9-1-1988

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PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 1, September 1988

THE SPACE OF MINIMAL PRIME IDEALS OF $C(X)$ NEED NOT BE BASICALLY DISCONNECTED

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(Communicated by Doug W. Curtis)

ABSTRACT. Problems posed twenty and twenty-five years ago by M. Henriksen and M. Jerison are solved by showing that the space of minimal prime ideals of the ring $C(X)$ of continuous real-valued functions on a compact (Hausdorff) space need not be basically disconnected—or even an F-space.

If R is a commutative ring, let $\text{Spec}(R) = S(R)$ denote the set of prime ideals of R, $\text{Minspec}(R) = m(R)$ the subset of minimal elements of $S(R)$, and if R has an identity element $\text{Maxspec}(R) = M(R)$, the set of maximal elements of $S(R)$. We impose the *hull kernel* or *Zariski* topology on $S(R)$; that is the topology with base $\{h^c(a): a \in R\}$, where $h^c(a) = \{P \in S(R): a \notin P\}$, and we regard $m(R)$ and $M(R)$ as subspaces of $S(R)$. For background, see [**Ho**, **HJ**, and **K**].

Below, X will always denote a Tychonoff space. We are concerned particularly with $m(R)$ in case $R = C(X)$, the ring of all real-valued, continuous functions on X, and our aim is to present a solution to a problem posed by M. Henriksen and M. Jerison about $m(C(X))$ in 1961 and 1965; see [**HJ**]. Terms not defined explicitly below may be found in [**GJ**].

For $f \in C(X)$, let $Z(f) = \{x: f(x) = 0\}$, $\text{coz}(f) = X - Z(f)$, $\text{spt}(f) = \text{Cl}(\text{coz}(f))$, and zero (cozero) sets [supports] are sets of the form $Z(f)$ $(\text{coz}(f))$ $[\text{spt}(f)]$ for some $f \in C(X)$. If $x \in X$, $M_x = \{f \in C(X): x \in Z(f)\}$ and $O_x = \{f \in C(X): x \in \text{Int}(Z(f))\}$. X is called an *F-space* if each $P \in S(C(X))$ contains a unique $P' \in m(C(X))$. It is well known (see [**GJ**]) that $M(C(X)) = \{M_x: x \in X\}$ for arbitrary compact Hausdorff X, and that $m(C(X)) = \{O_x: x \in X\}$ if X is also an F-space.

A pair A, B of subsets of X are said to be *completely separated* if there is an $f \in C(X)$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$. Completely separated subsets have disjoint closures, and the converse holds if X is a normal space.

X is *basically* (*extremally*) *disconnected* if whenever A is an open set and B a cozero (open) set disjoint from A, then A and B are completely separated. As shown in [**GJ**], X is basically (extremally) disconnected iff the closure of each cozero set (open set) is open. Thus every extremally disconnected space is basically disconnected. It is also shown in [**GJ**] that X is an F-space iff disjoint cozero sets are completely separated, so basically disconnected spaces are all F-spaces. X is called an F'-space if disjoint cozero sets have disjoint closures. Every F-space is an F'-space but the converse may fail if X is nonnormal [**GH**].

Received by the editors October 22, 1986 and, in revised form, September 23, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54C40, 06F25.

As usual we let βX denote the Stone-Čech compactification of X and X^* denote $\beta X - X$. We let $\mathbf{N}$ denote the countably infinite discrete space. The following are established in [**HJ**]:

(a) $m(C(X))$ is countably compact (we have shown in [**DHKV**] that the closure of each weakly Lindelöf subset is compact),

(b) $m(C(X))$ is basically disconnected if it is locally compact,

(c) $m(C(\mathbf{N}^*))$ is nowhere locally compact,

(d) it follows easily from [**HJ**], 4.2 that the closure of a countable union of basic open sets (i.e., sets of the form $h^c(a)$) is a basic open set.

M. Henriksen and M. Jerison asked in 1961 if $m(C(\mathbf{N}^*))$ is basically disconnected, and in 1965 they asked whether for any X, $m(C(X))$ fails to be basically disconnected; see [**HJ**]. By condition (d) these seem natural questions. Below we answer the second question in the affirmative (in ZFC), and the first in the negative if Martin's Axiom (MA) holds; in particular, if the continuum hypothesis (CH) holds. For a statement of Martin's Axiom see [**Ku**].

A key part of the solution is the notion of a *P-set*: it is a nonempty compact subset K of X for which every countable intersection of neighborhoods of K is a neighborhood of K. Equivalently, K is a P-set iff it is nonempty, compact and completely separated from any cozero set disjoint from it. Clearly if a zero set is a P-set, then it is open. Many facts about P-sets are given in [**vM**].

A continuous surjection $g\colon Y \to Z$ is called *irreducible* or said to *map Y irreducibly onto Z* if no proper closed subset of Y is mapped onto Z by g. For every compact Y, there is a unique extremally disconnected space EY minimal with respect to having an irreducible map onto X, which we call the *absolute* of the space Y. It is the Stone space of the Boolean algebra of regular closed subsets of X; for background see [**Wo**].

The proof of the following folk-lemma is an exercise.

1. LEMMA. *If $g\colon Y \to Z$ is closed and irreducible and D is dense in Z then $f^{-1}[D]$ is dense in Y.*

As noted above, if X is an F-space then $m(C(X)) = \{O_x : x \in X\}$. Thus $x \to O_x$ naturally identifies X with $m(C(X))$.

2. LEMMA. *If X is a compact F-space then the map $x \to O_x$ is a homeomorphism from X with the topology whose base is $\{\mathrm{spt}(f) : f \in C(X)\}$ to $m(C(X))$ with the hull-kernel topology.*

PROOF. For $f \in C(X)$, $h^c(f) = \{O_x : f \notin O_x\} = \{O_x : x \notin \mathrm{Int}(Z(f))\} = \{O_x : x \in \mathrm{spt}(f)\}$, and as noted above, $\{h^c(f) : f \in C(X)\}$ is a base for the hull-kernel topology on $m(C(X))$.

From this characterization of the topology of $m(C(X))$ when X is a compact F-space we obtain:

3. THEOREM. *If X is a compact F-space in which every zero set is regular closed, and if X contains a P-set E which maps irreducibly onto $[0,1]$, then $m(C(X))$ is not an F'-space. In particular, $m(C(X))$ is not basically disconnected.*

PROOF. Suppose $p\colon E \to [0,1]$ is irreducible. Let $A = \{a_n : n \in \mathbf{N}\}$, $B = \{b_n : n \in \mathbf{N}\}$ be two disjoint, countably infinite dense subsets of $[0,1]$. By Lemma

1, $p^{-1}[A]$ and $p^{-1}[B]$ are disjoint dense subsets of E. Since E is compact, p has a continuous extension $g\colon X \to [0,1]$. For each positive integer n, both $g^{-1}(a_n)$ and $g^{-1}(b_n)$ are zero sets and by assumption, each has dense interior. By Lemma 2, $\mathrm{Int}(g^{-1}(a_n))$ is open, as well as closed, in the hull-kernel topology, so $L(A) = \bigcup\{\mathrm{Int}(g^{-1}(a_n))\colon n \in \mathbf{N}\}$, $L(B) = \bigcup\{\mathrm{Int}(g^{-1}(b_n))\colon n \in \mathbf{N}\}$, are nonempty cozero sets of $m(C(X))$, which are clearly disjoint. Our theorem will be proved once we establish that E is contained in the hull-kernel closure of both $L(A)$ and $L(B)$.

Suppose $x \in E$. By Lemma 2, each basic neighborhood of x takes the form $\mathrm{Cl}_X(U)$, where $U = \mathrm{coz}(h)$ for some $h \in C(X)$. Since E is a P-set and $\mathrm{Cl}_X(U)$ meets E, we know that U meets E. Since $p^{-1}[A]$ and $p^{-1}[B]$ are dense in E, U meets each of them and hence meets both $g^{-1}[A]$ and $g^{-1}[B]$. Since $g^{-1}[A]$, $g^{-1}[B]$ are unions of regular closed zero sets, it follows that U is not disjoint from either $L(A)$ or $L(B)$. Thus x is in the closure of both $L(A)$ and $L(B)$, so $m(C(X))$ is not an F'-space.

K. Kunen has shown that if MA holds then $\mathbf{N}^*$ contains a P-set homeomorphic to the absolute of $[0,1]$ (see [**Ku**, Theorem 1.2]). It is well known that $\mathbf{N}^*$ is an F-space in which every nonempty zero set has nonempty interior [**vM**, 1.6.2]. Thus the following corollary follows from Theorem 3.

4. COROLLARY (MA). *$m(C(\mathbf{N}^*))$ is not an F-space.*

There is also a compact X such that $m(C(X))$ is not an F-space whose existence does not depend on MA. To produce it we will need to prove the following.

5. LEMMA. *If Y is a zero set of $\mathbf{N}^*$ with nonempty boundary, then that boundary is a P-set of Y.*

PROOF. Let $Y = Z(f)$, $f \in C(X)$. By duality it will suffice to show that the union of a sequence $\{S_n : n \in \mathbf{N}\}$ of closed subsets of $\mathrm{Int}(Y)$ has closure contained in $\mathrm{Int}(Y)$. Construct successive clopen subsets $U_1, \ldots$ such that U_1 is empty and for each n, $S_n \cup U_n \subset U_{n+1} \subset \mathrm{Int}(Y)$. Next for each n let V_n be clopen in $\mathbf{N}^*$ such that $f^{-1}[[0, 1/(2n+1)]] \subset V_n \subset f^{-1}[[0, 1/2n)]$. Then $\{U_n\}$ and $\{V_n\}$ are sequences of clopen subsets of $\mathbf{N}^*$ such that $S_n \subset U_n \subset U_{n+1} \subset V_{m+1} \subset V_m$ and $\mathrm{Bd}(Y) \subset V_m$ for positive integers m, n. By [**W**, Chapter 3], there is a clopen subset W of $\mathbf{N}^*$ such that $U_n \subset W \subset V_m$ for all m, n; thus

$$\mathrm{Cl}\left(\bigcup\{U_n : n \in \mathbf{N}\}\right) \subset W \subset \bigcap\{V_n : n \in \mathbf{N}\} = Y,$$

and since W is open, $\mathrm{Cl}(\bigcup\{U_n : n \in \mathbf{N}\}) \subset \mathrm{Int}(Y)$.

Suppose S and T are spaces, A is a closed subspace of S and $f\colon A \to T$ is continuous. Recall that $S \cup_f T$ is the quotient space of the disjoint union of S and T obtained by identifying each $a \in A$ with $f(a) \in T$.

6. COROLLARY. *There is a zero set Z of $\mathbf{N}^*$ with a quotient space X such that $m(C(X))$ is not an F-space.*

PROOF. Let Z be a zero set of $\mathbf{N}^*$ with nonempty boundary. It is well known that the boundary of Z maps continuously onto $\beta\mathbf{N}$, hence onto $E[0,1]$. Let f be such a continuous map of Z onto $E[0,1]$. Since $\mathrm{Bd}(Z)$ is a nowhere dense P-set of Z by Lemma 5, the space $X = Z \cup_f E[0,1]$ is an F-space in which by [**vM**, 1.4.1 and 1.4.2] every zero set is a regular closed set. In X there is a copy of $E[0,1]$

that is a P-set, so the hypotheses of Theorem 3 are satisfied, and we conclude that $m(C(X))$ fails to be an F-space.

7. Remarks and open problems. To apply Theorem 3 to the space $\mathbf{N}^*$ seems to require the existence of a separable infinite P-set in $\mathbf{N}^*$. As is noted in [**vM**, problem 6] it is an open problem whether there is such a P-set in $\mathbf{N}^*$ unless MA or some set-theoretic axiom beyond ZFC holds. Hence the question of whether $m(C(\mathbf{N}^*))$ is an F-space remains open in ZFC.

Among other questions which come to mind from the examples given above: Suppose X is an F-space in which zero sets are regular closed, and suppose X fails to contain an infinite P-set. Must $m(C(X))$ be basically disconnected, or even an F'-space? Is there a compact X such that $m(C(X))$ is basically disconnected but not locally compact? Exactly when is $m(C(X))$ basically disconnected?

REFERENCES

[**DHKV**] A. Dow, M. Henriksen, R. Kopperman and J. Vermeer, *The countable annihilator condition and weakly Lindelöf subspaces of minimal prime ideals* (submitted).

[**GH**] L. Gillman and M. Henriksen, *Rings of continuous functions inn which finitely generated ideals are principal*, Trans. Amer. Math. Soc. **82** (1956), 366–391.

[**GJ**] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.

[**HJ**] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, General Topology and its Relations to Modern Analysis and Algebra (Prague, 1962), and Trans. Amer. Math. Soc. **115** (1965), 110–130.

[**Ho**] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969), 43–60.

[**K**] J. Kist, *Minimal prime ideals in commutative semigroups*, Proc. London Math. Soc. (3) **13** (1963), 31–50.

[**Ku**] K. Kunen, *Some points in* β**N**, Math. Proc. Cambridge Philos. Soc. **80** (1976), 385–398.

[**vM**] J. van Mill, *An introduction to* β**N**, Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984.

[**W**] R. Walker, *The Stone-Čech compactification*, Springer-Verlag, Berlin and New York, 1974.

[**Wo**] R. G. Woods, *A survey of absolutes of topological spaces*, Topological Structures II, Mathematical Centre Tracts, no. 116, 1979, pp. 323–362.

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