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# Multiple Solutions for a Semilinear Dirichlet Problem

Alfonso Castro
*Harvey Mudd College*

Jorge Cossio
*Universidad Nacional de Colombia*

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# MULTIPLE SOLUTIONS FOR A NONLINEAR DIRICHLET PROBLEM*

ALFONSO CASTRO† AND JORGE COSSIO‡

**Abstract.** The authors prove that a semilinear elliptic boundary value problem has five solutions when the range of the derivative of the nonlinearity includes at least the first two eigenvalues. Extensive use is made of Lyapunov–Schmidt reduction arguments, the mountain pass lemma, and characterizations of the local degree of critical points.

**Key words.** nonlinear elliptic equations, multiplicity of solutions, local degree, mountain pass lemma

**AMS subject classifications.** 35J65, 35J20

**1. Introduction.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, and

$$f'(\infty) = \lim_{|u|\to\infty} \frac{f(u)}{u} \in \mathbb{R}. \tag{1.1}$$

Let $\Omega$ be a smooth bounded region in $\mathbb{R}^n$, and $\Delta$ the Laplacian operator. Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \ldots$ be the eigenvalues of $-\Delta$ with Dirichlet boundary condition in $\Omega$.

The solvability of the boundary value problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has proven to be closely related to the position of the numbers $f'(0)$, $f'(\infty)$ with respect to the spectrum of $-\Delta$. In fact, Castro and Lazer in [11] showed that if the interval $(f'(0), f'(\infty)) \cup (f'(\infty), f'(0))$ contains the eigenvalues $\lambda_k, \ldots, \lambda_j$ and $f'(t) < \lambda_{j+1}$ for all $t \in \mathbb{R}$ then (1.2) has at least three solutions. The proofs in [11] are based on global Lyapunov–Schmidt arguments applied to variational problems. Subsequently Chang (see [12]) approached the same problems using Morse theory, and Hofer (see [14]) obtained the existence of five solutions when $f'(\infty) < \lambda_1$. For other results in the study of this problem we refer the reader to [3], [4], [6], [8], [10], [17], [18], and [19], among others.

Here we prove the following.

THEOREM A. *If $f'(0) < \lambda_1$, $f'(\infty) \in (\lambda_k, \lambda_{k+1})$ with $k \geq 2$, and $f'(t) \leq \gamma < \lambda_{k+1}$, then* (1.2) *has at least five solutions. Moreover, one of the following cases occur*:

(a) *$k$ is even and* (1.2) *has two solutions that change sign.*

(b) *$k$ is even and* (1.2) *has six solutions, three of which are of the same sign.*

(c) *$k$ is odd and* (1.2) *has two solutions that change sign.*

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† Department of Mathematics, University of North Texas, Denton, Texas 76203-5116 (acastro@unt.edu).

‡ Departamento de Matemáticas, Universidad Nacional de Colombia, Apartado Aéreo 3840, Medellín Colombia (jcossio@sigma.eafit.edu.co).

(d) *$k$ is odd and* (1.2) *has three solutions of the same sign.*

The assumption $k \geq 2$ is sharp; Theorem B of [11] gives sufficient conditions for (1.2) to have exactly three solutions when $k = 1$. We prove Theorem A by using Lyapunov–Schmidt arguments to reduce the solvability of (1.2) to a finite-dimensional problem, and then we use degree and index theories applied to critical points. We make intensive use of the fact that the Leray–Schauder degree is invariant under the Lyapunov–Schmidt reduction process. In order to calculate various indices and degrees we prove that in large regions the Leray–Schauder degree of maps arising in problems like (1.2) where $f'$ crosses the first eigenvalue

$$\left( \lim_{u \to -\infty} \frac{f(u)}{u} < \lambda_1 < \lim_{u \to \infty} \frac{f(u)}{u} \right)$$

is equal to zero. We also use "mountain pass arguments" of the Ambrosetti–Rabinowitz type (see [5]).

In §2 we recall the framework that allows studying solutions to (1.2) in terms of variational functionals and the Lyapunov–Schmidt reduction method. In §3 we calculate the index of the trivial solution when the nonlinearity crosses the first eigenvalue, establish the existence of positive and negative solutions, and compute their indices. In §4 we prove Theorem A.

## 2. Preliminaries and notation.

First we state a global version of the Lyapunov–Schmidt method. For the sake of completeness we recall that if $\Phi$ is a functional of class $C^1$ and $u_0$ is a critical point of $\Phi$ then $u_0$ is called of mountain pass type if for every open neighborhood $U$ of $u_0$ $\Phi^{-1}(-\infty, \Phi(u_0)) \cap U \neq \emptyset$ and $\Phi^{-1}(-\infty, \Phi(u_0)) \cap U$ is not path connected.

LEMMA 2.1. *Let $M$ be a real separable Hilbert space. Let $X$ and $Y$ be closed subspaces of $M$ such that $M = X \oplus Y$. Let $j \colon M \to \mathbb{R}$ be a functional of class $C^1$. If there are $m > 0$ and $\alpha > 1$ such that*

$$\langle \nabla j(x+y) - \nabla j(x+y_1), y - y_1 \rangle \geq m \|y - y_1\|^\alpha \quad \textit{for all} \quad x \in X,\ y,\ y_1 \in Y \tag{2.1}$$

*then we have the following.*

(i) *There exists a continuous function $\psi \colon X \to Y$ such that*

$$j(x + \psi(x)) = \min_{y \in Y} j(x+y).$$

*Moreover, $\psi(x)$ is the unique member of $Y$ such that*

$$\langle \nabla j(x + \psi(x)), y \rangle = 0 \quad \textit{for all} \quad y \in Y. \tag{2.2}$$

(ii) *The function $\tilde{j} \colon X \to \mathbb{R}$ defined by $\tilde{j}(x) = j(x + \psi(x))$ is of class $C^1$, and*

$$\langle \nabla \tilde{j}(x), x_1 \rangle = \langle \nabla j(x + \psi(x)), x_1 \rangle \quad \textit{for all} \quad x,\ x_1 \in X. \tag{2.3}$$

(iii) *An element $x \in X$ is a critical point of $\tilde{j}$ if and only if $x + \psi(x)$ is a critical point of $j$.*

(iv) *Let $\dim X < \infty$ and $P$ be the projection onto $X$ across $Y$. Let $S \subset X$ and $\Sigma \subset M$ be open bounded regions such that*

$$\{x + \psi(x); x \in S\} = \Sigma \cap \{x + \psi(x); x \in X\}.$$

*If $\nabla \tilde{j}(x) \neq 0$ for $x \in \partial S$ then*

$$d(\nabla \tilde{j}, S, 0) = d(\nabla j, \Sigma, 0),$$

*where $d$ denotes the Leray–Schauder degree.*

(v) *If $u_0 = x_0 + \psi(x_0)$ is a critical point of mountain pass type of $j$ then $x_0$ is a critical point of mountain pass type of $\tilde{j}$.*

*Proof.* The reader is referred to [9] for the proof of parts (i)–(iii). The proof of part (iv) follows by arguing as in Lemma 2.6 of [16]. Now we proceed with the proof of part (v).

Suppose $x_0$ is not of mountain pass type of $\tilde{j}$. Let $V$ be an open neighborhood of $x_0$ in $X$ such that either $\tilde{j}^{-1}(-\infty, \tilde{j}(x_0)) \cap V$ is empty or path connected. If $\tilde{j}^{-1}(-\infty, \tilde{j}(x_0)) \cap V$ is empty, by part (i) we see that $\{x + y; x \in V, y \in Y\} \cap j^{-1}(-\infty, j(u_0))$ is also empty. Thus $u_0$ is not of mountain pass type for $j$. On the other hand if $\tilde{j}^{-1}(-\infty, \tilde{j}(x_0)) \cap V$ is path connected, letting $W = \{x + y; x \in V, \|y - \psi(x)\| < 1\}$ and using again part (i) it is easily seen that $W \cap j^{-1}(-\infty, j(u_0))$ is also path connected. This concludes the proof of Lemma 2.1. $\square$

For each positive integer $m$ let $\varphi_m$ denote an eigenfunction corresponding to the eigenvalue $\lambda_m$. Let $H$ be the Sobolev space $H_0^1(\Omega)$ which is the completion of the inner product space consisting of real $C^1$ functions having support contained in $\Omega$ with inner product

$$\langle u, v\rangle = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx.$$

As it is well known, the set $\{\varphi_m\}$ can be assumed to be complete and orthonormal in $H$.

We say that $u \in H$ is a weak solution to (1.2) if for every $\varphi \in H$

$$\int_\Omega (\nabla u . \nabla\varphi - f(u)\,\varphi)\, dx = 0$$

By standard regularity for elliptic operators (see [11]) it follows that weak solutions are classical solutions when $f$ is continuous and sublinear, i.e., when $f$ is continuous and there is a positive constant $a$ such that

$$|f(u)| \le a(1 + |u|). \tag{2.4}$$

Let $J\colon H \to \mathbb{R}$ denote the functional defined by

$$J(u) = \int_\Omega \left(\frac{1}{2}\|\nabla u\|^2 - F(u)\right) dx, \tag{2.5}$$

where $F(\xi) = \int_0^\xi f(s)\, ds$. Since $f'(\infty) \in (\lambda_k, \lambda_{k+1})$, $f$ satisfies (2.4). Thus $J \in C^1(H, \mathbb{R})$ (see [19]) and

$$\langle \nabla J(u), \varphi\rangle = \int_\Omega (\nabla u . \nabla\varphi - f(u)\,\varphi)\, dx \quad \text{for} \quad \varphi \in H. \tag{2.6}$$

In particular $u$ is a weak solution of (1.2) if and only if $u$ is a critical point of $J$.

Let $X$ denote the subspace of $H$ spanned by $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$, $Y$ its orthogonal complement, and $J$ the functional defined by (2.5). We claim $J$ satisfies hypothesis (2.1). Indeed, from (2.6) and the mean value theorem

$$\langle \nabla J(x + y) - \nabla J(x + y_1), y - y_1\rangle = \|y - y_1\|^2 - \int_\Omega f'(\xi)(y - y_1)^2. \tag{2.7}$$

Denoting by $\| \quad \|_0$ the usual $L^2(\Omega)$ norm and using that $f'(\xi) \le \gamma < \lambda_{k+1}$, we have

$$\begin{aligned}\langle \nabla J(x+y) - \nabla J(x+y_1), y - y_1\rangle &\ge \|y-y_1\|^2 - \gamma\|y-y_1\|_0^2 \\ &\ge \left(1 - \frac{\gamma}{\lambda_{k+1}}\right)\|y-y_1\|^2,\end{aligned} \tag{2.8}$$

where we have used that $\|z\|^2 \ge \lambda_{k+1}\|z\|_0^2$ for all $z \in Y$. Thus (2.1) holds with $m = 1 - \gamma/(\lambda_{k+1})$ and $\alpha = 2$.

**3. Index of the trivial solution when the nonlinearity crosses the first eigenvalue.** For $\gamma > \lambda_1$ let $p(\gamma) := p$ be the homogeneous function defined by

$$p(x) = \begin{cases} \gamma x & \text{for } x \ge 0, \\ f'(0)x & \text{for } x < 0. \end{cases}$$

Let $P$ be the primitive of $p$ with $P(0) = 0$, and $\pi\colon H \to \mathbb{R}$ be the functional defined by

$$\pi(u) = \int_\Omega \left(\frac{1}{2}\|\nabla u\|^2 - P(u)\right) dx. \tag{3.1}$$

As observed in §2 (see (2.4)) $\pi$ is a functional of class $C^1$, and its critical points are the weak solutions to

$$\begin{cases} \Delta u + p(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Because $f'(0) < \lambda_1$ and the principal eigenvalue of the Laplacian in any subregion of $\Omega$ is bigger than or equal to $\lambda_1$, we see that if $u \ne 0$ is a weak solution to (3.2) then $u$ is a positive eigenfunction. Since this contradicts that $\gamma > \lambda_1$, we conclude that $u = 0$ is the only critical point of $\pi$.

LEMMA 3.1. *If $B$ is a ball in $H$ containing zero then $d(\nabla\pi, B, 0) = 0$.*

*Proof.* By the definition of the Leray–Schauder degree if $Z$ denotes the subspace spanned by $\varphi_1, \varphi_2, \ldots, \varphi_l$ with $l$ big enough

$$d(\nabla\pi, B, 0) = d(P\,\nabla\pi, B \cap Z, 0), \tag{3.3}$$

where $P$ denotes the orthogonal projection onto $Z$. Since $\gamma > \lambda_1$ we see that $h(t) := p(t) - \lambda_1 t > 0$ for $t \ne 0$. Because $\varphi_1$ is in $Z$ we have

$$\begin{aligned}\langle P\,\nabla\pi(x), \varphi_1\rangle &= \langle \nabla\pi(x), \varphi_1\rangle \\ &= \int_\Omega (\nabla x.\nabla\varphi_1 - \lambda_1\, x\,\varphi_1 - h(x)\,\varphi_1)\,dz \\ &= \int_\Omega (-h(x)\,\varphi_1)\,dz < 0 \quad \text{if} \quad x \in Z \cap \partial B,\end{aligned} \tag{3.4}$$

where we have used that $\varphi_1$ is positive in $\Omega$. From (3.4) we have now, for each $s \in [0,1]$ and $x \in Z \cap \partial B$,

$$\langle s\,P\,\nabla\pi(x) + (1-s)(-\varphi_1), \varphi_1\rangle < 0. \tag{3.5}$$

Hence by invariance under homotopy of the Leray–Schauder degree we have

$$d(P\,\nabla\pi, B \cap Z, 0) = d(K, B \cap Z, 0) = 0, \tag{3.6}$$

where $K(x) = -\varphi_1$ for all $x \in Z$. From (3.3) and (3.6) the lemma is proven. □

Let $f^+$ be the function defined by

$$f^+(\xi) = \begin{cases} f(\xi) & \text{if } \xi \geq 0, \\ f'(0)\xi & \text{if } \xi < 0. \end{cases}$$

Let $F^+(\xi) = \int_0^\xi f^+(s)\,ds$, and $J^+ : H \to \mathbb{R}$ be the functional defined by

$$J^+(u) = \int_\Omega \left( \frac{1}{2}\|\nabla u\|^2 - F^+(u) \right) dx. \tag{3.7}$$

Imitating the proof of Corollary 2.23 of [19] it readily follows that $J^+$ satisfies the hypotheses of the mountain pass theorem. Hence $J^+$ has a critical point $u^+$, which by the maximum principle is a positive solution to (1.2). Therefore, by Theorems 1 and 2 of [15], if the set of critical points of $J^+$ is discrete then at least one of them is of mountain pass type and has local degree $-1$. Similar arguments produce either infinitely many negative solutions to (1.2) or a negative solution $u^-$ which is a critical point of mountain pass type and has local degree $-1$.

Let $\gamma = f'(\infty)$ and $\pi$ as in Lemma 3.1. Since $f'(\infty)$ is not an eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions, for $\rho > 0$ big enough and $s \in [0,1]$ the function $s\nabla J^+ + (1-s)\nabla\pi$ has no zero on the sphere centered at 0 with radius $\rho$. Hence by Lemma 3.1 we have

$$d(\nabla J^+,\ B_\rho,\ 0) \ = \ 0 \tag{3.8}$$

for $\rho$ big enough. For future reference we summarize the above discussion into the following lemma.

LEMMA 3.2. *Under the hypotheses of Theorem* A, (1.2) *possesses a positive* (*respectively, a negative*) *solution. If the set of positive* (*respectively, negative*) *solutions is discrete then at least one of them is a critical point of mountain pass type and its local degree is* $-1$.

Since 0 is an isolated local minimum of $J^+$ and $J$ we have

$$d(\nabla J^+,\ B,\ 0) \ = \ 1 \ = \ d(\nabla J,\ B,\ 0), \tag{3.9}$$

where $B$ is a ball centered at zero containing no other critical point (see [2]). Hence if $\Sigma$ is a bounded region containing the positive solutions and no other critical point of $J$ we have

$$\begin{aligned} d(\nabla J,\ \Sigma,\ 0) \ &= \ d(\nabla J^+,\ \Sigma,\ 0) \\ &= \ d(\nabla J^+,\ B_\rho - \overline{B},\ 0) \\ &= \ d(\nabla J,\ B_\rho,\ 0) - d(\nabla J,\ B,\ 0) \\ &= \ -1. \end{aligned} \tag{3.10}$$

Similarly we see that if $\Sigma_1$ is a bounded region containing the negative solutions to (1.2) and no other critical point of $J$ then

$$d(\nabla J,\ \Sigma_1,\ 0) \ = \ -1. \tag{3.11}$$

## 4. Proof of Theorem A.

First, we show that there exists $u_0 \in H$ such that $\nabla J(u_0) = 0$ and, if isolated, then

$$d(\nabla J, V, 0) = (-1)^k \tag{4.1}$$

in any region $V$ containing no other critical point of $J$. In fact, by Lemma 2.1 and (2.8) there exists $\psi\colon X \to Y$ such that

$$J(x+\psi(x)) = \min_{y\in Y} J(x+y).$$

Moreover, $\psi(x)$ is the unique member of $Y$ such that

$$\langle \nabla J(x+\psi(x)), y\rangle = 0 \quad \text{for all} \quad y \in Y, \tag{4.2}$$

the function $\tilde{J}\colon X \to \mathbb{R}$ defined by $\tilde{J}(x) = J(x+\psi(x))$ is of class $C^1$, and

$$\langle \nabla \tilde{J}(x), x_1\rangle = \langle \nabla J(x+\psi(x)), x_1\rangle \quad \text{for all} \quad x, x_1 \in X. \tag{4.3}$$

We now claim that for $x \in X$

$$J(x) \to -\infty \quad \text{as} \quad \|x\| \to \infty. \tag{4.4}$$

Because $f'(\infty) \in (\lambda_k, \lambda_{k+1})$ there exists $b \in \mathbb{R}$ and $\overline{\gamma} > \lambda_k$ such that $F(\xi) \geq (\overline{\gamma}\xi^2/2) + b$. Hence

$$J(x) = \frac{1}{2}\|x\|^2 - \int_\Omega F(x) \leq \frac{1}{2}\|x\|^2 - \frac{\overline{\gamma}}{2}\int_\Omega x^2 - b|\Omega|.$$

Since $\langle x, x\rangle \leq \lambda_k \langle x, x\rangle_0$ for $x \in X$, we obtain

$$J(x) \leq \frac{1}{2}\|x\|^2\left(1 - \frac{\overline{\gamma}}{\lambda_k}\right) - b|\Omega| \longrightarrow -\infty \quad \text{as} \quad \|x\| \to \infty.$$

Because $\tilde{J}(x) \leq J(x)$, (4.4) implies that

$$\tilde{J}(x) \to -\infty \quad \text{as} \quad \|x\| \to \infty. \tag{4.5}$$

Since $\dim X < \infty$ there exists $x_0 \in X$ such that

$$\tilde{J}(x_0) = \max_{x\in X} J(x+\psi(x)).$$

Taking $u_0 = x_0 + \psi(x_0)$ we have (see Lemma 2.1) $\nabla J(u_0) = 0$. Suppose now that $x_0$ is an isolated critical point of $\tilde{J}$, hence $u_0$ is an isolated critical point of $J$. Since $-\tilde{J}$ has a local minimum at $x_0$, taking $W = \{x \in X; x+\psi(x) \in V\}$ then $d(\nabla\tilde{J}, W, 0) = (-1)^k$. Therefore by part (iv) of Lemma 2.1 we have (4.1).

*Suppose $k$ is even.* Let $R$ be large enough so that if $\nabla\tilde{J}(x) = 0$ then $\|x\| < R$. Because $f'(t) \leq \gamma < \lambda_{k+1}$, there exist positive numbers $c_1$ and $c_2$ such that for all $x \in X$ $\|\psi(x)\| \leq c_1 + c_2\|x\|$. Thus if $u = x + y$ is a critical point of $J$ then $\|x\| \leq R$ and $\|y\| \leq c_1 + c_2\|x\|$. Because $-\tilde{J}$ is coercive, $d(\nabla\tilde{J}, B_R, 0) = (-1)^k = 1$. Thus by part (iv) of Lemma 2.1 $d(\nabla J, C, 0) = 1$ where $C = \{x+y; \|x\| < R, \|y\| < c_1 + c_2 R\}$. Suppose that $K$, the set of critical points of $J$, is finite. Let $S_1$, $S_2$ and $S_3$ be disjoint open bounded regions in $H$ such that $\overline{S_1} \cap K = \{0\}$, $\overline{S_2} \cap K$ is the set of positive solutions to (1.2), and $\overline{S_3} \cap K$ is the set of negative solutions to (1.2). By (3.10) and (3.11) we have

$$d(\nabla J, S_2, 0) = d(\nabla J, S_3, 0) = -1. \tag{4.6}$$

If $u_0 = x_0 + \psi(x_0) \notin S_2 \cup S_3$ we let $S_4$ denote an open bounded region disjoint from $\overline{S_1 \cup S_2 \cup S_3}$ such that $\overline{S_4} \cap K = \{u_0\}$. By the excision property of the Leray–Schauder degree we have

$$\begin{aligned} 1 = d(\nabla J, C, 0) &= d(\nabla J, S_1, 0) + d(\nabla J, S_2, 0) + d(\nabla J, S_3, 0) + d(\nabla J, S_4, 0) \\ &\quad + d(\nabla J, C - \overline{(S_1 \cup S_2 \cup S_3 \cup S_4)}, 0) \\ &= 1 - 1 - 1 + 1 + d(\nabla J, C - \overline{(S_1 \cup S_2 \cup S_3 \cup S_4)}, 0). \end{aligned}$$

Thus, by the existence property of the Leray–Schauder degree we see that there exists $u_1 \in C - \overline{(S_1 \cup S_2 \cup S_3 \cup S_4)}$ such that $\nabla J(u_1) = 0$, which proves that (1.2) has at least five solutions. In this case both $u_0$ and $u_1$ change sign.

Suppose now that $u_0 \in S_2 \cup S_3$; without loss of generality we can assume that $u_0 \in S_2$. Let $S_4$ be a neighborhood of $u_0$ such that $\overline{S_4} \cap K = \{u_0\}$. By Lemma 3.2 there exists a critical point of mountain pass type $u_1 \in S_2$ such that if $S_5$ is a neighborhood of $u_1$ containing no other critical point of $J^+$ then $d(\nabla J, S_5, 0) = -1$. Thus

$$\begin{aligned} -1 = d(\nabla J, S_2, 0) &= d(\nabla J, S_4, 0) + d(\nabla J, S_5, 0) + d(\nabla J, S_2 - \overline{S_4 \cup S_5}, 0) \\ &= 1 - 1 + d(\nabla J, S_2 - \overline{S_4 \cup S_5}, 0). \end{aligned}$$

Thus, by the existence property of the Leray–Schauder degree there exists $u_2 \in S_2 - \overline{S_4 \cup S_5}$ with $\nabla J(u_2) = 0$. Finally,

$$\begin{aligned} 1 = d(\nabla J, C, 0) &= d(\nabla J, S_1, 0) + d(\nabla J, S_2, 0) + d(\nabla J, S_3, 0) \\ &\quad + d(\nabla J, C - \overline{(S_1 \cup S_2 \cup S_3)}, 0) \\ &= 1 - 1 - 1 + d(\nabla J, C - \overline{(S_1 \cup S_2 \cup S_3)}, 0). \end{aligned}$$

Thus there exists $u_3 \in C - \overline{(S_1 \cup S_2 \cup S_3)}$ with $\nabla J(u_3) = 0$. Thus the set $\{0, u_0, u_1, u_2, u_3\}$ together with a critical point of $J$ in $S_3$ shows that (1.2) has six solutions. Since $u_3 \notin S_2 \cup S_3$ and $u_0, u_1, u_2 \in S_2$, $u_3$ is a sign changing solution and $u_0, u_1, u_2$ have the same sign. This completes the proof of Theorem A when $k$ is even.

*Suppose $k$ is odd.* Let $S_i$, $i = 1, 2, 3$ be as above. If $u_0 \notin S_2 \cup S_3$ the proof follows very closely that of the case $k$ even; the details are left to the reader. Suppose $u_0 \in S_2 \cup S_3$, say, $u_0 \in S_2$. Because $u_0 > 0$ in $\Omega$ and $\partial u_0/\partial\eta < 0$ in $\partial\Omega$ (here $\partial/\partial\eta$ denotes the outward unit normal derivative), using that $X$ is finite-dimensional and standard regularity theory of elliptic operators it follows that for some $\epsilon > 0$ $x + \psi(x) > 0$ in $\Omega$ if $\|x - x_0\| < \epsilon$. Thus $\tilde{J}$ and $\tilde{J}^+$ coincide in $\{x; \|x - x_0\| < \epsilon\}$. Thus $\tilde{J}^+$ has a local maximum at $x_0$. Since we are assuming (1.2) to have only finitely many solutions, $x_0$ is a strict local maximum of $\tilde{J}^+$. Let $\delta > 0$ be such that $\tilde{J}^+(x) < \tilde{J}^+(x_0)$ if $\|x - x_0\| < \delta$. Since $k > 2$, $\{x; 0 < \|x - x_0\| < \delta\}$ is connected. Thus $x_0$ is not a critical point of mountain pass type. By Lemma 3.2 $J^+$ has a critical point of mountain pass type $u_1 = x_1 + \psi(x_1)$ such that if $V$ is a neighborhood of $u_1$ containing no other critical point of $J^+$ in its closure then $d(\nabla J^+, V, 0) = -1$. In particular, by part (v) of Lemma 2.1 $x_0 \neq x_1$. Let $V_0$ (respectively, $V_1$) be a neighborhood of $u_0$ (respectively, $u_1 = x_1 + \psi^+(x_1)$) containing no other critical point in its closure. Thus

$$\begin{aligned} -1 = d(\nabla J^+, S_2, 0) &= d(\nabla J^+, V_0, 0) + d(\nabla J^+, V_1, 0) + d(\nabla J^+, S_2 - \overline{(V_0 \cup V_1)}, 0) \\ &= -2 + d(\nabla J^+, S_2 - \overline{(V_0 \cup V_1)}, 0). \end{aligned}$$

Thus by the existence property of the Leray–Schauder degree there exists a third positive solution $u_2 \in S_2 - \overline{(V_0 \cup V_1)}$. Since by the existence property of the Leray–Schauder degree (1.2) has a solution $u_3 \in S_3$, we see that (1.2) has five solutions, namely $0, u_0, u_1, u_2, u_3$. Since $u_0, u_1, u_2 \in S_2$ they have the same sign. This proves Theorem A. $\square$

## REFERENCES


[1] S. AGMON, *The $L^p$ approach to the Dirichlet problem*, Ann. Scuola Norm. Sup. Pisa, 13 (1959), pp. 405–408.

[2] H. AMMAN, *A note on degree theory for gradient mappings*, Proc. Amer. Math. Soc., 85 (1982), pp. 591–595.

[3] H. AMMAN AND E. ZEHNDER, *Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations*, Ann. Scuola Norm. Sup. Pisa, 7 (1980), pp. 539–603.

[4] A. AMBROSETTI AND G. MANCINI, *Sharp nonuniqueness results for some nonlinear problems*, Nonlinear Anal. Theory Methods, 3 (1979), pp. 635–645.

[5] A. AMBROSETTI AND P. RABINOWITZ, *Dual variational methods in critical point theory*, J. Funct. Anal., 14 (1973), pp. 343–381.

[6] L. BOCCARDO AND T. GALLOUËT, *Homogenization for "Castro-Lazer Equation,"* Nonlinear Anal., 14 (1990), pp. 81–91.

[7] H. BRÉZIS, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, 1973.

[8] N. P. CÁC, *On nontrivial solutions of an asymptotically linear Dirichlet problem*, J. Differential Equations, 75 (1988), pp. 103–107.

[9] A. CASTRO, *Métodos de reducción via minimax*, Primer simposio Colombiano de Análisis Funcional, Medellín, Colombia, 1981.

[10] ———, *Métodos variacionales y análisis funcional no lineal*, X Coloquio Colombiano de Matemáticas, Paipa, Colombia, 1980.

[11] A. CASTRO AND A. C. LAZER, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem*, Ann. Mat. Pura Appl., (4) 120 (1979), pp. 113–137.

[12] K. C. CHANG, *Solutions of asymptotically linear operator equations via Morse theory*, Comm. Pure Appl. Math., 34 (1981), pp. 693–712.

[13] H. HOFER, *A geometric description of the neighborhood of a critical point given by the mountain-pass theorem*, J. London Math. Soc., 31 (1985), pp. 566–570.

[14] ———, *Variational and topological methods in partially ordered Hilbert spaces*, Math. Ann., 261 (1982), pp. 493–514.

[15] ———, *The topological degree at a critical point of mountain-pass type*, Proc. Sympos. Pure Math., 45 (1986), pp. 501–509.

[16] A. C. LAZER AND J. P. MCKENNA, *Multiplicity results for a class of semilinear elliptic and parabolic boundary value problems*, J. Math. Anal. Appl., 107 (1985), pp. 371–395.

[17] A. C. LAZER AND S. SOLIMINI, *Nontrivial solutions of operator equations and Morse indices of critical points of min-max type*, Nonlinear Anal., 12 (1988), pp. 761–775.

[18] S. LI AND J. Q. LIU, *Nontrivial critical points for asymptotically quadratic function*, Report of International Center for Theoretical Physics IC/390, Trieste, Italy, 1986.

[19] P. RABINOWITZ, *Minimax methods in critical point theory with applications to differential equations*, Regional Conference Series in Mathematics, number 65, American Mathematical Society, Providence, RI, 1986.