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Spectral equivalence of bosons and fermions in one-dimensional harmonic potentials

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Recently, Schmidt and Schnack [Physica A **260**, 479 (1998)], following earlier references, reiterate that the specific heat of N noninteracting bosons in a one-dimensional harmonic well equals that of N noninteracting fermions in the same potential. We show that this peculiar relationship between heat capacities results from a more dramatic equivalence between Bose and Fermi systems. Namely, we prove that the excitations of such Bose and Fermi systems are *spectrally equivalent*. Two complementary proofs of this equivalence are provided; one based on a combinatoric argument, the other from analysis of the underlying dynamical symmetry group.

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I. INTRODUCTION

With the advent of dilute atomic BEC (Refs. [1–3]) and degenerate dilute atomic Fermi gas [4–6], there is renewed interest in understanding aspects of quantum many-body theory in inhomogeneous (in particular, harmonically trapped) systems. Since trapped, cooled atoms have properties that are, in principle, controllable to a degree unavailable in other systems (e.g., clusters and nuclei), they present new opportunities to study quantum mechanics and many-body theory.

Although the ultracold dilute atomic gas systems are large compared to the coherence lengths, they are not homogeneous, due to the fact they are generally trapped in a (nearly) harmonic potential. In many of these systems the interparticle forces are significant. In this Brief Report, however, we ignore the interactions between atoms, with the aim to better understand the thermodynamic properties of N trapped *noninteracting* bosons and fermions. Recent work [7,8] describes surprising relations between the equilibrium thermodynamics of these two systems. It was shown, for example, that the heat capacity (as a function of temperature) of N noninteracting bosons in a one-dimensional harmonic potential is the same as that of N noninteracting fermions in an identical potential. Likewise, the respective energies and partition functions for these systems are closely related (see Refs. [8] and [9]).

These ''coincidences'' provide hints that a deeper underlying connection exists between Bose and Fermi gases in a one-dimensional (1D) harmonic well. In particular, the heat capacity and partition functions, as functions of the inverse temperature β, can be thought of as an ''imaginary time'' continuation of a Fourier transform of the spectrum. The fact that the heat capacities are the same for all temperatures suggests that there should be a state-for-state, level-for-level correspondence between these noninteracting many-body bosonic and fermionic systems. We show that this is indeed the case, and below describe two independent proofs of the spectral equivalence of excitations in these systems. The first is based on a combinatoric argument, while the second relies on properties of the dynamical symmetry group of these systems.

II. COMBINATORIC APPROACH

The spectral equivalence of one-dimensional, noninteracting harmonically-trapped bosonic and fermionic gases can be understood through a straightforward combinatoric analysis of energy-level multiplicities.

In a system of N noninteracting particles (bosons or fermions) in a harmonic well, let the energy level of the ith particle be specified by the integer e_i, with $E=\Sigma_{i=1}^{N}e_i$ the total energy of the system. (Note: In writing the energy e_i as an integer, we are setting $\hbar\omega=1$, and for notational convenience are ignoring the constant 1/2 associated with the single-particle ground-state energy.) Clearly there are many different microconfigurations possessing the same total energy E; we let $G_N(E)$ denote the multiplicity of states with fixed energy E. We will show that the multiplicity functions for bosons and fermions are equivalent. More precisely, we show that $G_N^{\mathrm{boson}}(E)=G_N^{\mathrm{fermion}}[E+N(N-1)/2]$, indicating that the multiplicities for the Bose and Fermi cases are identical provided each is measured relative to its respective ground-state energy [i.e., 0 for bosons and $N(N-1)/2$ for fermions]. This equivalence (between multiplicity functions) is sufficient to show that the excitation spectra of the bosonic and fermionic gases are, in fact, identical.

We begin with the Bose case. We imagine ordering the N particles from lowest energy to highest $(e_1,e_2,\ldots,e_N)$. The energy of the lowest-energy particle (e_1) can range from zero up to a maximum value of $[E/N]$, where the brackets [ ] denote the integer part of the expression enclosed. (It is readily seen that if the energy of the *lowest*-energy particle were to exceed this maximum value, then the sum of the energies of the N individual particles would exceed the total specified energy E of the system.)

For a fixed e_1, the remaining energy $E-e_1$ must be divided up among $N-1$ particles. So the possible values of e_2, which represents the lowest energy among the remaining $(N-1)$ particles, can range from e_1 to $[(E-e_1)/(N-1)]$.

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(As before, it is clear that if e_2 went outside this range, then the sum of the energies of the $N-1$ particles would exceed the prescribed value $E-e_1$.)

Proceeding in this fashion, we see that

$$G_N^{\mathrm{boson}}(E)=\sum_{e_1=0}^{[E/N]}\ \sum_{e_2=e_1}^{[(E-e_1)/(N-1)]}\cdots\sum_{e_{N-1}=\epsilon_{N-2}}^{[(E-e_1-e_2-\cdots-e_{N-2})/2]} 1.$$

A similar argument is used to construct the multiplicity function for the fermionic case. The fundamental distinction stems from the additional constraint that two fermions cannot occupy the same energy orbital, which in turn modifies the lower and upper bounds in the above summations, as we now describe. Consider first the lower bounds. From the exclusion principle, it immediately follows that the lower (fermionic) bounds must take the form $e_i=e_{i-1}+1$. The upper limits are found by noting that for a system of N fermions with total energy E, the energy of the lowest-energy fermion cannot exceed $[(E-N(N-1)/2)/N]$, as a straightforward calculation reveals. Consequently, we find

$$G_N^{\mathrm{fermion}}(E)=\sum_{e_1=0}^{[(E-N(N-1)/2)/N]}\ \sum_{e_2=e_1+1}^{[(E-e_1-(N-1)(N-2)/2)/(N-1)]}\cdots\sum_{e_{N-1}=e_{N-2}+1}^{[(E-e_1-e_2-\cdots-e_{N-2}-(2)(1)/2)/2]} 1.$$

Although neither the bosonic nor fermionic multiplicity function is easily evaluated, the equivalence of $G_N^{\mathrm{boson}}(E)$ and $G_N^{\mathrm{fermion}}[E+N(N-1)/2]$ is readily revealed via the following key coordinate transformation: In the fermionic summations above, introduce new coordinates $\hat{e}_i=e_i-i+1$. We claim that this will transform the fermionic sum into the corresponding Bose sum.

To see that this transformation achieves the desired result, first observe that under this transformation, the lower bounds in the fermionic summations ($e_{i+1}=e_i+1$) become ($\hat{e}_{i+1}=\hat{e}_i$), just as in the Bose case. Meanwhile, it is not difficult to verify that the upper limits in the fermionic summations

$$e_{i+1}=\left[\frac{E-e_1-e_2-\cdots-e_i-(N-i)(N-i-1)/2}{N-i}\right]$$

now take the form

$$\hat{e}_{i+1}=\left[\frac{E-N(N-1)/2-\hat{e}_1-\hat{e}_2-\cdots-\hat{e}_i}{N-i}\right],$$

which, again is the same as for the bosonic case (once we shift by the fermionic ground-state energy $E\rightarrow E+N(N-1)/2$.

This equivalence between the bosonic and fermionic multiplicity functions proves that the excitation spectrum of one-dimensional harmonically trapped N noninteracting bosons is identical to that of N noninteracting trapped fermions.

III. DYNAMICAL SYMMETRY GROUP APPROACH

We next present an alternative description of this isomorphism between bosonic and fermionic systems based on the dynamical symmetry group.

Classically, a system of N noninteracting particles in a one-dimensional harmonic potential is identical to that of a single particle in an N-dimensional isotropic harmonic potential. The system thus has an obvious spatial $O(N)$ symmetry we call ''angular momentum.'' However, it is apparent with more introspection that the system possesses a much larger dynamical symmetry group. Orbits in the N-dimensional isotropic harmonic potential do not precess. In analogy with the Kepler problem, we say that there is a conserved Runge-Lenz vector (which may be thought of as the axis of the orbit in configuration space), and we thus expect the symmetry group to be enlarged.

Since we will be interested in the quantization of the system, we describe this dynamical symmetry using operators of the associated quantum theory. To simplify notation, we take $\hbar\omega=1$ as before. Label the raising and lowering operators for the bosonic theory $a_i^\dagger$, a_i, respectively, with $i=1,...,N$. The canonical commutation relations (for the bosonic case) are $[a_i,a_j^\dagger]=\delta_{ij}$. The many-body Hamiltonian operator of this noninteracting system is $H=\Sigma_i a_i^\dagger a_i+\epsilon$, where ϵ is an overall constant that we suppress throughout.

We call the space of the eigenvalues of the $a_i^\dagger a_i$ the state space. Equivalently, the state space is the integer lattice in the $(+,\cdots,+)$ quadrant of N-dimensional Euclidean space. State space is not the Fock space, but is a useful auxiliary space from which we will construct the Fock space. Let e_i be orthonormal unit basis vectors in this Euclidean space associated with the eigenvalues of $a_i^\dagger a_i$. We name several distinguished vectors in this space, namely, the *level* vector $k=\Sigma_i e_i$ and the *root* vectors $l_i=e_i-e_{i+1}$ for $i=1,...,N-1$. We also define a spanning set of *weight* vectors r_i via $(r_i,l_j)=\delta_{i,j}$ together with $(r_i,k)=0$.

Note that the operators associated with l_i, namely, $a_i^\dagger a_i-a_{i+1}^\dagger a_{i+1}$, are independent, and commute with each other (being all diagonal) and with the Hamiltonian. The Hamiltonian corresponds to the level vector. Furthermore, to each pair of particles $l\neq j$, there is an associated su(2) subalgebra generated by $\{a_l^\dagger a_l-a_j^\dagger a_j, a_l^\dagger a_j+a_j^\dagger a_l, i(a_l^\dagger a_j-a_j^\dagger a_l)\}$. The application of the second or third operators in the above su(2) subalgebra ''shift'' the first operator's eigenvalue by a combination of root vectors. Finally, note that the matrix of inner products $M_{ij}=(l_i,l_j)$ of the root vectors is exactly the cartan matrix of su(N). Thus, we have identified the dynamical symmetry group of this system generated by the (trace-free part of the) products of $a_i^\dagger a_j$ to be su(N). (See Ref. [10].)

We now construct the Fock space for both fermions and bosons from the state space by realizing the respective antisymmetrizations and symmetrizations of the multiparticle Fock states as linear combinations of states in the state space that lie on the same Weyl group orbit. We make this correspondence precise with the following observations. Each state in the state space can be thought of as a particular product of single-particle states, its coordinates (the components of the l_i are integers) are simply the harmonic-oscillator level of each particle. Constructing the multiparticle state associated with that product of single-particle states consists of combining all the states from single-particle label permutations. The permutation group S_N is generated by primitive transpositions $(\ldots,n_i,n_{i+1},\ldots)\to(\ldots,n_{i+1},n_i,\ldots)$. Each of these primitive transpositions acts as a Weyl reflection (acting on all the roots) about the hyperplane perpendicular to the root l_i.

Thus, the Weyl group, $\mathcal{W}$ of the symmetry algebra su(N) is exactly the group of permutations of the single-particle states that make up the many-body state. Each element of the Weyl group preserves the level k. We specify a many-body state through an assignment of a highest weight vector r and level s (a natural number) for which $r+sk/N$ is a vector in the $(+,\cdots,+)$ quadrant (boundaries included). Explicitly, in terms of the vectors in the state space, the bosonic many-body Fock space has the basis $\Psi^{\text{boson}}_{r,s}$

$$\Psi^{\text{boson}}_{r,s}=\frac{1}{\sqrt{N!}}\sum_{\sigma\in\mathcal{W}}\left|\frac{s}{N}k+\sigma\cdot r\right\rangle$$

whereas the basis of the fermionic many-body Fock space is

$$\Psi^{\text{fermion}}_{r,s}=\frac{1}{\sqrt{N!}}\sum_{\sigma\in\mathcal{W}}(-1)^{\text{sgn}(\sigma)}\left|\frac{s}{N}k+\sigma\cdot r\right\rangle,$$

where the sgn(σ) is 1 if σ is an even permutation and -1 if it is an odd permutation. Note that according to this definition, only r vectors from the interior of the Weyl chamber are associated with a fermionic many-body state.

Succinctly stated, the multiparticle permutation symmetry of quantum mechanics maps the single-particle states of state space into the highest weight space of the symmetry algebra su(N). For bosons, the map covers the entire Weyl chamber (including the lattice points in the bounding hyperplanes) at each level. For fermions, the map covers only the interior lattice points of the Weyl chamber. Additionally, due to the constraint that $r+sk/N$ is in the $(+,\cdots,+)$ quadrant, at each level there are of course only a finite number of highest weight candidates.

The vector $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha$ (half the sum of positive roots) translates the vacuum of the bosonic Fock space to that of the fermionic Fock space at each level. Note also that ρ is thus orthogonal to the level vector k. It can be combined with the level vector to constitute a one-to-one map between the spectrum of the N boson and N fermion systems. Note that translation by the vector $\Gamma=(0,1,2,...,N-1)$ is precisely that map, and that $\Gamma=\rho+[(N-1)/2]k$. In particular, the coordinate change $\hat{e}_i=e_i-i+1$ used in the combinatoric proof can now be understood as translation by Γ. It is the only lattice vector that translates the lattice points of the Weyl chambers (including those in the bounding hyperplanes) exactly onto the *interior* of the Weyl chambers. Finally note that Γ has level $(\Gamma,k)=N(N-1)/2$, which is precisely the ground-state energy shift between the bosonic and fermionic system. Hence, we have confirmed our main finding that the entire spectrum of these bosonic and fermionic systems are isomorphic for any N up to an overall energy shift.

IV. REMARKS AND CONCLUSION

Although we have shown that the excitation spectra of one-dimensional Fermi and Bose systems are identical; we note that these systems are not related by an obvious supersymmetry. There may, however, exist a connection to the fermionic representation of affine lie algebra characters as described in Refs. [11–13]. We also note that the recent work of Schmidt and Schnack [7,8] indicates that the specific heats of Bose and Fermi systems in higher-spatial dimensions (specifically, odd dimensions) might also be equivalent, just as for the one-dimensional case considered here. Similar connections between bosonic and fermionic systems in two dimensions (or higher) have also been explored by Lee [14] and Pathria [15]. However we do expect that the spectral equivalence found here in the one-dimensional case will not persist in higher dimensions.

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