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Verification of Solutions to the Sensor Location Problem

Chandler May
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Abstract

Traffic congestion is a serious problem with large economic and environmental impacts. To reduce congestion (as a city planner) or simply to avoid congested channels (as a road user), one might like to accurately know the flow on roads in the traffic network. This information can be obtained from traffic sensors, devices that can be installed on roads or intersections to measure traffic flow. The *sensor location problem* is the problem of efficiently locating traffic sensors on intersections such that the flow on the entire network can be extrapolated from the readings of those sensors. I build on current research concerning the sensor location problem to develop conditions on a traffic network and sensor configuration such that the flow can be uniquely extrapolated from the sensors. Specifically, I partition the network by a method described by Morrison and Martonosi (2010) and establish a necessary and sufficient condition for uniquely extrapolatable flow on a part of that network that has certain flow characteristics. I also state a different sufficient but not necessary condition and include a novel proof thereof. Finally, I present several results illustrating the relationship between the inputs to a general network and the flow solution.

Acknowledgments

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Chapter 1

Introduction

Traffic congestion is a large and prevalent problem today. Individuals who are caught in a traffic jam suffer an opportunity cost, and the environment suffers due to the emissions of the delayed motor vehicles. The Texas Transportation Institute analyzed the effects of traffic congestion in the U.S. and found the following. Traffic congestion in America's 439 urban areas in 2007 cost Americans an extra 4.2 billion hours in travel time and an extra 2.8 billion gallons of fuel, for a combined monetary cost of $87.2 billion. This cost represents an increase of over 50% over the preceding decade. For each average peak-period traveler in 2007, this cost amounted to a full 36 hours in delay and 24 gallons in fuel (three weeks of fuel for the average U.S. resident) per year, summing to $760 in lost productivity and fuel costs. The effects were even worse in areas with over one million people (Schrank and Lomax, 2009).

With traffic congestion so damaging, it is not surprising that several methods have been developed to cope with it. Management and refinement of the traffic networks help considerably; capacity can be added to observably congested channels; alternative usage patterns can be encouraged and adopted; provision of alternate transportation options allows network users to customize their plans and potentially avoid bottlenecks; diverse urban-development patterns reduce the requirement for day-to-day travel; finally, it is helpful to adopt realistic expectations of real-world traffic patterns (Schrank and Lomax, 2009). I will consider the first of these approaches, improving the operation of the current networks.

As urban planners attempt to understand and optimize a traffic network, and as network users attempt to avoid congestion and minimize their time on the network, knowing the characteristics of the traffic flow on the network becomes very useful. An urban planner can use the information to prioritize his or her time and efforts and better determine the impacts of planning decisions. A network user can—likely with the help of a GPS receiver or similar device—pick the route that leads to his or her destination using the least time and frustration.

One method for obtaining this information involves observation of sensors that record the traffic flow directly on a road or intersection. These readings can be used for analysis of flow on individual roads and intersections of the network, and they can also be used to estimate the Origin/Destination trip matrix that is often used to analyze complete routes. Particularly, Bianco, Confessore, and Reverberi (2001) show how to improve the accuracy of Origin/Destination matrix estimation given the solution to the Sensor Location Problem, the problem of finding the minimum configuration of sensors required to derive flow on every link in the network. Bianco and colleagues (2001) provided a necessary and sufficient condition for the verification aspect of the Sensor Location Problem; that is, a characterization of when flow is uniquely determinable, given a network and a selection of sensors. Morrison and Martonosi (2010) provided a counterexample illustrating that the condition of Bianco and colleagues (2001) is in fact not sufficient, and presented a stronger necessary (albeit not generally sufficient) condition. I extend the work of Bianco and colleagues (2001) and Morrison and Martonosi (2010) by continuing the search for a necessary and sufficient condition concerning verification of a solution to the Sensor Location Problem.

I begin by describing the Sensor Location Problem more thoroughly in Chapter 2. In Chapter 3, I establish a necessary and sufficient condition for uniquely determinable flow in a special type of network. Next, in Chapter 4, I present several results concerning how the flow solution to a network is affected by the network's inputs. Finally, in Chapter 5, I conclude by mentioning other approaches that were considered and by suggesting directions for further research.

Chapter 2

The Sensor Location Problem

I present the Sensor Location Problem and prior work by Morrison and Martonosi (2010). I then posit three additional assumptions to the Sensor Location Problem in preparation for the discussion of my own results.

2.1 Formulation of the Problem

We represent a road network as a directed graph $G = (V, A)$ where each vertex represents an intersection, and arcs between vertices represent roads between intersections. We require that arcs exist symmetrically; that is, for each arc $vu \in A$ there also exists an arc $uv \in A$. We call the graph with this symmetric relationship a *two-way directed graph*. We represent the real-valued flow on an arc $vu \in A$ by f_{vu}. Our network is constrained by a system of flow-balance equations, one for each vertex $v \in V$,

$$\sum_{e \in v^-} f_e - \sum_{e \in v^+} f_e + S_v = 0, \tag{2.1}$$

where v^- is the set of arcs incoming to v, v^+ is the set of arcs outgoing from v, and S_v is the *balancing flow* at vertex v. Hence $S_v = 0$ if flow is conserved at v; otherwise, S_v represents the amount of flow that is produced at v. (If $S_v < 0$, then $-S_v$ is the amount of flow that is consumed.) We require that flow be conserved in the graph as a whole. That is,

$$\sum_{v \in V} S_v = 0. \tag{2.2}$$

If $S_v \neq 0$ then we call v a *centroid*. We denote by B the set of all centroids, and we assume that this set is known (but we do not assume knowledge of the balancing flows). Examples of centroids in a physical network are intersections near office buildings or apartment complexes on a weekday morning, when many people are going to work.

At each vertex, we assume to know the fraction of flow that leaves along each outgoing arc. For example, we might know that at the intersection of a highway and a side road, $\frac{2}{5}$ of the flow leaves (in each direction) along the highway, and $\frac{1}{10}$ of the flow leaves (in each direction) along the side road. This information could be obtained by paying someone to stand on a street corner and count cars for a short period of time. The information acquired will reduce the number of sensors needed to uniquely extrapolate the flow on the network, so it is potentially worth that cost.

More specifically, we assume knowledge of all turning ratios in the network, where the *turning ratio* $c_{vu} \geq 0$ for an arc $vu \in A$ is the ratio of the flow on vu to the total incoming flow to vertex v. That is,

$$f_{vu} = c_{vu} \sum_{e \in v^-} f_e. \tag{2.3}$$

Note that the flow f_{vu} on arc vu can be expressed in terms of any other flow f_{vw} on an arc vw outgoing from the same vertex v by taking a ratio of turning ratios:

$$f_{vu} = \frac{c_{vu}}{c_{vw}} f_{vw} \tag{2.4}$$

(assuming $c_{vw} > 0$). Now, picking a *canonical arc* vw for vertex v with $c_{vw} > 0$, we can define, for any other arc vu outgoing from v, the *turning factor* α_{vu} of vu with respect to vw as

$$\alpha_{vu} = \frac{c_{vu}}{c_{vw}}. \tag{2.5}$$

We can thus find the flow f_{vu} with respect to the flow f_{vw} on the canonical arc,

$$f_{vu} = \alpha_{vu} f_{vw}. \tag{2.6}$$

We also assume that there are sensors at some intersections (vertices). If there is a sensor at a given intersection, we immediately know the flows on all arcs into and out of the corresponding vertex, and we call that vertex a *monitored vertex*. We denote by M the set of all monitored vertices, and we assume M is known. We also assume knowledge of the balancing flow S_v for every monitored vertex $v \in M$. Using turning ratios, we can deduce the flow on arcs between two vertices that are each adjacent to monitored vertices. In this light, we call the set of vertices that are adjacent to a monitored vertex the *neighbor set* and denote it by $A(M)$. Because we know the flow on all arcs into or out of M and we can deduce the flow on all arcs between vertices in $A(M)$, the flow on every arc in the subgraph of G induced by $M \cup A(M)$ is known. We give these arcs a name:

Definition 2.1 (Bianco et al. (2001))**.** *The* combined cutset, C_M, *is the set of arcs in the subgraph of G induced by* $M \cup A(M)$.

We can now formally define the Sensor Location Problem:

Definition 2.2 (Bianco et al. (2001))**.** *Given a two-way directed graph* $G = (V, A)$, *a network-flow function* f, *and a set of centroids B, what is the smallest set M of monitored vertices such that knowledge of all turning ratios, the values of* f *on incoming and outgoing arcs of vertices in M, and balancing flows* S_v *for vertices in M uniquely determines* f *and the balancing flows* S_v *everywhere on G?*

The focus of Morrison and Martonosi (2010) is in verifying that a given set M of monitored vertices uniquely determines f and the balancing flows S_v. I continue in this vein. However, I first state several relevant results of Morrison and Martonosi (2010).

2.2 Previous Results

Morrison and Martonosi (2010) showed that the condition for uniquely determining the flow f and the balancing flows S_v, proposed by Bianco and colleagues (2001), is necessary but not sufficient in general. They then proposed a new, stronger necessary condition, based on the following definition, and showed that this condition is also sufficient given a certain assumption on G. I now summarize these results.

Definition 2.3 (Morrison and Martonosi (2010))**.** *A B-*path *is a path starting at a centroid and ending at a vertex in* $A(M)$.

The condition found by Morrison and Martonosi (2010) is as follows:

Theorem 2.1 (Morrison and Martonosi (2010))**.** *Let* $G = (V, A)$ *be a two-way directed graph with centroid set B, and let M be a set of monitored vertices. The flow on arcs in G and the balancing flow at the vertices in B can be uniquely determined everywhere only if there exists a set* $\mathcal{P}$ *of* $|B - M|$ *vertex disjoint B-paths.*

In order to prove this theorem, Morrison and Martonosi (2010) translated the condition into an algebraic framework which I now summarize. The following paragraph and five-step process are copied from Morrison and Martonosi (2010) with minor editorial revisions:

Let $\mathbf{E}$ be the $|V| \times |A|$ incidence matrix of G, where the $(u,e)^{th}$ entry of E is -1 if vertex u is the tail of arc e, 1 if u is the head of e, and 0 otherwise. Let $\mathbf{f}$ be the $|A| \times 1$ vector of arc flows and let $\mathbf{S}$ be the $|V| \times 1$ vector of balancing flows. Then the system of flow-balance equations is given by

$$\mathbf{Ef} + \mathbf{S} = \mathbf{x} \tag{2.7}$$

where $\mathbf{x} = \mathbf{0}$. Observe that the the sum of the equations represented by this system gives $\sum_{v \in V} S_v = 0$, so that constraint is extraneous. Now, using knowledge of B and the turning ratios, we reduce the system as follows:

1. For each vertex $u \in V$, we choose some arc e_u outgoing from u to be the canonical arc for u. Thus $f_{uv} = \alpha_{uv} f_{e_u}$ for each arc uv outgoing from u. This reformulation in terms of canonical flows reduces the number of variables in the system from $|A|$ to $|V|$.

2. Now the flow-balance matrix $\mathbf{E}$ reduces to the $|V| \times |V|$ matrix $\hat{\mathbf{E}}$, where row u still corresponds to the flow-balance equation for vertex u and column v now corresponds to the canonical arc e_v for vertex v. Specifically, the $(u,v)^{th}$ entry of $\hat{\mathbf{E}}$ is

$$\left[\hat{\mathbf{E}}\right]_{uv} = \begin{cases} \alpha_{vu} & \text{if } u \text{ and } v \text{ are adjacent} \\ -\sum_{w \text{ adjacent to } u} \alpha_{uw} & \text{if } u = v \\ 0 & \text{if } u \text{ and } v \text{ are not adjacent.} \end{cases}$$

3. We can also augment $\hat{\mathbf{E}}$ with $|B|$ columns for the unknown balancing flows of the centroids: The column for the centroid $u \in B$ has a 1 in row u and 0's everywhere else. Similarly, create the $(|V| + |B|) \times 1$ vector

$$\mathbf{g} = \begin{bmatrix} \mathbf{f} \\ \hat{\mathbf{S}} \end{bmatrix},$$

where $\hat{\mathbf{S}}$ is the $|B| \times 1$ vector of unknown balancing flows for the centroids, and augment $\mathbf{x}$ by $|B|$ 0's. Equation 2.7 then becomes

$$\hat{\mathbf{E}}\mathbf{g} = \mathbf{x}. \tag{2.8}$$

Note that $\hat{\mathbf{E}}$ now has $|V|$ rows and $|V| + |B|$ columns.

4. Now, for each vertex $m \in M$, we can remove row m and column m from $\hat{\mathbf{E}}$ because we know the flow along the canonical arc e_m. We then update the right-hand-side vector $\mathbf{x}$ by subtracting f_{e_m} times that removed m^{th} column from $\mathbf{x}$. That is, we subtract $\alpha_{mu} f_{e_m}$ from the u^{th} entry of $\mathbf{x}$ for each vertex u adjacent to m. We also remove the entry from $\mathbf{g}$ corresponding to f_{e_m}, and the m^{th} entry from $\mathbf{x}$ (because we have removed the corresponding column and row, respectively). Furthermore, if m is a centroid, we can remove the column of $\hat{\mathbf{E}}$ corresponding to its balancing flow, and remove the entry corresponding to S_m from $\mathbf{g}$.

5. For each vertex $a \in A(M)$, the outgoing flow from a to any monitored vertex $m \in M$ is known, so we can deduce the flow f_{e_a} on the canonical arc of a. We thus remove column a from $\hat{\mathbf{E}}$ and subtract $f_{e_a} = (1/\alpha_{am}) f_{am}$ times column a from the right-hand-side vector $\mathbf{x}$. That is, we subtract $\alpha_{au} f_{e_u}$ from the u^{th} entry of $\mathbf{x}$ for each vertex u adjacent to a, and we add $\sum_{w \text{ adjacent to } a} \alpha_{aw} f_{e_a}$ to the a^{th} entry of $\mathbf{x}$. We also remove the entry from $\mathbf{g}$ corresponding to f_{e_a}.

(Thus concludes the excerpt from Morrison and Martonosi (2010), with revisions.) We call the matrix that results from this procedure the *flow-calculation matrix* $\mathbf{F}$, and so Equation 2.8 becomes

$$\mathbf{F}\mathbf{g} = \mathbf{x}. \tag{2.9}$$

Note that $\mathbf{F}$ has $|V - M|$ rows and $|(V - M) - A(M)| + |B - M|$ columns; $\mathbf{g}$ has $|(V - M) - A(M)| + |B - M|$ rows; and $\mathbf{x}$ has $|V - M|$ rows. We have now accounted for everything we know: we have reduced the $|A|$ flow variables to $|V|$ variables using turning ratios; we have removed all equations and canonical-arc–flow variables corresponding to monitored vertices, as those are known; we have removed the flow-balance variables of noncentroids and of monitored vertices; we have removed all canonical-arc–flow variables for vertices in $A(M)$, because those can be deduced from turning ratios (the corresponding equations still carry information, because we don't know all flows into vertices in $A(M)$). So, under our assumptions, the flow and flow balance can be determined everywhere on the graph if and only if Equation 2.9 has a unique solution; that is, $\mathbf{F}$ has full column rank.

Now return momentarily to the graph-theoretic representation of the network. We know the balancing flows of vertices in M, and we know the flows on all arcs in C_M. Define the *unmonitored subgraph* G' of G as the subgraph of G where M and C_M have been removed, $G^{'} = (V - M, A - C_M)$. This subgraph may or may not be connected; in general, call the i^{th} connected component the i^{th} *unmonitored component* and denote it as $G_i^{'}$. Denote the centroids of $G_i^{'}$ as B_i and the subset of vertices from $A(M)$ in $G_i^{'}$ as $A(M)_i$.

If vertices u and v are in different unmonitored components $G_i^{'}$ and $G_j^{'}$, then any path from u to v in G must have passed through an arc in C_M. Because the columns corresponding to canonical arcs for vertices in $M \cup A(M)$ are removed in the creation of the flow-calculation matrix **F**, the terms representing flows on arcs in C_M are also removed. Because the rows corresponding to the monitored vertices are also removed in the creation of **F**, we can permute the rows and columns of **F**, arranging **F** in a block form by collecting rows and columns corresponding to vertices in the same unmonitored component. That is, if there are k distinct unmonitored components then we can arrange **F** into the form

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}^1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{F}^k \end{bmatrix},$$

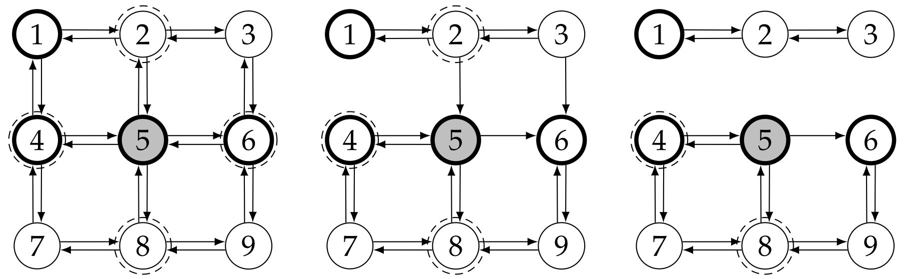
where $\mathbf{F}^i$ is the flow-calculation matrix for the unmonitored component $G_i^{'}$.

Morrison and Martonosi (2010) used this representation of the network to rephrase Theorem 2.1 into the following formulation. This formulation was proved, so Theorem 2.1 holds in turn.

Theorem 2.2 (Morrison and Martonosi (2010))**.** *Let G, B, and M be as in Theorem 2.1, with the graph partitioned into unmonitored components and the flow-calculation matrix partitioned into blocks as described. For each unmonitored component i, let C_i be the minimum vertex cut between $(B - M)_i$ and $A(M)_i$. (If $(B - M)_i$ is empty, then let $C_i = \emptyset$.)* $\operatorname{rank}(\mathbf{F}^i) = \#\{$*columns of* $\mathbf{F}^i\}$ *(and hence the flow on $G_i^{'}$ is calculable) only if $|C_i| = |(B - M)_i|$.*

Incidentally, as noted by Morrison and Martonosi (2010) in their proof of the theorem, if C_i is a minimum disconnecting set then we have $|C_i| \geq |(B - M)_i|$ almost immediately, by application of Menger's theorem.

The condition that $|C_i| = |(B - M)_i|$ is necessary for flow to be uniquely determinable everywhere. Morrison and Martonosi (2010) showed that it is also sufficient for an unmonitored component that is a tree. Specifically,



a. A strongly connected network. **b.** A weakly connected network. **c.** A network that is not connected.

Figure 2.1 These networks illustrate our additional assumptions to the Sensor Location Problem. Subfigures a and b satisfy assumption 1 whereas subfigure c does not. In subfigure a or b, if $c_{25} = 0$ then assumption 2 is not satisfied. In subfigure b, if $c_{32} = c_{36} = 0$ then assumption 3 is not satisfied. (Note that these cases are not the only ways in which the respective assumptions can be unsatisfied in these three networks.) In these figures and in subsequent figures, a vertex with a bold border is a centroid, a vertex with a dashed second border is in the neighbor set $A(M)$, and a grey vertex is a monitored vertex.

Theorem 2.3 (Morrison and Martonosi (2010)). *Let G, B, and M be as in Theorem 2.1, with the graph partitioned into blocks as described. For each unmonitored component i, let C_i be the minimum vertex cut between $(B-M)_i$ and $A(M)_i$. If the i^{th} component is a tree, then $\text{rank}(\mathbf{F}^i) = \#\{\text{columns of } \mathbf{F}^i\}$ (i.e., the flow on block i is calculable) if and only if $|C_i| = |(B-M)_i|$.*

With relevant prior results in hand, now, I proceed to describe several additional assumptions to the Sensor Location Problem.

2.3 Additional Assumptions

In order to make my results more rigorous and clear, I impose three new assumptions:

1. G is connected, ignoring directionality of the arcs; that is, G is connected but not necessarily strongly connected. (Figures 2.1a and 2.1b depict networks that satisfy this condition. Figure 2.1c shows a network that does not.)

2. For each $a \in A(M)$, there exists a monitored vertex $m \in M$ such that $am \in A$ and $c_{am} > 0$. (In either Figure 2.1a or Figure 2.1b, if $c_{25} = 0$ then this condition is not satisfied. Also, note that vertex 6 is not in $A(M)$ in Figure 2.1b or Figure 2.1c due to the removal of the arc from 6 to 5.)

3. For each vertex $v \in V$ there is at least one arc $e \in v^+$ such that $c_e > 0$. (For example, if c_{32} and c_{36} are zero in Figure 2.1b, then this condition is not satisfied.)

Assumption 1 provides mathematical simplicity without imposing any real constraints on the network; we can easily treat two components of a disconnected network as two separate networks. I impose assumption 2 to ensure that the canonical flow of any vertex in the neighbor set can be calculated by comparison to the known flow from that vertex to an adjacent monitored vertex, which is the reason we make the distinction of the neighbor set in the first place. If a vertex has incoming arcs from a monitored vertex but no outgoing arc (with a positive turning ratio) to any monitored vertex, we do not want to include it in the neighbor set because we cannot determine its flow solely from its turning factors and its adjacency to one or more monitored vertices. Without assumption 3, there would not exist a sensible choice for the canonical arc of v, considering that the turning ratio of the canonical arc appears in a denominator when flows on other arcs are expressed in terms of the flow on the canonical arc.

We are now ready to proceed to the results.

Chapter 3

Flow Calculability: Centroid-Free Components

Morrison and Martonosi (2010) proved a necessary condition for uniquely calculable flow on an unmonitored component of a graph, and proceeded to show that this condition is also sufficient when the component is a tree. I provide two conditions for the general unmonitored component with no centroids. The first condition, developed in Sections 3.1 and 3.2, is sufficient but not necessary for uniquely determinable flow. The proof I devised for this condition is long and unintuitive; it is listed in Appendix A. The second condition, developed in Section 3.3, is both necessary and sufficient; it is my main result. The first condition is included for record keeping and as potential inspiration to future research; the second condition is clearly stronger and is accompanied by a comparatively elegant proof.

3.1 Condition One: A Simple Case

Suppose an unmonitored component $G_i^{'}$ is the complete graph K_{n+1} on the $n+1$ vertices $V_i = \{1, 2, \ldots, n+1\}$. Suppose further that $A(M)_i = \{n+1\}$, and that the turning factors are all the same, $\alpha_{uv} = \alpha > 0$ for all $uv \in A_i$. Let the arcs outgoing from vertices $1, \ldots, n$ to vertex $n+1$ be canonical.

Then

$$\mathbf{F}^i = \begin{bmatrix} -1-(n-1)\alpha & \alpha & \cdots & \alpha \\ \alpha & -1-(n-1)\alpha & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & -1-(n-1)\alpha \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

For $k \in \{1, \ldots, n\}$, by adding $-1/\alpha$ times row k to row $n+1$ and then scaling by

$$\left[1 + \frac{1}{\alpha}\left(1 + (n-1)\alpha\right)\right]^{-1},$$

we can generate the k^{th} standard basis vector $\mathbf{e}_k^T$ of $\mathbb{R}^n$ as a linear combination of the rows. Hence this graph has uniquely determinable flow.

However, this situation is a far cry from the general case. In particular, our turning factors are not necessarily all the same; moreover, they are allowed to be zero (given that the assumptions in Section 2.3 are satisfied). We also allow arbitrarily many vertices in $A(M)$. Proof of uniquely determinable flow in the general case is much more involved. I now proceed to state conditions on a general unmonitored component such that flow is uniquely determinable; I then develop intuition towards the meaning of those conditions.

3.2 Condition One: Result and Illustration

Before I present the result and an illustration of its implications, I recall a definition:

Definition 3.1. *An* arborescence *is a directed graph in which there is a vertex v, called the root, such that for any other vertex u there is exactly one directed path from v to u.*

A reverse arborescence *is a directed graph in which there is a vertex v, called the root, such that for any other vertex u there is exactly one directed path from u to v.*

Figure 3.1 depicts embeddings of several reverse arborescences. Note that if these three reverse arborescences are taken to be the components of a single graph, that graph is a *forest* of reverse arborescences.

I now state my result and illustrate it with an example.

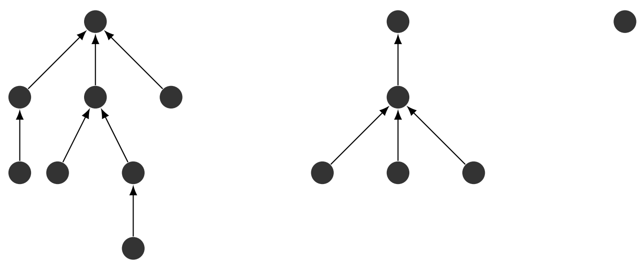


Figure 3.1 Examples of reverse arborescences.

Theorem 3.1. *Let G, B, and M be as in Theorem 2.1 with the graph partitioned into unmonitored components and the flow-calculation matrix partitioned into blocks as described. For each unmonitored component G_i', define n_i to be the number of columns of $\mathbf{F}^i$. If $(B-M)_i$ is empty, then* $\operatorname{rank}(\mathbf{F}^i) = n_i$ *(and hence the flow on G_i' is calculable) if there exists a set of n_i arcs, one emanating from each of the n_i vertices in $V_i - A(M)_i$, such that the turning factors for those arcs are positive and the subgraph of G_i' induced on those arcs is a forest of reverse arborescences.*

The proof of this theorem is too long for inclusion here; rather, it is provided in Appendix A.

To gain insight into this result, consider the unmonitored component G_i' depicted in Figure 3.2. Note that there are no centroids, so $(B-M)_i$ is empty. There are six vertices in this component, two of which are in $A(M)_i$, so $n_i = |V_i - A(M)_i| + |(B-M)_i| = |\{1,2,3,4\}| + |\emptyset| = 4$. Thus the flow on this component is uniquely determined if there is a set of four arcs, one emanating from each of the four vertices in $V_i - A(M)_i = \{1,2,3,4\}$, such that the turning factors for those arcs are positive and the subgraph induced on those arcs is a forest of reverse arborescences.

Two such selections of arcs are depicted (by their induced subgraphs) in Figure 3.3a and Figure 3.3b, respectively. We have thus found a choice of n_i arcs, one emanating from each of the vertices in $V_i - A(M)_i$, such that the subgraph induced on those arcs is a forest of reverse arborescences. (Indeed, we have found two choices, and there are more.) Thus, if α_{12}, α_{26}, α_{34}, and α_{42} (corresponding to Figure 3.3a) are all positive, or if α_{12}, α_{26}, α_{34}, and α_{45} (corresponding to Figure 3.3b) are all positive, then Theorem 3.1 implies that the flow on this unmonitored component is uniquely determined.

Figure 3.3c shows a selection of arcs whose induced subgraph is *not* a forest of reverse arborescences because there is a cycle. Note that there is at least one arc to a vertex in $A(M)_i$ in both Figure 3.3a and Figure 3.3b,

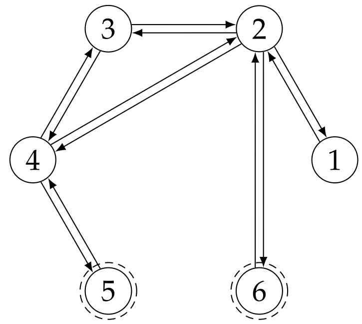
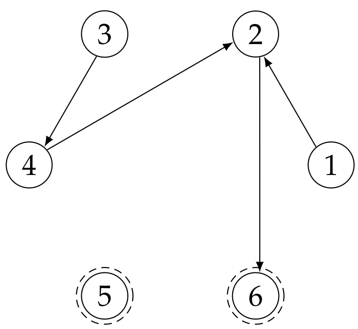
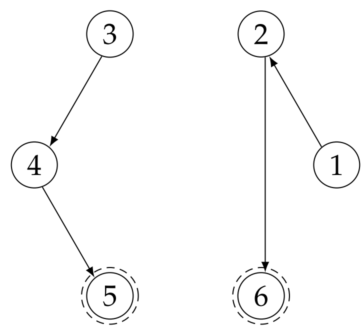


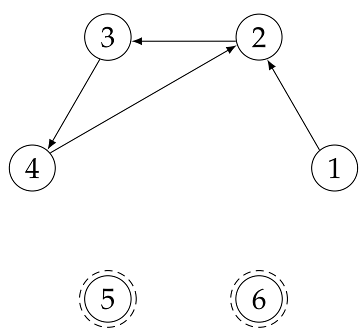
Figure 3.2 An example unmonitored component, used to illustrate the sufficient condition on centroid-free unmonitored components.



a. A forest of reverse arborescences.

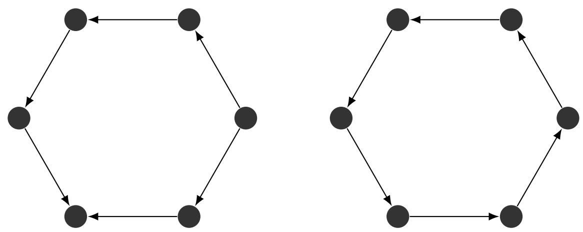


b. Another forest of reverse arborescences.



c. Cyclic; not a forest of reverse arborescences.

Figure 3.3 Induced subgraphs of the unmonitored component illustrating the centroid-free sufficient condition.



a. An undirected cycle. **b.** A directed cycle.

Figure 3.4 Cycles illustrating the role of $A(M)$ in the centroid-free necessary and sufficient condition.

whereas there is no such arc in Figure 3.3c. This is not a coincidence. If there is no arc to any vertex in $A(M)_i$ then there must be n_i arcs between the n_i vertices in $V_i - A(M)_i$. We know that an undirected graph with n_i vertices and n_i edges would necessarily contain a cycle, so the n_i vertices and (directed) arcs in our induced subgraph form a cycle *if the directionality of the arcs is ignored*. However, because there is exactly one arc emanating from each vertex, every cycle in the induced subgraph must be a directed cycle. (To see this, consider the example undirected cycle of Figure 3.4a in contrast to the directed cycle of Figure 3.4b. In the latter, there is one arc emanating from each of the six vertices. In the former, there is not. In particular, there are zero or two arcs emanating from the vertices at which the cycle "switches direction.") Thus, because there are n_i arcs between the n_i vertices in $V_i - A(M)_i$, there is a directed cycle, and so the subgraph is not a forest of reverse arborescences. This means that the sufficient condition of Theorem 3.1 requires connections to $A(M)_i$; it will never imply that an unmonitored component disjoint from $A(M)$ (or an unmonitored component for which the connection to $A(M)$ does not carry enough information, e.g. because of insufficient positive turning factors on arcs to $A(M)$) has uniquely calculable flow.

3.3 Condition Two

I now state and justify a necessary and sufficient condition for unique flow calculability on a centroid-free unmonitored component. The proof of this condition uses a more graph theoretical approach than the proof for Theorem 3.1; it is also significantly more intuitive.

We assume that all turning factors are positive, and we do *not* require that arcs exist symmetrically, i.e. if there is an arc $(u, v) \in A$ we no longer require there to be an arc $(v, u) \in A$. Note that a turning factor of zero in the original formulation of the Sensor Location Problem is equivalent to removal of that arc, so the assumption that all turning factors are positive, together with the relaxation of symmetry in the arcs, yields no real difference from the original formulation.

In order to state my condition, I first define a certain structure, which I will call a *trap*, on an unmonitored component.

Definition 3.2. *Consider an unmonitored component $G_i^{'}$ and its neighbor set $A(M)_i$. The unmonitored component is a* trap *if its vertices can be partitioned into two sets, $V_{I,i}$ and $V_{O,i}$, such that the following conditions hold:*

1. *$V_{I,i}$ and $A(M)_i$ are disjoint;*

2. *$V_{I,i}$ has at least two elements; and*

3. *if there are any arcs between $V_{O,i}$ and $V_{I,i}$, they must be from $V_{O,i}$ to $V_{I,i}$.*

This structure is a degenerate case of a centroid-free unmonitored component. As I will show, it allows many solutions to the system. I now state this result.

Theorem 3.2. *Let G, B, and M be as in Theorem 2.1 with the graph partitioned into unmonitored components $G_i^{'}$ and the flow-calculation matrix partitioned into blocks $\mathbf{F}^i$ as described. Assume G is connected. For each unmonitored component $G_i^{'}$ where $(B - M)_i$ is empty, the flow on $G_i^{'}$ is uniquely calculable if and only if $A(M)_i$ is nonempty and $G_i^{'}$ is not a trap.*

I henceforth drop the superscript (subscript) i for ease of notation. I proceed directly to the proof, in which I establish the converses of the forward and reverse directions (in turn).

3.3.1 Converse of Forward Direction

We first assume there are two distinct flow vectors $\mathbf{f}$ and $\mathbf{f}'$ that solve the system of flow-balance equations, and show that either $A(M)$ is empty or $G^{'}$ is a trap.

Denote the component of $\mathbf{f}$ corresponding to the canonical flow of some vertex v as f_{e_v}. (Denote the components of $\mathbf{f}'$ similarly.) Then without loss of generality, for some $v_k \in V$, we have $f'_{e_{v_k}} > f_{e_{v_k}}$. Note here that there must be an incoming arc to v_k, otherwise flow balance is violated at v_k.

Let V_I be the set of vertices v such that $f'_{e_v} > f_{e_v}$, and let $V_O = V - V_I$. Immediately, we see $v_k \in V_I$. Observe that $A(M) \subseteq V_O$ because the canonical flow of every vertex in the neighbor set is fixed. One of the following conditions must hold:

1. V_O is empty;
2. there are no arcs between V_I and V_O;
3. there is at least one arc from V_I to V_O (and possibly arcs from V_O to V_I); or
4. there is at least one arc from V_O to V_I but there are no arcs from V_I to V_O.

These conditions are illustrated by Figures 3.5a, 3.5b, 3.5c, and 3.5d respectively.

We consider conditions 2 and 3 in turn, showing that each leads to a contradiction. We then develop the implications of conditions 1 and 4.

Suppose condition 2 holds. Because $A(M)$ and V_I are disjoint, V_I must be disconnected from the rest of the original graph G, so we immediately have a contradiction.

Now suppose condition 3 holds. We will apply flow balance to find that this condition leads to a contradiction. Let V_I^+ be the set of arcs (v, w) where $v \in V_I$ and $w \in V_O$, and let V_I^- be the set of arcs (w, v) (again, where $v \in V_I$ and $w \in V_O$). Because V_I contains *all* vertices whose canonical flows are greater in $\mathbf{f}'$ than in $\mathbf{f}$, no arcs from V_O to V_I have greater flows in $\mathbf{f}'$ than in $\mathbf{f}$. In other words, $f'_e \leq f_e$ for all arcs $e \in V_I^-$. So

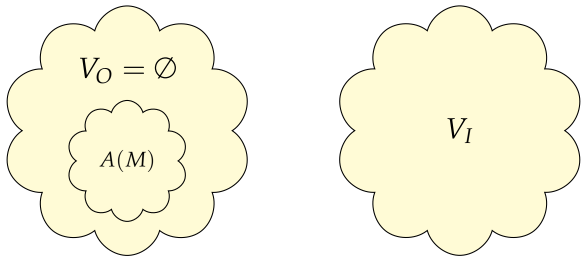
$$\sum_{e \in V_I^+} f'_e - \sum_{e \in V_I^-} f'_e \geq \sum_{e \in V_I^+} f'_e - \sum_{e \in V_I^-} f_e. \tag{3.1}$$

Condition 3 implies that V_I^+ is nonempty. Thus, by the fact that the canonical flows of all vertices in V_I are greater in $\mathbf{f}'$ than in $\mathbf{f}$,

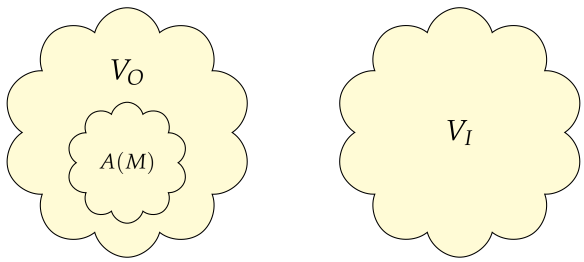
$$\sum_{e \in V_I^+} f'_e > \sum_{e \in V_I^+} f_e.$$

In particular, let (v_i, v_j) be an arc in V_I^+, so $f'_{v_i v_j} > f_{v_i v_j}$. Then

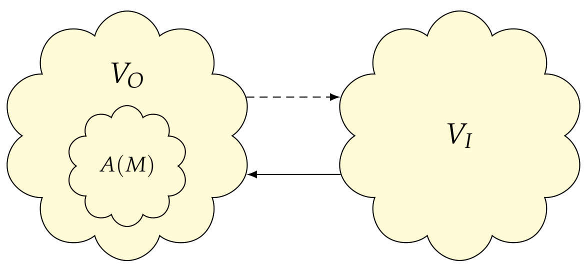
$$\sum_{e \in V_I^+} f'_e \geq (f'_{v_i v_j} - f_{v_i v_j}) + \sum_{e \in V_I^+} f_e,$$



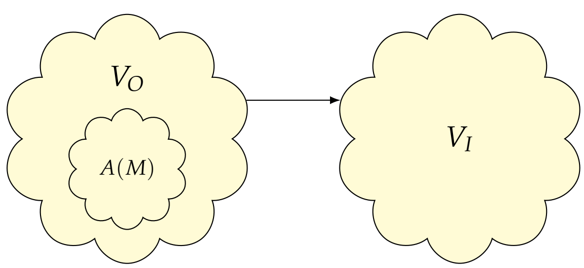
a. Condition 1: V_O is empty.



b. Condition 2: there are no arcs between V_I and V_O.



c. Condition 3: there is an arc from V_I to V_O.



d. Condition 4: there is an arc from V_O to V_I, but not vice versa.

Figure 3.5 Illustrations of the four conditions used in the proof of the centroid-free necessary and sufficient condition.

where $f'_{v_iv_j} - f_{v_iv_j} > 0$. Plugging this into the right-hand side of Equation 3.1, we get

$$\sum_{e \in V_I^+} f'_e - \sum_{e \in V_I^-} f'_e \geq (f'_{v_iv_j} - f_{v_iv_j}) + \sum_{e \in V_I^+} f_e - \sum_{e \in V_I^-} f_e. \tag{3.2}$$

By flow balance,

$$\sum_{e \in V_I^+} f_e - \sum_{e \in V_I^-} f_e = 0,$$

so from Equation 3.2 we get

$$\sum_{e \in V_I^+} f'_e - \sum_{e \in V_I^-} f'_e \geq (f'_{v_iv_j} - f_{v_iv_j}) > 0,$$

violating flow balance for $\mathbf{f}'$. Condition 3 therefore leads to a contradiction.

Thus if there are two distinct solutions, condition 1 or condition 4 must hold. If condition 1 holds then V_I is the entire unmonitored component. But V_I and $A(M)$ are disjoint, so $A(M)$ is empty.

Now suppose condition 4 holds, so V_I^+ is empty and V_I^- is nonempty. By flow balance,

$$\sum_{e \in V_I^+} f_e - \sum_{e \in V_I^-} f_e = 0,$$

so we require $f_e = 0$ for all arcs $e \in V_I^-$.

Now we will show that for any vertex $v \in V_O$ such that there is a path from v to some vertex $w \in V_I$, the canonical flow f_{e_v} of v is zero. Let v be such a vertex, with a path to a vertex $w \in V_I$. (Recall that we have assumed all turning factors are nonzero. In particular, the turning factors for the arcs forming this path are nonzero.) Let the path be $(v_0, v_1, \ldots, v_n)$ where $v = v_0$ and $w = v_n$, and suppose by way of contradiction that $f_{e_v} > 0$. If $f_{e_{v_j}} > 0$ for some j where $0 \leq j < n-1$, then there is a nonzero flow into vertex v_{j+1} (the next vertex on the path). In order for flow balance to hold, then, there must be a nonzero flow *out* of vertex v_{j+1} as well; that is, $f_{e_{v_{j+1}}} > 0$. By induction, v_{n-1} has a nonzero canonical flow, so there is a positive flow from $v_{n-1} \in V_O$ to $w \in V_I$. We thus have a contradiction, hence the canonical flow of any vertex $v \in V_O$ such that there is a path from v to a vertex in V_I must be zero.

Thus we justify that the unmonitored component is a trap: Due to condition 4, there are no arcs from V_I to V_O, and the only arcs between $V - V_I$ and V_I are those from V_O to V_I (which necessarily carry zero flow). We know V_I and $A(M)$ are disjoint. Lastly, if V_I had only one element, namely

v_k, then we would immediately have a contradiction: The canonical flow of v_k would be positive, implying that v_k has an outgoing arc (by assumption), violating condition 4. So $|V_I| \geq 2$, hence the unmonitored component is indeed a trap, with V_I and V_O taking the same meanings as they do in the definition.

Because conditions 2 and 3 led to contradictions, my argument of the converse of the forward direction is complete: If there is not a unique solution to the flow-balance equations for this unmonitored component then either $A(M)$ is empty or the unmonitored component is a trap.

3.3.2 Converse of Reverse Direction

I now argue that the converse of the reverse direction also holds. We assume that either $A(M)$ is empty or the unmonitored component is a trap; we will show that in either case there is not a unique solution satisfying flow balance.

If $A(M)$ is empty then $\mathbf{F}$ is square; because its rows sum to zero, we immediately see that there is not a unique solution.

Let $\mathbf{g}$ be a flow solution to the flow balance system $\mathbf{Fg} = \mathbf{x}$, and let g_{e_v} represent the component of that solution for the canonical arc e_v of some vertex v. If the unmonitored component is a trap then we require

$$\sum_{v \in V_I} \mathbf{F}_v g_{e_v} = \mathbf{0}$$

(where $\mathbf{F}_v$ is the column of $\mathbf{F}$ corresponding to the canonical flow of vertex v). This requirement is a consequence of

- flow balance;
- the fact that there are no arcs from V_I to $A(M)$ or $V - V_I$, which implies that the entries of the column vector $(\mathbf{Fg})$ that correspond to vertices in V_I depend on the turning factors and flows only of vertices in V_I; and
- the observation that $f_e = 0$ for all arcs $e \in V_I^-$ (and in particular $f_e = 0$ for all arcs from $A(M)$ to V_I), which implies that the entries in the right-hand side of the system $\mathbf{Fg} = \mathbf{x}$ that correspond to vertices in V_I are zero.

Thus, if we define $\mathbf{g}'$ as the flow vector with components

$$g'_{e_v} = \begin{cases} 2g_{e_v}, & v \in V_I \\ g_{e_v}, & v \in V - V_I, \end{cases}$$

then $\mathbf{g}'$ is a distinct solution to the flow-balance equations.

I have shown that if $A(M)$ is empty or the unmonitored component is a trap then there is not a unique solution, as desired. This statement is the converse of the reverse direction. Note that $A(M)$ is empty if and only if there were no monitored vertices in the original network or there were no unmonitored vertices in the original network. This equivalence depends on our assumption that the original network is connected.

Because the converses of the forward and reverse directions hold, we conclude that Theorem 3.2 holds. That is, $A(M)$ is nonempty and G' is not a trap if and only if there is a unique solution for this centroid-free unmonitored component. So concludes the proof and my results.

Note that a trap represents an unrealistic physical network. Travel between the two vertex sets of the network would be extremely constrained; one could go from $V_{O,i}$ to $V_{I,i}$ but would not be able to return to $V_{O,i}$. We will also find that if the unmonitored component is centroid-free, as we have been assuming, then flow balance requires that there is zero flow out of $V_{O,i}$; hence, there is no travel between the two vertex sets whatsoever. In light of these observations, Theorem 3.2 suggests that a centroid-free unmonitored component is practically guaranteed to have uniquely calculable flow.

It is my hope that my rigorous treatment of the centroid-free case will encourage research into the more general and difficult centroid-existing case, and advance our understanding of the Sensor Location Problem. In the next chapter, I analyze the effects of the turning factors and known flows of a network on the network's flow solution. These findings illuminate the general case and serve as a guide to future research.

Chapter 4

Role of Turning Factors

I have so far focused on centroid-free unmonitored components of a network. I now proceed to describe several observations and results for a general unmonitored component $G_i^{'}$ which can have centroids.

4.1 Flow Uniqueness

Morrison and Martonosi (2010) proved a necessary condition for uniqueness of the flow solution. This condition was phrased in terms of B-paths in the unmonitored component. I now show that flow uniqueness may depend not only on such structural properties of the graph, but also on the turning factors. For example, consider Figure 4.1, a reproduction of Figure 5 from Morrison and Martonosi (2010). If the monitored flows into and out of vertex 5 are all one and the turning factors are also all one, then this network has more than one solution. In one solution, all flows are one and the balancing flows for vertices 1 and 2 are zero. In another solution, $f_{13} = f_{14} = \frac{3}{2}$, $f_{23} = f_{24} = \frac{1}{2}$, all other flows are one, and the balancing flows are $S_1 = 1$ and $S_2 = -1$.

Suppose the monitored flows into and out of vertex 5 are all one and all turning factors are one except $\alpha_{23} = 3$ and $\alpha_{14} = 2$. (The canonical arcs can arbitrarily be any arcs whose turning factors are one.) Then the network has a unique solution given by $f_{13} = \frac{4}{5}$, $f_{14} = \frac{8}{5}$, $f_{23} = \frac{6}{5}$, $f_{24} = \frac{2}{5}$, all other flows are one, and the balancing flows are $S_1 = \frac{2}{5}$ and $S_2 = -\frac{2}{5}$.

In fact, for this network, there is a unique solution if and only if

$$\frac{c_{24}}{c_{23}} \neq \frac{c_{14}}{c_{13}},$$

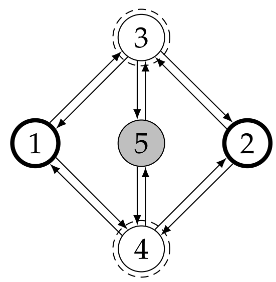


Figure 4.1 Network illustrating the dependence of unique flow calculability on turning factors and the possibility of negative flow solutions.

or equivalently,

$$\frac{\alpha_{24}}{\alpha_{23}} \neq \frac{\alpha_{14}}{\alpha_{13}},$$

where the turning factors for the canonical arcs are one. This condition can be verified by computing the flow-calculation matrix $\mathbf{F}^i$ for the unmonitored component depicted in Figure 4.1 and row-reducing. (It is assumed that all turning factors are nonzero.)

Thus, in general, flow determinability of an unmonitored component may depend on the configuration of its turning ratios. This result indicates that a necessary and sufficient condition for unique flow determinability on a general graph must involve a condition on the turning ratios, or equivalently, the turning factors.

4.2 Flow Negativity

Now consider another set of inputs (monitored flows and turning factors) to the unmonitored component depicted in Figure 4.1. Suppose the monitored flows into and out of vertex 5 are all one and all turning factors are one except $\alpha_{24} = 3$ and $\alpha_{14} = 2$. Then the network has a unique solution given by $f_{13} = 4$, $f_{14} = 8$, $f_{23} = -2$, $f_{24} = -6$, all other flows are one, and the balancing flows are $S_1 = 10$ and $S_2 = -10$. The solution is uniquely determinable, but it contains negative flows, hence it is physically unrealistic. It is not correct to interpret negative flow on an arc as flow moving in the opposite direction because the turning factors lose their intended meaning when the flow is negative.

The physical infeasibility of the solution thus indicates that the inputs to the network were physically invalid. That is, the monitored flows and turning factors we used in this example are not realistic; they would never have emerged from accurate observations on a physical traffic network. The fact that these inputs were perfectly valid in a strictly *mathematical* sense suggests that our model of the traffic network is more general and lenient than it needs to be. Accordingly, it may be easier to find a necessary and sufficient condition for unique flow determinability if we assume we are given inputs that yield a physically valid solution; that is, a solution whose flows are strictly positive.

Logistic regression was performed on the turning factors of the network in Figure 4.1, using the existence of a negative flow in the unique solution as the response variable, in hopes to characterize when negative flows arose. (Turning factor configurations yielding nonunique solutions were discarded.) This analysis did not provide much new information or insight. The behavior of the solution largely appeared to depend on the relationship of α_{24} and α_{13}, when $(2,3)$ and $(1,4)$ were taken to be the canonical arcs for vertices 2 and 1 (respectively). However, this observation is not altogether surprising considering the importance of those parameters in unique flow determinability.

4.3 Perturbation Analysis

Recall that the condition for uniquely determinable flow in the network of Figure 4.1 was

$$\frac{\alpha_{24}}{\alpha_{23}} \neq \frac{\alpha_{14}}{\alpha_{13}}.$$

Thus it would seem that "most" configurations of the turning factors yield uniquely determinable flow for the network in Figure 4.1. More precisely, if turning factors were selected at random, they would be very unlikely to satisfy

$$\frac{\alpha_{24}}{\alpha_{23}} = \frac{\alpha_{14}}{\alpha_{13}}.$$

Moreover, one might expect that real-world turning factors are "messy" in a sense; intuitively, they would be rational numbers with long decimal expansions and no clear relationship to one another, and they would be unlikely to satisfy such an equality. Therefore, if other networks have a similar condition for uniquely determinable flow to that of Figure 4.1, then we might expect turning factor inputs from the real world to yield uniquely determinable solutions, at least as long as the necessary condition of Theorem 2.1 is satisfied (namely, that there are $|(B-M)_i|$ vertex-disjoint B-

paths). In the cases where that necessary condition is satisfied but flow is not uniquely determinable, we could force a unique solution by perturbing the turning factors such that they do not satisfy the equality characterizing nonunique flow.

With this idea in mind, I performed a rudimentary perturbation analysis of the turning factors for the unmonitored components depicted in Figure 4.2. For each unmonitored component, the flow-calculation matrix block $\mathbf{F}^i$ was constructed twice, the first time using turning factors of one, and the second time using turning factors drawn from a normal distribution with mean 1 and standard deviation 0.01. (This standard deviation was used because it is relatively small but still numerically practical.) The rank of $\mathbf{F}^i$ was computed in each case to determine whether the flow solution was unique.

When all turning factors were set to 1, the unmonitored components depicted in Figures 4.1, 4.2c, 4.2d, 4.2f, 4.2g, and 4.2h did not have uniquely calculable flow; the other unmonitored components (Figures 4.2a, 4.2b, and 4.2e) did have uniquely calculable flow. However, when each turning factor was drawn from a normal distribution with mean 1 and standard deviation 0.01, all of the unmonitored components depicted in Figures 4.1 and 4.2 had uniquely calculable flow. Observe that each of those unmonitored components has $|(B - M)_i|$ vertex-disjoint B-paths; that is, it satisfies the necessary condition of Theorem 2.1. In other words, each unmonitored component tested satisfies that necessary condition for uniquely calculable flow, and adding a small random perturbation to each turning factor was sufficient to yield uniquely calculable flow (whether or not the original flow solution, from setting all turning factors to one, was unique). I conjecture that any unmonitored component with $|(B - M)_i|$ vertex-disjoint B-paths has uniquely calculable flow when the turning factors are randomly perturbed, e.g. by a normal distribution centered at zero with small standard deviation.

Practically, this analysis suggests that as long as the necessary condition of Theorem 2.1 is satisfied, we may be able to force a unique solution in an unmonitored component by subtly perturbing the turning factors. Moreover, because real-world turning factors are unlikely to satisfy neat algebraic relationships like

$$\frac{\alpha_{24}}{\alpha_{23}} = \frac{\alpha_{14}}{\alpha_{13}}$$

(the condition for flow uniqueness in Figure 4.1, likely resembling a simplified form of the condition for flow uniqueness on a general graph), there

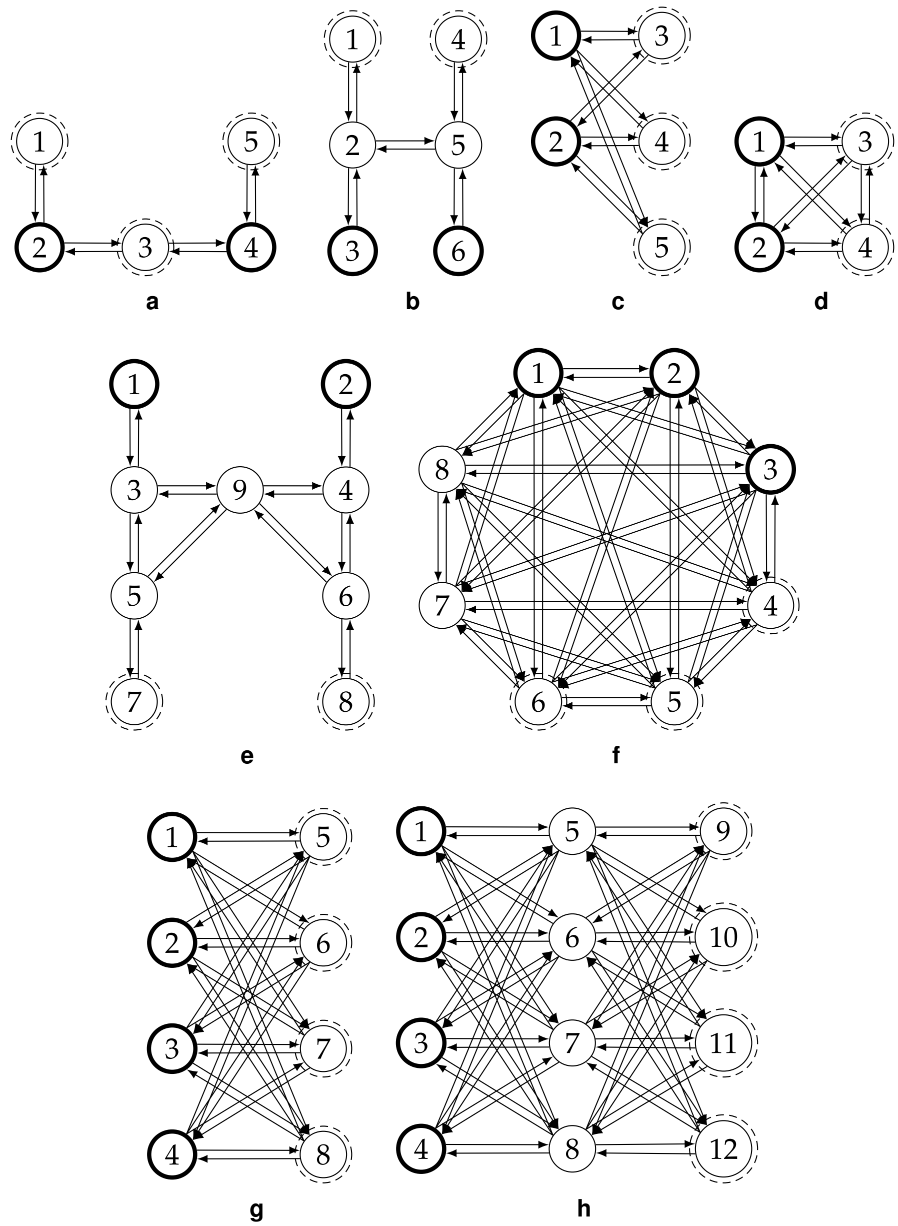


Figure 4.2 Networks analyzed in turning factor perturbation analysis.

is hope that most inputs of real-world turning factors yield uniquely determinable flow even without perturbation.

If perturbation of turning factors is to be used as a technique for forcing a unique solution, two aspects of the method must first be more thoroughly investigated. First, the desired result of the perturbation—namely, forcing a uniquely determinable solution—is currently only a conjecture. It would be helpful to rigorously prove conditions under which perturbation does yield a uniquely determinable solution. Second, and perhaps more important, is the issue of sensitivity. If a small perturbation in the turning factors creates a relatively large change in the flow solution, then the usefulness and meaning of the perturbed solution is doubtful. Based on my simulations, I believe the flow solution is considerably sensitive to the turning factors. Thus turning factor perturbation would need to be performed with great care, if at all.

Chapter 5

Conclusion

I have reviewed results from Morrison and Martonosi (2010) and Bianco and colleagues (2001) about flow determinability of SLP, including a necessary and sufficient condition for a unique solution on unmonitored components that are trees. I then developed a sufficient condition for uniquely determinable flow on centroid-free unmonitored components, heavily using a modified flow-calculation matrix in my proof (listed in Appendix A). Next I developed a necessary and sufficient condition on centroid-free unmonitored components with a considerably more intuitive interpretation and proof. In the last section, I presented several results illustrating the importance of the turning factors and monitored flows in uniqueness and positivity of the flow solution. Much remains to be done in the verification aspect of the Sensor Location Problem; I now describe other approaches that were considered in hopes of informing further research.

I conjecture that if the set of $|B - M|$ vertex-disjoint B-paths (of the necessary condition of Theorem 2.1) are in some sense unique, then the unmonitored component in question has uniquely determinable flow. More precisely, there exists a set of $|B - M|$ vertices in $A(M)$, call it V_A, such that there is exactly one set of vertex-disjoint B-paths between $B - M$ and V_A. My intuition is that if an unmonitored component has two solutions and such a unique set of B-paths, the additional flow added by a centroid in one solution must be in "rerouted" to another centroid. If flow balance is maintained, all of these "reroutings" of flow can be used to trace an alternate set of B-paths, yielding a contradiction.

Alternatively, one could consider decomposing a graph into simpler subgraphs. If H and K are subgraphs of G, and flow is uniquely determinable on H and K (using the same monitored vertices and centroid set), then the subgraph of G given by a union (or intersection) of H and K may also have uniquely determinable flow. If this conjecture, or a modification thereof, is true, then one could potentially show uniquely determinable flow of the general graph by constructing it as unions (or intersections) of simpler graphs with uniquely determinable flow.

In Section 2.3, I added three assumptions to the Sensor Location Problem:

1. G is connected, ignoring directionality of the arcs; that is, G is connected but not necessarily strongly connected.

2. For each $a \in A(M)$, there exists a monitored vertex $m \in M$ such that $am \in A$ and $c_{am} > 0$.

3. For each vertex $v \in V$ there is at least one arc $e \in v^+$ such that $c_e > 0$.

It may be possible to circumvent these assumptions by carefully handling the deficiencies they address. (For example, it might be possible to reformulate the case where there is no arc $e \in v^+$ such that $c_e > 0$ by eliminating v from the graph altogether and labeling all vertices previously adjacent to v as centroids.) This approach has not yet been attempted to any significant extent.

The Sensor Location Problem is an interesting framework. It provides plenty of material for mathematical analysis and exploration, and it is also highly applicable to the real world. I have proved an equivalent condition for unique flow determinability of a simple case, one whose physical manifestation would be absurd: the lack of centroids would imply a closed community of people ceaselessly driving in various circuits. I have also studied how turning factors and monitored flows affect the flow solution. I have revealed that the turning factors play a significant role in uniqueness of the flow solution, and that even if there is a unique solution, it may have negative flows. Future work should therefore pay close attention to the turning factors, and may benefit from considering the restriction of the Sensor Location Problem to inputs that yield strictly positive flows in the solution. In addition, I have discussed the results of a brief perturbation analysis on the turning factors, results that shed an optimistic light on obtaining uniquely determinable flow in real-world applications of the Sensor Location Problem. I hope that these insights will be used to inform further research on the Sensor Location Problem, hence facilitating practical solutions to traffic congestion.

Appendix A

Proof of Sufficient Condition for Flow Calculability on Centroid-Free Components

Here I describe a slight alteration of the network representation. I will use this reformulation to justify Theorem 3.1, which establishes a sufficient condition for uniquely calculable flow on a centroid-free unmonitored component. This condition is weaker than the necessary and sufficient condition of Theorem 3.2, and its proof is far less intuitive. The proof of Theorem 3.1 is included for record keeping and for its value as inspiration and entertainment for future researchers.

A.1 Revised Formulation of the Sensor Location Problem

I focus on a centroid-free unmonitored component of a network. To prove Theorem 3.1, I first describe a modification of the centroid-free blocks $\mathbf{F}^i$ of the flow-calculation matrix $\mathbf{F}$ of Morrison and Martonosi (2010). Take a block $\mathbf{F}^i$ representing an unmonitored component G_i' that has no centroids. Recall that each row of $\mathbf{F}^i$ corresponds to a vertex, and each column of $\mathbf{F}^i$ corresponds to a canonical arc, and hence a vertex. For clarity of notation, relabel the vertices represented by the rows of $\mathbf{F}^i$ as $1, 2, \ldots, m$ and the vertices represented by the columns as $1, 2, \ldots, n$. Note that there are at least as many columns as rows:

$$n = |(V - M)_i - A(M)_i| \geq |(V - M)_i| = m.$$

For $v \in \{1, 2, \ldots, n\}$, define $R_v = \{1, 2, \ldots, v-1, v+1, \ldots, m\}$. To build intuition, assume for the time being that the unmonitored component this block represents is a complete graph, namely K_n. That is, for each pair of distinct vertices $u, v \in \{1, 2, \ldots, n\}$, both arcs (v, u) and (u, v) exist. Then the $(u, v)^{th}$ element of $\mathbf{F}^i$ is given by

$$\left[\mathbf{F}^i\right]_{uv} = \begin{cases} -\sum\limits_{w \in R_1} \alpha_{uw} & \text{if } u = v \\ \alpha_{vu} & \text{otherwise,} \end{cases}$$

so

$$\mathbf{F}^i = \begin{bmatrix} -\sum\limits_{v \in R_1} \alpha_{1v} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & -\sum\limits_{v \in R_2} \alpha_{2v} & \cdots & \alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & -\sum\limits_{v \in R_n} \alpha_{nv} \\ \alpha_{1(n+1)} & \alpha_{2(n+1)} & \cdots & \alpha_{n(n+1)} \\ \vdots & \vdots & & \vdots \\ \alpha_{1m} & \alpha_{2m} & \cdots & \alpha_{nm} \end{bmatrix}. \tag{A.1}$$

Now return to the general case: Let $\mathbf{F}^i$ represent a component with no centroids, but which is not necessarily completely connected. Define

$$\hat{\mathbf{F}}^i_+ = \begin{bmatrix} -\sum\limits_{v \in R_1} \alpha_{1v} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & -\sum\limits_{v \in R_2} \alpha_{2v} & \cdots & \alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & -\sum\limits_{v \in R_n} \alpha_{nv} \\ \alpha_{1(n+1)} & \alpha_{2(n+1)} & \cdots & \alpha_{n(n+1)} \\ \vdots & \vdots & & \vdots \\ \alpha_{1m} & \alpha_{2m} & \cdots & \alpha_{nm} \end{bmatrix},$$

with the understanding that $\alpha_{uv} = 0$ whenever $uv \notin A$. Symbolically, the definition of $\hat{\mathbf{F}}^i_+$ is the same as the right-hand–side of Equation A.1; that is, it has the same visual appearance. However, this definition represents a generally different unmonitored component. In particular, although turning factors representing all possible arcs are present in $\hat{\mathbf{F}}^i_+$, many may be zero, representing arcs that are not present in the actual component (or possibly indicating arcs with turning ratios of zero). Moreover, when the numerical values for the turning factors are plugged in, we will have $\hat{\mathbf{F}}^i_+ = \mathbf{F}^i$ for this component. Thus the unmonitored component $G_i^{'}$ has uniquely determinable flow and flow balance if and only if $\hat{\mathbf{F}}^i_+$ has full column rank.

We are now ready to proceed to the proof of Theorem 3.1.

A.2 Result and Proof

I now recall Theorem 3.1. I then proceed to lay the foundation for my proof and provide a high-level description of the argument.

Theorem. *Let G, B, and M be as in Theorem 2.1 with the graph partitioned into unmonitored components and the flow-calculation matrix partitioned into blocks as described. For each unmonitored component $G_i^{'}$, define n_i to be the number of columns of $\mathbf{F}^i$. If $(B - M)_i$ is empty, then* $\operatorname{rank}(\mathbf{F}^i) = n_i$ *(and hence the flow on $G_i^{'}$ is calculable) if there exists a set of n_i arcs, one emanating from each of the n_i vertices in $V_i - A(M)_i$, such that the turning factors for those arcs are positive and the subgraph of $G_i^{'}$ induced on those arcs is a forest of reverse arborescences.*

To prove this theorem, we consider the matrix $\hat{\mathbf{F}}^i_+$ for a given unmonitored component $G_i^{'}$, and henceforth drop the superscript (subscript) i for ease of notation.

Assume that $(B - M)$ is empty and that $a = |A(M)|$. So $\hat{\mathbf{F}}_+$ has $n + a$ rows. Let the rows and columns of $\hat{\mathbf{F}}_+$ be permuted such that the top n rows correspond to the n vertices not in $A(M)$ and such that the n columns are arranged in the same order. Label those vertices $1, 2, \ldots, n$, and label the a vertices in $A(M)$ as $n + 1, n + 2, \ldots, n + a$. Define $\hat{\mathbf{F}}$ as the $n \times n$ matrix consisting of the top n rows of $\hat{\mathbf{F}}_+$. Then, if we can show that $\hat{\mathbf{F}}$ is invertible, $\hat{\mathbf{F}}_+$ will have full column rank, and so the unmonitored component will have uniquely determinable flow.

Specifically, $\hat{\mathbf{F}}$ has the form

$$\hat{\mathbf{F}} = \begin{bmatrix} -\sum\limits_{u \in R_1} \alpha_{1u} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & -\sum\limits_{u \in R_2} \alpha_{2u} & \cdots & \alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & -\sum\limits_{u \in R_n} \alpha_{nu} \end{bmatrix}$$

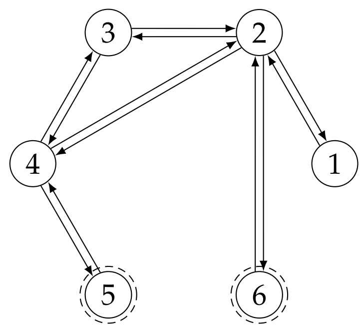


Figure A.1 An example unmonitored component, used to illustrate the sufficient condition on centroid-free unmonitored components.

(where $\alpha_{vu} = \alpha_{uv} = 0$ is equivalent to there being no arc between vertices u and v).

For $k_1 \in R_1, k_2 \in R_2, \ldots, k_n \in R_n$, define the *determinant part* $\hat{\mathbf{F}}_{k_1 \cdots k_n}$ as $\hat{\mathbf{F}}$ except with α_{vu} set to 0 for all pairs u, v where $u \neq k_v$. Then

$$\det\left(\hat{\mathbf{F}}\right) = \sum_{k_1 \in R_1, \ldots, k_n \in R_n} \det\left(\hat{\mathbf{F}}_{k_1 \cdots k_n}\right). \tag{A.2}$$

I justify this decomposition of $\det\left(\hat{\mathbf{F}}\right)$ in Section A.2.2. In Section A.2.3, I show that the nonzero summands in the decomposition all have the same sign. Then, in Section A.2.4, I show that at least one of the summands is nonzero—hence the sum is nonzero, so $\hat{\mathbf{F}}$ has full row rank—if there exist $k_1 \in R_1, \ldots, k_n \in R_n$ such that $\alpha_{1k_1}, \ldots, \alpha_{nk_n}$ are all positive and the subgraph of $G^{'}$ with arc set $\{(1, k_1), \ldots, (n, k_n)\}$ is a forest of reverse arborescences.

In order to establish intuition, I begin with an example.

A.2.1 Example

Suppose we have the centroid-free unmonitored component depicted in Figure A.1 (reproduced from Figure 3.2). In this section I will describe a sketch of the proof for this graph. In subsequent sections I will lay out the details of the general proof. For the graph in Figure A.1,

$$\hat{\mathbf{F}}_+ = \begin{bmatrix} -\sum\limits_{v \in R_1} \alpha_{1v} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{12} & -\sum\limits_{v \in R_2} \alpha_{2v} & \alpha_{32} & \alpha_{42} \\ \alpha_{13} & \alpha_{23} & -\sum\limits_{v \in R_3} \alpha_{3v} & \alpha_{43} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & -\sum\limits_{v \in R_4} \alpha_{4v} \\ \alpha_{15} & \alpha_{25} & \alpha_{35} & \alpha_{45} \\ \alpha_{16} & \alpha_{26} & \alpha_{36} & \alpha_{46} \end{bmatrix},$$

so

$$\hat{\mathbf{F}} = \begin{bmatrix} -\sum\limits_{v \in R_1} \alpha_{1v} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{12} & -\sum\limits_{v \in R_2} \alpha_{2v} & \alpha_{32} & \alpha_{42} \\ \alpha_{13} & \alpha_{23} & -\sum\limits_{v \in R_3} \alpha_{3v} & \alpha_{43} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & -\sum\limits_{v \in R_4} \alpha_{4v} \end{bmatrix}.$$

In $\hat{\mathbf{F}}$, if we replace the turning factors for nonexistent arcs with zeros (which we can do because those turning factors are zero by definition), we get the following simplified matrix:

$$\begin{bmatrix} -\alpha_{12} & \alpha_{21} & 0 & 0 \\ \alpha_{12} & -\alpha_{21} - \alpha_{23} - \alpha_{24} - \alpha_{26} & \alpha_{32} & \alpha_{42} \\ 0 & \alpha_{23} & -\alpha_{32} - \alpha_{34} & \alpha_{43} \\ 0 & \alpha_{24} & \alpha_{34} & -\alpha_{42} - \alpha_{43} - \alpha_{45} \end{bmatrix}.$$

(It is more convenient in the general case to not substitute zeros for those turning factors in $\hat{\mathbf{F}}$, but in this example we will make the substitution.)

The determinant of $\hat{\mathbf{F}}$ is given by

$$\begin{aligned} \det\left(\hat{\mathbf{F}}\right) = &(-\alpha_{12})(-\alpha_{21} - \alpha_{23} - \alpha_{24} - \alpha_{26})(-\alpha_{32} - \alpha_{34})(-\alpha_{42} - \alpha_{43} - \alpha_{45}) \\ &- (-\alpha_{12})(-\alpha_{21} - \alpha_{23} - \alpha_{24} - \alpha_{26})(\alpha_{34})(\alpha_{43}) \\ &- (-\alpha_{12})(\alpha_{23})(\alpha_{32})(-\alpha_{42} - \alpha_{43} - \alpha_{45}) \\ &+ (-\alpha_{12})(\alpha_{23})(\alpha_{34})(\alpha_{42}) \\ &+ (-\alpha_{12})(\alpha_{24})(\alpha_{32})(\alpha_{43}) \\ &- (-\alpha_{12})(\alpha_{24})(-\alpha_{32} - \alpha_{34})(\alpha_{42}) \\ &- (\alpha_{12})(\alpha_{21})(-\alpha_{32} - \alpha_{34})(-\alpha_{42} - \alpha_{43} - \alpha_{45}) \\ &+ (\alpha_{12})(\alpha_{21})(\alpha_{34})(\alpha_{43}) \end{aligned}$$

(using the determinant definition of Shilov (1977)—see Section A.2.2). Expanding the sums of negative turning factors that came from the main diagonal entries and collecting terms, we get

$$
\begin{aligned}
\det\left(\hat{\mathbf{F}}\right) = {} & (1-1)\alpha_{12}\alpha_{21}\alpha_{32}\alpha_{42} + (1-1)\alpha_{12}\alpha_{21}\alpha_{32}\alpha_{43} \\
& + (1-1)\alpha_{12}\alpha_{21}\alpha_{32}\alpha_{45} + (1-1)\alpha_{12}\alpha_{21}\alpha_{34}\alpha_{42} \\
& + (2-2)\alpha_{12}\alpha_{21}\alpha_{34}\alpha_{43} + (1-1)\alpha_{12}\alpha_{21}\alpha_{34}\alpha_{45} \\
& + (1-1)\alpha_{12}\alpha_{23}\alpha_{32}\alpha_{42} + (1-1)\alpha_{12}\alpha_{23}\alpha_{32}\alpha_{43} \\
& + (1-1)\alpha_{12}\alpha_{23}\alpha_{32}\alpha_{45} + (1-1)\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{42} \\
& + (1-1)\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{43} + (1-0)\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{45} \\
& + (1-1)\alpha_{12}\alpha_{24}\alpha_{32}\alpha_{42} + (1-1)\alpha_{12}\alpha_{24}\alpha_{32}\alpha_{43} \\
& + (1-0)\alpha_{12}\alpha_{24}\alpha_{32}\alpha_{45} + (1-1)\alpha_{12}\alpha_{24}\alpha_{34}\alpha_{42} \\
& + (1-1)\alpha_{12}\alpha_{24}\alpha_{34}\alpha_{43} + (1-0)\alpha_{12}\alpha_{24}\alpha_{34}\alpha_{45} \\
& + (1-0)\alpha_{12}\alpha_{26}\alpha_{32}\alpha_{42} + (1-0)\alpha_{12}\alpha_{26}\alpha_{32}\alpha_{43} \\
& + (1-0)\alpha_{12}\alpha_{26}\alpha_{32}\alpha_{45} + (1-0)\alpha_{12}\alpha_{26}\alpha_{34}\alpha_{42} \\
& + (1-1)\alpha_{12}\alpha_{26}\alpha_{34}\alpha_{43} + (1-0)\alpha_{12}\alpha_{26}\alpha_{34}\alpha_{45} \\
= {} & \alpha_{12}\alpha_{23}\alpha_{34}\alpha_{45} + \alpha_{12}\alpha_{24}\alpha_{32}\alpha_{45} \\
& + \alpha_{12}\alpha_{24}\alpha_{34}\alpha_{45} + \alpha_{12}\alpha_{26}\alpha_{32}\alpha_{42} \\
& + \alpha_{12}\alpha_{26}\alpha_{32}\alpha_{43} + \alpha_{12}\alpha_{26}\alpha_{32}\alpha_{45} \\
& + \alpha_{12}\alpha_{26}\alpha_{34}\alpha_{42} + \alpha_{12}\alpha_{26}\alpha_{34}\alpha_{45}.
\end{aligned}
$$

Because no turning factors are negative, if all four turning factors in one of those eight terms in the previous expansion are positive then the determinant of $\hat{\mathbf{F}}$ is positive, so the component has uniquely determinable flow. But let's investigate the coefficients of those terms. Consider, for example, the term $\alpha_{12}\alpha_{21}\alpha_{34}\alpha_{43}$ with coefficient $2-2=0$. Note that the determinant of the following matrix (which is $\hat{\mathbf{F}}$ with all turning factors removed except α_{12}, α_{21}, α_{34}, and α_{43}) is zero:

$$
\begin{bmatrix}
-\alpha_{12} & \alpha_{21} & 0 & 0 \\
\alpha_{12} & -\alpha_{21} & 0 & 0 \\
0 & 0 & -\alpha_{34} & \alpha_{43} \\
0 & 0 & \alpha_{34} & -\alpha_{43}
\end{bmatrix}.
$$

(This matrix is equal to $\hat{\mathbf{F}}_{k_1k_2k_3k_4}$ where $k_1 = 2$, $k_2 = 1$, $k_3 = 4$, and $k_4 = 3$.) In particular, when we take the determinant of the matrix using the determinant definition of Shilov (1977), we get

$$\alpha_{12}\alpha_{21}\alpha_{34}\alpha_{43} + \alpha_{12}\alpha_{21}\alpha_{34}\alpha_{43} - \alpha_{12}\alpha_{21}\alpha_{34}\alpha_{43} - \alpha_{12}\alpha_{21}\alpha_{34}\alpha_{43} = 0.$$

Note that there are two positive terms and two negative terms. This correspondence is not a coincidence. In general, the determinant of the matrix given by $\hat{\mathbf{F}}$ with all turning factors except α_{1k_1}, α_{2k_2}, α_{3k_3}, and α_{4k_4} removed—that is, the determinant of $\hat{\mathbf{F}}_{k_1k_2k_3k_4}$—is equal to $\alpha_{1k_1}\alpha_{2k_2}\alpha_{3k_3}\alpha_{4k_4}$ times its coefficient in the previous expansion. Hence

$$\det\left(\hat{\mathbf{F}}\right) = \sum_{k_1,k_2,k_3,k_4} \det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right).$$

We saw previously that each individual summand $\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right)$ is either zero or positive. This nonnegativity is not a coincidence either. Consider $\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right)$ for $k_1 = 2$, $k_2 = 6$, $k_3 = 4$, and $k_4 = 2$:

$$\det\left(\begin{bmatrix} -\alpha_{12} & 0 & 0 & 0 \\ \alpha_{12} & -\alpha_{26} & 0 & \alpha_{42} \\ 0 & 0 & -\alpha_{34} & 0 \\ 0 & 0 & \alpha_{34} & -\alpha_{42} \end{bmatrix}\right).$$

Observe that we can take successive Laplace expansions along columns to quickly determine the value of the determinant:

$$\begin{aligned}
\det\left(\begin{bmatrix} -\alpha_{12} & 0 & 0 & 0 \\ \alpha_{12} & -\alpha_{26} & 0 & \alpha_{42} \\ 0 & 0 & -\alpha_{34} & 0 \\ 0 & 0 & \alpha_{34} & -\alpha_{42} \end{bmatrix}\right)& \\
&= (-\alpha_{26})\det\left(\begin{bmatrix} -\alpha_{12} & 0 & 0 \\ 0 & -\alpha_{34} & 0 \\ 0 & \alpha_{34} & -\alpha_{42} \end{bmatrix}\right) \\
&= (-\alpha_{26})(-\alpha_{12})\det\left(\begin{bmatrix} -\alpha_{34} & 0 \\ \alpha_{34} & -\alpha_{42} \end{bmatrix}\right) \\
&= (-\alpha_{26})(-\alpha_{12})(-\alpha_{42})\det\left(\begin{bmatrix} -\alpha_{34} \end{bmatrix}\right) \\
&= (-\alpha_{26})(-\alpha_{12})(-\alpha_{42})(-\alpha_{34}) \\
&= \alpha_{26}\alpha_{12}\alpha_{42}\alpha_{34}.
\end{aligned}$$

Now consider $\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right)$ for some k_1, k_2, k_3, k_4. Either $\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right)$ is zero or it is not; assume it is not. In this case there must be a column j of $\hat{\mathbf{F}}_{k_1k_2k_3k_4}$, with $1 \leq j \leq 4$, such that $-\alpha_{jk_j}$ is in the $(j,j)^{th}$ entry and there are zeros in the other entries, and such that $\alpha_{jk_j} > 0$. Otherwise the rows of $\hat{\mathbf{F}}_{k_1k_2k_3k_4}$ will sum to zero, by construction—if there are two nonzero elements in a column, one is the negative of a turning factor (on the main diagonal) and the other is the same turning factor but with a positive sign. If the rows sum to zero, we have a contradiction, because we assumed $\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right) \neq 0$. So there must exist a column containing only one nonzero entry. Take the Laplace expansion along this column to get an equation of the form

$$\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right) = (-\alpha_{jk_j})\det\left(\hat{\mathbf{F}}^{(1)}_{k_1k_2k_3k_4}\right)$$

where $\hat{\mathbf{F}}^{(1)}_{k_1k_2k_3k_4}$ is a 3×3 submatrix. Now there must exist a column containing only one nonzero entry in $\hat{\mathbf{F}}^{(1)}_{k_1k_2k_3k_4}$, by the same reasoning as before. In fact, that nonzero entry will be the negative of a turning factor because the main diagonal entries (except for the $(j,j)^{th}$ entry) will not have been removed by the Laplace expansion. Take the Laplace expansion along this column to get

$$\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right) = (-\alpha_{jk_j})(-\alpha_{hk_h})\det\left(\hat{\mathbf{F}}^{(2)}_{k_1k_2k_3k_4}\right)$$

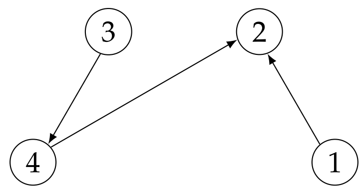
for some h with $\alpha_{hk_h} > 0$ and some 2×2 submatrix $\hat{\mathbf{F}}^{(2)}_{k_1k_2k_3k_4}$. Continue in this manner to obtain that $\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right)$ is the product of four positive turning factors, each with a negative sign in front; hence $\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right)$ is positive. Thus, for any k_1, k_2, k_3, k_4, $\det\left(\hat{\mathbf{F}}_{k_1k_2k_3k_4}\right)$ is either zero or positive.

We have now justified that $\det\left(\hat{\mathbf{F}}\right)$ is a sum of determinants, each of which is either zero or positive. Now, if we can show that the condition of Theorem 3.1 is sufficient for one of the summands to be positive, we will have shown that flow is uniquely calculable for this example.

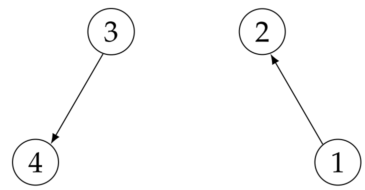
Consider a graph-theoretic interpretation of $\hat{\mathbf{F}}_{k_1k_2k_3k_4}$, for given k_1, k_2, k_3, and k_4. Specifically, let the graph corresponding to $\hat{\mathbf{F}}_{k_1k_2k_3k_4}$ be the graph on the vertices $V - A(M)$ (the original vertices, less those in the neighbor set) with only arcs $1k_1$, $2k_2$, $3k_3$, and $4k_4$. For example, take $k_1 = 2$, $k_2 = 6$, $k_3 = 4$, and $k_4 = 2$. Then

$$\hat{\mathbf{F}}_{k_1k_2k_3k_4} = \begin{bmatrix} -\alpha_{12} & 0 & 0 & 0 \\ \alpha_{12} & -\alpha_{26} & 0 & \alpha_{42} \\ 0 & 0 & -\alpha_{34} & 0 \\ 0 & 0 & \alpha_{34} & -\alpha_{42} \end{bmatrix}$$

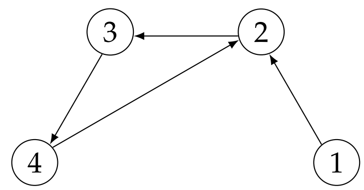
and the corresponding graph is shown in Figure A.2a. (Note that the arc from 2 to 6 is not present because vertex 6, a vertex in the neighbor set, was removed.) This graph is clearly a reverse arborescence (trivially, a forest of



a. $k_1 = 2, k_2 = 6, k_3 = 4, k_4 = 2.$



b. $k_1 = 2, k_2 = 6, k_3 = 4, k_4 = 5.$



c. $k_1 = 2, k_2 = 3, k_3 = 4, k_4 = 2.$

Figure A.2 Graphs of determinant parts used in the centroid-free sufficient condition example.

reverse arborescences). The graph for $k_1 = 2$, $k_2 = 6$, $k_3 = 4$, and $k_4 = 5$ is given in Figure A.2b, and is also a forest of reverse arborescences (as we expect because the corresponding summand was positive, assuming the appropriate turning factors are positive). The graph for $k_1 = 2$, $k_2 = 3$, $k_3 = 4$, and $k_4 = 2$ is displayed in Figure A.2c. The corresponding determinant part in this case was zero, and as we see, the graph is not a forest of reverse arborescences (in particular, it is cyclic).

Justification of the proposed sufficient condition is as follows. I previously showed that a determinant part $\hat{\mathbf{F}}_{k_1 k_2 k_3 k_4}$ is nonzero if (and only if) successive Laplace expansions reduce it to the product of the entries on its main diagonal, and those entries (the turning factors) are nonzero. For the determinant part corresponding to Figure A.2a, the first Laplace expansion is along column 2. This expansion reduces the determinant to the product of $-\alpha_{26}$ and the 3×3 submatrix given by removal of the second row and second column. To obtain the graph corresponding to this submatrix, we (intuitively) remove vertex 2. The succession of Laplace expansions is manifested in the graph as a succession of vertex removals. Note that a vertex has out-degree zero if and only if the corresponding column in the matrix has only one nonzero element (the negative of the turning factor on the main diagonal). A determinant part thus has nonzero determinant if and only if its turning factors are positive and we can successively remove vertices with out-degree zero until we have removed all vertices. If at any step there are no vertices with out-degree zero then the corresponding submatrix of the Laplace expansion has determinant zero, contradicting our assumption.

If a graph is a forest of reverse arborescences, then it has at least one root, so it has at least one vertex with out-degree zero. Moreover, when a root is removed, the resulting graph is again a forest of reverse arborescences, so it has at least one vertex with out-degree zero. Thus if the graph corresponding to a determinant part is a forest of reverse arborescences, and the corresponding turning factors are all positive, then the determinant part is nonzero, so $\hat{\mathbf{F}}$ has nonzero determinant—implying uniquely calculable flow.

Note, finally, that there is a bijection between the choices of four arcs, one emanating from each of the four vertices in $V - A(M)$, and the determinant parts. Each determinant part corresponds to such a selection, and each selection corresponds to a determinant part. Therefore if there is a set of four arcs, one emanating from each of the four vertices in $V - A(M)$, such that the turning factors for those arcs are positive and the subgraph induced on those arcs is a forest of reverse arborescences, then this component has uniquely calculable flow as desired.

The proof sketch for the example is complete. With this sketch in mind, I now turn to the task of proving Theorem 3.1 in the general case. I begin by rigorously justifying the decomposition of the determinant of $\hat{\mathbf{F}}$ into the previously described determinant parts $\hat{\mathbf{F}}_{k_1 \cdots k_n}$.

A.2.2 Determinant Decomposition

Let $(u_1, u_2, \ldots, u_n)$ be a permutation of $(1, 2, \ldots, n)$ and let $N(u_1, u_2, \ldots, u_n)$ be the number of inversions in $(u_1, u_2, \ldots, u_n)$. That is,

$$N(u_1, u_2, \ldots, u_n) = \sum_{g=1}^{n-1} \sum_{h=g+1}^{n} \theta(u_h - u_g)$$

where $\theta(x)$ is 1 if $x < 0$ and 0 otherwise. Now define P as the set of all permutations of $(1, 2, \ldots, n)$, define $\left[\hat{\mathbf{F}}\right]_{uv}$ to be the entry in row u and column v of $\hat{\mathbf{F}}$, and define $\left[\hat{\mathbf{F}}_{k_1 \cdots k_n}\right]_{uv}$ similarly to be the entry in row u and column v of the determinant component $\hat{\mathbf{F}}_{k_1 \cdots k_n}$. Then, by the definition of the matrix determinant introduced by Shilov (1977),

$$\det\left(\hat{\mathbf{F}}\right) = \sum_{(u_1,\ldots,u_n)\in P} (-1)^{N(u_1,\ldots,u_n)} \prod_{v=1}^{n} \left[\hat{\mathbf{F}}\right]_{u_v v}. \tag{A.3}$$

Likewise,

$$\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) = \sum_{(u_1,\ldots,u_n)\in P} (-1)^{N(u_1,\ldots,u_n)} \prod_{v=1}^{n} \left[\hat{\mathbf{F}}_{k_1\cdots k_n}\right]_{u_v v}. \tag{A.4}$$

Now note that $[\hat{\mathbf{F}}]_{u_v v}$ is $-\sum_{k\in R_v} \alpha_{vk}$ when $u_v = v$ and α_{vu_v} otherwise. Thus, when the terms in the sum of Equation A.3 are completely expanded, we will have

$$\det\left(\hat{\mathbf{F}}\right) = \sum_{k_1\in R_1,\ldots,k_n\in R_n} \sigma(k_1,\ldots,k_n) \prod_{v=1}^{n} \alpha_{vk_v} \tag{A.5}$$

for some function $\sigma(k_1,\ldots,k_n)$ whose codomain is the integers.

Take some $k_1 \in R_1,\ldots,k_n \in R_n$, and some $v \in \{1,\ldots,n\}$. Consider where α_{vk_v} appears in $\hat{\mathbf{F}}$. If $k_v > n$ then α_{vk_v} appears in the sum $\left[\hat{\mathbf{F}}\right]_{vv} = -\sum_{g\in R_v} \alpha_{vg}$ and nowhere else. If $k_v \leq n$ then α_{vk_v} appears twice in $\hat{\mathbf{F}}$: once in the sum $\left[\hat{\mathbf{F}}\right]_{vv} = -\sum_{g\in R_v} \alpha_{vg}$ and once in $\left[\hat{\mathbf{F}}\right]_{k_v v} = \alpha_{vk_v}$. With this observation in mind, define

$$\mu(v,k) = \begin{cases} \{v\}, & k > n \\ \{v,k\}, & k \leq n \end{cases}.$$

Now $\mu(v,k_v)$ is the set of rows such that α_{vk_v} appears in those rows in column v of $\hat{\mathbf{F}}$ (and nowhere else). Thus the sum of all $\prod_{v=1}^{n} \alpha_{vk_v}$ terms in Equation A.3 is given by

$$\sum_{(u_1,\ldots,u_n)\in\tau(k_1,\ldots,k_n)} (-1)^{N(u_1,\ldots,u_n)} \prod_{v=1}^{n} \alpha_{vk_v} \tag{A.6}$$

where

$$\begin{aligned}\tau(k_1,\ldots,k_n) = \{(u_1,\ldots,u_n)\colon u_1 \in \mu(1,k_1),\ldots,u_n \in \mu(n,k_n),\\ (u_1,\ldots,u_n) \in P\}.\end{aligned}$$

In other words $\tau(k_1,\ldots,k_n)$ is the set of all permutations $(u_1,\ldots,u_n)$ where $u_1 \in \mu(1,k_1),\ldots,u_n \in \mu(n,k_n)$. We thus find that

$$\sigma(k_1,\ldots,k_n) = \sum_{(u_1,\ldots,u_n)\in\tau(k_1,\ldots,k_n)} (-1)^{N(u_1,\ldots,u_n)}.$$

Therefore

$$\det\left(\hat{\mathbf{F}}\right) = \sum_{k_1 \in R_1, \ldots, k_n \in R_n} \left[\sum_{(u_1,\ldots,u_n) \in \tau(k_1,\ldots,k_n)} (-1)^{N(u_1,\ldots,u_n)} \prod_{v=1}^{n} \alpha_{vk_v} \right].$$

The inner sum resembles Equation A.3. In fact,

$$\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) = \sum_{(u_1,\ldots,u_n) \in \tau(k_1,\ldots,k_n)} (-1)^{N(u_1,\ldots,u_n)} \prod_{v=1}^{n} \alpha_{vk_v},$$

so we have established Equation A.2.

A.2.3 Sign of the Determinant Parts

Define sign (x) as 1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$. We will now show that for given $k_1 \in R_1, \ldots, k_n \in R_n$, either $\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^n$ or $\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) = 0$.

Suppose $k_1 \in R_1, \ldots, k_n \in R_n$ are given. Assume $\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) \neq 0$. We will show that, in this case, $\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^n$. To do so, we will first show by induction on g (where $1 \leq g < n$) that

$$\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^g \operatorname{sign}\left(\det\left(\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right)\right) \tag{A.7}$$

where $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ is a $(n-g) \times (n-g)$ submatrix of $\hat{\mathbf{F}}_{k_1\cdots k_n}$ (consisting of $n-g$ rows and the corresponding $n-g$ columns of $\hat{\mathbf{F}}_{k_1\cdots k_n}$). We will find that $\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}^{(n-1)}_{k_1\cdots k_n}\right)\right) = -1$ for our choice of $\hat{\mathbf{F}}^{(n-1)}_{k_1\cdots k_n}$, so that

$$\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^n$$

as desired.

We start with the base case, $g = 1$. There must be some column number v of $\hat{\mathbf{F}}_{k_1\cdots k_n}$ such that $n < k_v \leq n + a$ and $\left[\hat{\mathbf{F}}_{k_1\cdots k_n}\right]_{vv} = -\alpha_{vk_v} < 0$ and $\left[\hat{\mathbf{F}}_{k_1\cdots k_n}\right]_{uv} = 0$ for $u \neq v$. Otherwise the rows would sum to zero, giving a zero determinant and contradicting our assumption. Taking the Laplace expansion along column v, then, we obtain

$$\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) = -\alpha_{vb} \det\left(\hat{\mathbf{F}}^{(1)}_{k_1\cdots k_n}\right)$$

where $\hat{\mathbf{F}}^{(1)}_{k_1\cdots k_n}$ is the $(n-1) \times (n-1)$ matrix consisting of rows and columns $1, \ldots, v-1, v+1, \ldots, n$ of $\hat{\mathbf{F}}_{k_1\cdots k_n}$. Thus the induction hypothesis, expressed by Equation A.7, is satisfied when $g = 1$.

Now we assume that, for some g where $1 \leq g < n-1$, the induction hypothesis holds; that is,

$$\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^g \operatorname{sign}\left(\det\left(\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right)\right).$$

There must be some column number v of $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ such that $\left[\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right]_{vv} = -\alpha_{v'k_{v'}} < 0$ (where v' is the number of the column of $\hat{\mathbf{F}}_{k_1\cdots k_n}$ corresponding to column v in $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$) and $\left[\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right]_{uv} = 0$ for $u \neq v$. Otherwise the rows would sum to zero, giving

$$\det\left(\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right) = 0,$$

and, by the induction hypothesis,

$$\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) = 0,$$

which would contradict our assumption. Taking the Laplace expansion along column v, then, we obtain

$$\det\left(\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right) = -\alpha_{v'k_{v'}} \det\left(\hat{\mathbf{F}}^{(g+1)}_{k_1\cdots k_n}\right); \tag{A.8}$$

thus

$$\begin{aligned}\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) &= (-1)^g \operatorname{sign}\left(\det\left(\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right)\right) \\ &= (-1)^{g+1} \operatorname{sign}\left(\det\left(\hat{\mathbf{F}}^{(g+1)}_{k_1\cdots k_n}\right)\right)\end{aligned}$$

as desired. By the principle of induction, then,

$$\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^{n-1} \operatorname{sign}\left(\det\left(\hat{\mathbf{F}}^{(n-1)}_{k_1\cdots k_n}\right)\right).$$

By Equation A.8, we can see that $\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}^{(n-1)}_{k_1\cdots k_n}\right)\right) = -1$. Therefore

$$\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^n.$$

So, for given $k_1 \in R_1, \ldots, k_n \in R_n$, either $\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^n$ or $\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) = 0$. Now, if we can show that there exist $k_1 \in R_1, \ldots, k_n \in R_n$ such that $\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) \neq 0$ (that is, $\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^n$), then there will be at least one determinant part with the same sign as $(-1)^n$, and the other terms will either have the same sign or be zero. Hence we will have $\det\left(\hat{\mathbf{F}}\right) \neq 0$ as desired.

A.2.4 Existence of an Invertible Determinant Part

To show that there exists an invertible determinant part if and only if there exist $k_1 \in R_1, \ldots, k_n \in R_n$ such that $\alpha_{1k_1}, \ldots, \alpha_{nk_n}$ are nonzero and the subgraph of $G^{'}$ with arc set $\{(1,k_1), \ldots, (n,k_n)\}$ is a forest of reverse arborescences, I will develop a graph-theoretic interpretation of $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ and the induction process on g. For convenience, define $\hat{\mathbf{F}}^{(0)}_{k_1\cdots k_n} = \hat{\mathbf{F}}_{k_1\cdots k_n}$. If we interpret the elements of $\hat{\mathbf{F}}_{k_1\cdots k_n}$ as turning factors of arcs on $V - A(M)$ in the same way that we interpret the elements of $G^{'}$ as turning factors of arcs on V, then $\hat{\mathbf{F}}_{k_1\cdots k_n}$ represents the subgraph of $G^{'}$ in which, for each vertex $v \in V - A(M)$, we have picked the arc (v, k_v) and discarded the other arcs emanating from v (if $k_v > n$, we have discarded all arcs emanating from v). We interpret $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ similarly; it represents the subgraph $\hat{\mathbf{F}}_{k_1\cdots k_n}$ minus the g vertices corresponding to the g rows and columns that have been deleted.

First, we prove the reverse direction of the equivalence. Suppose $k_1 \in R_1, \ldots, k_n \in R_n$ are given. Assume $\alpha_{1k_1}, \ldots, \alpha_{nk_n}$ are nonzero and the subgraph of $G^{'}$ with arc set $\{(1,k_1), \ldots, (n,k_n)\}$ is a forest of reverse arborescences. We will show by induction on g that $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ represents a forest of reverse arborescences for $0 \le g \le n-1$ and that Equation A.7 holds for $1 \le g \le n-1$. Hence $\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) \ne 0$ by the principle of induction.

We start with the base case, $g = 0$. We have assumed that $\hat{\mathbf{F}}_{k_1\cdots k_n}$ represents a forest of reverse arborescences, so $\hat{\mathbf{F}}^{(0)}_{k_1\cdots k_n}$ trivially represents a forest of reverse arborescences.

Now assume $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ represents a forest of reverse arborescences for an integer g where $0 \le g < n-1$. Then there must exist at least one root of a reverse arborescence, call it v' (and let v be the column of $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ to which it corresponds). Because v' is a root, there are no arcs emanating from it, so the column v of $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ contains all zeros, except that $\left[\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right]_{vv} = -\alpha_{v'k_{v'}}$. Thus Equation A.7 holds for $g+1$. Because $\hat{\mathbf{F}}^{(g+1)}_{k_1\cdots k_n}$ is the matrix produced by the Laplace expansion along column v of $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$, the graph represented by $\hat{\mathbf{F}}^{(g+1)}_{k_1\cdots k_n}$ is the same except that vertex v' has been removed (along with every arc into or out of it). If the reverse arborescence including v' included more than one vertex, then the components (there may be more than one) resulting from the deletion of v' are also reverse arborescences. If v' was the only vertex remaining in its reverse arborescence, then, because $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ represented a forest of reverse arborescences and $g < n-1$, there must be at least one other reverse arborescence remaining in the forest (after the deletion of v'). Thus the graph represented by $\hat{\mathbf{F}}^{(g+1)}_{k_1\cdots k_n}$ is also a forest of reverse arborescences, as desired.

By the principle of induction, then, $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ represents a forest of reverse arborescences for $0 \leq g \leq n-1$ and Equation A.7 holds for $1 \leq g \leq n-1$. Therefore $\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) \neq 0$ as desired.

Now we prove the forward direction. Suppose $k_1 \in R_1, \ldots, k_n \in R_n$ are given. Assume $\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) \neq 0$; that is, $\hat{\mathbf{F}}_{k_1\cdots k_n}$ is invertible. Then $\alpha_{1k_1}, \ldots, \alpha_{nk_n}$ must be nonzero, otherwise there would be a zero column in $\hat{\mathbf{F}}_{k_1\cdots k_n}$, contradicting our assumption that $\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) \neq 0$. Then Equation A.7 holds for $1 \leq g \leq n-1$; in particular, there is a column number v of $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ such that $\left[\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right]_{vv} = -\alpha_{v'k_{v'}} < 0$ (where v' is the number of the column of $\hat{\mathbf{F}}_{k_1\cdots k_n}$ corresponding to column v in $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$) and $\left[\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}\right]_{uv} = 0$ for $u \neq v$.

Consider an induction like that of Section A.2.3, but in reverse. $\hat{\mathbf{F}}^{(n-1)}_{k_1\cdots k_n}$ represents a single isolated vertex, which is trivially a forest of reverse arborescences. Assume $\hat{\mathbf{F}}^{(g+1)}_{k_1\cdots k_n}$ represents a forest of reverse arborescences, for $0 \leq g < n-1$. Then, to get back to $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$, a new vertex is added, and arcs are added from roots of existing reverse arborescences (potentially many, one, or zero of them) to the new root. If we started with a forest of reverse arborescences and we added a new vertex (possibly with arcs from previous roots to this new vertex), then $\hat{\mathbf{F}}^{(g)}_{k_1\cdots k_n}$ is a forest of reverse arborescences as well. By induction, then, $\hat{\mathbf{F}}_{k_1\cdots k_n} = \hat{\mathbf{F}}^{(0)}_{k_1\cdots k_n}$ is a forest of reverse arborescences as desired.

I have thus shown that there exists a determinant part that is invertible. I previously showed that for each determinant part $\hat{\mathbf{F}}_{k_1\cdots k_n}$, either $\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right) = 0$ or $\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^n$. Putting these two results together, we have

$$\operatorname{sign}\left(\det\left(\hat{\mathbf{F}}\right)\right) = \operatorname{sign}\left(\sum_{k_1\in R_1,\ldots,k_n\in R_n} \det\left(\hat{\mathbf{F}}_{k_1\cdots k_n}\right)\right) = (-1)^n,$$

and, in particular, $\det\left(\hat{\mathbf{F}}\right) \neq 0$. Thus $\hat{\mathbf{F}}$ is invertible, so $\hat{\mathbf{F}}_+$ has full column rank, which implies the unmonitored flow has uniquely determinable flow. So my condition on the structure of the graph—that there are n arcs, one emanating from each vertex in $V - A(M)$, such that the turning factors for those arcs are positive and the subgraph induced on those arcs is a forest of reverse arborescences—is sufficient for uniquely determinable flow. This concludes the proof.

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