1-1-1998

A Minmax Principle, Index of the Critical Point, and Existence of Sign Changing Solutions to Elliptic Boundary Value Problems

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Electronic Journal of Differential Equations, Vol. **1998**(1998), No. 02, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

A minmax principle, index of the critical point, and existence of sign-changing solutions to elliptic boundary value problems *

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Abstract

In this article we apply the minmax principle we developed in [6] to obtain sign-changing solutions for superlinear and asymptotically linear Dirichlet problems. We prove that, when isolated, the local degree of any solution given by this minmax principle is +1. By combining the results of [6] with the degree-theoretic results of Castro and Cossio in [5], in the case where the nonlinearity is asymptotically linear, we provide sufficient conditions for: i) the existence of at least four solutions (one of which changes sign exactly once), ii) the existence of at least five solutions (two of which change sign), and iii) the existence of precisely two sign-changing solutions.

For a superlinear problem in thin annuli we prove: i) the existence of a non-radial sign-changing solution when the annulus is sufficiently thin, and ii) the existence of arbitrarily many sign-changing non-radial solutions when, in addition, the annulus is two dimensional.

The reader is referred to [7] where the existence of non-radial sign-changing solutions is established when the underlying region is a ball.

1 Introduction

Let Ω be a smooth bounded region in $\mathbb{R}^N$. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that $f(0)=0$ and $f'(0)<\lambda_1$, where $\lambda_1<\lambda_2\leq\ldots$ are the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω. Let $F:\mathbb{R}\to\mathbb{R}$ be given by $F(u)=\int_0^u f(s)\,ds$. We assume that f', f, and F have *subcritical* growth, i.e., that there exist $A>0$ and $p\in[1,(N+2)/(N-2))$ such that

$$|f'(u)|\leq A(|u|^{p-1}+1)\quad \text{for } u \text{ in } \mathbb{R}\,. \tag{1.1}$$

When necessary we will assume the following additional hypotheses:

(h_1) $\lim_{|u|\to\infty} f(u)/u=\infty$, i.e., f is *superlinear*.

* *1991 Mathematics Subject Classifications:* 35J20, 35J25, 35J60.
Key words and phrases: Dirichlet problem, sign-changing solution.

Submitted September 17, 1997. Published January 30, 1998.
Partially supported by NSF grant DMS-9215027, and Colciencias-BID.

(h_2) $f'(u) > f(u)/u$ for all $u \neq 0$.

(h_3) There exist $m \in (0,1)$ and $\rho > 0$ such that $\frac{m}{2}uf(u) - F(u) \geq 0$ for $|u| > \rho$.

From these hypotheses it follows that there exists a positive constant K such that

$$\alpha t f(\alpha t) \geq K\alpha^{2/m} t f(t), \tag{1.2}$$

for $\alpha \geq 1$ and $|t| > \rho$. The proof of this inequality is deferred to Section 5.

Let H denote the Sobolev space $H_0^{1,2}(\Omega)$ (see [1]). Let $J : H \to \mathbb{R}$ be defined by

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(u)\right) dx, \tag{1.3}$$

so that

$$\langle \nabla J(u), v\rangle = \int_\Omega (\nabla u \cdot \nabla v - vf(u))\, dx, \text{ for all } v \in H\,.$$

Because of (1.1), we see that $J \in C^2(H, \mathbb{R})$ (see [21]). Letting $\gamma : H \to \mathbb{R}$ be defined by $\gamma(u) = \langle \nabla J(u), u\rangle = \int_\Omega \{|\nabla u|^2 - uf(u)\}\, dx$, one sees that

$$\gamma'(u)(v) = \langle \nabla\gamma(u), v\rangle = 2\int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Omega f(u)v\, dx - \int_\Omega f'(u)uv\, dx\,. \tag{1.4}$$

Recall that for $u \in H$, $u_+(x) = \max\{u(x), 0\} \in H$ and $u_-(x) = \min\{u(x), 0\} \in H$ (see [19]). We say that $u \in H$ *changes sign* if $u_+ \neq 0$ and $u_- \neq 0$. For $u \neq 0$ we say that u is *positive* (and write $u > 0$) if $u_- = 0$, and similarly, u is *negative* ($u < 0$) if $u_+ = 0$. As noted in [6], the transformations $u \to u_+$ and $u \to u_-$ are continuous from H into H. Let

$$S = \{u \in H - \{0\} : \gamma(u) = 0\} \quad\text{and}\quad S_1 = \{u \in S : u_+ \neq 0, u_- \neq 0, \gamma(u_+) = 0\}.$$

In [6] we proved the following *minmax principle*:

Theorem 1.1 *If $(h_1) - (h_3)$ hold, then there exists $w \in H \cap C^2(\Omega)$ such that $J'(w) = 0$ and $J(w) = \min\{J(u) : u \in S_1\}$. In addition, w changes sign exactly once, i.e., $\{x : w(x) > 0\}$ and $\{x : w(x) < 0\}$ are connected. Moreover, there exist $w_1 > 0$ and $w_2 < 0$ such that $J(w_1) = \min\{J(u) : u \in S, u = u_+\}$, $J(w_2) = \min\{J(u) : u \in S, u = u_-\}$, $J'(w_1) = J'(w_2) = 0$, w_1 and w_2 are local minima of $J|_S$, and $J(w) \geq J(w_1) + J(w_2)$.*

By the definition of weak solution and regularity theory for second order elliptic boundary value problems (see [19] and [15]), the critical points of J are the solutions to the boundary value problem

$$\begin{aligned} \Delta u + f(u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1.5}$$

We note that nontrivial solutions to (1.5) are in S (a closed subset of H) and sign-changing solutions to (1.5) are in S_1 (a closed subset of S).

When defined, we denote by $d(v,W,0)$ the Leray-Schauder degree of the vector field v on the bounded region W with respect to 0 (see [11]). In Section 2 we prove that if the critical point w given by Theorem 1.1 is isolated then its Leray-Schauder index (degree of ∇J with respect to zero in any region containing w but no other critical point of J) is +1. More precisely we prove the following result.

Theorem 1.2 *Let w be as in Theorem 1.1. If $A \subset H$ is a bounded region containing w and no other critical point of J in its closure, then*

$$d(\nabla J, A, 0) = +1.$$

In Section 3 we consider arbitrary smooth bounded regions Ω in the case where f is asymptotically linear, i.e., we assume $f'(+\infty) \equiv \lim_{u\to+\infty} f'(u) \in \mathbb{R}$, $f'(-\infty) \equiv \lim_{u\to-\infty} f'(u) \in \mathbb{R}$. In addition we assume that $tf''(t) > 0$ for $t \neq 0$. The latter hypothesis implies (h_2). Because we assume f to be asymptotically linear it satisfies (h_3) but not (h_1). By again applying Theorem 1.1 we establish the following result.

Theorem 1.3 *If $tf''(t) > 0$ for $t \neq 0$ and $f'(-\infty), f'(+\infty) \in (\lambda_2, \infty)$, then (1.5) has at least four solutions. One of these solutions changes sign exactly once and, if isolated, its local Leray-Schauder degree is +1.*

We emphasize that the latter theorem includes the case where (1.5) has *jumping nonlinearities*, i.e., the interval $(f'(-\infty), f'(+\infty)) \cup (f'(+\infty), f'(-\infty))$ contains an eigenvalue λ_k. In turn, Theorem 1.3 allows us to extend the results of [5] by proving:

Theorem 1.4 *If $tf''(t) > 0$ for $t \neq 0$ and $f'(-\infty), f'(+\infty) \in (\lambda_k, \lambda_{k+1})$ for $k \geq 2$, then (1.5) has at least five solutions, two of which change sign. Moreover, one of these two sign-changing solutions changes sign exactly once.*

In addition, we show that Theorem 1.4 is sharp in the sense that no more than two sign-changing solutions need exist. In fact we have:

Theorem 1.5 *If $k = 2$ in Theorem 1.4, then (1.5) has precisely two solutions which change sign; both change sign exactly once.*

The reader is referred to [7] where the authors showed the existence of non-radial sign-changing solutions when Ω is a ball in $\mathbb{R}^N$ and f is asymptotically linear. More precisely, let $\lambda_1^r < \lambda_2^r < \ldots$ be the eigenvalues of $-\Delta$ acting on radial functions of $H_0^1(\Omega)$ and recall that $\lambda_1 = \lambda_1^r$ and $\lambda_2 < \lambda_2^r$. In [7] we proved the following theorem.

Theorem 1.6 *If $tf''(t) > 0$ for $t \neq 0$, $f'(\infty) \in (\lambda_k, \lambda_{k+1})$ with $k \geq 2$, $f'(t) \leq \gamma < \lambda_{k+1}$ for all $t \in \mathbb{R}$, and $\lambda_1 < \lambda_k < \lambda_{k+1} \leq \lambda_2^r$, then the boundary value problem (1.5) has at least two solutions which are non-radial and change sign. Moreover, one of these two sign-changing solutions changes sign exactly once.*

For related results on asymptotically linear problems we refer the reader to [3] and [10].

In Section 4 we consider the case in which f is superlinear and $\Omega \equiv \Omega(\epsilon)$ is the *thin annulus* given by $\Omega = \{x; 1-\epsilon < \|x\| < 1\}$, where ϵ is a small positive number. We prove the following theorems.

Theorem 1.7 *Let (h_1) - (h_3) hold and let Ω be as above. There exists $\epsilon_1 > 0$ such that if $0 < \epsilon < \epsilon_1$ then (1.5) has a sign-changing non-radial solution.*

For the special case $N = 2$ we further prove the following result.

Theorem 1.8 *Let (h_1)–(h_3) hold and let Ω be as above. If $N = 2$ then for any positive integer k there exists $\epsilon_1(k) > 0$ such that if $0 < \epsilon < \epsilon_1(k)$ then (1.5) has k sign-changing non-radial solutions.*

2 The Leray-Schauder Index of the Critical Point

Throughout this section w denotes a critical point of J satisfying the variational characterization of Theorem 1.1. We further assume that w is an isolated critical point. We let X denote the linear subspace of H generated by $\{w_+, w_-\}$. Since w is a sign-changing function, X is a two dimensional subspace. We denote by Y the orthogonal complement of X in H.

By the definition of J and (h_2) we have

$$\begin{aligned} \langle J''(w)w_+, w_+\rangle &= \int_\Omega \left(\nabla w_+ \cdot \nabla w_+ - f'(w)w_+^2\right)\, dx \\ &= \int_\Omega \left(w_+ f(w) - f'(w)w_+^2\right) dx \qquad (2.1) \\ &= \int_\Omega w_+^2 \left(\frac{f(w_+)}{w_+} - f'(w_+)\right) dx < 0\,. \end{aligned}$$

Similarly $\langle J''(w)w_-, w_-\rangle < 0$. Since w_- and w_+ are orthogonal in H, $J''(w)$ is negative definite on X. By the continuity of J'' and the assumption that w is an isolated critical point, we may assume that there exist $\epsilon > 0$ and $K > 0$ such that $\nabla J(u) \neq 0$ if $0 < \|u - w\| < \sqrt{2}\epsilon$ and $\langle J''(w + x + y)v, v\rangle \leq -K\|v\|^2$ for all $x \in X$, $x \in B(0, \epsilon)$, and $y \in B(0, \epsilon)$. Since $\nabla J(w) = 0$ we may assume, without loss of generality, that

$$J(x + w) < J(w) \quad \text{for } \|x\| = \epsilon. \qquad (2.2)$$

Lemma 2.1 *There exists $\delta \in (0, \epsilon)$ such that if $y \in B(0, \delta) \cap Y$ and $\|x\| = \epsilon$ with $x \in X$ then $J(w + y + x) < J(w)$.*

Proof. We prove this lemma by contradiction. Suppose $\{y_j\} \subset Y$ and $\{x_j\} \subset X$ are sequences with $\lim_{j\to\infty} y_j = 0$, $\|x_j\| = \epsilon$ for all j, and $J(w + y_j + x_j) \geq J(w)$. Since X is finite-dimensional, without loss of generality we may assume that $\lim_{j\to\infty} x_j = \hat{x}$ and $\|\hat{x}\| = \epsilon$. By the continuity of J we have $J(w + \hat{x}) \geq$

$J(w)$, but this is a contradiction since $\nabla J(w) = 0$ and $J''(w + x)$ is negative definite for all $x \in X$ with $\|x\| \leq \epsilon$. □

Lemma 2.2 *There exists a continuous function $\phi : B(0, \delta_1) \cap Y \to B(0, \epsilon_1) \cap X$ such that $\tilde{J}(y) := J(w + y + \phi(y)) = \max_{||x||<\epsilon_1} J(w + y + x)$ is of class C^1. Furthermore, $w + x + y$ is a critical point of J if and only if $x = \phi(y)$ and y is a critical point of $\tilde{J}$. Moreover, $d(\nabla\tilde{J}, B(0, \delta_1) \cap Y, 0) = d(\nabla J, \Sigma, 0)$.*

Proof. By Lemma 2.1, for $y \in B(0, \delta) \cap Y$ there exists $\hat{x} \in B(0, \epsilon) \cap X$ such that $J(w+y+\hat{x}) = \max\{J(w+y+x); \|x\| < \epsilon, x \in X\}$. Hence $\langle \nabla J(w+y+\hat{x}), x_1\rangle = 0$ for all $x_1 \in X$. Assuming that $\langle \nabla J(w + y + x_0), x_1\rangle = 0$ for all $x_1 \in X$, we have $0 = \langle \nabla J(w+y+\hat{x}) - \nabla J(w+y+x_0), \hat{x} - x_0\rangle = \langle J''(w+y+x')(\hat{x} - x_0), (\hat{x} - x_0)\rangle \leq -K\|\hat{x} - x_0\|^2$. This proves the uniqueness of $\hat{x}$. Thus we may write $\hat{x} = \phi(y)$. Arguing as in [20] using the implicit function theorem one sees that ϕ is a function of class C^1. For the proof that $w + x + y$ is a critical point of J, with $\|x\| < \epsilon$ and $\|y\| < \epsilon$, if and only if $x = \phi(y)$ and $w + y$ is a critical point of $\tilde{J}$, we refer the reader to [2] and [4].

For the proof of the last assertion of the lemma we refer the reader to Theorem 3 of [17] and Lemma 2.6 of [20]. □

Lemma 2.3 *For each $y \in B(0, \delta) \cap Y$, the set $S_1 \cap \{w + y + x; \|x\| < \epsilon\}$ is nonempty.*

Proof. For $\|y\| \leq \delta$ let $P(y, s, t) = (\langle \nabla J(w + y + sw_+ + tw_-), w + y + sw_+ + tw_-\rangle, \langle \nabla J(w + y + sw_+ + tw_-), (w + y + sw_+ + tw_-)_+\rangle)$. It is easily seen that $P(0, s, t) = 0$ if and only if $(s, t) = (0, 0)$. Therefore there exists $\rho > 0$ such that

$$\|P(0, s, t)\| \geq \rho \quad \text{if} \quad \|sw_+ + tw_-\| = \epsilon. \tag{2.3}$$

Since $P(0, s, t)$ is equal to

$$(\langle \nabla J(w + sw_+ + tw_-), (1 + s)w_+ + (1 + t)w_-\rangle, \langle \nabla J(w + sw_+ + tw_-), (1 + s)w_+\rangle)\,,$$

we see that $f \equiv P(0, \cdot, \cdot)$ is a differentiable function. An elementary calculation shows that $\det(f'(0, 0)) = -\langle J''(w)w_+, w_+\rangle\langle J''(w)w_-, w_-\rangle < 0$. Thus $d(f, \{(s, t); \|sw_+ + tw_-\| \leq \epsilon\}, 0) = -1$. Also by (2.3) there exists $\delta_1 \in (0, \delta)$ such that if $\|y\| \leq \delta_1$ then $\|P(y, s, t)\| \geq \rho/2$ for $\|sw_+ + tw_-\| = \epsilon$. By the existence and homotopy invariance properties of the Brouwer degree, for $\|y\| \leq \delta_1$ there exists (s, t) such that $P(y, s, t) = 0$. This and the definition of S and S_1 prove the lemma. □

Proof of Theorem 1.2 Arguing as in Theorem 3 of [17] or Lemma 2.6 of [20] one sees that

$$\begin{aligned} d(\nabla J, B(w, \epsilon), 0) &= d(\nabla\tilde{J}, B(0, \delta_1) \cap Y, 0) \cdot (-1)^{\dim X} \\ &= d(\nabla\tilde{J}, B(0, \delta_1) \cap Y, 0). \end{aligned} \tag{2.4}$$

On the other hand, by Lemma 2.3, for each $y \in B(0,\delta_1)$ there exists $x \in B(0,\epsilon)$ such that $w+y+x \in S_1$. Hence $\tilde{J}(y) = J(w+y+\phi(y)) \geq J(w+y+x) > J(w) = \tilde{J}(w)$. Since this shows that $\tilde{J}$ has a local minimum at 0 we have $d(\nabla\tilde{J}, B(0,\delta_1)\cap Y, 0) = 1$ (see [2] or [9]). Hence $d(\nabla J, B(w,\epsilon), 0) = 1$. By the excision property of the Leray-Schauder degree, if Σ is a bounded region containing w but no other critical point we have $d(\nabla J, \Sigma, 0) = d(\nabla J, \Sigma - B(w,\epsilon), 0) + d(\nabla J, B(w,\epsilon), 0) = 0 + 1 = 1$. This proves the theorem.

3 Asymptotically Linear Problems on General Regions

Proof of Theorem 1.3 Assume that $tf''(t) > 0$ for $t \neq 0$, and that $f'(+\infty)$, and $f'(-\infty)$ are in $(\lambda_2, +\infty)$. As pointed out in [5], the latter assumptions imply that (1.5) has a positive and a negative solution. Since 0 is also a solution to (1.5) it remains only to show the existence of a sign-changing solution.

Let $\sigma \in (1, 1+(2/N))$. For $n = 1, 2, \ldots$ let

$$f_n(t) = \begin{cases} f(t) & |t| < n \\ f(n) + f'(n)(t-n) + (t-n)^\sigma & t \geq n \\ f(-n) + f'(-n)(t+n) + (t+n)^\sigma & t \leq -n\,. \end{cases} \tag{3.1}$$

Since $\sigma > 1$ there exists $C_1 \in \mathbb{R}$ such that $|f(t)| \leq C_1(|t|^\sigma + 1)$ for all $t \in \mathbb{R}$. Also, since $nf'(n) > f(n)$ (see (h_2)), $f_n(t) \leq f'(+\infty)t + (t-n)^\sigma \leq (f'(\infty)+1)t^\sigma$ for $t > n$. Similarly $f_n(t) \geq -(f'(-\infty)+1)|t|^\sigma$ for $t < -n$. Therefore,

$$|f_n(t)| \leq C_2(|t|^\sigma + 1) \text{ for all } t \in \mathbb{R}\text{,and all positive integer } n\,, \tag{3.2}$$

where $C_2 = \max\{C_1, f'(-\infty)+1, f'(+\infty)+1\}$.

We let $F_n(t) = \int_0^t f_n(s)\,ds$. Let $g_n(t) = tf_n(t) - 2F_n(t) - af_n(t) + tf_n(a)$. Using the convexity of f_n on $(0,\infty)$, we see that for $0 < a < t$ we have $g_n'(t) > 0$. Thus $tf_n(t) - 2F_n(t) \geq af_n(t) + tf_n(a)$. Since $f'(0) < \lambda_1$, there exists $a_1 > 0$ such that $f(a_1) = \lambda_1 a_1$. Let $\epsilon \in (0, \min\{f'(+\infty) - \lambda_2, f'(-\infty) - \lambda_2\})$. Because $f'(+\infty) > \lambda_2$, there exists $b_1 > a_1$ such that $f(t) > (f'(+\infty) - \epsilon)t$ for all $t > b_1$. Again by the convexity of f_n on $(0,\infty)$, for $t > b_1$ we have

$$tf_n(t) - 2F_n(t) \geq a_1 f_n(t) - tf_n(a_1) \geq a_1 f_n(t)[1 - (\lambda_1/(f'(+\infty) - \epsilon))]. \tag{3.3}$$

Similarly, there exists $a_2 < 0$, and $b_2 < 0$ such that if $t < b_2$ then

$$tf_n(t) - 2F_n(t) \geq a_2 f_n(t)[1 - (\lambda_1/(f'(-\infty) - \epsilon))]. \tag{3.4}$$

Combining (3.3) and (3.4) we see that

$$tf_n(t) - 2F_n(t) \geq a|f_n(t)| + D, \tag{3.5}$$

where $a = \min\{a_1, a_2\}$ and

$$D = \min\Big\{\min\{tf(t) - F(t) - a|f(t)|; t \in [-b_2, b_1]\}, \min\{tf_n(t) - F_n(t) - a|f_n(t)|; t \in [-b_2, b_1], n \in [1, b_1 - b_2]\}\Big\}. \tag{3.6}$$

For $u \in H$ we let $J_n(u) = \int_\Omega \{\frac{1}{2}|\nabla u|^2 - F_n(u)\}\,dx$. By Theorem 1.1, the equation

$$\begin{aligned} \Delta u + f_n(u) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.7}$$

has a solution w_n which changes sign exactly once. Also,

$$J_n(w_n) = \min\{J_n(u) : u \in H, \langle \nabla J_n(u), u_\pm \rangle = 0, u_+ \neq 0, u_- \neq 0\}.$$

Let us see that, for n large enough, w_n is a solution to (1.5). Let $\phi \in C^1(\bar\Omega) \subset H$ be an eigenfunction of $-\Delta$ with zero Dirichlet boundary condition corresponding to the second eigenvalue λ_2. By the Courant-Weinstein minmax principle (see [12], pp. 452), ϕ changes sign exactly once. Since $f'(+\infty) > \lambda_2$, $\lim_{t\to+\infty} J(t\phi_+) = -\infty$. Thus there exists $\alpha > 0$ such that $\gamma(\alpha\phi_+) = 0$ (see [6], Lemma 2.1). Similarly, there exists $\beta > 0$ such that $\gamma(\beta\phi_-) = 0$. Let $\bar m \geq ||\phi||_\infty(\alpha + \beta)$ be a positive integer. For $n \geq \bar m$ we have $\nabla J(\alpha\phi_+) = \nabla J_n(\alpha\phi_+)$ and $\nabla J(\beta\phi_-) = \nabla J_n(\beta\phi_-)$. Hence, $J_n(w_n) \leq J_n(\alpha\phi_+ + \beta\phi_-) = J(\alpha\phi_+ + \beta\phi_-)$. Thus,

$$J_n(w_n) \leq M \equiv \max\{J_1(w_1), \ldots, J_{\bar m - 1}(w_{\bar m - 1}), J(\alpha\phi_+ + \beta\phi_-)\}. \tag{3.8}$$

Let us see that there exists a positive integer K such that

$$||w_n||_\infty \leq n \quad \text{for all} \quad n > K. \tag{3.9}$$

This will establish that, for all $n > K$, w_n is a solution to (1.5) that changes sign exactly once. For the sake of simplicity of notation we write $w_n = w$.

From (3.5) and (3.8) we have

$$\begin{aligned} M &\geq J_n(w_n) = \int_\Omega \left(\frac{1}{2}\|\nabla w_n\|^2 - F_n(w_n)\right) \\ &= \frac{1}{2}\int_\Omega (w_n\, f_n(w_n) - 2F_n(w_n)) \\ &\geq \frac{D}{2}|\Omega| + \frac{a}{2}\int_\Omega |f_n(w_n)|\,. \end{aligned} \tag{3.10}$$

Let $\nu = 2N/(N-2)$ if $N \geq 3$ and $\nu = 4$ if $N = 2$. By the Sobolev embedding theorem there exists a real number $C(\Omega)$ such that

$$\left(\int_\Omega |u|^\nu\right)^{2/\nu} \leq C(\Omega)\int_\Omega \|\nabla(u)\|^2 \quad \text{for all} \quad u \in H. \tag{3.11}$$

Multiplying (3.7) by $|w|^{N-1}w$ and integrating by parts we infer

$$\begin{aligned} \int_\Omega |w|^{N-1} w f_n(w) &= N\int_\Omega |w|^{N-1}\|\nabla(w)\|^2 \\ &= \frac{4N}{(N+1)^2}\int_\Omega \Big(\|\nabla(|w|^{(N+1)/2})\|\Big)^2 \\ &\geq M_2\left(\int_\Omega |w|^{(N+1)\nu/2}\right)^{2/\nu}, \end{aligned} \tag{3.12}$$

where $M_2 = 4N/(C(\Omega)(N+1)^2)$. Let $s = (\nu(N+1)-2N-2\sigma)/(\nu(N+1)-2\sigma)$. By the definition of ν and σ we have $s \in (0,1)$ and $\sigma(1-s) < 1$. Let $p = 1/s$ and $q = p/(p-1)$. Thus Hölder's inequality, (3.2), (3.10), and (3.12) imply

$$\begin{aligned} &(\int_\Omega |w|^{(N+1)\,\nu/2})^{2/\nu} \\ &= (1/M_2)\int_\Omega |f_n(w)|^s\,|w|^N\,|f_n(w)|^{1-s} \\ &\leq (1/M_2)(\int_\Omega |f_n(w)|)^{1/p}\ (C_2\int_\Omega (|w|^{Nq+\sigma} + |w|^{Nq}))^{(1-s)} \\ &\leq M_3(\int_\Omega |w|^{Nq+\sigma} + (\int_\Omega |w|^{Nq+\sigma})^{\frac{Nq}{Nq+\sigma}}|\Omega|^r)^{1-s} \\ &\leq M_4(\int_\Omega |w|^{(N+1)\nu/2})^{(1-s)} + M_5(\int_\Omega |w|^{(N+1)\nu/2})^{((1-s)Nq)/(Nq+\sigma)}, \end{aligned} \tag{3.13}$$

where M_3, M_4, and M_5 are constants independent of n. Therefore,

$$\left(\int_\Omega |w|^{(N+1)\nu/2}\right)^{2/(\nu(N+1))} \leq M_6\,, \tag{3.14}$$

with M_6 independent of n, since $2/\nu > (1-s) > (1-s)\,Nq/Nq+\sigma$.

This and (3.2) imply that $\{f_n(w_n); n = 1,2,\ldots\}$ is bounded in the space $L^{(N+1)\nu/(2\sigma)}(\Omega)$. Hence, by a priori estimates for elliptic boundary value problems, $\{w_n; n = 1,2,\ldots\}$ is bounded in the Sobolev space $W^{2,(N+1)\nu/(2\sigma)}(\Omega)$. Since by the choice of σ, $(N+1)\nu/(2\sigma) > (N/2)$, we see by the Sobolev embedding theorem (see [1]) that $\{w_n; n = 1,2,\ldots\}$ is bounded in $L^\infty(\Omega)$. This proves that for n sufficiently large we have $|w_n(x)| \leq n$ for all $x \in \Omega$. Thus by the definition of f_n the function w_n is actually a solution to (1.5). This shows that (1.5) has a solution that changes sign exactly once. Finally, if w_n is an isolated critical point of J then it is also an isolated critical point of J_n. Thus, by Theorem 1.2, its Leray-Schauder index is $+1$. This proves Theorem 1.3.

Proof of Theorem 1.4 Because $f'(+\infty), f'(-\infty) \in (\lambda_k, \lambda_{k+1})$, using arguments from [5], one sees that there exists $r_1 > 0$ such that if $\nabla J(u) = 0$ then $\|u\| < r_1$. Moreover J has at least five critical points and

$$d(\nabla J, B(0,r_1), 0) = (-1)^k. \tag{3.15}$$

Since $f'(0) < \lambda_1$ the functional J has a local minimum at 0 and 0 is an isolated critical point of J. Let $r_2 \in (0, r_1)$ be such 0 is the only critical point of J in $B(0, r_2)$. Then

$$d(\nabla J, B(0, r_2), 0) = 1. \tag{3.16}$$

Because $k > 1$ and $f'(0) < \lambda_1$, if P is any region containing the positive solutions to (1.5) and no other critical point of J, then

$$d(\nabla J, P, 0) = -1. \tag{3.17}$$

Similarly

$$d(\nabla J, N, 0) = -1, \tag{3.18}$$

where N is any subregion containing the negative solutions to (1.5) and no other critical point of J. If we assume that w is the only solution to (1.5) that changes sign, by Theorem 1.2 we have $d(\nabla J, B(0, r_1) - [B(0, r_2) \cup P \cup N], 0) = 1$. Thus

$$\begin{aligned} (-1)^k &= d(\nabla J, B(0, r_1) - (B(0, r_2) \cup P \cup N, 0) \\ &\quad + d(\nabla J, P, 0) + d(\nabla J, N, 0) + d(\nabla J, B(0, r_2)\,, \end{aligned} \tag{3.19}$$

which contradicts (3.15)–(3.17), and this proves the theorem.

Proof of Theorem 1.5 Let z be any sign-changing solution. Since by assumption $uf''(u) > 0$ for $u \neq 0$, it follows that

$$\begin{aligned} \langle D^2 J(z) z_\pm, z_\pm \rangle &= \int_\Omega |\nabla z_\pm|^2 - f'(z) z_\pm^2 \, dx \\ &= \int_\Omega z_\pm f(z) - f'(z) z_\pm^2 \, dx \\ &= \int_\Omega (z_\pm^2) \left\{ \frac{f(z_\pm)}{z_\pm} - f'(z_\pm) \right\} dx < 0. \end{aligned}$$

Thus, $D^2 J$ is negative definite on the two-dimensional subspace spanned by $\{z_+, z_-\}$. On the other hand, since $f'(t) < \lambda_3$ for all $t \in \mathbb{R}$ we see that $D^2 J(\zeta)$ is positive definite on the subspace spanned by $\{\phi_3, \phi_4, \ldots\}$. Thus, $D^2 J(z)$ is non-degenerate and $\deg(\nabla J, B(z, \delta), 0) = (-1)^2 = 1$ for any sign-changing solution z, where δ is sufficiently small. In particular, every sign-changing solution changes sign exactly once (otherwise the dimension of the negative-definite space would be greater than 2.) Also, since $\deg(\nabla J(z), B(0, R) - [B(0, \epsilon) \cup P \cup (-P)]\,, 0) = 2$ (see [5]), there are exactly two sign-changing solutions. This concludes the proof of Theorem 1.5.

4 A Superlinear Problem on Thin Annuli

The purpose of this section is to prove Theorems 1.7 and 1.8. Given $\epsilon \in (0, 1)$ and $k \in \mathbb{N}$, let $\Omega^\epsilon \equiv \{x \in \mathbb{R}^N : 1 - \epsilon < \|x\| < 1\}$ and $\Omega_k^\epsilon = \{x \in \Omega^\epsilon : \theta \in (0, \frac{\pi}{k})\}$, where $(r,\phi_1,\cdots,\phi_{N-2},\theta)\equiv(r,\Phi,\theta)$ denote the spherical coordinates of $x\in\mathbb{R}^N$ given by

$$\begin{aligned} r &= (x_1^2+\cdots+x_N^2)^{1/2}\\ x_1 &= r\cos(\phi_1)\\ &\vdots\\ x_{N-1} &= r\sin(\phi_1)\cdots\sin(\phi_{N-2})\cos(\theta)\\ x_N &= r\sin(\phi_1)\cdots\sin(\phi_{N-2})\sin(\theta)\,. \end{aligned}\tag{4.1}$$

We recall that $\phi_i\in[0,\pi]$ whereas $\theta\in[0,2\pi)$. Also we define

$$H_k^\epsilon=\{u\in H^{1,2}(\Omega_k^\epsilon):u(x)=0\ \text{ if }\ \|x\|\in\{1-\epsilon,1\}\}\,.$$

For $u\in H_k^\epsilon$ we define

$$J_k^\epsilon(u)=\textstyle\int_{\Omega_k^\epsilon}(\frac{|\nabla u|^2}{2}-F(u))dx,\qquad \gamma_k^\epsilon(u)=(J_k^\epsilon)'(u)(u)=<\nabla J_k^\epsilon(u),u>$$

$$S(\epsilon,k)=\{u\in H_k^\epsilon-\{0\}:\gamma_k^\epsilon(u)=0\}\,,$$

$$S_1(\epsilon,k)=\{u\in S(\epsilon,k):u_+,u_-\in S(\epsilon,k)\}\,.$$

Imitating the proof of Poincaré's inequality one sees that

$$\int_{\Omega_k^\epsilon}u^2(x)dx\le 4\epsilon^2\int_{\Omega_k^\epsilon}|\nabla u(x)|^2dx\ \text{ for all }\ \ u\in H_k^\epsilon.\tag{4.2}$$

Let $\lambda_1(\epsilon,k)$ denote the smallest eigenvalue of $-\Delta$ subject to the boundary condition

$$u(1-\epsilon,\Phi,\theta)=u(1,\Phi,\theta)=\frac{\partial u}{\partial\eta}u(r,\Phi,0)=\frac{\partial u}{\partial\eta}u(r,\Phi,\pi/k)=0.\tag{4.3}$$

From (4.2) and (4.3) we see that $\lambda_1(\epsilon,k)$ tends to infinity as ϵ tends to 0. Thus there exists $\epsilon_0>0$ such that if $\epsilon\in(0,\epsilon_0)$ then

$$f'(0)<\lambda_1(\epsilon,k).\tag{4.4}$$

Hence, as in [6], one sees that that for $\epsilon<\epsilon_0$ the functional J_k^ϵ has a critical point $w_{\epsilon,k}$ that satisfies $J_k^\epsilon(w_{\epsilon,k})=\min_{S_1(\epsilon,k)}J_k^\epsilon$ and changes sign. By regularity theory for second order elliptic operators (see [18]), it follows that $w_{\epsilon,k}$ is a classical solution to

$$\begin{aligned} &\text{(a)} & \Delta u+f(u) &= 0\quad \text{in } \Omega_k^\epsilon\\ &\text{(b)} & u(r,\Phi,\theta) &= 0\quad \text{for }\ r\in\{1-\epsilon,1\}\\ &\text{(c)} & u_\theta(r,\Phi,\theta) &= 0\quad \text{for } k>0,\quad \theta\in\{0,\tfrac{\pi}{k}\}. \end{aligned}\tag{4.5}$$

Now we extend *evenly* $w_{\epsilon,k}$ to Ω^ϵ by

$$u_{\epsilon,k}(r,\Phi,\theta)=\begin{cases} w_{\epsilon,k}(r,\Phi,\theta) & \text{if}\quad \theta\in[0,\frac{\pi}{k}]\\ w_{\epsilon,k}(r,\Phi,\frac{2\pi}{k}-\theta) & \text{if}\quad \theta\in[\frac{\pi}{k},\frac{2\pi}{k}]\\ u_{\epsilon,k}(r,\Phi,\theta) & \text{if}\quad \theta=\frac{s\pi}{k}+t\quad\text{with}\ \ s\in\mathbb{N}\\ & \qquad\text{and}\ \ t\in[0,\frac{2\pi}{k}]\,. \end{cases}\tag{4.6}$$

For $j \in N$, we will denote by $u_{\epsilon,k,2jk}$ the restriction of $u_{\epsilon,2jk}$ to Ω_k^ϵ. We note that $u_{\epsilon,k}$ is a solution to (1.5) in $\Omega = \Omega^\epsilon$, whereas $u_{\epsilon,k,2jk} \in H_k^\epsilon$ satisfies (4.5).

Lemma 4.1 *If $\frac{\partial}{\partial\theta} w_{\epsilon,2jk} \not\equiv 0$, then $u_{\epsilon,k,2jk} \neq w_{\epsilon,k}$.*

Proof. Let $\theta_0 \in (0, \pi/(2jk))$ be such that $\frac{\partial}{\partial\theta} w_{\epsilon,2jk}(r,\Phi,\theta_0) \neq 0$ for some $(r,\Phi,\theta_0) \in \Omega_{2jk}^\epsilon$. Define

$$y(r,\Phi,\theta) = \begin{cases} u_{\epsilon,k,2jk}(r,\Phi,\theta+\theta_0) & \text{for } \theta \in [0, \frac{\pi}{k} - \theta_0) \\ u_{\epsilon,k,2jk}(r,\Phi,\theta - \frac{\pi}{2k} + \theta_0) & \text{for } \theta \in [\frac{\pi}{k} - \theta_0, \frac{\pi}{k}]\,. \end{cases}$$

Since $u_{\epsilon,k,2jk}(r,\Phi,0) = u_{\epsilon,k,2jk}(r,\Phi,\frac{\pi}{k})$ and

$$\frac{\partial}{\partial\theta} u_{\epsilon,k,2jk}(r,\Phi,0) = \frac{\partial}{\partial\theta} u_{\epsilon,k,2jk}(r,\Phi,\frac{\pi}{k}) = 0\,,$$

we see that y is a function of class C^1. In particular $y \in H_k^\epsilon$. Since $w_{\epsilon,2jk}$ changes sign, and by invariance of the integral $J_k^\epsilon(y) = J_k^\epsilon(u_{\epsilon,k,2jk})$, we have $y \in S_1(\epsilon,k)$. However, since y does not satisfy the boundary condition (4.5) (c), it follows that $J_k^\epsilon(u_{\epsilon,k,2jk}) = J_k^\epsilon(y) > J_k^\epsilon(w_{\epsilon,k})$. This proves the lemma. □

Lemma 4.2 *For each positive integer k, there exists $\epsilon_1(k)$ such that if $\epsilon \leq \epsilon_1(k)$ then $J_k^\epsilon(w_{\epsilon,k}) < J_k^\epsilon(v)$ for any sign-changing radial solution v to (1.5).*

Proof. Let $v(x) = v(\|x\|)$ be a radial sign-changing solution to (1.5). Since $v(1-\epsilon) = 0$ we see that

$$\int_{1-\epsilon}^1 (v_\pm)^2 r^{N-1} dr \leq 4\epsilon^2 \int_{1-\epsilon}^1 (v_\pm)_r^2 r^{N-1} dr. \tag{4.7}$$

for $\epsilon \in (0, 1/2)$.

Let k be a given positive integer. Let j be *an even positive integer to be chosen independent of* (ϵ, k). Let

$$\hat{z}(r,\Phi,\theta) = \begin{cases} v(r)\sin(\Phi)\sin(jk\theta) & \text{for } (r,\Phi,\theta) \text{ if } \quad \theta \in (0, \frac{\pi}{jk}) \\ 0 & \text{for } \theta \in (\frac{\pi}{jk}, \frac{\pi}{k}), \end{cases}$$

where $\sin(\Phi) = \sin(\phi_1)\cdots\sin(\phi_{N-2})$ if $N > 2$ and $\sin(\Phi) = 1$ if $N = 2$. Since v changes sign, so does $\hat{z}$. By the chain rule and (4.1) we have

$$\begin{aligned} &|\nabla \hat{z}_\pm(r,\Phi,\theta)|^2 \\ &= (v_\pm)_r^2(r)(\sin(\Phi)\sin(jk\theta))^2 \\ &\quad + \left(r^{-1}(v_\pm)(r)\sin(jk\theta)\right)^2 \Sigma_{i=1}^{N-2}\Big(\frac{\sin(\Phi)\cos(\phi_i)}{\sin(\phi_i)\sin(\phi_1)\cdots\sin(\phi_{i-2})}\Big)^2 \\ &\quad + r^{-2}(v_\pm)^2(r)\Big(\sin(\Phi) jk \frac{\cos(jk\theta)}{\sin(\phi_1)\cdots\sin(\phi_{N-2})}\Big)^2 \end{aligned} \tag{4.8}$$

if $\theta \in (0, \frac{\pi}{jk})$; otherwise $\nabla\hat{z} = 0$.Thus

$$|\nabla\hat{z}_{\pm}(r,\Phi,\theta)|^2 \le (v_{\pm})_r^2(r) + (N-2)r^{-2}(v_{\pm})^2(r) + (jkr^{-1}(v_{\pm})(r))^2\,.$$

This and (4.7) imply

$$\begin{aligned}\int_{\Omega_k^\epsilon} |\nabla(\hat{z})_+|^2 dx &\le \int_{\Omega_{jk}^\epsilon} \left((v_+)_r^2(r) + (v_+)^2(r)((N-2)+j^2k^2)r^{-2}\right) dx \\ &\le (1+(16/9)((N-2)+(jk)^2)4\epsilon^2)\int_{\Omega_{jk}^\epsilon}(v_+)_r^2 dx \qquad (4.9)\\ &\le 2\int_{\Omega_{jk}^\epsilon}(v_+)_r^2 dx\end{aligned}$$

for (see (4.4))

$$\epsilon \le \min\{\epsilon_0, 1/4, \frac{3}{8((N-2)+j^2k^2)^{1/2}}\}. \qquad (4.10)$$

Similarly $\int_{\Omega_k^\epsilon} |\nabla(\hat{z})_-|^2 dx \le 2\int_{\Omega_{jk}^\epsilon}(v_-)_r^2 dx$. Because of $(h_1)-(h_2)$ there exist positive numbers α and β such that $\gamma_k^\epsilon(\alpha\hat{z}_+) = \gamma_k^\epsilon(\beta\hat{z}_-) = 0$. Let $\rho > 0$ and m be as in (h_3). Let

$$D = \{(r,\Phi,\theta); v(r)\ge\rho,\ \phi_i \in (\frac{\pi}{4},\frac{3\pi}{4}) \text{ for } \ i=1,\dots,N-2,\ \theta\in(\frac{\pi}{4jk},\frac{3\pi}{4jk})\}\,.$$

Suppose that $\alpha > 2^{(N-1)}(4m+2)/m$. Thus for $(r,\Phi,\theta)\in D$ we have $\alpha\sin(\Phi)\sin(jk\theta) \ge (4m+2)/m$. Using this, the fact that $g(t) = tf(t)$ defines a function bounded from below, and Lemma 5.2, we conclude

$$\begin{aligned}&\int_{\Omega_k^\epsilon} \alpha(\hat{z})_+ f(\alpha(\hat{z}_+))\,dx \qquad (4.11)\\ &= \int_{\Omega_{jk}^\epsilon} \alpha\sin(jk\theta)\sin(\Phi)v_+(r)f(\alpha\sin(jk\theta)\sin(\Phi)v_+(r))dx\\ &\ge E|\Omega_{jk}^\epsilon| + K_1(\alpha)^{\frac{2}{m}}\int_D v_+(r)f(v(r))\,dx\\ &\ge E|\Omega_{jk}^\epsilon| + K_1(\alpha)^{\frac{2}{m}}(\int_{v_+\ge\rho} v_+f(v_+)r^{N-1}dr)(\int_\Sigma \sin(\Phi)d\Phi d\theta)\,,\end{aligned}$$

where $E = \inf\{g(t); t\in\mathbb{R}\}$, $K_1 = K2^{(1-N)/m}$ with K as in Lemma 5.2, and $\Sigma = \{(\Phi,\theta); (\pi/4)\le\phi_i\le(3\pi/4) \text{ for } i=1,\dots,N-2, (\pi/4jk)\le\theta\le(3\pi/4jk)\}$. Now from Lemma 5.1 we have, denoting $r^{N-1}dr$ by $d\hat{r}$,

$$\begin{aligned}&\int_{v_+(r)\ge\rho} v_+(r)f(v(r))\,d\hat{r} \qquad (4.12)\\ &= \int_{v_+(r)\ge 0} v_+(r)f(v(r))d\hat{r} - \int_{v_+(r)\le\rho} v_+(r)f(v(r))\,d\hat{r}\end{aligned}$$

$$
\begin{aligned}
&= [1-\frac{\int_{v_+(r)\leq\rho} v_+(r)f(v(r))d\hat{r}}{\int_{v_+(r)\geq 0} v_+(r)f(v(r))\,d\hat{r}}]\int_{v_+(r)\geq 0} v_+(r)f(v(r))\,d\hat{r}\\
&= [1-\frac{\int_{v_+(r)\leq\rho} v_+(r)f(v(r))d\hat{r}}{\int_{v_+(r)\geq 0}(v'(r))^2 d\hat{r}}]\int_{v_+(r)\geq 0} v_+(r)f(v(r))\,d\hat{r}\\
&\geq [1-\frac{4^{N-1}\int_{v_+(r)\leq\rho} v_+(r)f(v(r))d\hat{r}}{3^{N-1}\int_{v_+(r)\geq 0}(v'(r))^2 d\hat{r}}]\int_{v_+(r)\geq 0} v_+(r)f(v(r))\,d\hat{r}\\
&\geq [1-C_1\epsilon^{2(p+1)/(p-1)}]\int_{v_+(r)\geq 0} v_+(r)f(v(r))\,d\hat{r}\,,
\end{aligned}
$$

where $C_1=(4/3)^{N-1}\max\{|tf(t)|;|t|\leq\rho\}/C$ and C is as in Lemma 5.1. Thus for ϵ satisfying (4.10) and

$$
\epsilon\leq\left(\frac{1}{2C_1}\right)^{(p-1)/(2(p+1))} \tag{4.13}
$$

we have

$$
\int_{v_+(r)\geq\rho} v_+(r)f(v(r))d\hat{r}\geq(1/2)\int_{v_+(r)\geq 0} v_+(r)f(v(r))d\hat{r}. \tag{4.14}
$$

Let $T=\{(\Phi,\theta);\,0\leq\phi_i\leq\pi$ for $i=1,\dots,N-2,\ 0\leq\theta\leq(\pi/jk)\}$. Thus by (4.14) we obtain

$$
\begin{aligned}
&K_1\alpha^{2/m}\int_{\Omega^\epsilon_{jk}} v_+f(v_+)\,dx \qquad (4.15)\\
&= K_1\alpha^{2/m}(\int_{v\geq 0} v_+f(v)\,d\hat{r})(\int_T\sin(\Phi)\,d\Phi\,d\theta)\\
&\leq K_1\alpha^{2/m}2(\int_{v\geq\rho} v_+f(v)\,d\hat{r})(\int_T\sin(\Phi)\,d\Phi\,d\theta)\\
&= K_1\alpha^{2/m}2(\int_{v\geq\rho} v_+f(v)\,d\hat{r})(2^{N/2}\int_\Sigma\sin(\Phi)\,d\Phi\,d\theta)\\
&= 2^{\frac{N+2}{2}}\int_D K_1\alpha^{2/m}v_+f(v)\,dx\,.
\end{aligned}
$$

This and (4.11) imply

$$
\begin{aligned}
K_1\alpha^{2/m}\int_{\Omega^\epsilon_{jk}} v_+f(v_+)\,dx &\leq 2^{(N+2)/2}[\int_{\Omega^\epsilon_{jk}}\alpha(\hat{z})f(\alpha(\hat{z}_+))\,dx-E|\Omega^\epsilon_{jk}|]\\
&\leq 2^{(N+2)/2}[\int_{\Omega^\epsilon_{jk}}|\alpha\nabla\hat{z}_+|^2\,dx-E\frac{K(N)\epsilon^N}{jk}] \qquad (4.16)\\
&\leq 2^{(N+2)/2}[2\alpha^2\int_{\Omega^\epsilon_{jk}} v_+f(v_+)\,dx-E\frac{K(N)\epsilon^N}{jk}],
\end{aligned}
$$

where, in addition, we have used the fact that $|\Omega^\epsilon_{jk}| \leq \frac{K(N)\epsilon^N}{jk}$ with $K(N)$ a constant depending only on N. On the other hand, using Lemma 5.1 we obtain

$$\begin{aligned}\textstyle\int_{\Omega^\epsilon_{jk}} v_+ f(v_+)\,dx &= (\textstyle\int_{v\geq 0} v_+ f(v)\,d\hat{r})(\int_T \sin(\Phi)\,d\Phi d\theta)\\ &\geq (3/4)^{N-1} C\epsilon^{-(3+p)/(p-1)} 2^{N-2}\tfrac{\pi}{jk}\\ &\equiv C_2\epsilon^{-(3+p)/(p-1)}(jk)^{-1},\end{aligned} \tag{4.17}$$

with C_2 independent of (ϵ, j, k). Replacing (4.17) in (4.16) and setting $E_1 = -EK(N)$, we have

$$\alpha \leq \max\left\{\left(\frac{2^{\frac{N+2}{2}+2}}{K_1}\right)^{m/(2-2m)}, \left(\frac{2^{\frac{N+2}{2}+1}E_1}{C_2K_1}\right)^{m/2}\right\} \equiv K_2. \tag{4.18}$$

Similarly, $\beta \leq K_2$. Because of (h_1) the function F is bounded below, say, $F(t) \geq M \in \mathbb{R}$ for all $t \in \mathbb{R}$. Let $z = \alpha\hat{z}_+ + \beta\hat{z}_-$. Then

$$\begin{aligned}J^\epsilon_k(z) &= \int_{\Omega^\epsilon_k}\{\frac{|\nabla(z)|^2}{2} - F(z)\}dx\\ &= \int_{\Omega^\epsilon_{jk}}\left(\frac{|\nabla(z)|^2}{2} - F(z)\right)dx\\ &\leq \int_{\Omega^\epsilon_{jk}} |\nabla(z)|^2/2\,dx - M|\Omega^\epsilon_{jk}|\\ &= \frac{1}{2}\int_{\Omega^\epsilon_{jk}}\left(\alpha^2|\nabla(\hat{z}_+)|^2 + \beta^2|\nabla(\hat{z}_-)|^2\right)dx - M|\Omega^\epsilon_{jk}|\,.\end{aligned} \tag{4.19}$$

Since $j\int_{\Omega^\epsilon_{jk}}(v_r)^2dx = \int_{\Omega^\epsilon_k}(v_r)^2dx$, by Lemma 5.1 (see also (4.9)) we have

$$J^\epsilon_k(z) \leq \frac{K_2^2}{2j}J^\epsilon_k(v) \tag{4.20}$$

for

$$\epsilon \leq \min\left\{1/4, \left(\frac{N4^{N-1}CK_2^2}{2M2^{N-2}\pi 3^{N-1}}\right)^{(p-10/((N-1)p+3-N)}\right\}. \tag{4.21}$$

Choosing $j \geq K_2^2$, by the variational characterization of $w_{\epsilon,k}$ we see that it cannot be radially symmetric. By the definition of K_2 it is clear that j can be chosen independent of (ϵ, k), which proves the lemma. □

Proof of Theorem 1.7 Let $\epsilon \in (0, \epsilon_1(1))$ with $\epsilon_1(1)$ as in Lemma 4.2. By Lemma 4.2 $w_{\epsilon,1}$ is non-radial and changes sign. Extending evenly (see (4.6)) $w_{\epsilon,1}$ to Ω^ϵ we see that this extension is a non-radial sign-changing solution to (1.5), which proves the theorem.

Proof of Theorem 1.8 Let $\epsilon \in (0, \epsilon_1(2^k))$. By Lemma 4.2, $w_{\epsilon,2^k}, w_{\epsilon,2^{k-1}}, \ldots, w_{\epsilon,2}$ are k non-radial sign-changing functions. Since $N = 2$, if u is non-radial then $\partial u/\partial\theta \not\equiv 0$. This and Lemma 4.1 imply that $u_{\epsilon,2^i,2^j\cdot 2^i} \neq w_{\epsilon,2^i}$ for $i = 1,\dots,k-1$, $i+j \leq k$, $j \geq 1$. Thus extending $w_{\epsilon,2^k}, w_{\epsilon,2^{k-1}}, \dots w_{\epsilon,2}$ evenly to Ω^ϵ we have k different non-radial sign-changing solutions to (1.5), which proves the theorem.

5 Auxiliary lemmas

Lemma 5.1 *There exist positive real numbers C and $\Lambda \in (0,1)$ such that if $v \not\equiv 0$ satisfies*

$$\begin{gathered} v'' + \tfrac{N-1}{r}v' + f(v) = 0, \quad \textit{for } \Lambda \leq r_1 < r < r_2 \leq 1 \\ v(r_1) = v(r_2) = 0. \end{gathered} \tag{5.1}$$

then

$$\int_{r_1}^{r_2} (v'(r))^2 dr \geq C(r_2 - r_1)^{-(3+p)/(p-1)}.$$

Proof. An elementary calculation shows that for $r_1 < r < r_2$, the function $w(t) = t^{-(N-2)/2}\sin(\pi\ln(t/r_1)/\ln(r_2/r_1))$ satisfies

$$\begin{gathered} w'' + \tfrac{N-1}{r}w' + r^{-2}((\tfrac{\pi}{\ln(r_2/r_1)})^2 + ((N-2)/2)^2)w = 0 \\ w(r_1) = w(r_2) = 0\,. \end{gathered} \tag{5.2}$$

Thus by the Sturm comparison theorem there exists $\xi \in (r_1, r_2)$ with $f(v(\xi))/v(\xi) \geq (\pi/\ln(r_2/r_1))^2$. Thus if $r_1 \geq \max\{.75, 1 - \frac{\pi}{4\sqrt{A}}\}$ then by (1) we have

$$\begin{aligned} |v(\xi)|^{p-1} &\geq \frac{\pi^2}{A(ln((r_2/r_1))^2} - 1 \geq \frac{\pi^2 r_1^2}{A(r_2-r_1)^2} - 1 \\ &\geq \frac{9\pi^2}{16A(r_2-r_1)^2} - 1 \geq \frac{\pi^2}{2A(r_2-r_1)^2}\,. \end{aligned} \tag{5.3}$$

Now integrating v on $[r_1, \xi]$ we conclude

$$\begin{aligned} (\pi^2/(2A))^{1/(p-1)}|r_2 - r_1|^{-2/(p-1)} &\leq |v(\xi)| \leq |\int_{r_1}^{\xi} v'(s)ds| \\ &\leq (\int_{r_1}^{r_2} (v'(s))^2 ds)^{1/2}(r_2 - r_1)^{1/2}\,. \end{aligned} \tag{5.4}$$

Taking $\Lambda = \max\{.75, 1 - \frac{\pi}{4\sqrt{A}}\}$ and $C = (\pi^2/(2A))^{1/(p-1)}$, the lemma is proven. □

As stated in the introduction, now we prove inequality(1.2).

Lemma 5.2 *There exists $K > 0$ such that $svf(sv) > Ks^{2/m}vf(v)$ for $|v| > \rho$ and $s > 2$.*

Proof. From hypothesis (h_1) we may assume, without loss of generality, that $F(v) \geq 0$ for $|v| > \rho$. This and (h_2) imply imply that $f'(v) > 0$ for $|v| > \rho$. Hence if $|v| > \rho$, $s > 1 + 2(m+1)/m$, and we let $k = [s] - 1$, then

$$\begin{aligned}
F(sv) &\geq F((s-1)v) + vf((s-1)v) \\
&= F((s-1)v) + \frac{1}{s-1}(s-1)vf((s-1)v) \\
&\geq (1 + \frac{2}{m(s-1)})F((s-1)v) \geq \cdots \\
&\geq \Pi_{j=1}^{k-1}(1 + \frac{2}{m(s-j)})F((s-k+1)v) := \Pi F((s-k+1)v) \\
&\geq \Pi(F((s-k)v) + vf((s-k)v)) \geq \Pi v f((s-k)v) \geq \Pi v f(v)\,.
\end{aligned}$$

Now by assumption (h_3) we see that

$$svf(sv) \geq \frac{2}{m}\Pi v f(v). \tag{5.5}$$

Since $s > 2$ and $s - k + 1 < 3$, we have

$$\begin{aligned}
\ln \Pi &= \sum_{j=1}^{k-1} \ln(1 + \frac{2}{m(s-j)}) \\
&> \int_1^{k-1} \left(\ln(m(s-x)+2) - \ln(m(s-x))\right)\, dx \\
&= \frac{1}{m}\left\{\int_{m(s-k+1)+2}^{m(s-1)+2} \ln r\, dr - \int_{m(s-k+1)}^{m(s-1)} \ln r\, dr\right\} \\
&= \frac{1}{m}\left\{\int_{m(s-1)}^{m(s-1)+2} \ln r\, dr - \int_{m(s-k+1)}^{m(s-k+1)+2} \ln r\, dr\right\} \qquad (5.6) \\
&\geq \frac{2}{m}\left\{\ln(m(s-1)) - \ln(m(s-k+1)+2)\right\} \\
&= \frac{2}{m}\ln\left(\frac{m(s-1)}{m(s-k+1)+2}\right) > \ln\left(\frac{ms}{6m+4}\right)^{2/m}.
\end{aligned}$$

By letting $K = \frac{2}{m}\left(\frac{m}{6m+4}\right)^{2/m}$ and combining (5.5) with (5.6), the proof is complete. □

Acknowledgment The authors want to express their gratitude to Professor Djairo de Figueiredo for his comments concerning Theorem 1.3.

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