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QUASI F-COVERS OF TYCHONOFF SPACES

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ABSTRACT. A Tychonoff topological space is called a quasi F-space if each dense cozero-set of X is C^*-embedded in X. In Canad. J. Math. **32** (1980), 657–685 Dashiell, Hager, and Henriksen construct the "minimal quasi F-cover" $QF(X)$ of a compact space X as an inverse limit space, and identify the ring $C(QF(X))$ as the order-Cauchy completion of the ring $C^*(X)$. In *On perfect irreducible preimages*, Topology Proc. **9** (1984), 173–189, Vermeer constructed the minimal quasi F-cover of an arbitrary Tychonoff space.

In this paper the minimal quasi F-cover of a compact space X is constructed as the space of ultrafilters on a certain sublattice of the Boolean algebra of regular closed subsets of X. The relationship between $QF(X)$ and $QF(\beta X)$ is studied in detail, and broad conditions under which $\beta(QF(X)) = QF(\beta X)$ are obtained, together with examples of spaces for which the relationship fails. (Here βX denotes the Stone-Čech compactification of X.) The role of $QF(X)$ as a "projective object" in certain topological categories is investigated.

1. Introduction. A Tychonoff space X is called a *quasi F-space* if each dense cozero-set of X is C^*-embedded in X. Several recent papers (see [**DHH**, **HdP$_3$**, **P**, **V$_1$**, **ZK**] have noted that every space X has a "minimal quasi F-cover" $(QF(X), \Phi_X)$ which can be characterized as follows:

1.1. THEOREM. *For each Tychonoff space X there exists a space $QF(X)$ and a function Φ_X: $QF(X) \to X$ such that*

(i) *$QF(X)$ is a quasi F-space,*

(ii) *Φ_X is a perfect irreducible continuous surjection from $QF(X)$ onto X.* [*Recalls that a perfect surjection is irreducible if it maps proper closed subsets of the domain onto proper subsets of the range*].

(iii) *If K is any other quasi F-space and if Ψ is a perfect irreducible continuous function from K onto X, then there exists a continuous surjection f: $K \to QF(X)$ such that $\Phi_X \circ f = \Psi$.*

(iv) *The pair $(QF(X), \Phi_X)$ is unique in the following sense: if Y is a quasi F-space, if μ: $Y \to X$ is a perfect irreducible continuous surjection, and if there exists a continuous function g: $K \to Y$ such that $\mu \circ g = \Psi$ whenever K and Ψ are as in* (iii), *then there is a homeomorphism h: $Y \to QF(X)$ such that $\Phi_X \circ h = \mu$.*

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The uniqueness of $(QF(X), \Phi_X)$ described in 1.1(iv) follows from 1.1(iii) by way of the following result, which appears in 1.37 of **[V$_2$]** and which we use in §4.

1.2. THEOREM. *Let f: $X \to Y$, g: $X \to Y$, and h: $Y \to Z$ be perfect irreducible continuous surjections. Let X, Y, Z, be Hausdorff spaces. If $h \circ f = h \circ g$ then $f = g$.*

A proof of 1.2 appears in 8.4(g) of **[PW]**. It generalizes the lemma of **[Ha]**.

The papers **[DHH, HdP$_3$, and P]** essentially consider only the case in which X is a compact space. (One should note that in **[DHH]**, when the authors speak of "the quasi F-cover of the Tychonoff space X," they mean by it the quasi F-cover of its Stone-Čech compactification βX.) In **[V$_1$]** the quasi F-cover of an arbitrary Tychonoff space is constructed, while in **[ZK]** the authors construct the so-called "sequential absolute $a_s X$" of a Tychonoff space X (but provide no proofs of their results). This "sequential absolute" coincides with the quasi F-cover when X is compact, but (as we shall see in §3) does not do so in general. See also §5.

The pair $(QF(X), \Phi_X)$ is called the *minimal quasi F-cover* of X. More generally, a pair (Y, f) is called a *cover* of X if f is a perfect continuous irreducible surjection from Y onto X. Thus $(QF(X), \Phi_X)$ is a quasi F-cover of X by 1.1(i), (ii). The word "minimal" indicates that $(QF(X), \Phi_X)$ possesses the property described in 1.1(iii).

Several sorts of construction of $QF(X)$ have appeared in the literature. In **[DHH**, and **V$_1$]** $QF(X)$ is constructed as an inverse limit, and in **[DHH** and **HdP$_3$]** it is constructed as a space of maximal ideals. In all but **[V$_1$]** X is assumed to be compact. Park **[P]** constructed $QF(X)$ as a space of maximal filters on a certain lattice, but only for a very restricted class of compact spaces X. In **[ZK]**, points of $QF(X)$ are constructed as equivalence classes of zero-sets of dense cozero-sets of X (in the case where X is compact). A discussion of the construction of $QF(X)$ also occurs in §8.4 of **[PW]**.

The minimal quasi F-cover of X is obviously analogous in its properties to the well-known "abolute" (EX, k_X) of a space X (see **[PW** or **W]** for background on absolutes). One standard way of constructing EX is to let its points be the convergent ultrafilters on the Boolean algebra $\mathscr{R}(X)$ of regular closed subsets of X, with a topology inherited from that of the Stone space of $\mathscr{R}(X)$. This led the authors to ask if $QF(X)$ could also be represented as a "space of ultrafilters" on some suitably chosen sublattice of $\mathscr{R}(X)$. In §2 we construct the quasi F-cover of a compact space X in precisely this way, and derive additional properties of the map Φ_X. In §3 we investigate under what conditions on the Tychonoff space X it is true that $(QF(X), \Phi_X) = (\Phi_{\beta X}^{\leftarrow}[X], \Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X])$ (in the sense of 1.1(iv)), and when it is true that $QF(\beta X) = \beta(QF(X))$ (in the sense that $\Phi_{\beta X}^{\leftarrow}[X]$ is C^*-embedded in $QF(\beta X)$). Here, of course, βX denotes the Stone-Čech compactification of X. We do not obtain complete answers to these questions, but we do obtain both general theorems and limiting counterexamples.

All topological spaces discussed are assumed to be Tychonoff—the word "space" will mean "Tychonoff topological space." We will use the notation and terminology of the Gillman-Jerison text **[GJ]** without further comment. In particular, $\mathscr{Z}(X)$ denotes the set of zero-sets of the space X.

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2. The quasi F-cover of a compact space. In this section we construct the quasi F-cover of a compact space as a space of ultrafilters on a sublattice of $\mathscr{R}(X)$.

Recall that a subset A of a space X is called *regular closed* if $A = \operatorname{cl}_X \operatorname{int}_X A$. It is well known (see [**W**] for example) that the set $\mathscr{R}(X)$ of all regular closed subsets of X is a complete Boolean algebra with respect to the following operations:

$A \vee B = A \cup B$,

$A \wedge B = \operatorname{cl}_X \operatorname{int}_X(A \cap B)$,

$A' = \operatorname{cl}_X(X \setminus A)$.

If X is compact, the *absolute* EX of X is the Stone space of $\mathscr{R}(X)$ (see [**W**]).

Every Boolean algebra is a lattice; hence $\mathscr{R}(X)$ is, where the associated partial order on $\mathscr{R}(X)$ is set inclusion. If $\mathscr{L}$ is a sublattice of $\mathscr{R}(X)$, then we can topologize the set $T(\mathscr{L})$ of maximal filters (henceforth called *ultrafilters*) on $\mathscr{L}$ in the same way as one defines the Stone space of a Boolean algebra. Specifically, we have the following.

2.1. THEOREM. *Let X be a space and let $\mathscr{L}$ be a sublattice of $(\mathscr{R}(X), \subseteq)$ with $\varnothing$, $X \in \mathscr{L}$. If $L \in \mathscr{L}$, let $L^* = \{\alpha \in T(\mathscr{L}): L \in \alpha\}$. Then*

(a) *A filter α on $\mathscr{L}$ is an ultrafilter iff, for each $L \in \mathscr{L} \setminus \alpha$, there exists $A_L \in \alpha$ such that $L \wedge A_L = \varnothing$.*

(b) *$\{L^*: L \in \mathscr{L}\}$ is a closed base for a compact topology τ on $T(\mathscr{L})$.*

(c) *The space $(T(\mathscr{L}), \tau)$ is Hausdorff iff the following condition holds*:

(*) *If $A, B \in \mathscr{L}$ and $A \wedge B = \varnothing$ then there exist $C, D \in \mathscr{L}$ such that $A \wedge C = \varnothing$, $B \wedge D = \varnothing$, and $(C \cup D)^* = T(\mathscr{L})$.*

(d) *If X is compact and $\mathscr{L}$ is a base for the closed sets of X, then the map Φ: $T(\mathscr{L}) \to X$ given by*

$$\Phi(\alpha) \text{ is the unique point in } \bigcap\{L: L \in \alpha\}$$

is a well-defined perfect irreducible continuous surjection, and $\Phi[L^*] = L$ for each $L \in \mathscr{L}$.

PROOF. (a) Let α be an ultrafilter on $\mathscr{L}$, let $L_0 \in \mathscr{L}$, and suppose that $L_0 \wedge L \neq \varnothing$ for each $L \in \alpha$. Then $\{m \in \mathscr{L}: \exists L \in \alpha \text{ such that } L_0 \wedge L \subseteq m\}$ is a filter on $\mathscr{L}$ containing $\alpha \cup \{L_0\}$. By the maximality of α, $L_0 \in \alpha$. Conversely, if α is a nonmaximal filter on $\mathscr{L}$, find an ultrafilter β on $\mathscr{L}$ such that $\alpha \subsetneqq \beta$. If $L_0 \in \beta \setminus \alpha$ then $L_0 \wedge L \neq \varnothing$ for each $L \in \alpha$.

(b) To show that $\{L^*: L \in \mathscr{L}\}$ is a closed base for a topology τ on $T(\mathscr{L})$, it suffices (since $\mathscr{L}$ is a sublattice of $\mathscr{R}(X)$ and hence closed under finite unions) to show that if $L_1, L_2 \in \mathscr{L}$ then $(L_1 \cup L_2)^* = L_1^* \cup L_2^*$. If $\alpha \in L_1^* \cup L_2^*$ then $\alpha \in L_1^*$ or $\alpha \in L_2^*$ so either $L_1 \in \alpha$ or $L_2 \in \alpha$; either way $L_1 \cup L_2 \in \alpha$ so $\alpha \in (L_1 \cup L_2)^*$. Conversely, if $\alpha \notin L_1^* \cup L_2^*$ then $L_1 \notin \alpha$ and $L_2 \notin \alpha$ so by (a) there exist $A_1, A_2 \in \alpha$ such that $L_i \wedge A_i = \varnothing$ $(i = 1, 2)$. Evidently $(L_1 \cup L_2) \wedge (A_1 \wedge A_2) = \varnothing$ and as $A_1 \wedge A_2 \in \alpha$, it follows that $\alpha \notin (L_1 \cup L_2)^*$.

Note also that if $A_1, \dots, A_n \in \mathscr{L}$ then

(i) $(\bigwedge_{i=1}^n A_i)^* = \bigcap_{i=1}^n A_i^*$.

To see this note that $\alpha \in (\bigwedge_{i=1}^n A_i)^*$ iff $\bigwedge_{i=1}^n A_i \in \alpha$ iff $A_i \in \alpha$ for each i iff $\alpha \in \bigcap_{i=1}^n A_i^*$.

To show that τ is a compact topology, suppose that $\{L_i^*: i \in I\}$ has the finite intersection property (where $\{L_i: i \in I\} \subseteq \mathscr{L}$). If F is a finite subset of I, then there exists $\alpha \in \bigcap\{L_i^*: i \in F\}$. Thus $\varnothing \neq \bigwedge\{L_i: i \in F\}$ as $\bigwedge\{L_i: i \in F\} \in \alpha$. Thus $\{m \in \mathscr{L}$: there exists a finite subset F of I with $\bigwedge\{L_i: i \in F\} \subseteq m\}$ is a filter on $\mathscr{L}$, and hence contained in an ultrafilter δ on $\mathscr{L}$. Evidently $\delta \in \bigcap\{L_i^*: i \in i\}$ so $\bigcap\{L_i^*: i \in I\} \neq \varnothing$. It follows that $(T(\mathscr{L}), \tau)$ is compact.

(c) Suppose that $\mathscr{L}$ satisfies $(*)$ and that α and δ are distinct points of $T(\mathscr{L})$. By (a) there exist $A \in \alpha$ and $B \in \delta$ such that $A \wedge B = \varnothing$. By $(*)$ there exist $C, D \in \mathscr{L}$ such that

$$A \wedge C = B \wedge D = \varnothing \quad \text{and} \quad C \vee D = C \cup D = X.$$

Evidently $\alpha \in T(\mathscr{L}) \setminus C^*$, $\delta \in T(\mathscr{L}) \setminus D^*$, and

$$\begin{aligned}(T(\mathscr{L}) \setminus C^*) \cap (T(\mathscr{L}) \setminus D^*) &= T(\mathscr{L}) - (C \cup D)^* \quad (\text{see (b)}) \\ &= \varnothing \quad \big(\text{as } (C \cup D)^* T(\mathscr{L})\big).\end{aligned}$$

Thus τ is a Hausdorff topology.

Conversely, suppose that $(T(\mathscr{L}), \tau)$ is Hausdorff and that $A, B \in \mathscr{L}$ with $A \wedge B = \varnothing$. Then $A^* \cap B^* = \varnothing$, so as $T(\mathscr{L})$ is compact Hausdorff follows from (i) that there exist, for each $\alpha \in A^*$, sets $G(\alpha)$ and $K(\alpha)$ in $\mathscr{L}$ such that $\alpha \in T(\mathscr{L}) \setminus G(\alpha)^*$, $B^* \subseteq T(\mathscr{L}) \setminus K(\alpha)^*$, and

(ii) $(T(\mathscr{L}) \setminus G(\alpha)^*) \cap (T(\mathscr{L}) \setminus K(\alpha)^*) = \varnothing$. As A^* is compact there exist $\alpha_1, \dots, \alpha_n \in A^*$ such that

$$A^* \subseteq \bigcup\left\{T(\mathscr{L}) \setminus G(\alpha_i)^*: 1 \leqslant i \leqslant n\right\} = T(\mathscr{L}) \setminus \left(\bigwedge_{i=1}^n G(\alpha_i)\right)^*.$$

Set $C = \bigwedge_{i=1}^n G(\alpha_i)$ and $D = \bigcup_{i=1}^n K(\alpha_i)$. Obviously $A^* \cap C^* = B^* \cap D^* = \varnothing$, so $A \wedge C = B \wedge D = \varnothing$. Note that

$$\begin{aligned}(C \cup D)^* = C^* \cup D^* &= \left[\bigwedge_{i=1}^n G(\alpha_i)\right]^* \cup \left[\bigcup_{j=1}^n K(\alpha_j)\right]^* \\ &= \left[\bigcap_{i=1}^n G(\alpha_i)^*\right] \cup \left[\bigcup_{j=1}^n K(\alpha_j)^*\right] \\ &= \bigcap_{i=1}^n \left[\bigcup_{j=1}^n G(\alpha_i)^* \cup K(\alpha_j)^*\right] \quad (\text{by (i)}).\end{aligned}$$

By (ii), $G(\alpha_i)^* \cup K(\alpha_i)^* = T(\mathscr{L})$ for $i = 1$ to n, so $(C \cup D)^* = T(\mathscr{L})$. Hence C and D are the required members of $\mathscr{L}$.

(d) Let X be compact and let $\alpha \in T(\mathscr{L})$. Then α (regarded as a family of closed subsets of X) has the finite intersection property, so $\bigcap\{A: A \in \alpha\}$ (henceforth denoted by $\bigcap\alpha$) is nonempty. If x and y are distinct points in $\bigcap\alpha$, then as $\mathscr{L}$ is a base for the closed sets of the compact space X, there exist $L_1, L_2 \in \mathscr{L}$ such that $x \in \operatorname{int}_X L_1$, $y \in \operatorname{int}_X L_2$, and $L_1 \cap L_2 = \varnothing$. (If H and K are disjoint closed neighborhoods of x and y, respectively, find a finite subfamily $\mathscr{F}$ of $\mathscr{L}$ such that $H \subseteq \bigcap\mathscr{F} \subseteq X \setminus K$, and let $L_1 = \bigwedge\mathscr{F}$. Now define L_2 similarly.) As $x \in \bigcap\alpha$, we see that if $A \in \alpha$ then $A \cap \operatorname{int}_X L_1 \neq \varnothing$ and so $A \wedge L_1 \neq \varnothing$. Hence by 2.1(a) $L_1 \in \alpha$; similarly $L_2 \in \alpha$, which contradicts the fact that $L_1 \cap L_2 = \varnothing$. Thus $|\bigcap\alpha| = 1$, so $\Phi(\alpha)$ is indeed well defined.

Now suppose $L \in \mathscr{L}$. If $\alpha \in L^*$ then $L \in \alpha$ so $\Phi(\alpha) \in L$. Thus $\Phi[L^*] \subseteq L$. Conversely, if $x \in L$ consider the family $\mathscr{F} = \{m \in \mathscr{L}\colon x \in \operatorname{int}_X m\} \cup \{L\}$. Since $\mathscr{L}$ is a closed base for X, a compactness argument like that used above shows that $\bigcap\mathscr{F} = \{x\}$. Any finite subfamily of $\mathscr{F}$ has a nonempty infimum in $\mathscr{L}$, so by Zorn's lemma there exists $\alpha \in T(\mathscr{L})$ such that $\mathscr{F} \subseteq \alpha$. Then $\alpha \in L^*$ as $L \in \alpha$, and $\Phi(\alpha) \in \bigcap\alpha \subseteq \bigcap\mathscr{F} = \{x\}$, so $\Phi(\alpha) = x$. Thus $\Phi[L^*] = L$. In particular $\Phi[T(\mathscr{L})] = \Phi[X^*] = X$ (since X belongs to every ultrafilter), so Φ is surjective. If $L^* \neq T(\mathscr{L})$, then $L \neq X$ so $\Phi[L^*] = L \neq X$. As $\{L^*\colon L \in \mathscr{L}\}$ is a closed base for $T(\mathscr{L})$, it follows that Φ is irreducible.

Let $\alpha_0 \in T(\mathscr{L})$ and suppose $\Phi(\alpha_0) \in V$, where V is open in X. By the regularity of X there exists $L_0 \in \mathscr{L}$ such that $\Phi(\alpha_0) \in X \setminus L_0 \subseteq \operatorname{cl}_X(X \setminus L_0) \subseteq V$. Thus $L_0 \notin \alpha_0$ so $\alpha_0 \in T(\mathscr{L}) \setminus L_0^*$. We claim that $\Phi[T(\mathscr{L}) \setminus L_0^*] \subseteq V$; this will verify the continuity of Φ at the arbitrarily chosen point α_0. If $\delta \in T(\mathscr{L}) \setminus L_0^*$ then $L_0 \notin \delta$ so there exists $D \in \delta$ such that $D \wedge L_0 = \varnothing$ (see part (a)). Thus $D \cap \operatorname{int}_X L_0 = \varnothing$, so $D \subseteq \operatorname{cl}_X(X \setminus L_0) \subseteq V$. As $\Phi(\delta) \in D$, it follows that $\Phi(\delta) \in V$. Our claim is verified, and Φ is continuous. As $T(\mathscr{L})$ and X are compact, the continuity of Φ implies that Φ is perfect. $\square$

Note that if $\mathscr{L}$ is a base for the closed sets of X, then $(C \cup D)^* = T(\mathscr{L})$ iff $C \cup D = X$, and we can make that change in the statement of 2.1(c). However, this equivalence will fail in general if $\mathscr{L}$ is not a base for the closed sets.

2.2. Lemma. *Let $\mathscr{B}$ be a base for the closed subsets of the space X. Assume that $\mathscr{B}$ is closed under finite unions and intersections. Let $\mathscr{B}^\# = \{\operatorname{cl}_X \operatorname{int}_X B\colon B \in \mathscr{B}\} \cup \{X, \varnothing\}$. Then $\mathscr{B}^\#$ is a sublattice of $\mathscr{R}(X)$ and is a base for the closed subsets of X.*

Proof. If $B_1, B_2 \in \mathscr{B}$ then

$$(\operatorname{cl}_X \operatorname{int}_X B_1) \vee (\operatorname{cl}_X \operatorname{int}_X B_2) = \operatorname{cl}_X \operatorname{int}_X B_1 \cup \operatorname{cl}_X \operatorname{int}_X B_2 = \operatorname{cl}_X \operatorname{int}_X (B_1 \cup B_2),$$

which belongs to $\mathscr{B}^\#$ as $B_1 \cup B_2 \in \mathscr{B}$. Also,

$$\operatorname{cl}_X \operatorname{int}_X B_1 \wedge \operatorname{cl}_X \operatorname{int}_X B_2 = \operatorname{cl}_X \operatorname{int}_X (B_1 \cap B_2)$$

which belongs to $\mathscr{B}^\#$ as $B_1 \cap B_2 \in \mathscr{B}$. Thus $\mathscr{B}^\#$ is a sublattice of $\mathscr{R}(X)$. If A is closed in X and $p \notin A$, find an open set $V(p)$ of X such that $A(p) \subseteq V(p) \subseteq \operatorname{cl}_X V(p) \subseteq X \setminus \{p\}$. As $\mathscr{B}$ is a closed base, there exists $B(p) \in \mathscr{B}$ such that $\operatorname{cl}_X V(p) \subseteq B(p) \subseteq X \setminus \{p\}$. Obviously $\bigcap\{\operatorname{cl}_X \operatorname{int}_X B(p)\colon p \in X \setminus A\} = A$, so $\mathscr{B}^\#$ is a closed base for X. $\square$

We will make use of the following result, which appears as 1.1 of [**BH**].

2.3. LEMMA. *If C is a cozero-set of a space X and V is a cozero-set of C, then V is a cozero-set of X.* $\square$

As usual, $\mathscr{Z}(X)$ denotes the family of zero-sets of the space X.

2.4. THEOREM. *If X is a space, then $\mathscr{Z}(X)^{\#}$ is a sublattice of $\mathscr{R}(X)$ that is a base for the closed subsets of X, and $T(\mathscr{Z}(X)^{\#})$ is a compact Hausdorff space.*

PROOF. Since $\mathscr{Z}(X)$ is a closed base for X closed under finite unions and intersections, the first assertion follows from 2.2. The second assertion will follow from 2.1 once we have shown that $T(\mathscr{Z}(X)^{\#})$ is Hausdorff. To show this, by 2.1(c) it suffices to show that if $Z_1, Z_2 \in \mathscr{Z}(X)$ and $(\operatorname{cl}_X \operatorname{int}_X Z_1) \wedge (\operatorname{cl}_X \operatorname{int}_X Z_2) = \varnothing$, then there exist $Z_3, Z_4 \in \mathscr{Z}(X)$ such that

$$(\operatorname{cl}_X \operatorname{int}_X Z_3) \wedge (\operatorname{cl}_X \operatorname{int}_X Z_1) = \varnothing = (\operatorname{cl}_X \operatorname{int}_X Z_2) \wedge (\operatorname{cl}_X \operatorname{int}_X Z_4),$$

and $\operatorname{cl}_X \operatorname{int}_X Z_3 \cup \operatorname{cl}_X \operatorname{int}_X Z_4 = X$.

If $(\operatorname{cl}_X \operatorname{int}_X Z_1) \wedge (\operatorname{cl}_X \operatorname{int}_X Z_2) = \varnothing$, then $\operatorname{int}_X(Z_1 \cap Z_2) = \varnothing$, so $X \setminus (Z_1 \cap Z_2)$ is a dense cozero-set C of X. As $Z_1 \cap C$ and $Z_2 \cap C$ are disjoint zero-sets of C, there exist disjoint cozero-sets V_1 and V_2 of C such that $Z_1 \cap C \subseteq V_1$ and $Z_2 \cap C \subseteq V_2$ (see 1.15 of **[GJ]**). By 2.3 $Z_3 = X \setminus V_1$ and $Z_4 = X \setminus V_2$ belong to $\mathscr{Z}(X)$. Evidently $Z_3 \cup Z_4 = X$, and from this it follows quickly that $\operatorname{cl}_X \operatorname{int}_X Z_3 \cup \operatorname{cl}_X \operatorname{int}_X Z_4 = X$. Furthermore $(\operatorname{cl}_X \operatorname{int}_X Z_1) \wedge (\operatorname{cl}_X \operatorname{int}_X Z_3)$ is a regular closed set contained in $Z_1 \cap Z_3$, which is a subset of $Z_1 \cap Z_2$ by our choice of V_1. Since $\operatorname{int}_X(Z_1 \cap Z_2) = \varnothing$, it follows that $(\operatorname{cl}_X \operatorname{int}_X Z_1) \wedge (\operatorname{cl}_X \operatorname{int}_X Z_3) = \varnothing$. Similarly $(\operatorname{cl}_X \operatorname{int}_X Z_2) \wedge (\operatorname{cl}_X \operatorname{int}_X Z_4) = \varnothing$; thus $T(\mathscr{Z}(X)^{\#})$ is Hausdorff. $\square$

2.5 COROLLARY. *If X is a compact space, and if Φ: $T(\mathscr{Z}(X)^{\#}) \to X$ is defined as in Theorem 2.1(d), then Φ is a perfect irreducible continuous surjection and $\Phi[(\operatorname{cl}_X \operatorname{int}_X Z)^*] = \operatorname{cl}_X \operatorname{int}_X Z$ for each $Z \in \mathscr{Z}(X)$.*

PROOF. See 2.1(d). $\square$

In general, if $\mathscr{L}$ is a sublattice of $\mathscr{R}(X)$ that is a base for the closed sets of X, the space $T(\mathscr{L})$ need not be Hausdorff. For example, for each space X the family $\{\operatorname{cl}_X C$: C is a cozero set of $X\}$ is easily verified to be a sublattice of $\mathscr{R}(X)$ that is a base for the closed sets. However, we have the following

2.6. EXAMPLE. Let Y be any locally compact, σ-compact noncompact space. Then $\beta Y \setminus Y$ is a compact F-space with no proper dense cozero-sets (see 3.1 of **[FG]**). Let S and T be any two compact spaces possessing proper dense cozero-sets C_1 and C_2, respectively. Let A be the direct sum $S \oplus T \oplus (\beta Y \setminus Y)$. Choose $p \in S \setminus C_1$, $q \in T \setminus C_2$, and let r be any point of $\beta Y \setminus Y$. Let X be the quotient space obtained from A by identifying the set $\{p, q, r\}$ to a point s. Then X is compact and C_4 and C_2 are cozero-sets of X. Let $\mathscr{L} = \{\operatorname{cl}_X V$: V is a cozero-set of $X\}$. Now $\operatorname{cl}_X C_1 \wedge \operatorname{cl}_X C_2 = \varnothing$ as $\operatorname{cl}_X C_1 \cap \operatorname{cl}_X C_2 = \{s\}$. Thus if $T(\mathscr{L})$ were Hausdorff, by 2.1(c) there would exist cozero-sets C_3 and C_4 of X such that $\operatorname{cl}_X C_1 \wedge \operatorname{cl}_X C_3 = \varnothing$, $\operatorname{cl}_X C_2 \wedge \operatorname{cl}_X C_4 = \varnothing$, and $\operatorname{cl}_X C_3 \cup \operatorname{cl}_X C_4 = X$. If $s \in C_3$ then $C_3 \cap C_1 \neq \varnothing$ as $s \in \operatorname{cl}_X C_1$. Hence we must have $s \notin C_3$; similarly $s \notin C_4$. Let $(C_3 \cup C_4) \cap (\beta Y \setminus Y) = C_5$;

then $C_5 \subseteq (\beta Y \setminus Y) \setminus \{s\}$ (here we identify $((\beta Y \setminus Y) \setminus \{r\}) \cup \{s\}$ with its homeomorph $\beta Y \setminus Y$). Obviously C_5 is a proper cozero-set of $\beta Y \setminus Y$, and as $\mathrm{cl}_X C_3 \cup \mathrm{cl}_X C_4 = X$, it follows that C_5 is a proper dense cozero-set of $\beta Y \setminus Y$, which is a contradiction. Thus $T(\mathscr{L})$ is not Hausdorff. $\square$

We will show that the pair $(T(\mathscr{Z}(X)^{\#}), \Phi)$, where Φ is as defined in 2.1(d), is the minimal quasi F-cover of the compact space X. First we recall some well-known facts about irreducible maps. The first listed below is easily proved; a proof of the second appears in 2.3 of **[W]**, and the third follows immediately from the second. These also appear in 6.5(b), (d) of **[PW]**.

2.7. PROPOSITION. *If $f\colon X \to Y$ is an irreducible perfect continuous surjection, then*

(a) *If S is dense in Y then $f^{\leftarrow}[S]$ is dense in X.*

(b) *The map $A \mapsto f[A]$ is a Boolean algebra isomorphism from $\mathscr{R}(X)$ onto $\mathscr{R}(Y)$.*

(c) *If V is open in X then $\mathrm{int}_Y f[V]$ is dense in $f[V]$.*

2.8. DEFINITION. Let X and Y be spaces and let $f\colon X \to Y$ be a perfect irreducible mapping. Then f is called *$\mathscr{Z}^{\#}$-irreducible* if $\{f[A]\colon A \in \mathscr{Z}(X)^{\#}\} = \mathscr{Z}(Y)^{\#}$. (For the history of this latter concept, see §5.)

Observe that $\{f[A]\colon A \in \mathscr{Z}(X)^{\#}\} \supseteq \mathscr{Z}(Y)^{\#}$ for any perfect irreducible map; this follows from 2.7(a) and the fact that if $Z \in \mathscr{Z}(Y)$ then $f^{\leftarrow}[Z] \in \mathscr{Z}(X)$. Also, note that by virtue of 2.7(b), the mapping f is $\mathscr{Z}^{\#}$-irreducible iff $A \mapsto f[A]$ is a lattice isomorphism from the lattice $(\mathscr{Z}(X)^{\#}, \subseteq)$ onto the lattice $(\mathscr{Z}(Y)^{\#}, \subseteq)$. The next theorem provides the "nuts and bolts" of the relationship between X and $T(\mathscr{Z}(X)^{\#})$, for compact spaces X.

2.9. THEOREM. *Let X be a compact space and consider the map $\Phi\colon T(\mathscr{Z}(X)^{\#}) \to X$ as defined in* 2.1(d). *Then*

(a) *(If C is a cozero-set of $T(\mathscr{Z}(X)^{\#})$, there exists a cozero-set V of X such that $\Phi^{\leftarrow}[V]$ is dense in C.*

(b) *If $S \in \mathscr{Z}(T(\mathscr{Z}(X)^{\#}))$, then there exists $Z \in \mathscr{Z}(X)$ such that*
$$\mathrm{cl}_{T(\mathscr{Z}(X)^{\#})}\,\mathrm{int}_{T(\mathscr{Z}(X)^{\#})} S = \mathrm{cl}_{T(\mathscr{Z}(X)^{\#})}\,\Phi^{\leftarrow}[\mathrm{int}_X Z].$$

(c) *Φ is a $\mathscr{Z}^{\#}$-irreducible map (and thus $A \to \Phi[A]$ is a lattice isomorphism from $\mathscr{Z}(T(\mathscr{Z}(X)^{\#}))^{\#}$ onto $\mathscr{Z}(X)^{\#}$).*

(d) *If $Z \in \mathscr{Z}(X)$ then $(\mathrm{cl}_X \mathrm{int}_X Z)^* = \mathrm{cl}_{T(\mathscr{Z}(X)^{\#})}\Phi^{\leftarrow}[\mathrm{int}_X Z]$.*

(e) *$\mathscr{Z}(T(\mathscr{Z}(X)^{\#}))^{\#} = \{A^*\colon A \in \mathscr{Z}(X)^{\#}\}$.*

PROOF. (a) Denote $T(\mathscr{Z}(X)^{\#})$ by K. As C is a cozero-set (and hence an F_σ-set) of a compact space, it is Lindelöf. Thus there exists $\{Z_n\colon n \in \omega\} \subseteq \mathscr{Z}(X)$ such that $C = \bigcup\{K \setminus (\mathrm{cl}_X \mathrm{int}_X Z_n)^*\colon n \in \omega\}$. Let $V = \bigcup\{X \setminus Z_n\colon n \in \omega\}$. Then V is a countable union of cozero-sets of X and hence is a cozero-set of X. If $\alpha \in \Phi^{\leftarrow}[V]$ there exists $k \in \omega$ such that $\Phi(\alpha) \in X \setminus Z_k$. Thus $\mathrm{cl}_X \mathrm{int}_X Z_k \notin \alpha$, so $\alpha \in K \setminus (\mathrm{cl}_X \mathrm{int}_X Z_k)^* \subseteq C$. Thus $\Phi^{\leftarrow}[V] \subseteq C$.

To see that $\Phi^{\leftarrow}[V]$ is dense in C, suppose that $A \in \mathscr{Z}(X)$ and $C \cap [K \setminus (\mathrm{cl}_X \mathrm{int}_X A)^*] \neq \varnothing$ (here we are using the notation of 2.1). There exists $k \in \omega$ such that
$$\left[K \setminus (\mathrm{cl}_X \mathrm{int}_X Z_k)^*\right] \cap \left[K \setminus (\mathrm{cl}_X \mathrm{int}_X A)^*\right] \neq \varnothing,$$

from which it quickly follows that $\mathrm{cl}_X \mathrm{int}_X A \cup \mathrm{cl}_X \mathrm{int}_X Z_k \neq X$. One easily deduces that $A \cup Z_k \neq X$. Choose $x_0 \in X \setminus (A \cup Z_k)$; as Φ is surjective (see 2.1(d)),

$$\begin{aligned}\varnothing \neq \Phi^{\leftarrow}(x_0) &\subseteq \Phi^{\leftarrow}[X \setminus Z_k] \cap \Phi^{\leftarrow}[X \setminus A] \\ &\subseteq \Phi^{\leftarrow}[X \setminus Z_k] \cap \left[K \setminus (\mathrm{cl}_X \mathrm{int}_X A)^*\right] \quad \text{(see 2.1(d))} \\ &\subseteq \Phi^{\leftarrow}[V] \cap \left[K \setminus (\mathrm{cl}_X \mathrm{int}_X A)^*\right].\end{aligned}$$

(b) By (a), if $S \in \mathscr{Z}(K)$ then there exists $Z \in \mathscr{Z}(X)$ such that $\Phi^{\leftarrow}[X \setminus Z]$ is dense in $K \setminus S$. Thus $\mathrm{cl}_K(K \setminus S) = \mathrm{cl}_K \Phi^{\leftarrow}[X \setminus Z]$. Note that $\mathrm{cl}_K \mathrm{int}_K S = (\mathrm{cl}_K(K \setminus S))'$. (Recall that if $A \in \mathscr{R}(K)$ then A' denotes the Boolean-algebra complement of A in the Boolean algebra $\mathscr{R}(K)$.) Since by 2.7(b) the map $A \to \Phi[A]$ preserves Boolean-algebraic complements, it follows that

$$\begin{aligned}\Phi[\mathrm{cl}_K \mathrm{int}_K S] &= \Phi\left[(\mathrm{cl}_K(K \setminus S))'\right] = \left(\Phi[\mathrm{cl}_K(K \setminus S)]\right)' \\ &= \Phi\left[\mathrm{cl}_K \Phi^{\leftarrow}(X \setminus Z)\right]' = \left(\mathrm{cl}_X(X \setminus Z)\right)' = \mathrm{cl}_X \mathrm{int}_X Z.\end{aligned}$$

But $\Phi[\mathrm{cl}_K \Phi^{\leftarrow}[\mathrm{int}_X Z]] = \mathrm{cl}_X \mathrm{int}_X Z$ obviously; since by 2.7(b) $A \mapsto \Phi[A]$ is one-to-one, it follows that $\mathrm{cl}_K \mathrm{int}_K S = \mathrm{cl}_K \Phi^{\leftarrow}[\mathrm{int}_X Z]$.

(c) If $Z \in \mathscr{Z}(X)$ then $\Phi^{\leftarrow}[Z] \in \mathscr{Z}(K)$ and $\mathrm{cl}_K \mathrm{int}_K \Phi^{\leftarrow}[Z] \subseteq \Phi^{\leftarrow}[Z]$. Thus $\Phi[\mathrm{cl}_K \mathrm{int}_K \Phi^{\leftarrow}[Z]] \subseteq Z$. By 2.7(b) $\Phi[\mathrm{cl}_K \mathrm{int}_K \Phi^{\leftarrow}[Z]]$ is a regular closed subset of X contained in Z, so $\Phi[\mathrm{cl}_K \mathrm{int}_K \Phi^{\leftarrow}[Z]] \subseteq \mathrm{cl}_X \mathrm{int}_X Z$. Conversely, $\Phi^{\leftarrow}[\mathrm{int}_X Z] \subseteq \mathrm{int}_K \Phi^{\leftarrow}[Z]$, so $\Phi[\mathrm{cl}_K \Phi^{\leftarrow}[\mathrm{int}_X Z]] \subseteq \Phi[\mathrm{cl}_K \mathrm{int}_K \Phi^{\leftarrow}[Z]]$. But $\Phi[\mathrm{cl}_K \Phi^{\leftarrow}[\mathrm{int}_X Z]] = \mathrm{cl}_X \mathrm{int}_X Z$ as Φ is closed and continuous. Thus $\Phi[\mathrm{cl}_K \mathrm{int}_K \Phi^{\leftarrow}[Z]] = \mathrm{cl}_X \mathrm{int}_X Z$ and it follows that $\{f[A]\colon A \in \mathscr{Z}(K)^{\#}\} \supseteq \mathscr{Z}(X)^{\#}$. Conversely, let $S \in \mathscr{Z}(K)$. By (b) there exists $Z \in \mathscr{Z}(X)$ such that $\mathrm{cl}_K \mathrm{int}_K S = \mathrm{cl}_K \Phi^{\leftarrow}[\mathrm{int}_X Z]$. Thus $\Phi[\mathrm{cl}_K \mathrm{int}_K S] = \mathrm{cl}_X \mathrm{int}_X Z$, and $\{f[A]\colon A \in \mathscr{Z}(K)^{\#}\} \subseteq \mathscr{Z}(X)^{\#}$. Hence Φ is $\mathscr{Z}^{\#}$-irreducible.

(d) If $\alpha \in (\mathrm{cl}_X \mathrm{int}_X Z)^*$ and $\alpha \in K \setminus (\mathrm{cl}_X \mathrm{int}_X F)^*$, where Z and F belong to $\mathscr{Z}(X)$, then $\mathrm{cl}_X \mathrm{int}_X Z \in \alpha$ and $\mathrm{cl}_X \mathrm{int}_X F \notin \alpha$. By 2.1(a) there exists $A \in \mathscr{Z}(X)$ such that $\mathrm{cl}_X \mathrm{int}_X A \in \alpha$ and $\mathrm{int}_X A \cap \mathrm{int}_X F = \varnothing$. Thus $(\mathrm{cl}_X \mathrm{int}_X A) \wedge (\mathrm{cl}_X \mathrm{int}_X Z) \neq \varnothing$ so $\mathrm{int}_X A \cap \mathrm{int}_X Z \neq \varnothing$. Thus there exists $x_0 \in \mathrm{int}_X Z \cap (X - \mathrm{cl}_X \mathrm{int}_X F)$. If $\delta \in \Phi^{\leftarrow}(x_0)$ then $\delta \in \Phi^{\leftarrow}[\mathrm{int}_X Z] \cap (K \setminus (\mathrm{cl}_X \mathrm{int}_X F)^*)$. Thus each neighborhood of α meets $\Phi^{\leftarrow}[\mathrm{int}_X Z]$ and $\alpha \in \mathrm{cl}_K \Phi^{\leftarrow}[\mathrm{int}_X Z]$.

Conversely, suppose $\alpha \in \mathrm{cl}_K \Phi^{\leftarrow}[\mathrm{int}_X Z]$. If $\alpha \notin (\mathrm{cl}_X \mathrm{int}_X Z)^*$ it would follow that $(K \setminus (\mathrm{cl}_X \mathrm{int}_X Z)^*) \cap \Phi^{\leftarrow}[\mathrm{int}_X Z] \neq \varnothing$. This would be a contradiction, for if $\Phi(\delta) \in \mathrm{int}_X Z$ then $(\mathrm{cl}_X \mathrm{int}_X Z) \wedge A \neq \varnothing$ for each $A \in \delta$ and so $\mathrm{cl}_X \mathrm{int}_X Z \in \delta$, i.e. $\delta \in (\mathrm{cl}_X \mathrm{int}_X Z)^*$, by 2.1(a). Thus $\alpha \in (\mathrm{cl}_X \mathrm{int}_X Z)^*$ and $\mathrm{cl}_K \Phi^{\leftarrow}[\mathrm{int}_X Z] = (\mathrm{cl}_X \mathrm{int}_X Z)^*$.

(e) This follows directly by combining (b), (c), and (d). □

REMARK. Note that in the proof of (a) above we used the compactness of X to assert that cozero-sets of X are Lindelöf. It is this detail of the proof that does not generalize to arbitrary Tychonoff spaces.

To prove that $T(\mathscr{Z}(X)^{\#})$ is a quasi F-space, we will use the following characterization of quasi F-spaces. An equivalent, but different version of this lemma is established in [**HdP$_1$**]. For the sake of completeness we include a proof.

2.10. LEMMA. *The following are equivalent for a Tychonoff space X:*

(a) *X is a quasi F-space.*

(b) *If $Z_1, Z_2 \in \mathscr{Z}(X)$ and $\operatorname{int}_X Z_1 \cap \operatorname{int}_X Z_2 = \varnothing$ then $\operatorname{cl}_X \operatorname{int}_X Z_1 \cap \operatorname{cl}_X \operatorname{int}_X Z_2 = \varnothing$.*

PROOF. (a) $\Rightarrow$ (b) If $Z_1, Z_2 \in \mathscr{Z}(X)$ and $\operatorname{int}_X Z_1 \cap \operatorname{int}_X Z_2 = \varnothing$, then it follows that $X \setminus (Z_1 \cap Z_2)$ is a dense cozero-set C of X and hence by hypothesis is C^*-embedded in X. Now $C \cap \operatorname{int}_X Z_1$ and $C \cap \operatorname{int}_X Z_2$ are contained in the disjoint zero-sets $Z_1 \cap C$ and $Z_2 \cap C$ of C. But $\operatorname{cl}_X(Z_1 \cap C) \cap \operatorname{cl}_X(Z_2 \cap C) = \varnothing$ as C is C^*-embedded in X (see 6.4 of **[GJ]**), so

$$\varnothing = \operatorname{cl}_X(C \cap \operatorname{int}_X Z_1) \cap \operatorname{cl}_X(C \cap \operatorname{int}_X Z_2) = \operatorname{cl}_X(\operatorname{int}_X Z_1) \cap \operatorname{cl}_X(\operatorname{int}_X Z_2).$$

(b) $\Rightarrow$ (a) Let C be a dense cozero-set of X. To show that C is C^*-embedded in X it suffices by 6.4 of **[GJ]** to show that the disjoint zero-sets Z_1 and Z_2 of C have disjoint X-closures. There exist cozero-sets V_1 and V_2 of C such that $Z_1 \subseteq V_1$, $Z_2 \subseteq V_2$, $Z_1 \cap \operatorname{cl}_C V_2 = \varnothing$, $Z_2 \cap \operatorname{cl}_C V_1 = \varnothing$, and $V_1 \cup V_2 = C$ (see 1.15 of **[GJ]**). Let $S_1 = X \setminus V_1$ and $S_2 = X \setminus V_2$. By 2.3, $S_1, S_2 \in \mathscr{Z}(X)$ and $\operatorname{int}_X S_1 \cap \operatorname{int}_X S_2 = \varnothing$ as $V_1 \cup V_2 = C$. By hypothesis $\operatorname{cl}_X \operatorname{int}_X S_1 \cap \operatorname{cl}_X \operatorname{int}_X S_2 = \varnothing$; obviously $\operatorname{cl}_X Z_1 \subseteq \operatorname{cl}_X \operatorname{int}_X S_2$ and $\operatorname{cl}_X Z_2 \subseteq \operatorname{cl}_X \operatorname{int}_X S_1$. Thus $\operatorname{cl}_X Z_1 \cap \operatorname{cl}_X Z_2 = \varnothing$, so by 6.4 of **[GJ]** C is C^*-embedded in X. $\square$

2.11. THEOREM. *If X is a compact space then $T(\mathscr{Z}(X)^{\#})$ is a quasi F-space.*

PROOF. Denote $T(\mathscr{Z}(X)^{\#})$ by K. By 2.10 it suffices to show that if $Z_1, Z_2 \in \mathscr{Z}(K)$ and $\operatorname{int}_K Z_1 \cap \operatorname{int}_K Z_2 = \varnothing$ then $\operatorname{cl}_K \operatorname{int}_K Z_1 \cap \operatorname{cl}_K \operatorname{int}_K Z_2 = \varnothing$. By 2.9(b) there exists $S_i \in \mathscr{Z}(X)$ such that $\operatorname{cl}_K \operatorname{int}_K Z_i = \operatorname{cl}_K \Phi^{\leftarrow}[\operatorname{int}_X S_i]$ $(i = 1, 2)$. Since

$$\varnothing = \operatorname{cl}_K(\operatorname{int}_K Z_1 \cap \operatorname{int}_K Z_2) = \operatorname{cl}_K \operatorname{int}_K Z_1 \wedge \operatorname{cl}_K \operatorname{int}_K Z_2,$$

it follows from 2.7(b) that

$$\begin{aligned}\varnothing &= \Phi[\operatorname{cl}_K \operatorname{int}_K Z_1 \wedge \operatorname{cl}_K \operatorname{int}_K Z_2] = \Phi[\operatorname{cl}_K \operatorname{int}_K Z_1] \wedge \Phi[\operatorname{cl}_K \operatorname{int}_K Z_2] \\ &= \operatorname{cl}_X \operatorname{int}_X S_1 \wedge \operatorname{cl}_X \operatorname{int}_X S_2 \quad \text{(by the above)}.\end{aligned}$$

It follows that $(\operatorname{cl}_X \operatorname{int}_X S_1)^* \cap (\operatorname{cl}_X \operatorname{int}_X S_2)^* = \varnothing$. But by 2.9(d) and the above we see that $\operatorname{cl}_K \operatorname{int}_K Z_i = (\operatorname{cl}_X \operatorname{int}_X S_i)^*$. It follows that $\operatorname{cl}_K \operatorname{int}_K Z_1 \cap \operatorname{cl}_K \operatorname{int}_K Z_2 = \varnothing$ as required. $\square$

Now we know that if X is any compact space then $T(\mathscr{Z}(X)^{\#})$ is a quasi F-space that can be mapped onto X by a perfect irreducible continuous surjection Φ. It remains to show that $(T(\mathscr{Z}(X))^{\#}, \Phi)$ satisfies the minimality condition described in 1.1(iii). Our proof is modelled on one originally constructed by Flaschmeyer **[F]** in his study of absolutes. (Flaschmeyer's proof is presented in 3.8 of **[W]** and in 6.11(d) of **[PW]**.)

2.12. THEOREM. *Let X be a compact space. The pair $(T(\mathscr{Z}(X)^{\#}), \Phi)$ is the minimal quasi F-cover of X. Explicitly:*

(a) *$T(\mathscr{Z}(X)^{\#})$ is a quasi F-space.*

(b) *Φ is a perfect irreducible continuous surjection from $T(\mathscr{Z}(X)^{\#})$ onto X.*

(c) *If Y is a quasi F-space and $f\colon Y \to X$ is a perfect irreducible continuous surjection, then there exists a perfect irreducible continuous surjection $g\colon Y \to T(\mathscr{Z}(X)^{\#})$ such that $\Phi \circ g = f$.*

PROOF. Part (a) is 2.11 and (b) is 2.5. It remains to prove (c). Denote $T(\mathscr{Z}(X)^{\#})$ by K.

Let Π_Y: $K \times Y \to Y$ and Π_K: $K \times Y \to K$ be the projection maps. Let $S = \{(\alpha, y) \in K \times Y: \Phi(\alpha) = f(y)\}$. Then S is closed in $K \times Y$, for if $\alpha \in K$, $y \in Y$, and $\Phi(\alpha) \neq f(y)$ there exist disjoint open sets V and W of X containing $\Phi(\alpha)$ and $f(y)$, respectively. Then $\Phi^{\leftarrow}[V] \times f^{\leftarrow}[W]$ is a $K \times Y$-neighborhood of (α, y) disjoint from S.

As f is perfect and X is compact, Y must be compact. Hence S is compact and so $\Pi_Y | S$ is a perfect map. Note that $(\Pi_Y | S)[S] = Y$, for if $y \in Y$ then as Φ is surjective $\Phi^{\leftarrow}(f(y)) \neq \varnothing$, and if $\delta \in \Phi^{\leftarrow}(f(y))$ then $(\delta, y) \in S$ and $\Pi_Y((\delta, y)) = y$. Similarly $\Pi_K | S$ is a perfect continuous surjection from S onto K.

We will show that $\Pi_Y | S$ is a homeomorphism from S onto Y. As $\Pi_Y | S$ is perfect it suffices to show $\Pi_Y | S$ is one-to-one. Suppose not; then there exist $y_0 \in Y$ and $\alpha_1, \alpha_2 \in K$ such that $(\alpha_1, y_0) \in S$, $(\alpha_2, y_0) \in S$ and $\alpha_1 \neq \alpha_2$. Then there exist disjoint zero-sets Z_1 and Z_2 of K such that $\alpha_i \in \operatorname{int}_K Z_i$ $(i = 1, 2)$. By 2.9(b) there exist zero-sets J_1 and J_2 of X such that $\Phi^{\leftarrow}[\operatorname{int}_X J_i]$ is a dense subset of $\operatorname{int}_K Z_i$ $(i = 1, 2)$. As $f(y_0) = \Phi(\alpha_i)$ $(i = 1, 2)$ it follows that $y_0 \in f^{\leftarrow}[\Phi[\operatorname{int}_K Z_i]]$ $(i = 1, 2)$. As f is irreducible and $\Phi^{\leftarrow}[\operatorname{int}_X J_i]$ is dense in $\operatorname{int}_K Z_i$, it follows from 2.7(a) that $\operatorname{cl}_Y f^{\leftarrow}[\Phi(\operatorname{int}_K Z_i)] = \operatorname{cl}_Y f^{\leftarrow}[\operatorname{int}_X J_i]$. Thus $y_0 \in \operatorname{cl}_Y f^{\leftarrow}[\operatorname{int}_X J_1] \cap \operatorname{cl}_Y f^{\leftarrow}[\operatorname{int}_X J_2]$ and so $y_0 \in \operatorname{cl}_Y \operatorname{int}_Y f^{\leftarrow}[J_i] \cap \operatorname{cl}_Y \operatorname{int}_Y f^{\leftarrow}[J_2]$. As $f^{\leftarrow}[J_i] \in Z(Y)$ and Y is a quasi F-space, it follows from 2.10 that $\operatorname{int}_Y f^{\leftarrow}[J_1] \cap \operatorname{int}_Y f^{\leftarrow}[J_2] \neq \varnothing$. By 2.7(c) this implies that $\operatorname{int}_X(J_1 \cap J_2) \neq \varnothing$. But $\Phi^{\leftarrow}[\operatorname{int}_X(J_1 \cap J_2)] \subseteq \operatorname{int}_K Z_1 \cap \operatorname{int}_K Z_2 = \varnothing$, so we have a contradiction. Thus $\Pi_Y | S$ is a homeomorphism as claimed.

If we define g to be $\Pi_K \circ (\Pi_Y | S)^{\leftarrow}$, then g is well defined, perfect, and continuous. If $y \in Y$ there is a unique $\alpha \in K$ such that $(\alpha, y) = (\Pi_Y | S)^{\leftarrow}(y)$. As $(\alpha, y) \in S$ it follows that $\Phi(\alpha) = f(y)$; thus $\Phi(g(y)) = \Phi(\Pi_K(\alpha, y)) = \Phi(\alpha) = f(y)$, so $\Phi \circ g = f$. The irreducibility of g follows immediately from that of f. $\square$

Now that we have established that $(T(\mathscr{Z}(X)^{\#}), \Phi)$ (as defined in 2.4 and 2.5) is the quasi F-cover of the compact space X "up to uniqueness" (as defined in 1.1(iv)), we will cease using the notation $T(\mathscr{Z}(X)^{\#})$ and start referring to this space by the symbol $QF(X)$. Similarly, the map Φ described in 2.5 will henceforth be denoted by Φ_X.

Not only is $(QF(X), \Phi_X)$ the minimal quasi F-cover of X (if X is compact), it is the largest cover of X that is mapped onto X by a $\mathscr{Z}^{\#}$-irreducible map. Precisely, we have the following result. (This result was discovered independently by one of us; however, it had previously been obtained, although not published, by Dashiell [**D**$_3$]. See also §5.)

2.13. THEOREM. *Let* (Y, f) *be a cover of the compact space* X (*as defined in the Introduction*) *and let* f *be a* $\mathscr{Z}^{\#}$*-irreducible map* (*see* 2.8). *Then there exists a* $\mathscr{Z}^{\#}$*-irreducible function* k: $QF(X) \to Y$ *such that* $f \circ k = \Phi_X$.

PROOF. It is well known that the composition of two perfect irreducible continuous surjections is a perfect irreducible continuous surjection. It hence is evident from 2.8 and the comment immediately following it that the composition of two $\mathscr{Z}^{\#}$-irreducible functions is $\mathscr{Z}^{\#}$-irreducible. Hence $f \circ \Phi_Y$ is a $\mathscr{Z}^{\#}$-irreducible map from the quasi F-space $QF(Y)$ onto X. By 2.12(c) there exists a perfect irreducible continuous surjection $g\colon QF(Y) \to QF(X)$ such that $\Phi_X \circ g = f \circ \Phi_Y$. We will show that g is a homeomorphism; as g is a continuous closed surjection, it suffices to show that g is one-to-one.

Let a and b be distinct points of $QF(Y)$. There exist disjoint zero-sets Z_1 and Z_2 of $QF(Y)$ such that $a \in \operatorname{int}_{QF(Y)} Z_1$, $b \in \operatorname{int}_{QF(Y)} Z_2$. As $f \circ \Phi_Y$ is $\mathscr{Z}^{\#}$-irreducible, there exist zero-sets S_1 and S_2 of X such that $f \circ \Phi_Y[\operatorname{cl}_{QF(Y)} \operatorname{int}_{QF(Y)} Z_i] = \operatorname{cl}_X \operatorname{int}_X S_i$ $(i = 1, 2)$. As $A \to f \circ \Phi_Y[A]$ is a Boolean algebra isomorphism it follows that $\operatorname{cl}_X \operatorname{int}_X S_1 \wedge \operatorname{cl}_X \operatorname{int}_X S_2 = \varnothing$; evidently $f \circ \Phi_Y(a) \in \operatorname{cl}_X \operatorname{int}_X S_1$ and $f \circ \Phi_Y(b) \in \operatorname{cl}_X \operatorname{int}_X S_2$. By 2.9(c) there exist zero-sets T_1 and T_2 of $QF(X)$ such that $\operatorname{cl}_X \operatorname{int}_X S_i = \Phi_X[\operatorname{cl}_{QF(X)} \operatorname{int}_{QF(X)} T_i]$ $(i = 1, 2)$. But as $\Phi_X \circ g = f \circ \Phi_Y$, we know that $\operatorname{cl}_X \operatorname{int}_X S_i = \Phi_X[g[\operatorname{cl}_{QF(Y)} \operatorname{int}_{QF(Y)} Z_i]]$ $(i = 1, 2)$; hence by 2.7(b), $\operatorname{cl}_{QF(X)} \operatorname{int}_{QF(X)} T_i = g[\operatorname{cl}_{QF(Y)} \operatorname{int}_{QF(Y)} Z_i]$. As $\operatorname{cl}_X \operatorname{int}_X S_1 \wedge \operatorname{cl}_X \operatorname{int}_X S_2 = \varnothing$, it follows from 2.7(b) that $\operatorname{cl}_{QF(X)} \operatorname{int}_{QF(X)} T_1 \wedge \operatorname{cl}_{QF(X)} \operatorname{int}_{QF(X)} T_2 = \varnothing$, and so $\operatorname{int}_{QF(X)} T_1 \cap \operatorname{int}_{QF(X)} T_2 = \varnothing$. As $QF(X)$ is a quasi F-space, it follows from 2.10 that $\operatorname{cl}_{QF(X)} \operatorname{int}_{QF(X)} T_1 \cap \operatorname{cl}_{QF(X)} \operatorname{int}_{QF(X)} T_2 = \varnothing$. But evidently $g(a) \in \operatorname{cl}_{QF(X)} \operatorname{int}_{QF(X)} T_1$ and $g(b) \in \operatorname{cl}_{QF(X)} \operatorname{int}_{QF(X)} T_2$, so $g(a) \neq g(b)$. Thus g is one-to-one and thus a homeomorphism. Then $\Phi_Y \circ (g^{\leftarrow})$ is the required function k. $\square$

We are now in a position to compare the properties of quasi F-spaces and extremally disconnected spaces, and the properties of $QF(X)$ and EX.

2.14. THEOREM. (a) *The following are equivalent for a Tychonoff space X:*

(i) *X is extremally disconnected.*

(ii) *$A \wedge B = A \cap B$ for each $A, B \in \mathscr{R}(X)$.*

(b) *The following are equivalent for a Tychonoff space X:*

(i) *X is a quasi F-space.*

(ii) *$A \wedge B = A \cap B$ for each $A, B \in \mathscr{Z}(X)^{\#}$.*

(c) *The compact extremally disconnected spaces X and Y are homeomorphic iff the lattices $\mathscr{R}(X)$ and $\mathscr{R}(Y)$ are lattice-isomorphic.*

(d) *The compact quasi F-spaces X and Y are homeomorphic iff $\mathscr{Z}(X)^{\#}$ and $\mathscr{Z}(Y)^{\#}$ are lattice-isomorphic.*

PROOF. (a) This is well known and follows from the fact that if $A \in \mathscr{R}(X)$, then $A \wedge \operatorname{cl}_X(X \setminus A) = A \cap \operatorname{cl}_X(X \setminus A)$ if A is clopen.

(b) Suppose $A \wedge B = A \cap B$ for each $A, B \in \mathscr{Z}(X)^{\#}$. If $Z_1, Z_2 \in \mathscr{Z}(X)$ and $\operatorname{int}_X Z_1 \cap \operatorname{int}_X Z_2 = \varnothing$ then $\operatorname{cl}_X \operatorname{int}_X Z_1 \wedge \operatorname{cl}_X \operatorname{int}_X Z_2 = \varnothing$. By hypothesis it follows that $\operatorname{cl}_X \operatorname{int}_X Z_1 \cap \operatorname{cl}_X \operatorname{int}_X Z_2 = \varnothing$. Thus by 2.10 X is a quasi F-space.

Conversely, suppose that $A \wedge B \neq A \cap B$ for some $A, B \in \mathscr{Z}(X)^{\#}$. Choose $p \in A \cap B \setminus (A \wedge B)$. As X is Tychonoff there exists $C \in \mathscr{Z}(X)^{\#}$ such that $p \in \operatorname{int}_X C$ and $C \cap (A \wedge B) = \varnothing$. Now $A \wedge C$ and $B \wedge C$ are in $\mathscr{Z}(X)^{\#}$ and it is easily verified that $p \in (A \wedge C) \cap (B \wedge C)$ while $(A \wedge C) \wedge (B \wedge C) = \varnothing$. Thus $\operatorname{int}_X(A \wedge C) \cap \operatorname{int}_X(B \wedge C) = \varnothing$ and it follows from 2.10 that X is not a quasi F-space.

(c) This is well known.

(d) This follows from 2.12 as the lattice structure of $\mathscr{Z}(X)^{\#}$ determines the topological structure of $T(\mathscr{Z}(X)^{\#})$. $\square$

2.15. DEFINITION. A cozero-set C of a Tychonoff space X is called a *complemented cozero-set* if there exists another cozero-set V of X such that $C \cap V = \varnothing$ and $C \cup V$ is dense in X.

Recall (see 1H of [**GJ**]) that a Tychonoff space X is *basically disconnected* if each cozero-set of X has an open closure. Basically disconnected spaces must have an open base of open-and-closed subsets. For background see 16 0 of [**GJ**]; in other words they must be zero dimensional. We characterize those compact spaces X for which $QF(X)$ is basically disconnected.

2.16. THEOREM. *The following are equivalent for a compact space X:*
(a) *Each cozero-set of X is complemented.*
(b) *$QF(X)$ is basically disconnected.*

PROOF. (a) $\Rightarrow$ (b) Let $Z \in \mathscr{Z}(X)$. Then there exists a cozero-set V of X such that $(X \setminus Z) \cap V = \varnothing$ and $(X \setminus Z) \cup V$ is dense in X. Let $S = X \setminus V$. It quickly follows that $\mathrm{cl}_X \mathrm{int}_X S \cup \mathrm{cl}_X \mathrm{int}_X Z = X$ and that $(\mathrm{cl}_X \mathrm{int}_X S) \wedge (\mathrm{cl}_X \mathrm{int}_X Z) = \varnothing$. It follows that $(\mathrm{cl}_X \mathrm{int}_X Z)^* = QF(X) \setminus (\mathrm{cl}_X \mathrm{int}_X S)^*$. Thus by 2.9(e) each member of $\mathscr{Z}(QF(X))^\#$ is a clopen subset of $QF(X)$. But if C is a cozero-set of $QF(X)$ then $\mathrm{cl}_{QF(X)} C = QF(X) \setminus \mathrm{int}_{QF(X)}(QF(X) \setminus C)$, which, by the preceding remark, is clopen. Thus $QF(X)$ is basically disconnected.

(b) $\Rightarrow$ (a) Let C be a cozero-set of X. Then $(\mathrm{cl}_X \mathrm{int}_X(X \setminus C))^* \in \mathscr{Z}(QF(X))^\#$ by 2.9(e). As $QF(X)$ is basically disconnected, each member of $\mathscr{Z}(QF(X))^\#$ is clopen and hence has a clopen complement. As clopen sets are zero-sets, by 2.9(c) $\Phi_X[QF(X) \setminus (\mathrm{cl}_X \mathrm{int}_X(X \setminus C))^*] \in \mathscr{Z}(X)^\#$. But

$$\begin{aligned}
\Phi_X\left[QF(X) \setminus [\mathrm{cl}_X \mathrm{int}_X(X \setminus C)]^*\right] &= \Phi_X\left[\left([\mathrm{cl}_X \mathrm{int}_X(X \setminus C)]^*\right)'\right] \\
&= \left(\Phi_X\left[[\mathrm{cl}_X \mathrm{int}_X(X \setminus C)]^*\right]\right)' \quad (\text{by } 2.7(\text{b})) \\
&= \left(\mathrm{cl}_X \mathrm{int}_X(X \setminus C)\right)' \quad (\text{by } 2.1(\text{d})) \\
&= \mathrm{cl}_X C.
\end{aligned}$$

(As usual, in the above $'$ denotes complementation in the Boolean algebra of regular closed sets.) Thus there is a zero-set S of X such that $\mathrm{cl}_X \mathrm{int}_X S = \mathrm{cl}_X C$. It is routine to verify that the cozero-set $X \setminus S$ has the properties of V in 2.15. $\square$

In [**V$_3$**] it is shown that every Tychonoff space X has a minimal basically disconnected cover $(\Lambda X, g_X)$. If X is compact, it follows readily from 2.16 that $QF(X) = \Lambda X$ iff each cozero-set of X is complemented, which in turn is true iff $\mathscr{Z}(X)^\#$ is a sub-Boolean algebra (as distinct from sublattice) of $\mathscr{R}(X)$.

Recall that a Tychonoff space is an *F-space* if all its cozero-sets are C^*-embedded (see Chapter 14 of [**GJ**]). Obviously F-spaces are their own minimal quasi F-covers. There are compact F-spaces that are not zero dimensional (for example $\beta\mathbf{R} \setminus \mathbf{R}$) and compact zero-dimensional F-spaces that are not basically disconnected (for example $\beta\mathbf{N} \setminus \mathbf{N}$); see [**GJ**] for proofs of these assertions. Thus minimal quasi F-covers need not be zero dimensional, and zero-dimensional quasi F-covers need not be basically disconnected. It is also of interest to note that a compact space does *not* in general have a "minimal F-cover"—see 2.11 of [**V$_1$**].

3. Quasi F-covers of Tychonoff spaces. If one wishes to construct the absolute EX of the Tychonoff space X, one can proceed as follows. Construct the absolute $E(\beta X)$ of the Stone-Čech compactification βX; it can be mapped onto βX by a perfect irreducible continuous surjection $k_{\beta X}$. Because $k_{\beta X}$ is irreducible, $k_{\beta X}^{\leftarrow}[X]$ is a dense subset of $E(\beta X)$ and hence is both extremally disconnected and C^*-embedded in βX (see 6M of **[GJ]** or Chapter 6 of **[PW]**). The restriction $k_{\beta X} | k_{\beta X}^{\leftarrow}[X]$ is a perfect irreducible continuous surjection from $k_{\beta X}^{\leftarrow}[X]$ onto X, and it immediately follows that $k_{\beta X}^{\leftarrow}[X] = EX$, $k_{\beta X} | k_{\beta X}^{\leftarrow}[X] = k_X$, and $\beta(EX) = E(\beta X)$ (in the sense that up to homeomorphism EX is a dense C^*-embedded subspace of the compact space $E(\beta X)$). Thus EX can always be regarded in a natural way as a dense subspace of $E(\beta X)$. (See **[W]** or Chapter 6 of **[PW]** for a detailed discussion of these ideas.)

Because of the analogy between the construction of the absolute and the quasi F-cover of a compact space (as described in §2), it is reasonable to ask if the quasi F-cover $QF(X)$ of a Tychonoff space X can be related to the quasi F-cover $QF(\beta X)$ of βX in a similar fashion. The answer is "not always" (see examples 3.16 and 3.19 below); however, for a wide class of spaces it is true that $QF(\beta X) = \beta(QF(X))$ (see 3.5 and 3.8). It is also possible for $(\Phi_{\beta X}^{\leftarrow}[X], \Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X])$ to be $(QF(X), \Phi_X)$ and yet not have $QF(\beta X) = \beta(QF(X))$ (see 3.16). Finally, there are spaces X for which $\Phi_{\beta X}^{\leftarrow}[X]$ is not a quasi F-space (see 3.19). We do not as yet have a satisfactory method of constructing $QF(X)$ as a subspace of a space of ultrafilters on some sublattice of $\mathscr{R}(X)$ for an arbitrary Tychonoff space X. Henceforth we will call $(QF(X), \Phi_X)$ the quasi F-cover of X; i.e. we will delete the word "minimal."

3.1. DEFINITION. A dense subspace X of a Tychonoff space Y is said to be *$\mathscr{Z}^{\#}$-embedded* in Y if, for each $Z \in \mathscr{Z}(X)$, there exists $S \in \mathscr{Z}(Y)$ such that $\mathrm{cl}_X \mathrm{int}_X Z = X \cap \mathrm{cl}_Y \mathrm{int}_Y S$.

Evidently $\mathscr{Z}^{\#}$-embedding is analogous to the notion of z-embedding introduced by Blair (see **[BH]**). (Recall that X is z-embedded in Y if $\mathscr{Z}(X) = \{Z \cap X$: $Z \in \mathscr{Z}(Y)\}$.) Although $\mathscr{Z}^{\#}$-embedding is in general weaker than C^*-embedding, they are equivalent for quasi F-spaces. Precisely, we have the following:

3.2. PROPOSITION. *The following are equivalent for a dense subspace X of a quasi F-space Y:*

(a) *X is C^*-embedded in Y.*

(b) *X is z-embedded in Y.*

(c) *X is $\mathscr{Z}^{\#}$-embedded in Y.*

Furthermore, these (equivalent) conditions imply that X is a quasi F-space.

PROOF. It is well known that (a) implies (b). If (b) holds and $Z \in \mathscr{Z}(X)$, find $S \in \mathscr{Z}(Y)$ such that $Z = S \cap X$. It is routine to verify that $\mathrm{cl}_X \mathrm{int}_X Z = X \cap \mathrm{cl}_Y \mathrm{int}_Y S$, and so (b) implies (c).

Now suppose that X is dense and $\mathscr{Z}^{\#}$-embedded in the quasi F-space Y. Let Z_1 and Z_2 be disjoint zero-sets of X. Then there exist disjoint zero-sets S_1 and S_2 of X such that $Z_i \subseteq \mathrm{int}_X S_i$ $(i = 1,2)$; see 1.15 of **[GJ]**. As X is $\mathscr{Z}^{\#}$-embedded in Y there exist zero-sets T_1 and T_2 of Y such that $\mathrm{cl}_X \mathrm{int}_X S_i = X \cap \mathrm{cl}_Y \mathrm{int}_Y T_i$ $(i = 1,2)$. Since

$\operatorname{int}_X S_1 \cap \operatorname{int}_X S_2 = \varnothing$, it follows that $\operatorname{int}_Y T_1 \cap \operatorname{int}_Y T_2 = \varnothing$. Since Y is a quasi F-space it follows from 2.10 that $\operatorname{cl}_Y \operatorname{int}_Y T_1 \cap \operatorname{cl}_Y \operatorname{int}_Y T_2 = \varnothing$. Thus $\operatorname{cl}_Y Z_1 \cap \operatorname{cl}_Y Z_2 = \varnothing$, and it follows from 6.4 of **[GJ]** that X is C^*-embedded in Y. Thus (c) implies (a).

In 5.1 of **[DHH]** it is shown that a Tychonoff space X is a quasi F-space iff βX is. Our final claim then follows immediately from (a). □

3.3. PROPOSITION. *If X is a space and $\Phi_{\beta X}^{\leftarrow}[X]$ is a quasi F-space, then the pair $(\Phi_{\beta X}^{\leftarrow}[X], \Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X])$ is the quasi F-cover $(QF(X), \Phi_X)$ of X.*

PROOF. Denote $\Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X]$ by g and $\Phi_{\beta X}^{\leftarrow}[X]$ by T. Since g is the restriction of a perfect irreducible continuous surjection to a dense preimage, it follows that g is a perfect irreducible continuous surjection from T onto X (see 1.8(f) of **[PW]**). To verify that $(T, g) = (QF(X), \Phi_X)$ it remains to show that if K is any other quasi F-space and Ψ: $K \to X$ is a perfect irreducible continuous surjection, then there exists a continuous surjection f: $K \to T$ such that $g \circ f = \psi$ (see 1.1(iii), (iv) of the Introduction).

If K and Ψ are as described, then the Stone extension $\beta\Psi$ maps βK onto βX. By 5.1 of **[DHH]** (see above) βK is a quasi F-space, so by the minimality of $(QF(\beta X), \Phi_{\beta X})$ (see 1.1(iii)), there exists a continuous function h: $\beta K \to QF(\beta X)$ such that $\Phi_{\beta X} \circ h = \beta\Psi$.

If $\alpha \in K$ then $\Phi_{\beta X}(h(\alpha)) = (\beta\Psi)(\alpha) \in X$, so $h(\alpha) \in T$. If $\alpha \in \beta K \setminus K$ then $\Phi_{\beta X}(h(\alpha)) = (\beta\Psi)(\alpha) \in \beta X \setminus X$ since Ψ is perfect (see 1.5 of **[HI]** or 4.2(g) of **[PW]**). It follows by 1.5 of **[HI]** that $h|K$ is perfect and that $g \circ (h|K) = \Psi$. The irreducibility of $h|K$ follows immediately from that of Ψ and g, so $h|K$ is the required f. □

3.4. DEFINITION. The statement "$QF(\beta X) = \beta(QF(X))$" means:

(a) $(QF(X), \Phi_X) = (\Phi_{\beta X}^{\leftarrow}[X], \Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X])$ ("up to uniqueness" as described in (iv) of the Introduction), and

(b) $\Phi_{\beta X}^{\leftarrow}[X]$ is C^*-embedded in $QF(\beta X)$.

3.5. THEOREM. *The following are equivalent for a space X:*

(a) $QF(\beta X) = \beta(QF(X))$,

(b) $\Phi_{\beta X}^{\leftarrow}[X]$ *is $\mathscr{Z}^{\#}$-embedded in $QF(\beta X)$,*

(c) $\Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X]$ *is a $\mathscr{Z}^{\#}$-irreducible mapping onto X.*

PROOF. (b) ⇒ (a) By 3.2 $\Phi_{\beta X}^{\leftarrow}[X]$ is C^*-embedded in $QF(\beta X)$ and is a quasi F-space. Thus (a) follows from 3.3.

(a) ⇒ (c) Let $Z \in \mathscr{Z}(\Phi_{\beta X}^{\leftarrow}[X])$. By (a) $\Phi_{\beta X}^{\leftarrow}[X]$ is C^*-embedded in $QF(\beta X)$ so by 3.2 there exists $S \in \mathscr{Z}(QF(\beta X))$ such that

$$\operatorname{cl}_{\Phi_{\beta X}^{\leftarrow}[X]} \operatorname{int}_{\Phi_{\beta X}^{\leftarrow}[X]} Z = \Phi_{\beta X}^{\leftarrow}[X] \cap \operatorname{cl}_{QF(\beta X)} \operatorname{int}_{QF(\beta X)} S.$$

By 2.9(c) $\Phi_{\beta X}$ is $\mathscr{Z}^{\#}$-irreducible, so there exists $T \in \mathscr{Z}(\beta X)$ such that $\Phi_{\beta X}[\operatorname{cl}_{QF(\beta X)} \operatorname{int}_{QF(\beta X)} S] = \operatorname{cl}_{\beta X} \operatorname{int}_{\beta X} T$. Thus

$$\left(\Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X]\right)\left[\operatorname{cl}_{\Phi_{\beta X}^{\leftarrow}[X]} \operatorname{int}_{\Phi_{\beta X}^{\leftarrow}[X]} Z\right] = X \cap \operatorname{cl}_{\beta X} \operatorname{int}_{\beta X} T = \operatorname{cl}_X \operatorname{int}_X (T \cap X).$$

It now follows quickly that $\Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X]$ is $\mathscr{Z}^{\#}$-irreducible.

(c) ⇒ (b) Let $S \in \mathscr{Z}(\Phi_{\beta X}^{\leftarrow}[X])^{\#}$. By hypothesis there exists $T \in \mathscr{Z}(X)^{\#}$ such that $(\Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X])[S] = T$. As X is C^*-embedded (and thus $\mathscr{Z}^{\#}$-embedded) in βX, there exists $A \in \mathscr{Z}(\beta X)^{\#}$ such that $T = X \cap A$. As $\Phi_{\beta X}$ is $\mathscr{Z}^{\#}$-irreducible (see 2.9(c)) there exists $B \in \mathscr{Z}(QF(\beta X))^{\#}$ such that $\Phi_{\beta X}[B] = A$. But $\Phi_{\beta X}[\mathrm{cl}_{QF(\beta X)} S] = \mathrm{cl}_{\beta X} \Phi_X[S] = \mathrm{cl}_{\beta X} T = A$. As $\mathrm{cl}_{QF(\beta X)} S \in \mathscr{R}(QF(\beta X))$, it follows from 2.7(b) that $\mathrm{cl}_{QF(\beta X)} S = B$. Thus $S = B \cap \Phi_{\beta X}^{\leftarrow}[X]$, and it follows that $\Phi_{\beta X}^{\leftarrow}[X]$ is $\mathscr{Z}^{\#}$-embedded in $QF(\beta X)$. □

Theorem 3.5 gives a criterion for determining when $(\Phi_{\beta X}^{\leftarrow}[X], \Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X]) = (QF(X), \Phi_X)$. We now show that this criterion is is satisfied for a large class of spaces, namely the weakly Lindelöf spaces.

Recall (see **[CHN]**) that a Tychonoff space X is *weakly Lindelöf* if each open cover of X contains a countable subfamily $\mathscr{F}$ such that $\bigcup \mathscr{F}$ is dense in X. The following theorem gives two known properties of weakly Lindelöf spaces that we will need. The first is 1.5 of **[CHN]**; the second is an easily proved "folk theorem."

3.6. PROPOSITION. (a) *A cozero-set of a weakly Lindelöf space is weakly Lindelöf.*

(b) *Every space of countable cellularity is weakly Lindelöf.*

We also will need the following (to our knowledge) new results concerning weakly Lindelöf spaces.

3.7. PROPOSITION. (a) *If f: $X \to Y$ is a perfect continuous irreducible surjection and Y is weakly Lindelöf, then X is weakly Lindelöf.*

(b) *If X is weakly Lindelöf and a dense subspace of the Tychonoff space T, then X is $\mathscr{Z}^{\#}$-embedded in T.*

PROOF. (a) Let $\mathscr{C}$ be an open cover of X. If $V \subset X$, denote $Y \setminus f[X \setminus V]$ by V^*. Then $\mathscr{A} = \{V^*\colon V$ is a union of finitely many members of $\mathscr{C}\}$ is an open cover of Y. (This follows from the fact that f is a perfect continuous surjection; see 1.8(c) of **[PW]**, for example.) As Y is weakly Lindelöf, there exists $\{V_n^*\colon n \in \mathbf{N}\} \subseteq \mathscr{A}$ such that $\bigcup\{V_n^*\colon n \in \mathbf{N}\}$ is dense in Y. As f is irreducible, it follows from 2.7(a) that $\bigcup\{f^{\leftarrow}[V_n^*]\colon n \in \mathbf{N}\}$ is dense in X. Evidently $f^{\leftarrow}[V_n^*] \subseteq V_n$, so $\bigcup\{V_n\colon n \in \mathbf{N}\}$, which is obviously expressible as a union of countably many members of $\mathscr{C}$, is dense in X. It follows that X is weakly Lindelöf.

(b) Let $Z \in \mathscr{Z}(X)$ and choose an open set W of T such that $X \setminus Z = W \cap X$. Since T is Tychonoff there is a family $\{C_\alpha\colon \alpha \in I\}$ of cozero-sets of T such that $W = \bigcup\{C_\alpha\colon \alpha \in I\}$. By 3.6(a) $X \setminus Z$ is weakly Lindelöf so there exists a countable subset H of I such that $\bigcup\{C_\alpha \cap X\colon \alpha \in H\}$ is dense in $X \setminus Z$ (and hence in W). Let $C = \bigcup\{C_\alpha\colon \alpha \in H\}$; then C is a cozero-set of T (see 1.14 of **[GJ]**) and evidently $\mathrm{cl}_T(X \setminus Z) = \mathrm{cl}_T C$. It immediately follows that $\mathrm{cl}_X \mathrm{int}_X Z = X \cap \mathrm{cl}_T \mathrm{int}_T(T \setminus C)$, and as $T \setminus C \in \mathscr{Z}(T)$ our result follows. □

3.8. THEOREM. *Let X be a weakly Lindelöf space and let αX be any compactification of X. Then $(\Phi_{\alpha X}^{\leftarrow}[X], \Phi_{\alpha X} | \Phi_{\alpha X}^{\leftarrow}[X])$ is the quasi F-cover of X and $QF(\alpha X) = \beta(\Phi_{\alpha X}^{\leftarrow}[X])$. In particular $QF(\beta X) = \beta(QF(X))$.*

PROOF. As $\Phi_{\alpha X} | \Phi_{\alpha X}^{\leftarrow}[X]$: $\Phi_{\alpha X}^{\leftarrow}[X] \to X$ is a perfect irreducible continuous surjection, by 3.7(a) $\Phi_{\alpha X}^{\leftarrow}[X]$ is a weakly Lindelöf space that is dense in $QF(\alpha X)$. By 3.7(b) it is $\mathscr{Z}^{\#}$-embedded in αX, and hence by 3.2 is C^*-embedded in $QF(\alpha X)$. Thus $QF(\alpha X) = \beta(\Phi_{\alpha X}^{\leftarrow}[X])$ and $\Phi_{\alpha X}^{\leftarrow}[X]$ is a quasi F-space by 5.1 of **[DHH]**. The minimality of the pair $(\Phi_{\alpha X}^{\leftarrow}[X], \Phi_{\alpha X} | \Phi_{\alpha X}^{\leftarrow}[X])$ is verified by an argument identical to that used to prove 3.3. $\square$

As we noted in 3.6(b), all spaces of countable cellularity are weakly Lindelöf; so are all Lindelöf spaces. Thus the above theorem applies to a reasonably large class of spaces.

3.9. DEFINITION. An *almost P-set* (resp. *P-set*) of a space X is a compact subset K of X such that if $Z \in \mathscr{Z}(X)$ and $K \subseteq Z$, then $K \subseteq \operatorname{cl}_X \operatorname{int}_X Z$ (resp. $K \subseteq \operatorname{int}_X Z$). A point p of X is an *almost P-point* (resp. *P-point*) if $\{p\}$ is an almost P-set (resp. P-set). The set of almost P-points (resp. P-points) of X is denoted by $AP(X)$ (resp. $P(X)$).

P-sets and almost P-sets are discussed in **[DF]** and **[Vek$_1$]**; P-points are investigated in 4L of **[GJ]**. Note that it is possible for a set to be an almost P-set but not a P-set. Such sets will exist in any infinite compact space in which every zero-set is regular closed; $\beta\mathbf{N} \setminus \mathbf{N}$ is such a space (see **[FG** or **Wa]**). We now investigate the relationship between P-points (resp. almost P-points) of X and those of $QF(X)$ when X is compact. The following proposition is analogous to 3.4 of **[DF]** and 1.4 of **[vM]**.

3.10. PROPOSITION. *If K is a nonempty closed nowhere dense almost P-set of the quasi F-space X, then the quotient space $X \setminus K$ is a quasi F-space.*

PROOF. Let $Y = X/K$, let f: $X \to Y$ be the quotient map, and let $f[K] = \{p\}$. Note that Y is easily verified to be Tychonoff and f is easily verified to be perfect.

Let C be a dense cozero-set of Y. Then $p \in C$; for suppose not. Then $K \subseteq X \setminus f^{\leftarrow}[C]$, and since K is nowhere dense, f is irreducible. Thus by 2.7(a) $X \setminus f^{\leftarrow}[C]$ would be a nowhere dense zero-set of X containing K, contradicting the fact that K is an almost P-set. Thus $p \in C$, so $K \subseteq f^{\leftarrow}[C]$.

Let Z_1 and Z_2 be disjoint zero-sets of C. Thus $f^{\leftarrow}[Z_1]$ and $f^{\leftarrow}[Z_2]$ are disjoint zero-sets of $f^{\leftarrow}[C]$, and either $K \subseteq f^{\leftarrow}[Z_i]$ or $K \cap f^{\leftarrow}[Z_i] = \varnothing$ $(i = 1, 2)$. Thus $\operatorname{cl}_X f^{\leftarrow}[Z_1] \cap \operatorname{cl}_X f^{\leftarrow}[Z_2] = \varnothing$ since $f^{\leftarrow}[C]$ is a dense cozero-set of X and hence is C^*-embedded in X. As f is closed, continuous, and one-to-one except on K, it follows that $\operatorname{cl}_Y Z_1 \cap \operatorname{cl}_Y Z_2 = \varnothing$. Thus by 6.5 of **[GJ]** C is C^*-embedded in Y. $\square$

We now show that $AP(X)$ and $AP(QF(X))$ are "the same" when X is compact. Specifically:

3.11. THEOREM. *Let X be compact. Then $\Phi_X^{\leftarrow}[AP(X)] = AP(QF(X))$, and $\Phi_X | AP(QF(X))$ is a homeomorphism from $AP(QF(X))$ onto $AP(X)$.*

PROOF. Suppose $p \in AP(X)$, and suppose r and s are distinct points of $\Phi_X^{\leftarrow}(p)$. Denote $QF(X)$ by Q. There exist $S_1, S_2 \in \mathscr{Z}(Q)$ such that $r \in \operatorname{cl}_Q \operatorname{int}_Q S_1$,

$s \in \mathrm{cl}_Q \mathrm{int}_Q S_2$, and $S_1 \cap S_2 = \varnothing$. Find $Z_i \in \mathscr{Z}(X)$ such that $\mathrm{cl}_Q \mathrm{int}_Q S_i = \mathrm{cl}_Q \Phi_X^{\leftarrow}[\mathrm{int}_X Z_i]$ $(i = 1,2)$; see 2.9(b). Thus $p \in \mathrm{cl}_X \mathrm{int}_X Z_1 \cap \mathrm{cl}_X \mathrm{int}_X Z_2$, and by 2.7(b) $\mathrm{int}_X Z_1 \cap \mathrm{int}_X Z_2 = \varnothing$. Thus $p \in (Z_1 \cap Z_2) \setminus \mathrm{cl}_X \mathrm{int}_X (Z_1 \cap Z_2)$, contradicting the assumption that $p \in AP(X)$. Thus $|\Phi_X^{\leftarrow}(p)| = 1$.

Let $\{r\} = \Phi_X^{\leftarrow}(p)$, and suppose $r \in S \in \mathscr{Z}(Q)$. If $r \notin \mathrm{cl}_Q \mathrm{int}_Q S$, there exists $T \in \mathscr{Z}(Q)$ such that $r \in T$ and $T \cap \mathrm{cl}_Q \mathrm{int}_Q S = \varnothing$. Thus $T \cap S$ is a nowhere dense zero-set of Q. Set $V = Q \setminus (T \cap S)$. Then V is a dense σ-compact subset of Q (as cozero-sets are F_σ-sets and Q is compact), and $\Phi_X^{\leftarrow}(p) \cap V = \varnothing$. Thus $p \in X \setminus \Phi_X[V]$, so as $\Phi_X[V]$ is a dense σ-compact subset of X, there exists a nowhere dense zero-set H of X such that $p \in H$. Thus $p \in H \setminus \mathrm{cl}_X \mathrm{int}_X H$, which contradicts the fact that $p \in AP(X)$. Thus $\Phi_X^{\leftarrow}[AP(X)] \subseteq AP(Q)$.

Finally, let $r \in AP(Q)$ and suppose $\Phi_X(r) \in Z \in \mathscr{Z}(X)$. Then $r \in \Phi_X^{\leftarrow}[Z] \in \mathscr{Z}(Q)$, so $r \in \mathrm{cl}_Q \mathrm{int}_Q \Phi^{\leftarrow}[Z]$ as $r \in AP(Q)$. By 2.7(b) $\Phi_X[\mathrm{cl}_Q \mathrm{int}_Q \Phi_X^{\leftarrow}[Z]] \in \mathscr{R}(X)$, so $\Phi_X[\mathrm{cl}_Q \mathrm{int}_Q \Phi_X^{\leftarrow}[Z]] \subseteq \mathrm{cl}_X \mathrm{int}_X Z$. Thus $\Phi_X(r) \in \mathrm{cl}_X \mathrm{int}_X Z$, so $\Phi_X(r) \in AP(X)$. Thus $\Phi_X[AP(Q)] \subseteq AP(X)$, so $AP(Q) \subseteq \Phi_X^{\leftarrow}[AP(X)]$. The theorem follows, since $\Phi_X | AP(Q) = \Phi_X | \Phi_X^{\leftarrow}[AP(X)]$, and the restriction of a perfect map to a complete preimage is perfect (and thus closed); thus $\Phi_X | \Phi_X^{\leftarrow}[AP(X)]$ is closed, continuous, and one-to-one, and hence a homeomorphism. □

We note in passing the following related result.

3.12. PROPOSITION. *Let X be a compact space. If $p \in P(X)$ then the unique point in $\Phi_X^{\leftarrow}(p)$ is a P-point of $QF(X)$.*

PROOF. By 3.11 $|\Phi_X^{\leftarrow}(p)| = 1$; let $\Phi_X^{\leftarrow}(p) = \{r\}$. Let $r \in S \in \mathscr{Z}(Q)$ (as above, Q denotes $QF(X)$). It follows from 3.11 that $r \in \mathrm{cl}_Q \mathrm{int}_Q S$. By 2.9(b) there exists $Z \in \mathscr{Z}(X)$ such that $\mathrm{cl}_Q \mathrm{int}_Q S = \mathrm{cl}_Q \Phi_X^{\leftarrow}[\mathrm{int}_X Z]$. Thus $p = \Phi_X(r) \in \mathrm{cl}_X \mathrm{int}_X Z$. As $p \in P(X)$, it follows that $p \in \mathrm{int}_X Z$, and it follows from our choice of Z that $r \in \mathrm{int}_Q S$. Thus $r \in P(Q)$. □

As is customary, we think of the ordinal α as the set of all ordinals less than α. It is well known that $\beta\omega_1 = \omega_1 + 1$ (ordinal numbers are given the order topology). We will show that $(QF(\omega_1), \Phi_{\omega_1}) = (\Phi_{\beta\omega_1}^{\leftarrow}[\omega_1], \Phi_{\beta\omega_1} | \Phi_{\beta\omega_1}^{\leftarrow}[\omega_1])$ but that $QF(\beta\omega_1) \neq \beta(QF(\omega_1))$. We need preliminary results.

3.13. LEMMA. *Let B be a clopen set of the Tychonoff space X. Then $QF(X) = QF(B) \oplus QF(X \setminus B)$ (where $\oplus$ denotes topological sum).*

PROOF. Obviously $QF(X) = \Phi_X^{\leftarrow}[B] \oplus \Phi_X^{\leftarrow}[X \setminus B]$. Clopen subsets of quasi F-spaces are easily seen to be quasi F, and $\Phi_X | \Phi_X^{\leftarrow}[B]$ is perfect and irreducible as Φ_X is. If Ψ: $K \to B$ is a perfect irreducible continuous surjection from the quasi F-space K onto B, then $\Psi \oplus \Phi_{X \setminus B}$ is a perfect irreducible continuous surjection from the space $K \oplus QF(X \setminus B)$ onto X. As $K \oplus QF(X \setminus B)$ is easily seen to be quasi F, by the minimality of $QF(X)$ there is a continuous surjection g: $K \oplus QF(X \setminus B) \to QF(X)$ such that $\Phi_X \circ g = \psi \oplus \Phi_{X \setminus B}$. One easily checks that $\Phi_X | \Phi_X^{\leftarrow}[B] \circ g | K = \Psi$. This verifies the minimality of $(\Phi_X^{\leftarrow}[B], \Phi_X | \Phi_X^{\leftarrow}[B])$, and shows that $\Phi_X^{\leftarrow}[B] = QF(B)$. Similarly $\Phi_X^{\leftarrow}[X \setminus B] = QF(X \setminus B)$, and the lemma follows. □

3.14. LEMMA. *If each point of the Tychonoff space X has a clopen neighborhood whose quasi F-cover is extremally disconnected, then $QF(X)$ is extremally disconnected.*

PROOF. Let U and V be disjoint open sets of $QF(X)$ (which we denote by Q) and let $r \in \mathrm{cl}_Q U \cap \mathrm{cl}_Q V$. Find a clopen neighborhood B of $\Phi_X(r)$ in X whose quasi F-cover is extremally disconnected. It follows from 3.13 that $\Phi_X^{\leftarrow}[B] = QF(B)$ and hence $\Phi_X^{\leftarrow}[B]$ is extremally disconnected. Evidently

$$r \in \mathrm{cl}_{\Phi_X^{\leftarrow}[B]}(U \cap \Phi_X^{\leftarrow}[B]) \cap \mathrm{cl}_{\Phi_X^{\leftarrow}[B]}(V \cap \Phi_X^{\leftarrow}[B]),$$

which contradicts the extremal disconnectedness of $\Phi_X^{\leftarrow}[B]$. Thus $\mathrm{cl}_Q U \cap \mathrm{cl}_Q V = \varnothing$ and $QF(X)$ is extremally disconnected. $\square$

3.15. COROLLARY. *If each point of the space X has a clopen neighborhood of countable cellularity, then $QF(X)$ is extremally disconnected.*

PROOF. Observe that a quasi F-space of countable cellularity is extremally disconnected. (Obviously every dense and open $U \subset T$ is C^*-embedded, since it contains a dense (C^*-embedded) cozero set of T.) (In **[DHH]** 4.7 this was shown for compact spaces.) Therefore, 3.14 can be applied. $\square$

3.16. THEOREM. *$QF(\omega_1)$ is the absolute $E\omega_1$ of ω_1, and $\beta(QF(\omega_1)) \neq QF(\beta\omega_1)$.*

PROOF. Since ω_1 is locally countable, it follows from 3.15 that $QF(\omega_1) = E(\omega_1)$, which is the space $\{\alpha \in \beta D(\omega_1)$: α is in the closure of some countable subset of $D(\omega_1)\}$, where $D(\omega_1)$ denotes the discrete space of cardinality ω_1 (see **[War]**). Now $\beta\omega_1 \setminus \omega_1 = \{\omega_1\}$, and ω_1 is a P-point of $\beta\omega_1$. However,

$$k_{\beta\omega_1}^{\leftarrow}(\{\omega_1\}) = \beta D(\omega_1) \setminus E(\omega_1)$$

which is infinite (see 5.13 of **[Wa]**). (Here $k_{\beta\omega_1}$ denotes the canonical map from $E(\beta\omega_1)$ onto $\beta\omega_1$.) Note that $D(\omega_1) \subseteq E(\omega_1) \subseteq \beta D(\omega_1)$ from the above description of $E(\omega_1)$. Thus $E(\omega_1) = \beta D(\omega_1)$ (see 6.7 of **[GJ]**). But $\beta E(\omega_1) = E(\beta\omega_1)$, as noted in the first paragraph of §3, so $\beta D(\omega_1) = E(\beta\omega_1)$. Thus by 3.11 $\beta(QF(\omega_1)) \neq QF(\beta\omega_1)$. (In fact $QF(\beta\omega_1)$ is the one-point compactification of $E(\omega_1)$.) $\square$

We now have an example of a space X for which $(\Phi_{\beta X}^{\leftarrow}[X], \Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X]) = (QF(X), \Phi_X)$ but for which $QF(\beta_X) \neq \beta(QF(X))$. We next produce an example of a space X for which $\Phi_{\beta X}^{\leftarrow}[X]$ is not a quasi F-space. We need some preliminary concepts and results.

3.17. DEFINITION. A Tychonoff space X is called *weakly realcompact* if for each point $p \in \beta X \setminus X$ there exists a nowhere dense zero-set of βX containing p.

3.18. LEMMA. (a) *Realcompact and weakly Lindelöf spaces are weakly realcompact.*
(b) *X is weakly realcompact iff $AP(\beta X) \subseteq X$.*
(c) *If X is weakly realcompact and $X \subseteq T \subseteq \beta X$ then T is weakly realcompact.*

PROOF. (a) It is well known that if X is realcompact and if $p \in \beta X \setminus X$, then there exists $Z \in \mathscr{Z}(\beta X)$ such that $p \in Z$ and $Z \cap X = \varnothing$ (see 5.11(b) of **[PW]**). Thus X is weakly realcompact. If X is weakly Lindelöf and $p \in \beta X \setminus X$, then $\beta X \setminus \{p\}$ is a union of cozero-sets of βX which covers X. There is a countable subfamily $\{C_i\colon i \in \mathbf{N}\}$ of these cozero-sets such that $X \cap [\bigcup\{C_i\colon i \in \mathbf{N}\}]$ is dense in X. Then $\beta X \setminus \bigcup\{C_i\colon i \in \mathbf{N}\} \in \mathscr{Z}(\beta X)$, contains p, and is nowhere dense. Hence X is weakly realcompact.

(b) If X is weakly realcompact then each point of $\beta X \setminus X$ belongs to a nowhere dense zero-set of βX, and hence not to $AP(\beta X)$. Conversely, if $AP(\beta X) \subseteq X$ and $p \in \beta X \setminus X$, there exists $Z \in \mathscr{Z}(\beta X)$ such that $p \in Z \setminus \mathrm{cl}_{\beta X}\mathrm{int}_{\beta X} Z$. There exists $T \in \mathscr{Z}(\beta X)$ such that $p \in T$ and $T \cap \mathrm{cl}_{\beta X}\mathrm{int}_{\beta X} Z = \varnothing$. Thus $Z \cap T$ is a nowhere dense zero-set of βX containing p. □

(c) This is immediate once one observes that $\beta T = \beta X$ (see 6.7 of [**GJ**]).

3.19. Lemma. *Let X be weakly realcompact (resp. realcompact). If $\Phi_{\beta X}^{\leftarrow}[X]$ is not C^*-embedded in $QF(\beta X)$ then there exists a weakly realcompact (resp. realcompact) spacxe T such that $X \subset T \subset \beta X$ and $\Phi_{\beta T}^{\leftarrow}[T]$ is not a quasi F-space.*

Proof. First suppose that X is weakly realcompact. If the conclusion of the lemma fails in this case, by 3.18(c) it is true that $\Phi_{\beta X}^{\leftarrow}[T]$ must be a quasi F-space whenever $X \subseteq T \subseteq \beta X$. Note that $\beta X = \beta T$ as observed in the proof of 3.18(c) above, so $QF(\beta X) = QF(\beta T)$. If $p \in QF(\beta X) \setminus \Phi_{\beta X}^{\leftarrow}[X]$ then $\Phi_{\beta X}(p) \in \beta X \setminus X$ so there exists a nowhere dense zero-set $Z(p)$ of βX with $\Phi_X(p) \in Z(p)$. By assumption $\Phi_{\beta X}^{\leftarrow}[X \cup Z(p)]$ is a quasi F-space and as $\Phi_{\beta X}$ is irreducible,

$$\Phi_{\beta X}^{\leftarrow}[X \cup Z(p)] \setminus \Phi_{\beta X}^{\leftarrow}[Z(p)]$$

is a dense cozero-set of $\Phi_{\beta X}^{\leftarrow}[X \cup Z(p)]$ that is contained in $\Phi_{\beta X}^{\leftarrow}[X]$. Hence $\Phi_{\beta X}^{\leftarrow}[X \cup Z(p)] \setminus \Phi_{\beta X}^{\leftarrow}[Z(p)]$, and therefore $\Phi_{\beta X}^{\leftarrow}[X]$, is C^*-embedded in $\Phi_{\beta X}^{\leftarrow}[X \cup Z(p)]$. Thus $\Phi_{\beta X}^{\leftarrow}[X]$ is C^*-embedded in $\Phi_{\beta X}^{\leftarrow}[X] \cup \{p\}$ for each $p \in QF(\beta X) \setminus \Phi_{\beta X}^{\leftarrow}[X]$. It follows from 6H of [**GJ**] that $\Phi_{\beta X}^{\leftarrow}[X]$ is C^*-embedded in $QF(\beta X)$.

If X is realcompact, the argument proceeds in essentially the same manner. In the above we used the fact that $X \cup Z(p)$ is weakly realcompact; here we use the fact that $X \cup Z(p)$, being the union of a realcompact subspace and a compact subspace of βX, is realcompact (see 8.16 of [**GJ**]). □

3.20. Lemma. *Let K be a compact space. If $QF(K) = EK$ then $\mathscr{R}(K) = \mathscr{Z}(K)^\#$.*

Proof. If $QF(K) = EK$ then by 2.9(c) $k_K\colon EK \to K$ is $\mathscr{Z}^\#$-irreducible, and induces a lattice isomorphism $A \to k_K[A]$ from $\mathscr{Z}^\#(EK)$ onto $\mathscr{Z}(K)^\#$. But as every regular closed set of EK is clopen and hence a zero-set, and as $\{k_K[A]\colon A \in \mathscr{R}(EK)\} = \mathscr{R}(K)$ (see 2.7(b)), the result follows. □

In 3.20 we cannot drop the condition that K is compact, as 3.20 fails if K is replaced by ω_1; see 3.16. The converse of 3.20 is also true; see 4.4.

3.21. Example. Let $X = \omega_1 \times [0,1]$. We show that there exists a space T such that $X \subseteq T \subseteq \beta X$ and $\Phi_{\beta X}^{\leftarrow}[T]$ is not a quasi F-space. Note that since $\omega_1 \times [0,1]$ is a pseudocompact product (see 9.14 of [**GJ**]), it follows that $\beta X = \beta\omega_1 \times [0,1] = (\omega_1 + 1) \times [0,1]$, and so $\beta X \setminus X = \{\omega_1\} \times [0,1]$ (see 8.12 of [**Wa**]).

If $(\alpha, t) \in X$ then $[0, \alpha + 1] \times [0,1]$ is a clopen X-neighborhood of (α, t) with countable cellularity, so by 3.15 $QF(X)$ is extremally disconnected.

Partition the set of isolated points (i.e. nonlimit ordinals) of $\omega_1 + 1$ into two uncountable sets I and J, and let $A = (\mathrm{cl}_{\omega_1+1} J) \times [0,1]$. Then $A \in \mathscr{R}(\beta X)$. Suppose that $Z \in \mathscr{Z}(\beta X)$ and $A = \mathrm{cl}_{\beta X}\mathrm{int}_{\beta X} Z$. Then $\{\omega_1\} \times [0,1] \subseteq \mathrm{cl}_{\beta X}\mathrm{int}_{\beta X} Z$, so if $t \in [0,1]$ then $(\omega_1, t) \in \mathscr{Z}((\omega_1 + 1) \times \{t\})$. Thus as (ω_1, t) is a P-point of $(\omega_1 + 1) \times \{t\}$, there exists $\alpha_t < \omega_1$ such that $(\alpha, \omega_1) \times \{t\} \subseteq Z$.

Let $\delta = \sup\{\alpha_t\colon t \text{ is a rational number in } [0,1]\}$. Evidently $\delta < \omega_1$ as the rationals are countable, and it follows that $(\delta, \omega_1) \times ([0,1] \cap \mathbf{Q}) \subseteq Z$ ($\mathbf{Q}$ denotes the rationals). Thus $[\delta, \omega_1] \times [0,1] \subseteq \mathrm{cl}_{\beta X}\mathrm{int}_{\beta X} Z$. If $\gamma \in I$, then $\{\gamma\} \times [0,1]$ is disjoint from A, and if we choose γ to be in (δ, ω_1), then $\{\gamma\} \times [0,1] \subseteq \mathrm{cl}_{\beta X}\mathrm{int}_{\beta X} Z$. This is a contradiction, and so $A \notin \mathscr{Z}(\beta X)^{\#}$. It follows from 3.20 that $QF(\beta X)$ is not extremally disconnected. As $QF(X)$ is extremally disconnected, it follows that $\beta(QF(X)) \neq QF(\beta X)$.

Finally, note that X is weakly realcompact, for if $t \in [0,1]$ then $(\omega_1 + 1) \times \{t\}$ is a nowhere dense zero-set of βX that contains the point (ω_1, t) of $\beta X \setminus X$. Thus by 3.19 there exists a space T with $X \subseteq T \subseteq \beta X$ such that $\Phi_{\beta T}^{\leftarrow}[T]$ is not a quasi F-space. (In fact we can let $T = X \cup \{(\omega_1, t)\}$ for some $t \in [0,1]$.) This concludes the example. $\square$

In [**ZK**] the "sequential absolute" $a_s T$ of an arbitrary Tychonoff space T is constructed. If T is compact then $a_s T = QF(T)$. However, in Lemma 3 of [**ZK**] it is shown that if T is Tychonoff, then $a_s T = \Phi_{\beta T}^{\leftarrow}[QF(\beta T)]$. Thus the "sequential absolute" of Tychonoff space does not always coincide with its quasi F-cover.

4. Quasi F-spaces and projective objects. Next we present some category-theoretic interpretations of our results. Chapter 10 of [**Wa**] provides useful background on category theory in a topological setting.

4.1. Definition. A category $\mathscr{C}$ is called a *topological category* if:

(a) Its objects are topological spaces.

(b) If X is an object of $\mathscr{C}$ and $h\colon X \to Y$ is a homeomorphism then Y is an object of $\mathscr{C}$ and h is a morphism of $\mathscr{C}$.

(c) All morphisms of $\mathscr{C}$ are continuous functions (but not necessarily conversely).

The above definition is a slight modification of that appearing in §9.3 of [**PW**].

One original motivation for studying extremally disconnected spaces and absolutes was the desire to characterize projective objects (and projective covers) in various topological categories.

4.2. Definition. (a) An object X of a topological catetgory $\mathscr{C}$ is called *projective in* $\mathscr{C}$ if, whenever Y and Z are objects of $\mathscr{C}$, $f\colon X \to Y$ and $g\colon Z \to Y$ are morphisms of $\mathscr{C}$, and $g[Z] = Y$, there exists a morphism $h\colon X \to Z$ in $\mathscr{C}$ such that $g \circ h = f$.

(b) A *projective cover* of an object Y of a topological category $\mathscr{C}$ is a pair (X, f) where X is a projective object of $\mathscr{C}$ and $f\colon X \to Y$ is a morphism of $\mathscr{C}$ that is a closed irreducible surjection.

Gleason [**Gl**] showed that the projective objects in the category of compact spaces and continuous functions are precisely the extremally disconnected spaces, and that each compact space has an essentially unique projective cover, namely its absolute

(EX, k_X). Iliadis [**Il**] and Banaschewski [**B**] proved similar results for the category of Hausdorff spaces (resp. regular spaces) and perfect continuous functions. See Chapters 6 and 9 of [**PW**] for an extensive discussion of this topic.

Our purpose in this section is to show that the quasi F-spaces are the projective objects in the category of Tychonoff spaces and $\mathscr{Z}^{\#}$-irreducible functions (which we denote henceforth by $\mathscr{T}^{\#}$), and to investigate the conditions under which an object in this category has a projective cover.

4.3. THEOREM. *In $\mathscr{T}^{\#}$ the projective objects are precisely the quasi F-spaces.*

PROOF. Suppose that X is a quasi F-space and that $f\colon X \to Y$ and $g\colon Z \to Y$ are $\mathscr{Z}^{\#}$-irreducible functions (note that this automatically means that $g[Z] = Y$). Then the Stone extension $\beta f\colon \beta X \to \beta Y$ is $\mathscr{Z}^{\#}$-irreducible; to see this note that if $Z \in \mathscr{Z}(\beta X)$ then

$$\beta f\left[\mathrm{cl}_{\beta X}\,\mathrm{int}_{\beta X} Z\right] = \beta f\left[\mathrm{cl}_{\beta X}\,\mathrm{cl}_X\,\mathrm{int}_X(Z \cap X)\right] = \mathrm{cl}_{\beta Y} f\left[\mathrm{cl}_X\,\mathrm{int}_X(Z \cap X)\right].$$

As f is $\mathscr{Z}^{\#}$-irreducible and Y is z-embedded in βY, there exists $S \in \mathscr{Z}(\beta Y)$ such that $\mathrm{cl}_{\beta Y} f[\mathrm{cl}_X\,\mathrm{int}_X(Z \cap X)] = \mathrm{cl}_{\beta Y}\,\mathrm{int}_{\beta Y} S$. It follows that βf is $\mathscr{Z}^{\#}$-irreducible. Similarly $\beta g\colon \beta Z \to \beta Y$ is $\mathscr{Z}^{\#}$-irreducible. By 5.1 of [**DHH**] βX is a quasi F-space, so by 2.12 there is a perfect irreducible continuous surjection $h\colon \beta X \to QF(\beta Y)$ such that $\Phi_{\beta Y} \circ h = \beta f$, and by 2.13 there exists a $\mathscr{Z}^{\#}$-irreducible function $k\colon QF(Y) \to Z$ such that $\beta g \circ k = \Phi_{\beta Y}$. Let $h' = k \circ h \,|\, X$. Arguing as in the proof of 3.3 we see that h' is a perfect irreducible continuous surjection from X onto Z and that $f = g \circ h'$. As f and g are $\mathscr{Z}^{\#}$-irreducible, $A \mapsto f[A]$ and $B \mapsto g[B]$ are lattice isomorphisms from $\mathscr{Z}(X)^{\#}$ onto $\mathscr{Z}(Y)^{\#}$ and from $\mathscr{Z}(Z)^{\#}$ onto $\mathscr{Z}(Y)^{\#}$, respectively; thus by 2.7 $A \mapsto h'[A]$ is a lattice isomorphism from $\mathscr{Z}(X)^{\#}$ onto $\mathscr{Z}(Z)^{\#}$, and so h' is $\mathscr{Z}^{\#}$-irreducible. Thus X is projective.

Conversely, suppose that X is not a quasi F-space and let U be a dense cozero-set of X that is not C^*-embedded in X. Let $j\colon U \to X$ be the inclusion map embedding U in X and $\beta j\colon \beta U \to \beta X$ be its Stone extension. Let $Z = \beta j^{\leftarrow}[X]$ and $g = \beta j \,|\, Z$. Then g is a perfect continuous irreducible surjection from Z onto X. We claim that g is $\mathscr{Z}^{\#}$-irreducible. To see this, let $A \in \mathscr{Z}(Z)$. Arguing as in the first part of this proof, we see that

$$g[\mathrm{cl}_Z\,\mathrm{int}_Z A] = g\left[\mathrm{cl}_Z\,\mathrm{int}_U(A \cap U)\right] = \mathrm{cl}_X\left(\mathrm{int}_U[A \cap U]\right).$$

By 2.3 there is a zero-set B of X such that $A \cap U = B \cap U$, and it quickly follows that $g[\mathrm{cl}_Z\,\mathrm{int}_Z A] = \mathrm{cl}_X\,\mathrm{int}_X B$.

Let $i\colon X \to X$ be the identity map. If X were projective in $\mathscr{T}^{\#}$ there would have to exist a $\mathscr{Z}^{\#}$-irreducible function $h\colon X \to Z$ such that $g \circ h = i$, $g(h(x)) = x$ for each $x \in X$. As U is not C^*-embedded in X there exist disjoint zero-sets T_1 and T_2 of U such that $\mathrm{cl}_X T_1 \cap \mathrm{cl}_X T_2 \neq \varnothing$. Let $p \in \mathrm{cl}_X T_1 \cap \mathrm{cl}_X T_2$. Because $i\,|\,U$ and $g\,|\,U$ are the identity on U, $h[T_i] = T_i$ $(i = 1, 2)$. As h is continuous, it follows that $h(p) \in \mathrm{cl}_Z T_1 \cap \mathrm{cl}_Z T_2$, which contradicts the fact that U is C^*-embedded in Z. Thus X is not projective in $\mathscr{T}^{\#}$. $\square$

4.4. COROLLARY. *Let K be a compact space. Then $QF(K) = EK$ iff $\mathscr{R}(K) = \mathscr{Z}(K)^{\#}$.*

PROOF. If $QF(K) = EK$ then by 3.20 $\mathscr{R}(K) = \mathscr{Z}(K)^{\#}$. Conversely, if $\mathscr{R}(K) = \mathscr{Z}(K)^{\#}$ then k_K: $EK \to K$ is $\mathscr{Z}^{\#}$-irreducible (see the comment after 2.8) so by 4.3 there is a $\mathscr{Z}^{\#}$-irreducible mapping from $QF(K)$ onto EK. As EK is extremally disconnected, this mapping must be one-to-one (see 10.50 of [**Wa**]) and hence a homeomorphism. □

We now characterize those objects of $\mathscr{T}^{\#}$ that have projective covers in $\mathscr{T}^{\#}$. We also show that if projective covers exist they are unique.

4.5. THEOREM. (a) *The following are equivalent for a Tychonoff space X:*

(i) *X has a projective cover in $\mathscr{T}^{\#}$,*

(ii) $QF(\beta X) = \beta(QF(X))$.

(b) *If (Y, g) is a projective cover of X in $\mathscr{T}^{\#}$ then there is a homeomorphism h: $Y \to QF(X)$ such that $g = \Phi_X \circ h$.*

PROOF. (a) If X has a projective cover (Y, g) in $\mathscr{T}^{\#}$ then by 4.3 Y is a quasi F-space; hence βY is as well (see 5.1 of [**DHH**]). Arguing as in 4.3, we see that βg: $\beta Y \to \beta X$ is $\mathscr{Z}^{\#}$-irreducible. Thus by 2.13 there exists k: $QF(\beta X) \to \beta Y$ such that $\beta g \circ k = \Phi_{\beta X}$. By the minimality of $(QF(\beta X), \Phi_{\beta X})$ (see 1.1) there exists h: $\beta Y \to QF(\beta X)$ such that $\Phi_{\beta X} \circ h = \beta g$. Thus $\Phi_{\beta X} \circ h \circ k = \Phi_{\beta X} \circ 1_{\beta X}$ so by 1.2 $h \circ k = 1_{\beta X}$. Similarly $k \circ h = 1_{\beta Y}$ so h is a homeomorphism and $k = h^{\leftarrow}$. Thus $\beta Y = QF(\beta X)$ (up to homeomorphism) and $Y = (\beta g)^{\leftarrow}[X]$ as g is perfect. As Y is quasi F, it follows from 3.3 that $(Y, g) = (QF(X), \Phi_X)$ (up to homeomorphism) and so $\beta(QF(X)) = QF(\beta X)$.

Conversely, if $QF(\beta X) = \beta(QF(X))$ then by 3.5 $\Phi_{\beta X} | \Phi_{\beta X}^{\leftarrow}[X]$ is $\mathscr{Z}^{\#}$-irreducible, and so $(QF(X), \Phi_X)$ is a projective cover of X in $\mathscr{T}^{\#}$.

(b) This follows from the proof that (i) $\Rightarrow$ (ii) above. □

5. Motivation and historical remarks. A *Riesz space* (or *vector lattice*) $\mathscr{L} = (L, +, \vee, \wedge)$ is a real vector space with a lattice structure such that, if $a, b \in L$, $a \geqslant 0$, $b \geqslant 0$, and $\lambda \geqslant 0$ is a real number, then $a + b \geqslant 0$ and $\lambda a \geqslant 0$. If $na \leqslant b$ $(a \geqslant 0)$ for $n = 1, 2, \ldots,$ implies $a = 0$ then L is said to be *archimedean*. In the sequel we assume that all Riesz spaces under consideration are archimedean. For background on Riesz spaces, see [**LZ**].

For $a \in L$ put $|a| = a \vee (-a)$. If $\{p_n\}$ is a sequence in L such that $p_n \geqslant p_{n+1}$, we write $\{p_n\}\downarrow$ and if $p_n \leqslant p_{n+1}$ we write $\{p_n\}\uparrow$. If $\{p_n\}\downarrow$ and $\inf\{p_n : n \in \mathbf{N}\} = 0$, we write $p_n \downarrow 0$. If $(p_n - p)\downarrow 0$ for some $p \in L$, we write $p_n \downarrow p$. Reversing the inequalities yields the definition of $p_n \uparrow p$. If for a sequence $\{f_n\}$ in L, there is a sequence $\{p_n\}$ such that $p_n \downarrow 0$ and $|f_{n+k} - f_n| \leqslant p_n$, then $\{f_n\}$ is called an *order Cauchy sequence*. If every order Cauchy sequence converges, then L is said to be *order Cauchy complete*. In [**Pap**] F. Papengelou showed that L is order Cauchy complete if and only if

(5.1) If $\{f_n\}\uparrow$ and $\{g_n\}\downarrow$ are sequences in L such that $f_n \leqslant g_m$ whenever n, $m \in \mathbf{N}$ and $\bigwedge(g_n - f_n) = 0$, then there is an $h \in L$ such that $f_n \leqslant h \leqslant g_n$ $(n = 1, 2, \ldots)$.

Let X be a Tychonoff space. In 1968, G. Seever had shown in **[S]** that the Riesz space $C(X)$ is an F-space if and only if $C(X)$ satisfies:

(5.2) If $\{f_n\}\uparrow$ and $\{g_n\}\downarrow$ are sequences in $C(X)$ such that $f_n \leqslant g_m$ whenever $n, m \in \mathbf{N}$ then there is an $h \in C(X)$, such that $f_n \leqslant h \leqslant g_n$ for all $n \in \mathbf{N}$.

F. Dashiell studied Riesz spaces of continuous functions satisfying (5.1) as early as 1976. Initially he was unaware of Papengelou's work and called such Riesz spaces up-down semicomplete, a terminology used in **[D$_1$]** along with order Cauchy complete. The similarity between (5.1) and (5.2) also inspired the terminology quasi F-space and it was shown in **[D$_1$]** that for compact spaces X, $C(X)$ is order Cauchy complete if and only if $C(X)$ is quasi F. In **[DHH]** this is shown to hold for arbitrary Tychonoff spaces and it is shown that any Riesz space L has an essentially unique order Cauchy completion $\tilde{L}$ (for definitions, see **[DHH** or **HdP$_2$]**). Moreover if $L = \widetilde{C(X)}$—for compact X—then $\tilde{L} = C(K)$ for some compact quasi F-space K, and an example is supplied of a noncompact space X such that $C(X)$ is not a $C(Z)$, for any space Z.

Recall from **[DHH]**, that in case X is compact, the order Cauchy completion of $C(X)$ is $C(QF(X))$, and $QF(X)$ is constructed as an inverse limit of a directed family of spaces of the form βS_α, where each S_α is a countable intersection of dense cozero-sets of X (see **[DHH]**). In **[HdP$_2$]** the order Cauchy completion of $C(X)$ is obtained by identifying appropriate sets of points in the Dedekind-MacNeille completion.

To describe the construction of $QF(X)$ given in **[HdP$_2$]** we begin with some definitions. If $S \subset L$, put $S^d = \{x \in L\colon |x| \wedge |s| = 0 \text{ for all } s \in S\}$. An *l-ideal* of L is an ideal S of L such that if $s \in S$ and $|x| \leqslant |s|$ then $x \in S$. A *d-ideal* S of L is an l-ideal such that $(\{s\}^d)^d \subset S$, for all $s \in S$. In **[HdP$_2$]** it is shown that $QF(X)$ can be identified with the space of maximal (proper) d-ideals in $C(X)$ with the hull-kernel topology in case X is compact.

Yet another construction of the quasi F-cover of X is presented in **[ZK]**, where it is called the sequential absolute of X. The latter coincides with 'our' $QF(X)$ in case X is compact, but need not do so if X fails to be compact. The results in **[DHH]** were obtained in 1977 and 1978 and were presented at the Spring Topology Conference in Athens, Ohio in March 1979 (see **[H]**), but were not disseminated widely until **[DHH]** appeared in mid-1980. The results in **[D$_1$]** which motivated our work on $QF(X)$ were communicated to A. Hager and M. Henriksen in preprint form in 1976, but were not published until 1981. As a result of a letter from A. Veksler to A. Hager in 1981, and conversations between M. Henriksen and A. Veksler in Prague in August 1986, we learned that many of the results in **[D$_1$** and **DHH]** were obtained in the late 1970s independently by A. Koldunov and V. Zakharov. For a detailed list of the papers or seminar notes involved, see A. Hager's review of **[D$_1$]** in *Mathematical Reviews* **83d** (1983), #47043 and **[Vek$_2$]**. The ability of two research groups to work in ignorance of each other is a consequence of two different publication systems. In the West, publication is slow but once a paper appears it is rapidly disseminated. Quick publication in the form of internally circulated seminar notes or conference proceedings is common in the Soviet Union and Eastern Europe, but only in a form that is very slow to reach the West.

The idea of $\mathscr{Z}^{\#}$-irreducible map was communicated to A. Hager and M. Henriksen by F. Dashiell in a letter in the fall of 1978. This formulation was different from (but equivalent to) the definition used here, and he used it to prove our Theorem 2.13. Also, C. Neville used an equivalent version of $\mathscr{Z}^{\#}$-irreducibility in an unpublished manuscript, but he restricted his attention to strongly zero-dimensional spaces. He also made use of our notion of a complemented cozero-set, which appears also in a mildly disguised form in the work of H. Cohen [**C**]. The work of F. Dashiell described above was announced in [$\mathbf{D_2}$] and appears in an unfinished manuscript [$\mathbf{D_3}$]. Part of this was communicated to us by A. Hager when we informed him about our results. We are indebted to him for valuable discussions. A. Hager had the idea of using Dashiell's results to give a topological construction of the quasi F-cover of a compact space as early as the autumn of 1978 using the methods of [**Ha**].

REFERENCES

[**B**] B. Banaschewski, *Projective covers in certain categories*, General Topology and its Relation to Modern Analysis and Algebra. II (Prague, 1966), Academic Press, New York, 1967.

[**BH**] R. L. Blair and A. W. Hager, *Extensions of zero-sets and of real-valued functions*, Math. Z. **136** (1974), 41–57.

[**C**] H. Cohen, *The k-extremally disconnected spaces as projectives*, Canad. J. Math. **16** (1964), 253–260.

[**CHN**] W. W. Comfort, N. Hindman, and S. Negrepontis, *F'-spaces and their products with P-spaces*, Pacific J. Math. **28** (1969), 489–502.

[$\mathbf{D_1}$] F. Dashiell, *Non-weakly compact operators from $C(S)$ lattices with applications to Baire classes*, Trans. Amer. Math. Soc. **266** (1981), 397–413.

[$\mathbf{D_2}$] ______, *The quasi F-cover of a compact space and strongly irreducible surjections*, Abstracts Amer. Math. Soc. **3** (1982), 96.

[$\mathbf{D_3}$] ______, *The quasi F-cover of a compact space and strongly irreducible surjections*, unpublished manuscript.

[**DF**] A. Dow and O. Forster, *Absolute C^*-embedding of F-spaces*, Pacific J. Math. **98** (1982), 63–71.

[**DHH**] F. Dashiell, A. W. Hager and M. Henriksen, *Order-Cauchy completions of rings and vector lattices of continuous functions*, Canad. J. Math. **32** (1980), 657–685.

[**F**] J. Flaschmeyer, *Topologische Projektivräume*, Math. Nachr. **26** (1963), 57–66.

[**FG**] N. J. Fine and L. Gillman, *Extensions of continuous functions in βN*, Bull. Amer. Math. Soc. **66** (1960), 376–381.

[**GJ**] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.

[**H**] M. Henriksen, *A summary of results on order-Cauchy completions of rings and vector lattices of continuous functors*, Topology Proc. **4** (1979), 239–263.

[**Ha**] A. Hager, *The projective resolultion of a compact space*, Proc. Amer. Math. Soc. **28** (1971), 262–266.

[**HI**] M. Henriksen and J. Isbell, *Some properties of compactifications*, Duke Math. J. **25** (1957), 83–105.

[$\mathbf{HdP_1}$] C. B. Huijsmans and B. de Pagter, *On z-ideals and d-ideals in Riesz spaces*. II, Indag. Math. **42** (1980), 391–408.

[$\mathbf{HdP_2}$] ______, *On z-ideals and d-ideals in Riesz spaces*. III, Indag. Math. **43** (1981), 409 –422.

[$\mathbf{HdP_3}$] ______, *Maximal d-ideals in a Riesz space*, Canad. J. Math. **35** (19830, 1010–1029.

[**I**] S. Iliadis, *Absolutes of Hausdorff spaces*, Soviet Math. Dokl. **4** (1963), 295–298.

[**LZ**] W. Luxemburg and A. Zaanen, *Riesz spaces*, North-Holland, Amsterdam, 1971.

[vM] J van Mill, *An introduction to $\beta\omega$*, Handbook of Set-Theoretic Topology (Chapter 11), Elsevier, Amsterdam, 1984.

[Pap] F. Papengelou, *Order convergence and topological completion of commutative groups*, Math. Ann. **155** (1964), 81–107.

[**P**] Y. L. Park, *The quasi F-cover as a filter space*, Houston J. Math. **9** (1983), 101–109.

[**PW**] J. R. Porter and R. G. Woods, *Absolutes and extensions of Hausdorff spaces*, Springer Universitext series (to appear).

[**S**] G. L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. **133** (1968), 269–280.

[$\mathbf{V_1}$] J. Vermeer, *On perfect irreducible preimages*, Topology Proc. **9** (1984).

[V_2] ______, *Expansions of H-closed spaces*, Doctoral Dissertation, Vrije Universiteit, Amsterdam, The Netherlands, 1983.

[V_3] ______, *The smallest basically disconnected pre-image of a space*, Topology Appl. **17** (1984), 217–232.

[**Vek**$_1$] A. K. Veksler, *P'-points, P'-sets, P'-spaces. A new class of order continuous measures and functionals*, Soviet Math. Dokl. **4** (1973), 1443–1450.

[**Vek**$_2$] ______, *Absolutes and vector lattices*, Proc. Conf. Topology and Measure. IV (Trassenheide, G. D. R. 1983), 1984, pp. 217–235.

[**Wa**] R. C. Walker, *The Stone-Čech compactification*, Springer-Verlag, New York, 1974.

[**War**] N. M. Warren, *Properties of Stone-Čech compactifications of discrete spaces*, Proc. Amer. Math. Soc. **33** (1972), 599–606.

[**W**] R. G. Woods, *A survey of absolutes of topological spaces*, Topological Structures. II, Math. Centrum Tract no. 116, 1979, pp. 323–362.

[**ZK**] V. K. Zakharov and A. V. Koldunov, *The sequential absolute and its characterizations*, Soviet Math. Dokl. **22** (1980), 70–74.

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