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ON MINIMAL COMPLETELY REGULAR SPACES ASSOCIATED WITH A GIVEN RING OF CONTINUOUS FUNCTIONS

Melvin Henriksen

1. INTRODUCTION

Let $C(X)$ denote the ring of all continuous real-valued functions on a completely regular space X . If X and Y are completely regular spaces such that one is dense in the other, say X is dense in Y , and every $f \in C(X)$ has a (unique) extension $\bar{f} \in C(Y)$, then $C(X)$ and $C(Y)$ are said to be *strictly isomorphic*. In a recent paper [2], L. J. Heider asks if it is possible to associate with the completely regular space X a dense subspace μX minimal with respect to the property that $C(\mu X)$ and $C(X)$ are *strictly isomorphic*.¹

In this note, Heider's question is answered in the negative. It is shown, moreover, that if μX exists, then it consists of all of the isolated points of X , together with those nonisolated points p of X such that $C(X \sim \{p\})$ and $C(X)$ fail to be strictly isomorphic. Thus, if μX exists, it is unique.

2. PRELIMINARY REMARKS

Let $C(X)$ denote the ring of all continuous real-valued functions on a completely regular space X . Let $C^*(X)$ denote the subring of all bounded $f \in C(X)$. The following known facts are utilized below.

(2.1) Corresponding to each completely regular space X , there exists an essentially unique compact space βX , called the Stone-Ćech compactification of X , such that (i) X is dense in βX , and (ii) every $f \in C^*(X)$ has a (unique) extension $\bar{f} \in C^*(\beta X) = C(\beta X)$. Thus $C^*(X)$ and $C(\beta X)$ are isomorphic. (See, for example, [3] or [4, Chapter 5].)

(2.2) There exists an essentially unique subspace νX of βX such that (i) X is a \mathcal{Q} -space, (ii) X is dense in νX , and (iii) every $f \in C(X)$ has a (unique) extension $\bar{f} \in C(\nu X)$. Thus $C(X)$ and $C(\nu X)$ are isomorphic. (For the definition of \mathcal{Q} -space, and a proof of this theorem, see [1] or [3].)

(2.3) If X and Y are completely regular spaces such that $C(X)$ and $C(Y)$ are isomorphic, then Y is homeomorphic to a dense subspace of νX such that every real-valued function continuous on this subspace has a (unique) continuous extension over νX . [3, Theorem 65.]

(2.4) If Z is any compact space, and f is any continuous mapping of X into Z , then there exists a (unique) continuous extension \hat{f} of f over βX into Z . (See [5, Theorem 88].)

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1. Since the writing of this paper, Heider's problem has been generalized and solved independently by J. Daly and L. J. Heider.

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In the oral presentation of [2], Heider asked ". . . whether or not to each completely regular space X , there is associated a completely regular space μX such that μX and $\nu(\mu X)$ are homeomorphic, and $\mu X \subset Y \subset \nu X$ for every completely regular space Y such that νY is homeomorphic to νX ." By considering the special case $Y = X$ in Heider's formulation, we see at once that $\mu X \subset X$. Moreover, since $\nu(\mu X)$ and νX are homeomorphic, it follows from (2.3) that μX is homeomorphic to a dense subspace of X all of whose continuous real-valued functions have continuous extensions over X . Thus, it is natural to identify μX with its image in X under this homeomorphism; this identification leads to the formulation of Heider's problem given in the Introduction, namely: does there exist a dense subspace μX of X which is minimal with respect to the property that $C(\mu X)$ and $C(X)$ are strictly isomorphic?¹

We conclude this section with a definition.

Definition. If X is a completely regular space, let ηX denote the union of the set of isolated points of X and the set of nonisolated points p of X such that $C(X \sim \{p\})$ and $C(X)$ fail to be strictly isomorphic.

Thus, by (2.3), a nonisolated point p of X fails to be in ηX if and only if every $f \in C(X \sim \{p\})$ has a (unique) continuous extension over X .

3. UNIQUENESS OF μX

We begin this section with a theorem which will be used below, and which we believe to be of some independent interest.

THEOREM 3.1. *If Y is a dense subspace of a completely regular X such that the rings $C(Y)$ and $C(X)$ (respectively, $C^*(Y)$ and $C^*(X)$) are strictly isomorphic, then, for any (nonisolated) point $p \in Y$, the rings $C(Y \sim \{p\})$ (respectively, $C^*(Y \sim \{p\})$ and $C^*(X \sim \{p\})$) are strictly isomorphic.*

Proof. Except for the part of the theorem in parentheses, it is enough, by (2.3), to show that every $f \in C(Y \sim \{p\})$ has a (unique) extension $F \in C(X \sim \{p\})$. As for the part in parentheses, it will be evident from the construction that if $f \in C^*(Y \sim \{p\})$, then $F \in C^*(X \sim \{p\})$.

Let $\{U_\alpha\}_{\alpha \in A}$ be a base of neighborhoods in the space X of p . The index set A becomes a directed set if we let the statement $\beta \geq \alpha$ mean that $U_\beta \subset U_\alpha$. Since X is completely regular, for each $\alpha \in A$, there exists an $i_\alpha \in C^*(X)$ such that $i_\alpha(x) = 1$ for $x \in X \sim U_\alpha$, and i_α vanishes on a neighborhood of p . (To see this, let $h_\alpha \in C^*(X)$ be such that $h_\alpha(X \sim U_\alpha) = 1$, and $h_\alpha(p) = -1$. Then let $i_\alpha(x) = \max(h_\alpha(x), 0)$ for every $x \in X$.) Let f be the function defined on Y by letting $f_\alpha(y) = i_\alpha(y)f(y)$ for every $y \in Y \sim \{p\}$, and by letting $f_\alpha(p) = 0$. Clearly, $f_\alpha \in C(Y)$, and $f_\alpha(y) = f(y)$ for all y outside of U_α . Now, by hypothesis (and (2.3)), f_α has a unique extension $F_\alpha \in C(X)$.

For each $x \in X \sim \{p\}$, the set $\{F_\alpha(x)\}_{\alpha \in A}$ forms a real-valued net [4, Chapter 2]. For each $x \in X \sim \{p\}$, let $F(x) = \lim_\alpha F_\alpha(x)$. This limit exists since, if U_{α_x} is a basic neighborhood of p disjoint from x , it follows from $\beta \geq \alpha_x$ that

$$F_{\alpha_x}(x) = F_\beta(x) = F(x).$$

It is clear that F is an extension of f . We will show next that $F \in C(X \sim \{p\})$, by verifying that F is continuous at each $x_0 \in X \sim \{p\}$.

Let V_{x_0} , U_{α_0} denote disjoint neighborhoods (in X) respectively of x_0 and p . If $x \in V_{x_0}$, then for any $\beta \geq \alpha_0$, $F(x) = F_\beta(x)$. Hence the continuity of F at x_0 follows from the continuity of F_β at x_0 . This completes the proof of the theorem.

COROLLARY. *If Y is a dense subspace of the completely regular space X then, for any (nonisolated) point $p \in Y$, if νY and νX (respectively, βY and βX) are homeomorphic, then $\nu(Y \sim \{p\})$ and $\nu(X \sim \{p\})$ (respectively, $\beta(Y \sim \{p\})$ and $\beta(X \sim \{p\})$) are homeomorphic.*

It will be shown next that if μX exists, then it is unique.

THEOREM 3.2. *If with the completely regular space X there is associated at least one dense subspace μX minimal with respect to the property that $C(\mu X)$ and $C(X)$ are strictly isomorphic, then μX is unique. In fact, $\mu X = \eta X$.*

Proof. It follows from the definition of ηX , and from the fact that μX is dense in X , that each of these spaces contains all the isolated points of X . Hence we need only consider the nonisolated points of X . We will show first that $\mu X \subset \eta X$.

Let p be a nonisolated point of X contained in μX . By the minimality of μX , there exists an $f \in C(\mu X \sim \{p\})$ with no continuous extension over μX . But, by Theorem 3.1, f has an extension $F \in C(X \sim \{p\})$. If p were not in ηX , F would have a continuous extension over X , whose restriction to μX would in turn be a continuous extension of f over μX . Hence $p \in \eta X$, whence $\mu X \subset \eta X$.

Suppose there were a point $p \in \eta X \sim \mu X$. If $f \in C(X \sim \{p\})$, then since $C(\mu X)$ and $C(X)$ are isomorphic, the restriction of f to μX has a continuous extension over X . This latter would be a continuous extension of f over X , contrary to the assumption that $p \in \eta X$. Hence $\mu X = \eta X$. This completes the proof of the theorem.

COROLLARY. *A necessary and sufficient condition that μX exist (in which case it is equal to ηX) is that ηX be dense in X and that every $f \in C(\eta X)$ have a (unique) extension $\bar{f} \in C(X)$.*

As noted by Heider [2], $\mu X = \eta X = X$, provided every point of X is a G_δ .

4. THE SUBSPACE μX NEED NOT EXIST

In this section we give an example of a completely regular space X such that μX does not exist. In fact, for this X , ηX is dense in X , but $C(\eta X)$ and $C(X)$ are not isomorphic.

We begin by generalizing a result of Hewitt [3, p. 62].

THEOREM 4.1. *Let Y be a noncompact completely regular space, and suppose that $Y \subset X \subset \beta Y$ and that $\beta Y \rightarrow X$ has power less than $\exp \exp \aleph_0$. Then $\nu X = \beta X = \beta Y$. In particular, $C(X) = C^*(X)$.*

Proof. We will show first that $C(X) = C^*(X)$, thus verifying that $\nu X = \beta X$. (See (2.1) and (2.2).) For any $f \in C(X)$, let f^* denote its restriction to Y . As noted in [1], f^* may be regarded as a continuous mapping of Y into the one-point compactification $R \cup \{\infty\}$ of the real line R . By (2.4), f^* has a (unique) continuous extension \hat{f}^* over βY into $R \cup \{\infty\}$. Since Y is dense in X , the function \hat{f}^* is also an extension of f . Now the set $G = \{y \in Y: \hat{f}^*(y) = \infty\}$ is a closed G_δ of βY , and it is contained in $\beta Y \sim X \subset \beta Y \sim Y$. Hewitt has shown [3, Theorem 49] that every nonempty closed G_δ of βY contained in $\beta Y \sim Y$ has power at least $\exp \exp \aleph_0$. On the other hand it is evident, from the hypothesis, that G has power less than $\exp \exp \aleph_0$. Hence G is empty. So $f^* \in C^*(Y)$, and it follows that $f \in C^*(X)$. Thus $\nu X = \beta X$.

Since X is dense in βY , and βY is compact, in order to conclude that $\beta X = \beta Y$ it suffices, by (2.1), to show that every $f \in C^*(X)$ has a (unique) extension $\bar{f} \in C^*(\beta X)$. We may take \bar{f} to be the (unique) extension over βY of the restriction of f to Y . This completes the proof of the theorem.

Example. Let Y be any completely regular space that admits unbounded continuous real-valued functions, and such that $\eta Y = Y$. (In particular, Y could be any infinite discrete space.) Let $X = \beta Y$. For each $p \in \beta Y \sim Y$, it follows from Theorem 4.1 that $\nu(X \sim \{p\}) = X$. Hence, $\eta X \subset Y$, and since $\eta Y = Y$, it follows that $\eta X = Y$. But, although ηX is dense in X , no unbounded $f \in C(Y)$ has a continuous extension over the compact space X . Thus, by the corollary to Theorem 3.1, μX does not exist.

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