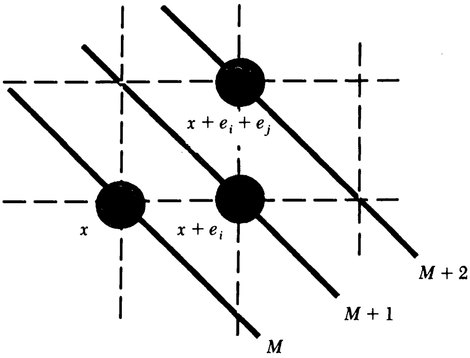

FIGURE 6
What we see when $t = 2$.

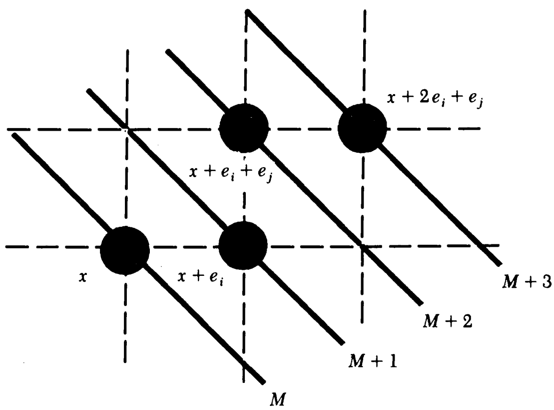

FIGURE 7
What we see when $t = 3$.

Returning to the problem at the outset of the article, we see that the 35-move trajectory must be optimal because the original configuration is translated a distance of 34 units, and the square configuration is not a speed-of-light configuration.

As a consequence of this theorem, we see that no speed-of-light configurations exist when the number of pieces is three or greater than four. In these cases, it is easy to create configurations that are translatable with speed $2/3$ (e.g., the snake configuration with p pieces). The question of whether the speed $2/3$ is *optimal* for the two- (and higher-) dimensional problem, when $p = 3$ or $p > 4$, remains open. More specifically, it remains unknown how to optimally translate six or nine pieces (arranged in a triangle or square) from the lower left-hand corner to the upper right-hand corner of the Go board.

Acknowledgments. We dedicate this paper to the Summer Studies in Mathematics program at Hampshire College, directed by Professor David Kelly, where this research took place. HCSSiM is a mathematics program for high-ability high school students. During the 1986 program, Benjamin led a week-long, open-ended course, where students explored properties of jumping games. Two students, Auslander and Wilkerson, proved the two-dimensional version of this theorem and presented it at an MAA section meeting in Baltimore in 1986. We also thank Professor Alan J. Goldman of Johns Hopkins University for suggesting this problem to us, Greg Levin for assistance with the figures, and the anonymous referees for many valuable suggestions.

REFERENCE

Readers may also enjoy *Wheels, Life and Other Mathematical Amusements* by Martin Gardner (W. H. Freeman and Co., 1983), particularly the chapters analyzing the games of "Halma" and "Life."