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What's Best?

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details to the reader. The first Hankel determinant is classical, the second is due to Radoux [**3**].

Theorem 4. *Let $D(n)$ denote the number of derangements of an n-element set, that is, the number of permutations without fixed points. The Hankel determinants of order $n+1$ of the matrices $[(i+j)!]$ and $[D(i+j)]$ are given by*

$$\det((i+j)!)_{0\le i,j\le n} = \det(D(i+j))_{0\le i,j\le n} = \left(\prod_{i=0}^{n} i!\right)^2.$$

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What's Best?

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Consider the following game. You are given 9 tokens and a weighted coin with heads probability $2/3$. You allocate the tokens so that some are assigned to heads and the rest assigned to tails. (See Figure 1 for two examples.) Next the coin is flipped, and if heads (tails) is flipped, then one token assigned to heads (tails) is removed. The coin is flipped until all tokens are removed. The challenge is to determine how to allocate the tokens so that they are removed as quickly as possible. Which of the allocations in Figure 1 is best?

Surprisingly, the answer depends on what you mean by best. Allocation $A = (6, 3)$ is attractive since it is proportional to the probabilities. Nonetheless, if you wish to minimize the average number of flips needed to remove all tokens, then $B = (7, 2)$ turns out to be best. On the other hand, when allocations A and B compete against each other using the same coin, you should bet on A to finish

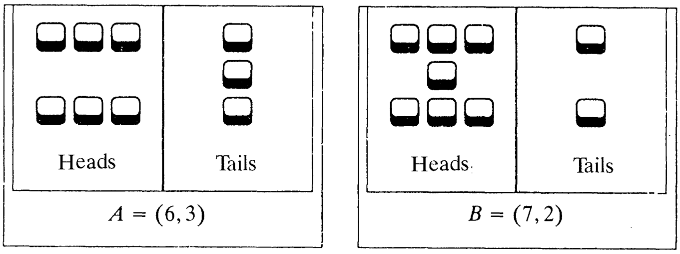

Figure 1. Using a coin with heads probability $2/3$, which card is cleared faster?

before B. In fact, when competing with any other allocation C of 9 tokens, A is favored to finish before C.

Why does A beat B more often than not? Let X denote the number of heads obtained by the coin (whose outcomes apply to both players) after the first 9 flips. If $X \leq 6$, then A must eventually win since both players have removed all of their tail tokens, and A has one fewer head token remaining. By the histogram of probabilities in Figure 2, we see $P(X \leq 6) = 0.6228$. The same argument shows that A is also favored to beat any allocation of 9 tokens with more than 7 heads. Likewise, allocation A will have a winning record against allocations of 9 tokens with fewer than 6 heads since A will win all contests whenever $X \geq 6$, which has probability 0.6503.

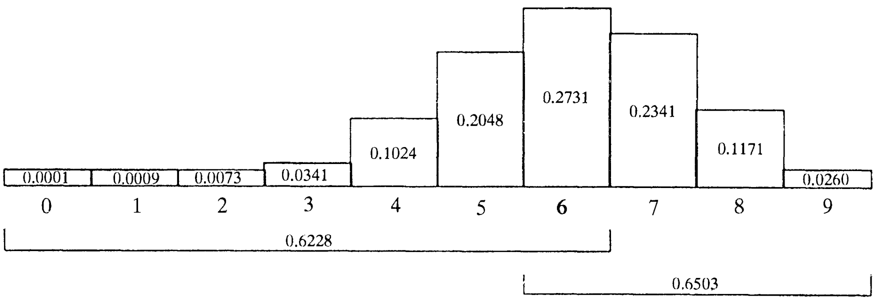

Figure 2. The median of the BINOMIAL$(9, \frac{2}{3})$ distribution.

What makes $(6, 3)$ best in competition is that 6 is the *median* of the binomial distribution with parameters 9 and $2/3$. In general, we have the following:

Theorem 1. *Using a coin with heads probability p, the best (heads, tails) allocation of t tokens is $(m, t - m)$, where m is the median* of the BINOMIAL(t, p) *distribution. It is best in the sense that it is favored to finish sooner than any other allocation of t tokens.*

Thus when faced with an opponent, the optimal number of heads is the .50 percentile of the BINOMIAL(t, p) distribution. However when playing alone, we have

Theorem 2. *Using a coin with heads probability p, the best (heads, tails) allocation of t tokens is $(m, t - m)$, where m is the p-th percentile* of the BINOMIAL(t, p) *distribution. It is best in the sense that it has the smallest expected number of flips among all allocations of t tokens.*

Theorem 2 asserts that the best allocation of 9 tokens is $(7,2)$ since $P(X \leq 6) = 0.6228 \leq 2/3 \leq 0.8569 = P(X \leq 7)$. A recursive calculation shows that on average, allocation $(7,2)$ uses 11.3 flips whereas $(6,3)$ uses 11.5 flips to clear.

To prove Theorem 2, let $E(n)$ denote the average clearing time of allocation $(n, t-n)$. Although a formula for $E(n)$ can be computed explicitly [**2**], it is not necessary for our proof. Instead, we show that $E(n+1) \geq E(n)$ if and only if n is greater than or equal to the p-th percentile of the BINOMIAL(t, p) distribution.

We compute $E(n+1) - E(n)$ by conditioning on the BINOMIAL(t, p) random variable X, the number of heads obtained in the first t flips of the coin. If $X \leq n$, then both allocations will have their "tails cleaned", and the resulting allocation $(n+1-X, 0)$ will need one more head than $(n-X, 0)$, which takes on average $1/p$ flips longer. On the other hand, if $X > n$, then the resulting allocation $(0, X-n-1)$ will finish on average $1/(1-p)$ flips before $(0, X-n)$. Thus

$$\begin{aligned} E(n+1) - E(n) &= \frac{1}{p} P(X \leq n) - \frac{1}{1-p}[1 - P(X \leq n)] \\ &= \frac{P(X \leq n) - p}{p(1-p)}, \end{aligned}$$

and the result follows.

In [**3**] it is shown that the median of the BINOMIAL(t, p) distribution can not differ from the mean tp by more than $\ln 2$. In particular, when tp is an integer, the mean, median, and mode are all equal. When t is large, the central limit theorem asserts that the p-th percentile is approximately $tp + z_p\sqrt{tp(1-p)}$, where z_p is the p-th percentile of the standard normal distribution. Thus, no matter what definition of "best" you prefer, in the long run, the best allocation will be approximately $(tp, t(1-p))$.

Suppose the game is played with a biased *die* with s sides instead of a biased coin? Although some necessary conditions are given in [**2**], there is no known closed form for the allocations with minimum average clearing time. To make matters more interesting [**1**], many values of t and $(p_1, p_2, \ldots, p_s)$ produce non-transitive situations in which allocations A, B, and C are favored against all *other* allocations, but A is favored against B, B is favored against C, and C is favored against A. What's best in this situation? Ask your opponent to allocate the tokens first!

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