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Positive solution curves of semipositone problems with concave nonlinearities*

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We consider the positive solutions to the semilinear equation:

$$-\Delta u(x)=\lambda f(u(x)) \quad \text{for } x\in\Omega,$$
$$u(x)=0 \quad \text{for } x\in\partial\Omega,$$

where Ω denotes a smooth bounded region in $\mathbb{R}^N (N>1)$ and $\lambda>0$. Here $f:[0,\infty)\to\mathbb{R}$ is assumed to be monotonically increasing, concave and such that $f(0)<0$ (semipositone). Assuming that $f'(\infty)\equiv\lim_{t\to\infty}f'(t)>0$, we establish the stability and uniqueness of large positive solutions in terms of $(f(t)/t)'$. When Ω is a ball, we determine the exact number of positive solutions for each $\lambda>0$. We also obtain the geometry of the branches of positive solutions completely and establish how they evolve. This work extends and complements that of [3, 7] where $f'(\infty)\leqq 0$.

1. Introduction

We consider the positive solutions to the semilinear equation:

$$-\Delta u(x)=\lambda f(u(x)) \quad \text{for } x\in\Omega, \tag{1.1}$$

$$u(x)=0 \quad \text{for } x\in\partial\Omega, \tag{1.2}$$

where Ω denotes a smooth bounded region in $\mathbb{R}^N(N>1)$ and $\lambda>0$. Here $f:[0,\infty)\to\mathbb{R}$ is assumed to be monotonically increasing, concave and such that

$$f(0)<0 \text{ (semipositone)}, \quad f(t)>0 \quad \text{for some } t>0. \tag{1.3}$$

We define F by $F(t)=\int_0^t f(s)\,ds$ and let β and θ denote the unique positive zeros of f and F, respectively.

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It is easy to show that either

$$(f(t)/t)' > 0 \quad \text{for all } t > 0, \tag{1.4}$$

or

$$\begin{gathered} \text{there exists an } \eta > 0 \quad \text{such that } \eta f'(\eta) = f(\eta); \quad (f(t)/t)' > 0 \\ \text{for all } t \in (0, \eta) \quad \text{and} \quad (f(t)/t)' < 0 \quad \text{for all } t \in (\eta, \infty), \end{gathered} \tag{1.5}$$

and, moreover, that $f'(\infty) := \lim_{t \to \infty} f'(t) > 0$ in the case where (1.4) holds. Therefore if $f'(\infty) = 0$, then f must satisfy (1.5). Moreover, (1.4)–(1.5) are determined by whether or not $h(t) \equiv f(t) - tf'(\infty)$ is negative. More precisely, we have:

REMARK 1.1. Condition (1.4) holds if and only if $h(t) < 0$ for all $t \in [0, \infty)$. Conversely, (1.5) holds if and only if h has a positive zero.

Here we consider the case where

$$f'(\infty) > 0. \tag{1.6}$$

The case when Ω is a ball and $f'(\infty) = 0$ has been completely classified in [**3**] (see Theorem 1.4). Also, the case when Ω is a ball, f is no longer monotone and $f'(\infty) \leqq 0$ has been completely classified in [**7**] (see Remark 1.5). For results when Ω is a general domain and $f'(\infty) = 0$, see [**6**].

NOTATION. Let μ_i, $i = 1, 2, \ldots$ denote the eigenvalues of

$$-\Delta\varphi(x) = \mu_i \varphi(x) \quad \text{for } x \in \Omega, \tag{1.7}$$

$$\varphi(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{1.8}$$

Our main results are:

THEOREM 1.2. *If* (1.6) *holds, then*:

(i) *for* λ*'s near* 0, (1.1)–(1.2) *has no positive solution*;

(ii) *if* (1.4) *holds, then positive solutions to* (1.1)–(1.2) *are unstable*;

(iii) *for* λ *in bounded intervals, large positive solutions to* (1.1)–(1.2) *are unique. If* (1.5) *holds, such solutions are stable.*

THEOREM 1.3. *Let* Ω *denote the unit ball centred at the origin in* $\mathbb{R}^N$. *Assume that* (1.6) *holds. Then there exist* $0 < \lambda_1 < \lambda_2 < \infty$ *such that*:

(i) *if* (1.4) *holds then* $\lambda_1 = \mu_1/f'(\infty)$ *and* (1.1)–(1.2) *has a positive solution if and only if* $\lambda_1 < \lambda \leqq \lambda_2$. *Moroever, such a solution is unstable* (*see Fig.* 1.1);

(ii) *if* (1.5) *holds, then* $\lambda_1 < \mu_1/f'(\infty)$ *and for* $\lambda = \lambda_1$ *the problem* (1.1)–(1.2) *has exactly one positive solution; it is unstable. If* $\mu_1/f'(\infty) < \lambda_2$, *then for* $\lambda \in (\lambda_1, \mu_1/f'(\infty))$ *the problem* (1.1)–(1.2) *has exactly one stable and one unstable positive solution. For* $\lambda \in [\mu_1/f'(\infty), \lambda_2]$, *the above problem has exactly one positive solution and it is unstable* (*see Fig.* 3.1). *If* $\mu_1/f'(\infty) \geqq \lambda_2$, *then for* $\lambda \in (\lambda_1, \lambda_2]$ *the problem* (1.1)–(1.2) *has exactly two positive solutions; one stable and one unstable. For* $\lambda \in (\lambda_2, \mu_1/f'(\infty))$, *the problem* (1.1)–(1.2) *has exactly one positive solution and it is stable* (*see Fig.* 3.1).

The following theorem and remark are from [**3, 7**] and we include them here for the sake of completeness.

THEOREM 1.4. *Assume that* $f'(\infty) = 0$. *Then there exist* $0 < \lambda_1 < \lambda_2 < \infty$ *such that for*

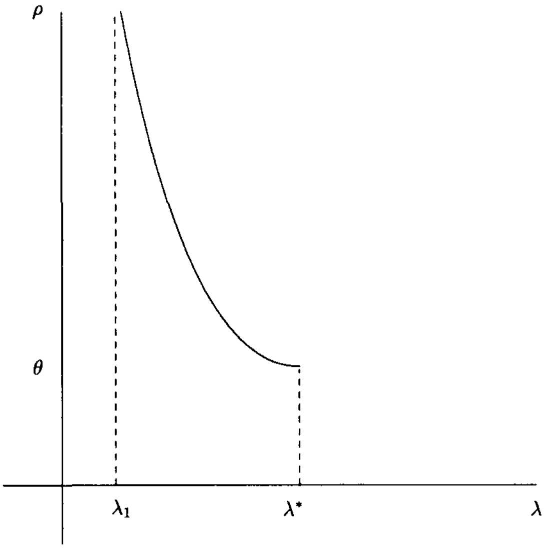

Figure 1.1. $f'(\infty) > 0$ and (1.4) holds.

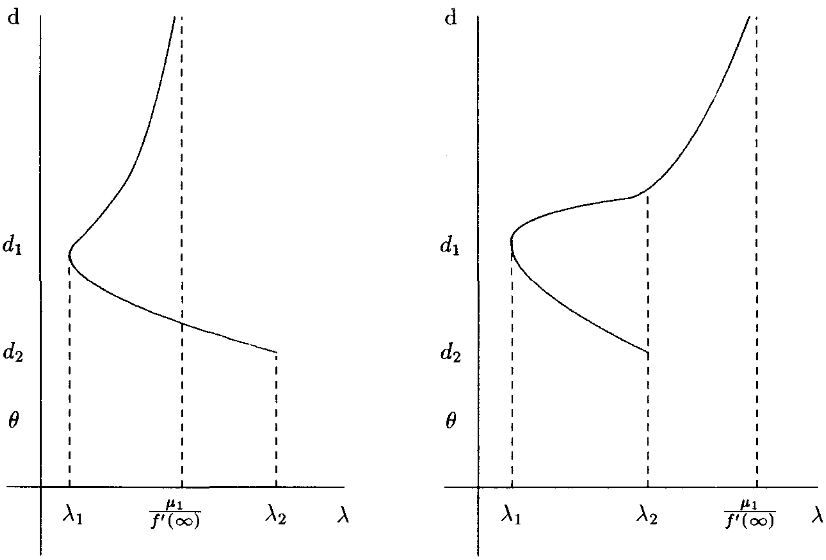

Figure 3.1. $f'(\infty) > 0$ and (1.5) holds.

$\lambda \in (\lambda_1, \lambda_2]$ the problem (1.1)–(1.2) *has exactly two positive solutions, one stable and one unstable. For $\lambda = \lambda_1$, the problem* (1.1)–(1.2) *has exactly one positive solution and it is stable* (*see Fig.* 3.2).

REMARK 1.5. Theorem 1.4 also holds if f is assumed to be concave with $f(0) < 0$ and $f(\gamma) = 0$ for some $\gamma > \theta$ (see [7]).

REMARK 1.6. The author of [**9**], while studying classes of concave nonlinearities, in

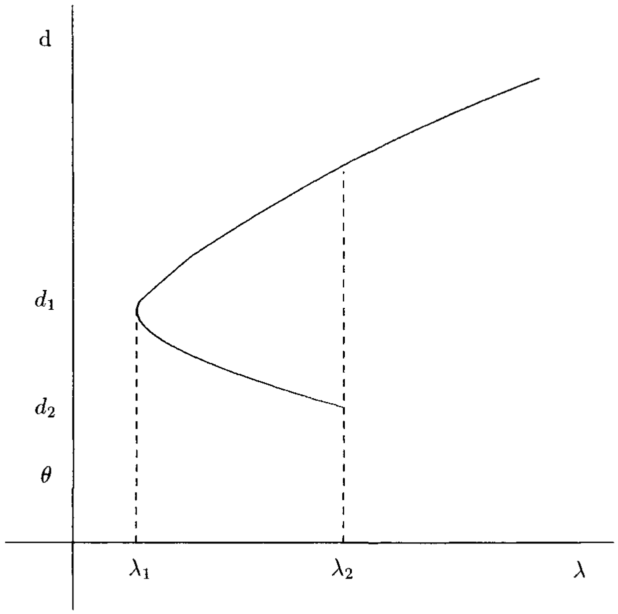

Figure 3.2. $f'(\infty) = 0$.

the case when $N = 1$, discusses the possibility of situations where there exist $0 < \lambda^* < \lambda^{**}$ such that for $0 < \lambda < \lambda^*$ there are no positive solutions and for $\lambda^* \leqq \lambda < \lambda^{**}$ there is a unique positive solution. Our results here prove that this does not occur. For proofs of the corresponding results for the case $N = 1$, see [**5**].

Our proofs use eigenvalue comparison arguments and bifurcation analysis. For other results about solutions of (1.1)–(1.2), the reader is referred to [**2**, **7**]. In order to prove Theorem 1.2, we use the fact that (1.1)–(1.2) has large positive solutions in arbitrary bounded regions Ω in $\mathbb{R}^N$. Our methods use the properties derived from bifurcation from infinity (see [**2**]).

2. Evolution of positive solution curves

From Theorems 1.2, 1.3 and 1.4, we now can deduce the evolution of the bifurcation curves as the nonlinearity changes from satisfying (1.4) to satisfying (1.5). For example, consider $f(s, t) = (1 - s)f_1(t) + sf_2(t)$ with f_1 satisfying (1.4) and f_2 satisfying (1.5). If $f_2'(\infty) > 0$, then the positive solution curves evolve from Figure 1.1 to Figure 3.1 as s varies from 0 to 1. Moreover, if $f_2'(\infty) = 0$, then the curves evolve from Figure 1.1 to Figure 3.2, passing through Figure 3.1.

3. Existence and stability of positive solutions in general regions

First we note that if f satisfies (1.6), then, by [**2**, Theorem 1], the problem (1.1)–(1.2) has positive solutions (λ, u) near $(\mu_1/f'(\infty), \infty)$. Let $\varphi_1 > 0$ be the eigenfunction corresponding to μ_1 in (1.7) with $\|\varphi_1\|_\infty = 1$. Since large solutions to (1.1)–(1.2) bifurcate from $(\mu_1/f'(\infty), \infty)$ and $\mu_1/f'(\infty)$ is a simple eigenvalue, by the results of [**8**] we see that there exists an $\varepsilon_* > 0$, $K_* > 0$ and $d_* > 0$ and continuous functions $\Lambda: [K_*, \infty) \to \mathbb{R}$ and $w: [K_*, \infty) \to \{u \in C^1(\Omega): \int_\Omega u\varphi_1 = 0\}$ such that if $\|u\|_\infty > d_*$, $u > 0$ and $|\lambda - \mu_1/f'(\infty)| < \varepsilon_*$ with (λ, u) is a solution to (1.1)–(1.2), then

$$u = K\varphi_1 + w(K), \quad \lambda = \Lambda(K), \tag{3.1}$$

for some $K \in [K_*, \infty)$. Also, $\Lambda(K) \to \mu_1/f'(\infty)$ as $K \to \infty$ and $\|w\|_{C^1} = o(K)$. That is, $\|u\|_\infty/K = O(1)$. Note that $\partial\varphi_1/\partial v > 0$ on $\partial\Omega$, where v denotes the inward unit normal vector. By the compactness of $\partial\Omega$, there exists a $C_1 > 0$ such that

$$\varphi_1(x) \geqq C_1 \operatorname{dist}(x, \partial\Omega), \tag{3.2}$$

for all $x \in \Omega$. On the other hand, since $\|w(K)\|_{C^1} = o(K)$ as $K \to \infty$, there exists a $K_0 \geqq K_*$ such that

$$|w(x)| \leqq (C_1/2) \operatorname{dist}(x, \partial\Omega)K,$$

for $K > K_0$. So, for $K > K_0$, we obtain

$$u(x) \geqq (C_1/2) \operatorname{dist}(x, \partial\Omega)K, \tag{3.3}$$

for all $x \in \Omega$.

REMARK 3.1. From (3.3), it follows that $\|u\|_\infty \to \infty$ uniformly on compact subsets of Ω as $K \to \infty$.

In the following lemmas, we determine the stability and the uniqueness of the positive solutions to (1.1)–(1.2) as a function of (1.4) and (1.5).

LEMMA 3.2. *If* (1.6) *holds, then there exists* $\varepsilon^* \in (0, \varepsilon_*)$ *and* $K^* > K_*$ *such that if* $|\lambda - \mu_1/f'(\infty)| < \varepsilon^*$, *then* (1.1)–(1.2) *has at most one positive solution with* $\|u\|_\infty > K^*$. *In particular, the function* Λ *in* (3.1) *is one-to-one on* $[K^*, \infty)$.

Proof. Let $u_1 = K_1\varphi_1 + w(K_1)$ and $u_2 = K_2\varphi_1 + w(K_2)$ be two positive solutions to (1.1)–(1.2) with K_1, K_2 large and for some λ close to $\mu_1/f'(\infty)$. Without loss of generality, we may assume that $K_1 > K_2$. We write $w_i = w(K_i)$, $i = 1, 2$. From (1.1), we have

$$-\Delta(u_1 - u_2) = \lambda(f(u_1) - f(u_2)). \tag{3.4}$$

Multiplying (3.4) by $w_1 - w_2$ and integrating by parts, we have

$$\int_\Omega |\nabla(w_1 - w_2)|^2 = \lambda \int_\Omega [f'(\infty)(w_1 - w_2) + (h(u_1) - h(u_2))](w_1 - w_2), \tag{3.5}$$

where $h(t) \equiv f(t) - tf'(\infty)$ (see Remark 1.1). From (3.3), we have

$$u_i(x) \geqq (C_1/2)K_i \operatorname{dist}(x, \partial\Omega), \tag{3.6}$$

for all $x \in \Omega$. By the Sobolev Embedding Theorem (see [**1**]), there exists a positive constant $C(\Omega) \equiv C_2$ such that

$$\left(\int_\Omega y^{2N/(N-2)}\right)^{(N-2)/2N} \leqq C_2\left(\int_\Omega |\nabla y|^2\right)^{\frac{1}{2}} \quad \text{for any } y \in H_0^1(\Omega). \tag{3.7}$$

Also, by the variational characterisation of eigenvalues to $-\Delta$, we know that

$$\int_\Omega |\nabla y|^2 \geqq \mu_2 \int_\Omega y^2, \tag{3.8}$$

for any $y \in H_0^1(\Omega)$ with y orthogonal to φ_1. Let $\varepsilon > 0$ be such that

$$3\varepsilon\lambda < \mu_2 - \mu_1 \quad \text{for} \left|\lambda - \frac{\mu_1}{f'(\infty)}\right| < \varepsilon_*.$$

Later we will restrict ε further. Let $M = M(\varepsilon)$ be such that if $t \geqq M$ then $h'(t) < \varepsilon$. By (3.3), there exists $C_3 > 0$ such that, for K sufficiently large,

$$|\{x \in \Omega : u(x) \leqq M\}| \leqq C_3/K. \tag{3.9}$$

Here $|\cdot|$ denotes the Lebesgue measure. Let $\varepsilon^* > 0$ and $K^* > K_*$ be such that if

$$\left|\lambda - \frac{\mu_1}{f'(\infty)}\right| < \varepsilon^*,$$

then

$$\frac{|\lambda f'(\infty) - \mu_1|}{\mu_2} + h'(0)C_2^2(2C_3/K^*)^{4/N} < \frac{\mu_2 - \mu_1}{3}. \tag{3.10}$$

Let $\Omega_1 = \{x \in \Omega : u_1(x) \leqq M, \text{ or } u_2 \geqq M\}$ and $\Omega_2 = \Omega \backslash \Omega_1$. Thus, by the Mean Value Theorem,

$$\begin{aligned}
&\int_{\Omega_1} (h(u_1) - h(u_2))(w_1 - w_2) \\
&\quad = \int_{\Omega_1} h'(\zeta)(u_1 - u_2)(w_1 - w_2) \\
&\quad \leqq \int_{\Omega_1} h'(0)|u_1 - u_2|\,|w_1 - w_2| \\
&\quad \leqq h'(0)\|w_1 - w_2\|_{L^2(\Omega_1)}((K_1 - K_2)|\Omega_1|^{\frac{1}{2}} + \|w_1 - w_2\|_{L^2(\Omega_1)}) \\
&\quad \leqq h'(0)\|w_1 - w_2\|_{L^{2N/(N-2)}(\Omega_1)}|\Omega_1|^{2/N} \\
&\qquad \times ((K_1 - K_2)|\Omega|^{\frac{1}{2}} + \|w_1 - w_2\|_{L^{2N/(N-2)}(\Omega_1)}|\Omega_1|^{2/N}).
\end{aligned} \tag{3.11}$$

On the other hand, by (3.9) for K_1, K_2 sufficiently large, we have

$$|\Omega_1| \leqq 2C_3(1/K_2). \tag{3.12}$$

Also,

$$\begin{aligned}
&\int_{\Omega_2} (h(u_1) - h(u_2))(w_1 - w_2) \\
&\quad \leqq \int_{\Omega_2} \varepsilon|u_1 - u_2|\,|w_1 - w_2| \\
&\quad \leqq \varepsilon\|w_1 - w_2\|_{L^2}((K_1 - K_2)|\Omega|^{\frac{1}{2}} + \|w_1 - w_2\|_{L^2})
\end{aligned}$$

$$\leqq \frac{\varepsilon}{\sqrt{\mu_2}} \|w_1 - w_2\|_{H^1}((K_1 - K_2)|\Omega|^{\frac{1}{2}} + (1/\sqrt{\mu_2})\|w_1 - w_2\|_{H^1}) \tag{3.13}$$

in view of (3.8). From (3.5), (3.7) and (3.11)–(3.13), we obtain

$$\left(1 - \frac{\lambda f'(\infty)}{\mu_2} - \lambda h'(0)C_2^2|\Omega_1|^{4/N} - \frac{\lambda\varepsilon}{\mu_2}\right)\|w_1 - w_2\|_{H^1}$$

$$\leqq \frac{\varepsilon}{\sqrt{\mu_2}}|K_1 - K_2|\,|\Omega|^{\frac{1}{2}} + C_2\lambda h'(0)|K_1 - K_2|\,|\Omega|^{\frac{1}{2}}|\Omega_1|^{2/N}. \tag{3.14}$$

By (3.7), (3.10) and the assumption that $3\varepsilon\lambda < \mu_2 - \mu_1$, we conclude that

$$\|w_1 - w_2\|_{L^{2N/(N-2)}} \leqq O((1/K_2)^{2/N} + \varepsilon)(K_1 - K_2), \tag{3.15}$$

as $K_1 \to \infty$ and $K_2 \to \infty$. Let $P: L^2(\Omega) \to L^2(\Omega)$ denote the orthogonal projection onto $\{u \in L^2(\Omega): \int_\Omega u\varphi_1 = 0\}$. Then from (1.1) we have

$$-\Delta(w_1 - w_2) = \lambda f'(\infty)(w_1 - w_2) + P(h(u_1) - h(u_2)). \tag{3.16}$$

Multiplying (3.4) by φ_1 and integrating by parts, we obtain that

$$\int_\Omega (h(u_1) - h(u_2))\varphi_1 = O(e^*)(K_1 - K_2). \tag{3.17}$$

Consider

$$\left(\int_\Omega [(h(u_1) - h(u_2))]^{2N/(N-2)}\right)^{(N-2)/2N}$$

$$\leqq \left(\int_{\Omega_1} [(h(u_1) - h(u_2))]^{2N/(N-2)} + \int_{\Omega_2} [(h(u_1) - h(u_2))]^{2N/(N-2)}\right)^{(N-2)/2N}$$

$$\leqq h'(0)\left(\int_{\Omega_1} [(K_1 - K_2)\varphi_1 + |w_1 - w_2|]^{2N/(N-2)}\right)^{(N-2)/2N}$$

$$+ \varepsilon\left(\int_{\Omega_2} [(K_1 - K_2)\varphi_1 + |w_1 - w_2|]^{2N/(N-2)}\right)^{(N-2)/2N}$$

$$\leqq (h'(0)|\Omega_1|^{(N-2)/2N} + \varepsilon|\Omega|^{(N-2)/2N})(K_1 - K_2) + (h'(0) + \varepsilon)\|w_1 - w_2\|_{L^{2N/(N-2)}}. \tag{3.18}$$

From (3.15)–(3.18) and by *a priori* estimates for solutions to elliptic boundary value problems, we obtain

$$\|w_1 - w_2\|_{H^{2,2N/(N-2)}} = (K_1 - K_2)O((1/K_2)^{2/N} + \varepsilon + (1/K_2)^{(N-2)/2N}).$$

Now, using a boot-strap argument and the Sobolev Embedding Theorem, this gives

$$\|w_1 - w_2\|_{C^1(\Omega)} = (K_1 - K_2)O((1/K_2)^{2/N} + \varepsilon + (1/K_2)^{(N-2)/2N}). \tag{3.19}$$

Thus there exists an ε^* and K^* such that if, for some λ with $|\lambda - \mu_1/f'(\infty)| < \varepsilon^*$, we have $u_1 = K_1\varphi_1 + w(K_1)$ and $u_2 = K_2\varphi_1 + w(K_2)$ as two positive solutions to (1.1)–(1.2) with $K_1 > K_2 > K^*$, then $u_1(x) > u_2(x)$ for all $x \in \Omega$ (see (3.3)). Now, we use the concavity of f to arrive at a contradiction. From (3.4) and the Mean Value

Theorem, we get

$$-\Delta(u_1-u_2)=\lambda f'(\zeta)(u_1-u_2), \tag{3.20}$$

where $u_1\geqq\zeta\geqq u_2$. Multiplying this by u_1 and integrating by parts, we obtain

$$0=\int_\Omega (f(u_1)-u_1f'(\zeta))(u_1-u_2)\leqq\int_\Omega (f(u_1)-u_1f'(u_1))(u_1-u_2) \tag{3.21}$$

using the fact that f is concave. If (1.4) holds, (3.21) shows that $u_1=u_2$. On the other hand, if f satisfies (1.5), we let $B_1>0$ be such that, for all $t\geqq\eta+1$,

$$f(t)-tf'(t)\geqq B_1. \tag{3.22}$$

From (3.9), if $A:=\{x:u_2(x)\leqq\eta+1\}$, then $\mu(A)=O(1/K_2)$. Also, from (3.19) we have $u_1(x)-u_2(x)\geqq(K_1-K_2)\operatorname{dist}(x,\partial\Omega)O(1)$. Hence $\int_\Omega u_1-u_2\geqq(K_1-K_2)O(1)$. Thus for K^* sufficiently large,

$$\begin{aligned}
&\int_\Omega [f(u_2)-f'(u_2)u_2](u_1-u_2)\\
&=\int_{\Omega\setminus A}[f(u_2)-f'(u_2)u_2](u_1-u_2)+\int_A [f(u_2)-f'(u_2)u_2](u_1-u_2)\\
&\geqq B_1\int_{\Omega\setminus A}(u_1-u_2)-(K_1-K_2)O(1/K_2^2)\\
&=B_1\int_\Omega (u_1-u_2)-(K_1-K_2)O(1/K_2^2)\geqq(K_1-K_2)O(1),
\end{aligned} \tag{3.23}$$

where we have used that $\int_A(u_1-u_2)\leqq(K_1-K_2)o(1)$. Also, multiplying (3.20) by u_2 and integrating by parts, we get

$$0=\int_\Omega [f(u_2)-u_2f'(\zeta)](u_1-u_2)\geqq\int_\Omega [f(u_2)-u_2f'(u_2)](u_1-u_2), \tag{3.24}$$

which contradicts (3.23). Thus the lemma is proved. □

LEMMA 3.3. *If* (1.4) *holds, then any positive solution to* (1.1)–(1.2) *is unstable. Moreover, there exist a continuous decreasing function* $d:(\mu_1/f'(\infty),(\mu_1/f'(\infty)+\varepsilon^*)\to(K^*,+\infty)$ *such that* $\lim_{\lambda\to\mu_1/f'(\infty)}d(\lambda)=\infty$. *If* $\|u\|_\infty\geqq K^*$ *and* $|\lambda-\mu_1/f'(\infty)|<\varepsilon^*$, *then* $u=u(\cdot,\lambda,d(\lambda))$. *That is, large positive solutions to* (1.1)–(1.2) *are unique for* λ *near* $\mu_1/f'(\infty)$.

Proof. Let u be a positive solution to (1.1)–(1.2) and let ρ_i, $i=1,2,\ldots$ denote eigenvalues of

$$-\Delta\psi(x)=\lambda f'(u(x))\psi(x)+\rho\psi(x)\quad\text{for } x\in\Omega, \tag{3.25}$$

$$\psi(x)=0\quad\text{for } x\in\partial\Omega. \tag{3.26}$$

Let ψ_1 be an eigenfunction corresponding to the smallest eigenvalue ρ_1 and chosen to be positive in Ω. Now multiplying (1.1) by ψ_1 and (3.25) by u, subtracting one from the other and integrating over Ω, we obtain

$$\lambda \int_\Omega [f'(u(x))u(x) - f(u(x))]\psi_1(x)\,dx = -\rho_1 \int_\Omega \psi_1(x)u(x)\,dx.$$

In view of (1.4), we conclude that $\rho_1 < 0$ and hence from the theory of linearised stability u is unstable. Finally, from the results of [2] and Lemma 3.2, the existence of the function d follows. □

LEMMA 3.4. *If* (1.5) *holds, then large positive solutions to* (1.1)–(1.2) *are stable. Moreover, there exists a continuous increasing function* $d:((\mu_1/f'(\infty)-\varepsilon^*, \mu_1/f'(\infty))\to(K^*, +\infty)$ *such that* $\lim_{\lambda\to\mu_1/f'(\infty)} d(\lambda)=\infty$. *If* $\|u\|_\infty \geqq K^*$ *and* $|\lambda-\mu_1/f'(\infty)|<\varepsilon^*$, *then* $u=u(\cdot,\lambda,d(\lambda))$. *That is, large positive solutions to* (1.1)–(1.2) *are unique for* λ *near* $\mu_1/f'(\infty)$.

Proof. Let ρ_1 and $\psi_1>0$ be as in (3.25)–(3.26). Suppose, on the contrary, that $\rho_1 \leqq 0$. Without loss of generality, we may assume that $\int_\Omega \psi_1 = 1$. Thus we have

$$\int_\Omega |\nabla\psi_1|^2 = \lambda \int_\Omega f'(u(x))\psi_1^2 + \rho_1 \int_\Omega \psi_1^2 \leqq \lambda f'(0) \int_\Omega \psi_1^2. \tag{3.27}$$

Since $\psi_1 \in H_0^1(\Omega)$, from (3.11) we have

$$\left(\int_\Omega \psi_1^{2N(N-2)}\right)^{(N-2)/2N} \leqq C\left(\int_\Omega |\nabla\psi_1|^2\right)^{\frac{1}{2}}. \tag{3.28}$$

Since $\psi_1 \in L^{2N/(N-2)}$, by the generalised Hölder inequality we have

$$\left(\int_\Omega \psi_1^2\right)^{\frac{1}{2}} \leqq \left(\int_\Omega \psi_1^{2N/(N-2)}\right)^{a((N-2)/2N)} \left(\int_\Omega \psi_1\right)^{1-a}, \tag{3.29}$$

where $a = N/(N+2)$. Now, from (3.27), (3.28) and (3.29), we obtain

$$\begin{aligned}\left(\int_\Omega \psi_1^{2N/(N-2)}\right)^{(N-2)/2N} &\leqq C(\lambda f'(0))^{\frac{1}{2}}\left(\int_\Omega \psi_1^2\right)^{\frac{1}{2}}\\ &\leqq C(\lambda f'(0))^{\frac{1}{2}}\left(\int_\Omega \psi_1^{2N/(N-2)}\right)^{a((N-2)/2N)},\end{aligned}$$

using the assumption that $\int_\Omega \psi_1 = 1$. Thus we obtain $B = B(\lambda, \Omega, f)$, with

$$\|\psi_1\|_{L^{2N/(N-2)}} \leqq B. \tag{3.30}$$

Now, using (1.1) and (3.25), we obtain

$$\lambda \int_\Omega [f(u(x)) - f'(u(x))u(x)]\psi_1(x)\,dx = \rho_1 \int_\Omega \psi_1(x)u(x)\,dx. \tag{3.31}$$

Let $\eta_1 > \eta$. Since $(f(t)/t)' < 0$ for $t > \eta$ and f is concave, there exists $B_1 > 0$ such that, for all $t \geqq \eta_1$, we have

$$f(t) - tf'(t) \geqq B_1. \tag{3.32}$$

From the bifurcation properties we have, if $A := \{x : u(x) \leqq \eta_1\}$, then $\mu(A) =$

$O(1/\|u\|_\infty)$ as $\|u\|_\infty \to \infty$ (see (3.4)). Thus

$$\int_A [f(u)-f'(u)u]\psi_1 \leqq K \int_A \psi_1$$
$$\leqq \left(\int_\Omega \psi_1^{2N/(N-2)}\right)^{a((N-2)/2N)} \left(\int_\Omega \psi_1\right)^{1-a} \tag{3.33}$$

and hence $\to 0$ as $\|u\| \to \infty$ (using the boundedness in (3.30)). Now, in view of (3.32) and (3.33), we get

$$\begin{aligned}\rho_1 \int_\Omega u\psi_1 &= \int_{\Omega\setminus A} [f(u)-f'(u)u]\psi_1 + \int_A [f(u)-f'(u)u]\psi_1\\ &\geqq B_1 \int_{\Omega\setminus A} \psi_1 - O(1/\|u\|_\infty)\\ &= B_1 \int_\Omega \psi_1 - \int_A \psi_1 - O(1/\|u\|_\infty)\\ &\geqq B_1 - O(1/\|u\|_\infty)\end{aligned}$$

and hence $\rho_1 > 0$ for $\|u\|_\infty$ large enough. Thus follows the stability of large positive solutions. The uniqueness of large positive solutions for λ near $\mu_1/f'(\infty)$ follows from an argument similar to that in Lemma 3.3. □

4. The radial case

When Ω is a ball, positive solutions to (1.1)–(1.2) are known to be radially symmetric. Without loss of generality we may assume Ω to be the unit ball centered at the origin. Thus it suffices to study the equation

$$u'' + ((N-1)/r)u' + \lambda f(u) = 0 \quad \text{for } r \in [0, 1], \tag{4.1}$$

$$u'(0) = 0, \tag{4.2}$$

$$u(1) = 0, \tag{4.3}$$

where $'$ denotes the differentiation with respect to $r = \|x\|$. For $d > 0$, we define $u(\cdot, \lambda, d)$ to be the solution to (4.1), (4.2) with $u(0, \lambda, d) = d$. We shall frequently write u rather than $u(\cdot, \lambda, d)$. It is well known and can be easily shown that if u is a positive solution to (4.1)–(4.3), then $u(0) > \theta$. Let $S = \{(\lambda, u) \in \mathbb{R} \times \mathscr{C}(\bar{\Omega}) : (\lambda, u)$ satisfies (4.1)–(4.3)$\}$. We note that studying the behaviour of S is equivalent to studying $\{(\lambda, d) : u(1, \lambda, d) = 0\}$. This follows from the continuous dependence of solutions to (4.1)–(4.3) on the initial conditions. *We identify S with the latter subset of* $\mathbb{R}^2$. Using a rescaling (see [**4**]) and the uniqueness of the solution to the initial value problem (4.1)–(4.2) when $u(0) = d$, we obtain

$$u(r\rho, \lambda, d) = u(r, \lambda\rho^2, d). \tag{4.4}$$

NOTATION. Let $\hat{\mu}_i$ denote the eigenvalues of the problem:

$$\varphi'' + ((N-1)/r)\phi' + \hat{\mu}\varphi = 0 \quad \text{in } (0, 1), \tag{4.5}$$

$$\varphi'(0) = 0, \quad \varphi(1) = 0. \tag{4.6}$$

Using a comparison argument and the rescaling in (4.4), we obtain the following nonexistence result.

LEMMA 4.1. *If f satisfies* (1.6), *then* (4.1)–(4.3) *does not have positive solutions for* $\lambda < \hat{\mu}_1/f'(0)$ *and for* $\lambda > \hat{\mu}_2/f'(\infty)$.

Proof. Let (λ, d) be such that $u(\cdot, \lambda, d)$ is a positive solution to (4.1)–(4.3). Let $\varphi_1 > 0$ be an eigenfunction to (4.5)–(4.6) corresponding to the smallest eigenvalue $\hat{\mu}_1$. Now multiplying (1.1) by φ_1 and integrating, we obtain

$$\int_\Omega \hat{\mu}_1 \varphi_1 u = \int_\Omega -\Delta\varphi_1 u = \int_\Omega \lambda f(u)\varphi_1 \leqq \int_\Omega \lambda f'(0)\varphi_1 u$$

from (4.5) and using the fact that $f(t) \leqq f'(0)t$ for all $t \geqq 0$. The above inequality is impossible if $\lambda \leqq \hat{\mu}_1/f'(0)$ and hence λ is bounded away from zero. Now, to prove the nonexistence of positive solutions for large λ, we proceed as follows. We extend f to the left of 0 in such a way that $f''(t) \leqq 0$ for all $t \in \mathbb{R}$. Let (λ, d) be such that $u(\cdot, \lambda, d)$ is a positive solution to (4.1)–(4.3). Using (4.4), we can choose $\zeta > \lambda$ such that $u(1, \zeta, d) = \beta$ and $u(\cdot, \zeta, d)$ has exactly two zeros in (0, 1). Thus $v(r) := u(\cdot, \zeta, d) - \beta$ satisfies

$$v'' + ((N-1)/r)v' + \zeta\left(\frac{f(u) - f(\beta)}{u - \beta}\right)v = 0,$$

$$v'(0) = 0, \quad v(1) = 0,$$

and v has exactly one zero in (0, 1). Comparing this with (4.5) for $\hat{\mu} = \hat{\mu}_2$, by Sturmian theory we conclude that there exists an $r_0 \in (0, 1)$ such that

$$\zeta\left(\frac{f(u(r_0)) - f(\beta)}{u(r_0) - \beta}\right) < \hat{\mu}_2. \tag{4.7}$$

Since

$$\frac{f(u(r_0)) - f(\beta)}{u(r_0) - \beta} = f'(a)$$

for some a, $f'(a)$ is bounded below by $f'(\infty)$ (by using the concavity of f). This with (4.7) gives that

$$\lambda < \zeta < \hat{\mu}_2/f'(\infty)$$

and hence the lemma is proved. □

LEMMA 4.2. *Let (λ_0, d_0) be such that $u_0 := u(\cdot, \lambda_0, d_0)$ is a positive solution to* (4.1)–(4.3) *satisfying $u_0'(1) = 0$. If (λ, d) is such that $u(\cdot, \lambda, d)$ is a positive solution to* (4.1)–(4.3), *then $d > d_0$.*

Proof. Let $u(\cdot, \lambda_1, d_1)$ be a positive solution to (4.1)–(4.3). Defining $u_i(r) := u(r/\sqrt{\lambda_i}, \lambda_i, d)$, from (4.5) we infer that

$$u_i'' + ((N-1)/r)u_i' + f(u_i) = 0, \tag{4.8}$$

$$u_i'(0) = 0, \quad u_i(\sqrt{\lambda_i}) = 0, \tag{4.9}$$

for $i = 0, 1$. Let $d_0 > d_1$. We first prove that u_0 and u_1 cannot meet above the β-level. For, let $u_0(r) > u_1(r)$ for $r \in [0, \bar{r})$ and let $u_0(\bar{r}) = u_1(\bar{r}) = a > \beta$. Thus $u_0'(\bar{r}) < u_1'(\bar{r}) < 0$. Since $u_0(r) > u_1(r) > \beta$ for $r \in [0, \bar{r})$, the concavity of f gives

$$(u_1(r) - \beta) f(u_0(r)) \leqq (u_0(r) - \beta) f(u_1(r)) \quad \text{on } [0, \bar{r});$$

that is,

$$(u_1(r) - \beta)(r^{N-1}u_0')' \geqq (u_0(r) - \beta)(r^{N-1}u_1')' \quad \text{on } [0, \bar{r});$$

that is,

$$[(u_1(r) - \beta) r^{N-1} u_0']' \geqq [(u_0(r) - \beta) r^{N-1} u_1']' \quad \text{on } [0, \bar{r}).$$

Integrating this over $(0, \bar{r})$, we get $u_0'(\bar{r}) \geqq u_1'(\bar{r})$, which is a contradiction. Now, let $u_1(r) > u_0(r)$ for $r \in (\bar{r}, \sqrt{\min\{\lambda_0, \lambda_1\}})$ and $u_0(\bar{r}) = u_1(\bar{r}) = a \leqq \beta$. Thus $u_0'(\bar{r}) < u_1'(\bar{r}) < 0$. Multiplying (4.8) by $r^{2N-2}u_i'$ and integrating over $(\bar{r}, \sqrt{\lambda_i})$, we obtain

$$\sqrt{\lambda_i}^{2N-2}(u_i'(\sqrt{\lambda_i}))^2 - \bar{r}^{2N-2}(u_i'(\bar{r}))^2 = 2\int_0^a r_i^{2N-2}(u) f(u)\, du,$$

where $r_i(u)$ represents the inverse function to u_i (i.e. $r_i : [0, d_i] \to [0, \sqrt{\lambda_i}]$ with $u_i(r_i(u)) = u$ for $0 \leqq u \leqq d_i$, $i = 0, 1$). This, in turn, implies that

$$\sqrt{\lambda_1}^{2N-2}(u_1'(\sqrt{\lambda_1}))^2 + \bar{r}^{2N-2}[(u_0'(\bar{r}))^2 - (u_1'(\bar{r}))^2]$$
$$= 2\int_0^a [r_1^{2N-2}(u) - r_0^{2N-2}(u)] f(u)\, du.$$

(Note that here we have used $u_0'(\sqrt{\lambda_0}) = 0$.) This is a contradiction, since the left side of the above equation is positive and the right side is negative. Thus we conclude that $d_1 > d_0$ and hence the lemma is proved. □

LEMMA 4.3. *If f satisfies* (1.6), *then* (4.1)–(4.3) *has at most one positive solution with* $u'(1, \lambda, d) = 0$.

Proof. Suppose on the contrary that $u(\cdot, \lambda_i, d_i)$ is a positive solution to (4.1)–(4.3) satisfying $u'(1, \lambda_i, d_i) = 0$ for $i = 0, 1$. Then by Lemma 4.2 we get $d_0 < d_1$ and $d_1 < d_0$, which are contradictory. Hence the lemma is proved. □

We denote the derivatives of u with respect to λ and d by u_λ and u_d, respectively. Differentiation of (4.4) with respect to ρ results in

$$u_\lambda(r, \lambda, d) = ru'(r, \lambda, d)/2\lambda. \tag{4.10}$$

Differentiating (4.1) with respect to d, we see that u_d satisfies the corresponding linearised problem:

$$u_d'' + ((N-1)/r)u_d' + \lambda f'(u)u_d = 0, \tag{4.11}$$

$$u_d(0) = 1, \quad u_d'(0) = 0. \tag{4.12}$$

The following result on the zeros of u_d is from [3]. We include it here for the sake of completeness.

LEMMA 4.4. *If u is a positive solution to* (4.1)–(4.3), *then u_d has at most one zero in* $[0, 1]$.

REMARK 4.5. Lemmas 4.1 and 4.2 hold for any monotonically increasing semipositone concave nonlinearity f. Also, note that Lemma 4.2 establishes that if $\lambda_0 < \lambda_1$, then $u_0(r) < u_1(r)$ in $[0, \sqrt{\lambda_0}]$.

5. Proofs of theorems

Proof of Theorem 1.2. The nonexistence of positive solutions to (1.1)–(1.2) follows from Lemma 4.1. The stability and the uniqueness results follow from Lemmas 3.2, 3.3 and 3.4. If $\{\lambda_j\}$ is bounded and $\{(\lambda_j, u_j)\}$ satisfies (1.1)–(1.2) with $u_j > 0$ in Ω and $\|u_j\|_\infty \to \infty$, then $\{u_j/\|u_j\|_\infty\}$ converges to a function $v > 0$ that satisfies $-\Delta v = \hat{\lambda} f'(\infty)v$, where $\hat{\lambda}$ is an accumulation point of $\{\lambda_j\}$. Since $\hat{\lambda} f'(\infty) = \mu_1$, $\hat{\lambda}$ is the only accumulation point of $\{\lambda_j\}$, and hence there follows the uniqueness of large positive solutions for λ's in bounded intervals. □

Proof of Theorem 1.3. If (1.4) holds, then from Lemmas 3.2 and 3.3 we obtain that for λ sufficiently close to $\mu_1/f'(\infty)$ there exists a unique unstable positive solution. In fact, there exist $c_1, \varepsilon_1 > 0$ and a continuous decreasing function $\sigma: [c_1, +\infty) \to [\mu_1/f'(\infty), \infty)$ such that $\lim_{d \to \infty} \sigma(d) = \mu_1/f'(\infty)$ and if $\|u\|_\infty \geqq c_1$ and $|\lambda - \mu_1/f'(\infty)| < \varepsilon_1$, then $u = u(\cdot, \sigma(\|u\|_\infty), \|u\|_\infty)$. Let $\Gamma \subset S$ denote the connected component of solutions to (4.1)–(4.3) containing $\{(\sigma(d), d): d \in [c_1, +\infty)\}$. Let $d_2 \equiv \inf\{c: \sigma$ can be extended as a continuous function from $[c, +\infty) \to (\mu_1/f'(\infty), \infty)$ such that $(\sigma(d), d) \subset \Gamma\}$. Note that $d_2 \geqq \theta$. We define $\lambda_2 \equiv \sup\{\sigma(d): d > d_2\}$. By Lemma 4.1, $\lambda_2 < \infty$. Also, $u_\lambda(1, \lambda_2, d_2) = 0$. For, otherwise, from (4.10) we have $u_\lambda(1, \lambda_2, d_2) < 0$, which implies that σ can be extended to the left of d_2, a contradiction to the definition of d_2. Now, using Lemmas 3.3 and 3.4, we have $u_d(1) < 0$ and hence $\sigma'(d) < 0$. Thus, for any $\lambda \in [\mu_1/f'(\infty), \lambda_2]$, there exists a unique d such that $(\lambda, d) \in \Gamma$. From the uniqueness of degenerate positive solution (see Lemma 4.3) we conclude that $\Gamma \equiv S$. This compltes the proof of (i).

If (1.5) holds, then from Lemmas 3.2 and 3.4 we obtain that for λ sufficiently close to $\mu_1/f'(\infty)$ there exists a unique stable positive solution. In fact, there exist $c_1, \varepsilon_1 > 0$ and a continuous increasing function $\sigma: [c_1, +\infty) \to [\mu_1/f'(\infty), \infty)$ such that $\lim_{d \to \infty} \sigma(d) = \mu_1/f'(\infty)$ and if $\|u\|_\infty \geqq c_1$ and $|\lambda - \mu_1/f'(\infty)| < \varepsilon_1$ then $u = u(\cdot, \sigma(\|u\|_\infty), \|u\|_\infty)$. Let $\Gamma \subset S$ denote the connected component of solutions to (4.1)–(4.3) containing $\{(\sigma(d), d): d \in [c_1, +\infty)\}$. Let $d_1 \equiv \inf\{c: \sigma$ can be extended as a continuous function from $[c, +\infty) \to (\mu_1/f'(\infty), \infty)$ such that $(\sigma(d), d) \subset \Gamma\}$. Note that $d_1 \geqq \theta$. We define $\lambda_1 \equiv \inf\{\sigma(d): d > d_1\}$. By Lemma 4.1, $\lambda_1 > 0$. Also, $u_d(1, \lambda_1, d_1) = 0$, for otherwise $u_d(1, \lambda_1, d_1) > 0$ which implies that σ can be extended to $[d_1 - \varepsilon, \infty)$, which contradicts the definition of d_1. By [3, Lemma 2], $u_\lambda(1, \lambda_1, d_1) < 0$ and $u_{dd}(1, \lambda_1, d_1) > 0$. These imply that there is a differentiable function $\Lambda: (d_1 - \varepsilon, d_1 + \varepsilon) \to \mathbb{R}$ such that $u(\cdot, \Lambda(d), d)$ is a solution to (4.1)–(4.2) for any $d \in (d_1 - \varepsilon, d_1 + \varepsilon)$. In addition, $\Lambda'(d_1) = 0$ with $\Lambda''(d_1) > 0$. We define $d_2 = \inf\{c: \Lambda$ can be extended as a continuous function from $[c, d_1) \to \mathbb{R}$ with $(\Lambda(d), d) \subset \Gamma\}$. Note that $d_2 \geqq \theta$. We define $\lambda_2 \equiv \sup\{\Lambda(d): d > d_2\}$. By Lemma 4.1, $\lambda_2 < \infty$. Also,

$u_\lambda(1, \lambda_2, d_2) = 0$. For, otherwise, from (4.10) we have $u_\lambda(1, \lambda_2, d_2) < 0$, which implies that Λ can be extended to the left of d_2, a contradiction to the definition of d_2. Now, using Lemmas 3.4 and 4.4, we have $u_d(1) < 0$ and hence $\Lambda'(d) < 0$. From the uniqueness of degenerate positive solution (see Lemma 4.3) we conclude that $\Gamma \equiv S$. This completes the proof of (ii). □

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