

A CONSTRUCTIVE METHOD FOR FINDING CRITICAL POINT OF THE  
GINZBURG-LANDAU ENERGY FUNCTIONAL

Parimah Kazemi

Dissertation Prepared for the Degree of

DOCTOR OF PHILOSOPHY

UNIVERSITY OF NORTH TEXAS

August 2008

APPROVED:

John W Neuberger, Major Professor

Robert J. Renka, Minor Professor

Santiago Betelu, Committee Mamber

Matt Douglass, Chair of the Department of  
Mathematics

Sandra L. Terrell, Dean of the Robert B. Toulouse  
School of Graduate Studies

Kazemi, Parimah. A Constructive Method for Finding Critical Point of the Ginzburg-Landau Energy Functional. Doctor of Philosophy (Mathematics), August 2008, 76 pp., 15 illustrations, bibliography, 18 titles.

In this work I present a constructive method for finding critical points of the Ginzburg-Landau energy functional using the method of Sobolev gradients. I give a description of the construction of the Sobolev gradient and obtain convergence results for continuous steepest descent with this gradient. I study the Ginzburg-Landau functional with magnetic field and the Ginzburg-Landau functional without magnetic field. I then present the numerical results I obtained by using steepest descent with the discretized Sobolev gradient.

Copyright 2008  
by  
Parimah Kazemi

## CONTENTS

LIST OF FIGURES	iv
CHAPTER 1. SUPERCONDUCTIVITY	1
CHAPTER 2. SOBOLEV SPACES	6
CHAPTER 3. SOBOLEV GRADIENTS	10
CHAPTER 4. ORTHOGONAL PROJECTIONS	14
CHAPTER 5. SIMPLIFIED GINZBURG-LANDAU	28
CHAPTER 6. THE GL ENERGY FUNCTIONAL WITH MAGNETIC FIELD ON A DOMAIN IN THE PLANE	40
CHAPTER 7. NUMERICAL RESULTS	60
7.1. Method	60
7.2. Experiments	64
7.3. Discussion	66
CHAPTER 8. CONCLUSION	73
BIBLIOGRAPHY	75

## LIST OF FIGURES

7.1 Plot of $\ u\ ^2$ for $\kappa = 50$ and $H_0 = 125$ . $d = 4$ and energy is 124.75.	66
7.2 Plot of $\ u\ ^2$ for $\kappa = 50$ and $H_0 = 150$ . $d = 12$ and energy=91.96.	66
7.3 Plot of $\ u\ ^2$ for $\kappa = 50$ and $H_0 = 200$ . $d = 16$ and energy=128.36.	67
7.4 Plot of $\ u\ ^2$ for $\kappa = 50$ and $H_0 = 225$ . $d = 20$ and energy=137.33.	67
7.5 Plot of $\ u\ ^2$ for $\kappa = 50$ and $H_0 = 250$ . $d = 24$ and energy=145.13.	67
7.6 Plot of $\ u\ ^2$ for $\kappa = 50$ and $H_0 = 275$ . $d = 28$ and energy=157.02.	68
7.7 Plot of $\ u\ ^2$ for $\kappa = 4$ , $H_0 = 6$ , and $n = 40$ .	68
7.8 Plot of $\ u\ ^2$ for $\kappa = 4$ , $H_0 = 6$ , and $n = 60$ .	68
7.9 Plot of $\ u\ ^2$ for $\kappa = 4$ , $H_0 = 6$ , and $n = 100$ .	69
7.10 Plot of $\ u\ ^2$ for $\kappa = 4$ , $H_0 = 6$ , and $n = 140$ .	69
7.11 Plot of $\ u\ ^2$ for $\kappa = 4$ , $H_0 = 6$ , and $n = 160$ .	69
7.12 Plot of $\ u\ ^2$ for $\kappa = 4$ , $H_0 = 6$ , and $n = 180$ .	70
7.13 Plot of $\ u\ ^2$ for $\kappa = 4$ , $H_0 = 6$ , $n = 837$ , and <i>energy</i> = 55.92.	70
7.14 Plot of energy vs. iteration for $\kappa = 4$ and $H_0 = 6$ .	70
7.15 Plot of energy vs. iteration for $\kappa = 4$ and $H_0 = 7$ .	71

## CHAPTER 1

### SUPERCONDUCTIVITY

Superconductivity was discovered in 1911 by H. Kamerlingh Onnes. He observed that the electrical resistance of certain metals, such as mercury, tin, and lead, disappeared completely if the temperature of the material was in a certain range, characteristic of the metal. In some experiments, it has been observed using nuclear resonance that  $10^5$  years is a lower bound for flow of current without measurable decrease [16].

Perfect conductivity is the phenomenon where current flows through a medium without any measurable decrease. Electric resistance depends on the shape of the crystal lattice of the conducting material and its temperature. All metals show a decrease in electrical resistance as they are cooled. In metals that are perfectly pure or have a perfect crystal lattice the lower bound for resistance is zero. Certain metals show no resistance at temperatures that are a few degrees above absolute zero even if they are impure. The temperature at which the transition is made is called the critical temperature and depends on the metal. If the metal is pure, the transition may be sharp whereas if the material is impure or has a disturbed crystal structure, the transition may be broader. Perfect conductivity is the first characteristic of a superconducting material [14].

Perfect diamagnetism is a second characteristic of superconductivity that was discovered by Meissner and Ochsenfeld in 1933. A sample in which there is no net flux density when a magnetic field is applied is said to exhibit perfect diamagnetism. A metal in a superconducting state never allows a magnetic flux density to exist in its interior. This is referred to as the Meissner effect. As a superconductor is cooled past the critical temperature, if there is a weak applied magnetic field, this is expelled from the material, and the material is in a superconducting state. Also, if a superconducting material is cooled below the critical temperature,

then a magnetic field does not penetrate into the material beyond a small penetration depth. However, if the applied magnetic field is strong enough, then a superconducting state will not be reached even if the sample is cooled below the critical temperature. Thus for certain metals there is a critical temperature range and a critical magnetic field range where the medium is in a superconducting state [14], [16].

During the next few decades, following the discovery of superconductivity, several groups came up with theories aimed at describing the above phenomenon. The major historical events are as follows.

- (i) The London Equations developed by brothers F. and H. London in 1935. The London equations use electrodynamic principles to describe perfect diamagnetism and perfect conductivity.
- (ii) In 1950, Ginzburg and Landau postulated the Ginzburg-Landau theory of superconductivity. Their theory used a complex wave functions  $\psi$  as an order parameter to describe the superconducting electrons.  $|\psi|^2$  gives the probability density of the superconducting electrons [3].
- (iii) In 1957 Bardeen, Cooper and Schrieffer came up with BCS theory.
- (iv) In his 1957 paper, "On the Magnetic Properties of Superconductors of the Second Group" , A. A. Abrikosov correctly distinguished between what is now known as type I and type II superconductors. He predicted the type of phase transition each type of superconductor undergoes. For type II superconductors he predicted the existence of the two critical magnetic fields that determine the three states type II superconductors can exist in. He also predicted a pattern for vortex formation near the second critical field [1].
- (v) In 1959, Gor'kov [5] showed that the Ginzburg-Landau theory was a limiting form of the BCS theory which is valid near the critical temperature.

In this work I will only be concerned with the Ginzburg-Landau theory. The postulates of the Ginzburg-Landau theory are

- (i) The behavior of the superconducting electrons may be described using a wavefunction. This implies that the probability density of the electrons is given by the square of the amplitude of the wavefunction.
- (ii) The free energy density,  $f$ , can be extended in a series of the form

$$(1) \quad f = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m_s}(|-i\hbar\nabla - \frac{e_s A}{c})\psi|^2 + \frac{|curl A|^2}{8\pi}$$

where  $\psi$  is a complex valued wavefunction,  $A$  is the magnetic vector potential,  $m_s$  and  $e_s$  are the mass and charge of electron pairs,  $\alpha < 0$  and  $\beta > 0$ , and  $f_n$  is the free energy of the normal state in the absence of a magnetic field[16].

The central Ginzburg-Landau problem is to find  $\psi$  and  $A$  so that the Gibbs free energy of the sample

$$(2) \quad \int_{\Omega} f - \frac{curl A \cdot H}{4\pi}$$

is minimized [11]. Here  $\Omega$  is the region occupied by the superconducting sample and  $H$  is the applied magnetic field. The conventional method is to use a variational approach to obtain the Ginzburg-Landau equations [11]

$$(3) \quad \frac{1}{2m_s}(-i\hbar\nabla - \frac{e_s}{c}A)^2\psi + \beta|\psi|^2\psi + \alpha\psi = 0 \text{ in } \Omega$$

and

$$(4) \quad j + c_1(\psi^*\nabla\psi - \psi\nabla\psi^*) + c_2|\psi|^2A = curl H \text{ in } \Omega$$

here  $j = curl(curl A)$  is the electrical current density,  $c_1 = \frac{2\pi i e_s \hbar}{m_s c}$  and  $c_2 = \frac{4\pi e_s^2}{m_s c^2}$ . The boundary conditions are

$$(5) \quad (i\hbar\nabla\psi + \frac{e_s}{c}A\psi) \cdot n = 0 \text{ on } \Gamma$$



and

$$(6) \quad \text{curl} A \times n = H \times n$$

where  $\Gamma$  is the boundary of  $\Omega$  and  $n$  is the unit outer normal [11].

In Chapter 3 I define the Sobolev gradient. The approach I present here treats the Gibbs free energy functional directly and hence avoids the Ginzburg-Landau equations completely. Since the definition of critical point I use is a point at which the Sobolev gradient is zero, I do not impose any boundary conditions on the minimization problem. A point which is a critical point satisfies the boundary conditions as a consequence of being a zero of the derivative of the energy functional. I also use a nondimensionalized version of equation (2). Several works derive this form of the functional, so I shall not give the derivation, one work in particular is [9]. If  $\Omega$  is a bounded region in  $\mathbb{R}^2$  that represents either a cross section of a superconductor or a thin film then the nondimensionalized version of equation (2) is

$$(7) \quad \phi(u, A) = \int_{\Omega} \frac{1}{2} |(\nabla - iA)u|^2 + \frac{1}{2} |\nabla \times A - H_0|^2 + \frac{\kappa^2}{4} (|u|^2 - 1)^2$$

The parameter  $\kappa$  is the ratio of the penetration depth to the coherence length and is known as the Ginzburg-Landau parameter,  $u$  is the order parameter, and  $A$  is the magnetic vector potential.

Superconductors can be classified as type I or type II. Superconducting state corresponds to the case where the modulus of the order parameter,  $|u|$ , is one everywhere. Normal state corresponds to the case where  $|u|$  is zero everywhere. In type I superconductors, the Ginzburg-Landau parameter,  $\kappa$  is less than  $1/\sqrt{2}$ . The literature on these superconductors is mainly concerned with the case where the domain occupied by the sample is all of  $\mathbb{R}^2$ . In this case it is predicted that there exists a critical value,  $H_C$ , for the external magnetic field,  $H_0$  so that if  $H_0 < H_C$  then the medium is in superconducting state and if  $H_0 > H_C$  the medium is in normal state. Type I superconductors undergo first order phase transitions. In type II superconductors, the parameter  $\kappa$  is greater than  $1/\sqrt{2}$ . For these superconductors,

the literature states that when the domain is all of  $\mathbb{R}^2$ , there exist two critical fields,  $H_{c1}$  and  $H_{c2}$  so that if  $H_0 < H_{c1}$  then the material is in superconducting state, if  $H_0 > H_{c2}$  then the material is in normal state. However, when  $H_{c1} < H_0 < H_{c2}$  the material exists in mixed state. In this state tubes of magnetic flux, called vortices, form in the interior of the region. The vortices correspond to zeros of the order parameter  $u$ . Type II superconductors, undergo phase two transition changes [1], [15], [16].

One of the main goals of this project is to numerically simulate the phase transition for type I and type II superconductors and to obtain numerical values for the critical magnetic fields. It has been predicted that for type II superconductors, as the external magnetic field increases beyond the first critical value, the vortices become more numerous and the space between them decreases. As the external magnetic field increases, the maximum of the modulus of the order parameter decreases. Also each vortex has associated with it one fluxoid. At the second critical field, the order parameter is predicted to vanish [11]. The numerical results that I obtained agree with all of these predictions.

When  $\kappa$  is large, the predicted values for the critical fields  $H_{c1}$  and  $H_{c2}$  when the domain is all of  $\mathbb{R}^2$  are

$$(8) \quad H_{c1} = \frac{\kappa |\ln(\kappa)|}{2}$$

and

$$(9) \quad H_{c2} = \kappa^2.$$

It has also been predicted that as the vortices form, they arrange themselves in a lattice. Two suggested shapes for the lattice are the Abrikosov lattice and a hexagonal lattice. It has also been predicted that near both critical points, the hexagonal lattice corresponds to a lower energy state and hence is more favorable [15]. A phase diagram is given in [15] to describe the transition from normal to superconducting state for both type I and type II superconductors.

## CHAPTER 2

### SOBOLEV SPACES

I begin the following with a statement of Hölder's inequality, the Cauchy-Schwarz inequality for Hilbert space, and the Riesz representation theorem for Hilbert spaces.

Proposition 2.1. *Let  $p$  and  $q$  be two integers so that  $1 \leq p, q$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $f \in L_p = L^p(\Omega)$  and  $g \in L_q = L^q(\Omega)$  then  $fg \in L_1 = L^1(\Omega)$  and*

$$\|fg\|_{L_1} \leq \|f\|_{L_p} \|g\|_{L_q}$$

Proposition 2.2. *Let  $V$  be an inner space. Then for all  $x, y \in V$*

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Theorem 2.3. *Let  $H$  be a real Hilbert space. If  $\phi$  is a continuous linear functional from  $H$  to  $\mathbb{R}$ , then there exists a unique  $x \in H$  so that*

$$\phi(y) = \langle x, y \rangle$$

*for all  $y \in H$ . From this it follows that the dual of  $H$  is isomorphic to  $H$ .*

I now give a brief discussion of Sobolev spaces. I list the necessary definitions and theorems that I will need without proof. I refer the reader to [2] for a more developed presentation of the theory of Sobolev spaces.

Let  $\Omega \subset \mathbb{R}^n$  be open,  $C^m(\Omega)$  is the space all functions  $f$  so that  $f$  along with the partial derivatives of  $f$  of order less than or equal to  $m$  are continuous.

Definition 2.4. For  $m$  a nonnegative integer and  $1 \leq p < \infty$  define the functional  $\|\cdot\|_{m,p}$  as follows: Let  $u \in C^m(\Omega)$  and  $\alpha$  a multi-index whose magnitude is less than or equal to  $m$ .

In other words  $\alpha = (k_1, \dots, k_n)$  where  $k_i$  is a positive integer that denotes the order of the partial derivative in the  $i^{th}$  independent variable so that

$$\sum_{i=1}^m k_i \leq m.$$

Define

$$(10) \quad \|u\|_{m,p} = \left( \sum_{0 \leq |\alpha| \leq m} |u_\alpha|_p^p \right)^{1/p}$$

where  $|\cdot|_p$  is the  $L^p(\Omega)$  norm and  $u_\alpha$  is the  $\alpha$  partial derivative.

$\|\cdot\|_{m,p}$  defines a norm on any space on which it is a finite valued provided that two functions are considered to be equivalent if they agree almost everywhere. The main case I are interested in is the case when  $n$  and  $p$  are 2 and  $m$  is 1. In this case

$$\|u\|_{1,2} = (\|u\|_2^2 + \|u_1\|_2^2 + \|u_2\|_2^2)^{1/2}.$$

Here and in the rest of the paper, in order to simplify notation, I denote the first partial derivative of  $u$  with respect to the first independent variable by  $u_1$  and the second partial derivative of  $u$  with respect to the second independent variable by  $u_2$ .

Definition 2.5. For  $1 \leq p < \infty$  define  $H^{m,p}(\Omega)$  to be the completion of  $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$  with respect to the norm  $\|\cdot\|_{m,p}$ .

I now give some characterizations of domains in  $\mathbb{R}^n$ .

Definition 2.6. For  $x \in \mathbb{R}^n$ , let  $B_1$  be an open ball in  $\mathbb{R}^n$  centered at  $x$  and  $B_2$  an open ball not containing  $x$ . The set  $B_1 \cap \{x + \lambda(y - x) : y \in B_2, \lambda \geq 0\}$  is called a finite cone in  $\mathbb{R}^n$  having vertex at  $x$ .

Definition 2.7. Let  $\Omega$  be an open set in  $\mathbb{R}^n$ .  $\Omega$  is said to have the cone property if there exists a finite cone  $C$  so that each point of  $x \in \Omega$  is the vertex of a finite cone  $C_x$  contained in  $\Omega$  and congruent to  $C$ .

The above properties are discussed in [2]. These characterizations of domains in Euclidean space are used in the Sobolev embedding theorem. I give a statement of the parts of the Sobolev embedding theorem and the Rellich-Kondrachov theorem that I will use and refer the reader to [2] for a proof and discussion.

Definition 2.8. Let  $X$  and  $Y$  be normed linear spaces.  $X$  is continuously embedded in  $Y$  ( $X \rightarrow Y$ ), if the points of  $X$  form a vector subspace of  $Y$  and there exists a constant  $c$  so that  $\|x\|_Y \leq c\|x\|_X$  for all  $x \in X$ .  $X$  is compactly embedded in  $Y$  if the identity operator from  $X$  to  $Y$  is compact.

If  $X$  and  $Y$  are normed linear spaces so that  $X$  is a linear subspace of  $Y$ , the identity operator is compact from  $X$  to  $Y$  if when  $\{x_n\}_{n \geq 1}$  is a bounded subset of  $X$  then  $\{x_n\}_{n \geq 1}$  has a subsequence that is convergent in  $Y$ . I will also make use of the following lemma.

Lemma 2.9. *If  $1 \leq p < \infty$  and  $a \geq 0, b \geq 0$ , then*

$$(11) \quad (a + b)^p \leq 2^{p-1}(a^p + b^p)$$

Theorem 2.10. *Suppose  $\Omega$  is an open set in  $\mathbb{R}^n$  satisfying the cone property. Let  $m$  be a nonnegative integers, and suppose  $1 \leq p < \infty$ . Then the following holds*

- (i)  $H^{m,p}(\Omega) \rightarrow L^q(\Omega)$  if  $mp = n$  and  $p \leq q < \infty$ .
- (ii)  $H^{m,p}(\Omega) \rightarrow C_B(\Omega)$  if  $mp > n$ .

*Here  $C_B(\Omega)$  is the set of continuous bounded functions on  $\Omega$  with the sup norm.*

For the case  $n = 2, p = 2, m = 1$ , note that the first conclusion of the above theorem implies that if the volume of  $\Omega$  is finite then  $H^{1,2}(\Omega)$  is continuously embedded in  $L^q(\Omega)$  for all  $q \geq 1$ . This is an immediate consequence of Hölder's inequality since if the volume of  $\Omega$  is finite and  $1 \leq p \leq q \leq \infty$  then  $L^q(\Omega) \rightarrow L^p(\Omega)$ .

Now suppose that  $f$  and  $g$  are in  $H^{1,2}(\Omega)$ ,  $i$  and  $j$  are two positive numbers so that there exist numbers  $k$  and  $l$  with  $ik > 1$  and  $jl > 1$  satisfying  $\frac{1}{k} + \frac{1}{l} = 1$ . Note that by using

Hölder's inequality and theorem 2.10, there is a constant  $d$  so that

$$(12) \quad \int_{\Omega} |f|^i |g|^j \leq \left( \int_{\Omega} |f|^{i'} \right)^{1/i'} \left( \int_{\Omega} |g|^{j'k} \right)^{1/k} = \|f\|_{L^{i'}}^i \|g\|_{L^{j'k}}^j \leq d \|f\|_H^i \|g\|_H^j$$

where  $L_q = L^q(\Omega)$ . This inequality along with theorem 2.10 will be used repeatedly in Chapters 4, 5, and 6. Also note that for  $\Omega$  bounded,  $m$  a nonnegative integer,  $1 \leq p, q < \infty$ ,  $H^{m,p}(\Omega)$  is dense in  $L^q(\Omega)$ . This is because for  $\Omega$  open in  $\mathbb{R}^n$ ,  $C_0(\Omega)$  is dense in  $L^q(\Omega)$  for  $1 \leq q < \infty$  [2]. By the Stone-Weierstrass theorem [2], if  $\Omega$  is bounded and  $f \in C_0(\Omega)$ , there exists a sequence of polynomials  $\{p_n\}_{n \geq 1}$  so that  $\lim_{n \rightarrow \infty} p_n = f$  in the sup norm. Since  $p_n \in H^{m,p}(\Omega)$  for all positive integers  $m$  and  $1 \leq p < \infty$ ,  $H^{m,p}(\Omega)$  is dense in  $C_0(\Omega)$  and hence in  $L^q(\Omega)$ . We have the following regarding compact embeddings.

**Theorem 2.11.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  satisfying the cone condition. Let  $m$  be a positive integer and  $1 \leq p < \infty$ . Then the following embeddings are compact:*

- (i)  $H^{m,p}(\Omega) \rightarrow L^q(\Omega)$  if  $mp = n$  and  $1 \leq q < \infty$ .
- (ii)  $H^{m,p}(\Omega) \rightarrow C_B(\Omega)$  if  $mp > n$ .

This theorem will be used in Chapter 4 to obtain compactness results for a class of transformations I will define.

## CHAPTER 3

### SOBOLEV GRADIENTS

In this section I define the Sobolev gradient and continuous steepest descent using the Sobolev gradient. Then I discuss the basic results regarding existence and uniqueness of the descent parameter. I follow the developments in [10] closely. The main concern of this chapter is convergence of the steepest descent parameter as defined in theorem 3.2 and the existence of critical points. I give some results regarding different types of convergence. I refer the reader to [10] for details and proofs on most of these results.

Definition 3.1. Let  $H$  be a Hilbert space and  $\phi$  a differentiable function from  $H$  to  $\mathbb{R}$ . By the Riesz representation theorem, for each  $x \in H$  there exists a unique member of  $H$ , denoted by  $\nabla_H \phi(x)$ , so that

$$(13) \quad \phi'(x)h = \langle h, \nabla_H \phi(x) \rangle_H.$$

Define the gradient of  $\phi$  at  $x$  to be  $\nabla_H \phi(x)$ .

Regarding the existence and uniqueness of the continuous steepest descent parameter, we have the following due to J. W. Neuberger.

Theorem 3.2. *Let  $H$  be a Hilbert space and  $\phi$  a differentiable function with domain  $H$  and range in the nonnegative real numbers. Suppose also that  $\nabla_H \phi : H \rightarrow H$  as defined in 13 is locally Lipschitzian. Then for  $w \in H$  there exists a unique function  $z : [0, \infty) \rightarrow H$  so that  $z(0) = w$  and  $z'(t) = -\nabla_H \phi(z(t))$ .*

Proof. Let  $z_0(t) = w$  and for  $n \geq 1$  define

$$(14) \quad z_n(t) = w - \int_0^t \nabla_H \phi(z_{n-1}).$$

Choose  $d$  and  $M \geq 1$  so that if  $\|x - w\|, \|y - w\| \leq d$  then

$$\|\nabla_H \phi(x) - \nabla_H \phi(y)\| < M\|x - y\|$$

and so that  $M$  is a bound for  $\|\nabla_H \phi\| : H \rightarrow \mathbb{R}$ .

I first show that  $z_n(t) \in B = B_d(w)$  for all  $t < d/M$  and all  $n$  by induction on  $n$ .  $z_0(t) \in B$  for all  $t$  by definition. Suppose  $z_n(t) \in B$  for  $t < d/M$ . Then for  $t < d/M$ ,

$$\|z_{n+1}(t) - w\| \leq \int_0^t \|\nabla_H \phi(z_n)\| \leq Mt < d.$$

Now for  $t < d/M$

$$\begin{aligned} \|z_n(t) - z_{n-1}(t)\| &\leq \int_0^t \|\nabla_H \phi(z_{n-1}) - \nabla_H \phi(z_{n-2})\| \leq \\ &\int_0^t M\|z_{n-1} - z_{n-2}\| \leq \frac{(Mt)^n}{n!} \leq \frac{(d)^n}{n!}. \end{aligned}$$

From this it follows that the sequence  $\{z_n\}_{n \geq 1}$  forms a uniform Cauchy sequence of functions on the interval  $[0, d/M)$  and hence the sequence of functions converges uniformly to a function which I call  $z$  on  $[0, d/M)$ . It also follows that  $z(0) = w$  and that  $z : [0, d/M) \rightarrow H$  is differentiable and that  $z'(t) = -\nabla_H \phi(z(t))$  by construction.

Let  $T$  be the largest positive number so that  $z(0) = w$ ,  $z'(t) = -\nabla_H \phi(z(t))$  for all  $t \in [0, T)$ . I show that  $\lim_{t \rightarrow T} z(t)$  exists. From this I can extend the domain of definition of  $z$  to include  $[T, T + \epsilon)$  thus  $T$  cannot be finite and hence  $z$  can be defined on  $[0, \infty)$ .

Suppose  $t_n \rightarrow T$  then

$$\|z(0) - z(t_m)\| \leq \int_0^{t_m} \|z'\| \leq (t_m)M$$

which by the Cauchy-Schwarz inequality is less than or equal to  $MT$ . This implies that  $\{z(t_n)\}_{n \geq 1}$  is uniformly Cauchy and hence converges uniformly to  $y \in H$ .

Furthermore, I claim that  $\lim_{t \rightarrow T} z(t) = y$  as otherwise there is  $\epsilon > 0$  and a sequence of numbers  $\{k_n\}_{n \geq 1}$  converging to  $T$  so that  $t_n < k_n < t_{n+1}$  and

$$\|z(k_n) - z(t_n)\| + \|z(t_n) - y\| \geq \|z(k_n) - y\| \geq \epsilon$$



for all  $n$ . For  $n$  large  $\|z(t_n) - y\| < \epsilon/2$  thus

$$\|z(k_n) - z(t_n)\| \geq \epsilon/2$$

for infinitely many  $n$  which is not possible as

$$\int_0^T \|z'\| = \int_0^T \|\nabla_H \phi(z)\| < \infty.$$

Uniqueness follows from basic existence and uniqueness results for ODEs.  $\square$

I call  $z$  the steepest descent parameter. A proof of this theorem in the case that  $\phi$  is  $C^{(2)}$  can be found in [10]. Note that if  $\phi$  is  $C^{(2)}$ , then it is locally Lipschitzian. Note that the conclusion of the above theorem implies that

$$(15) \quad (\phi(z))'(t) = \phi'(z(t))z'(t) = \langle z'(t), \nabla_H \phi(z(t)) \rangle_H = -\|\nabla_H \phi(z(t))\|_H^2$$

for all  $t \in [0, \infty)$ . If  $\phi \geq 0$ , we get that  $\phi(z)$  is a decreasing function from the nonnegative real numbers to the nonnegative real numbers. Therefore  $\phi(z)$  is a bounded function. From equation (15), it follows that

$$\int_0^\infty \|\nabla_H \phi(z)\|_H^2 = -\int_0^\infty (\phi(z))' < \phi(z(0)) < \infty$$

Thus  $\|\nabla_H \phi(z)\|_H \in L^2[0, \infty)$ . From this result I get the following result.

*Theorem 3.3. Under the hypotheses of theorem 3.2, there exists an unbounded sequence of real numbers  $\{t_n\}_{n \geq 1}$  so that  $\lim_{n \rightarrow \infty} \nabla_H \phi(z(t_n)) = 0$ . Where  $z$  is as in the above theorem.*

Proof. Let

$$a_n = \int_n^{n+1} \|\nabla_H \phi(z)\|_H^2.$$

Then we have that

$$\sum_{n=0}^{\infty} a_n < \infty,$$

so that for  $n$  a positive integer, there exists  $k_n$  so that  $a_{k_n} < 1/n^2$ . Hence

$$\int_{k_n}^{k_n+1} \|\nabla_H \phi(z)\|_H \leq \left( \int_{k_n}^{k_n+1} 1^2 \right)^{1/2} \left( \int_{k_n}^{k_n+1} \|\nabla_H \phi(z)\|_H^2 \right)^{1/2} = a_{k_n}^{1/2} < 1/n$$

This implies that for each  $n$  there is  $t_n \in [k_n, k_n + 1]$  so that  $\|\nabla_H \phi(z(t_n))\|_H < 1/n$ . Thus  $\{\nabla_H \phi(z(t_n))\}_{n \geq 1}$  converges to zero in  $H$ . □

## CHAPTER 4

### ORTHOGONAL PROJECTIONS

In this section I give a discussion of orthogonal projections. Suppose  $H$  is a Hilbert space and  $F$  is a closed linear subspace of  $H$ . Then there exists a linear symmetric transformation  $P$  from  $H$  onto  $F$  so that

- $|P| = 1$
- $P^2 = P$
- $Pf = f$  for all  $f \in F$

see [13] for a discussion.  $|P|$  denotes the operator norm of  $P$ . This projection is in fact the near point transformation. For  $h \in H$ ,  $Ph$  is the element of  $F$  that is closest to  $h$ .  $P$  is called the orthogonal projection of  $H$  onto  $F$ . If  $T$  is a closed densely defined linear transformation from a Hilbert space  $H$  to a Hilbert space  $K$ , then  $G_T = \left\{ \begin{pmatrix} x \\ Tx \end{pmatrix} : x \in D(T) \right\}$ , where  $D(T)$  denotes the domain of  $T$ , is closed in  $H \times K$ . For any linear operator  $T : H \rightarrow K$ , define  $T^t$  to be the transformation with domain

$$\{y \in K : \exists ! z \in H \text{ such that } \langle Tx, y \rangle_K = \langle x, z \rangle_H \forall x \in D(T)\}.$$

If  $y$  is in the domain of  $T^t$ , then  $T^t y$  is defined to be  $z$ . If  $T$  is closed and  $G_T \neq H \times K$ , then  $T^t$  is nontrivial since if  $\begin{pmatrix} f \\ g \end{pmatrix} \in H \times K$ , then there is  $x$  in the domain of  $T$  and  $\begin{pmatrix} z \\ y \end{pmatrix}$  in the orthogonal complement of  $G_T$ , so that  $x + z = f$ ,  $Tx + y = g$  and  $\langle x, z \rangle_H + \langle y, Tx \rangle_K = 0$ . This implies that  $y$  is in the domain of  $T^t$  as defined above and that  $T^t y = -z$ . If the domain of  $T^t$  is trivial then every element of  $H \times K$  can be written in the form  $\begin{pmatrix} x \\ Tx \end{pmatrix}$  which is not possible due to the assumption that  $G_T \neq H \times K$ . Define an operator  $T^t T$  with domain  $\{x \in H : Tx \text{ in the domain of } T^t\}$ . Then we have the following two results which can be found in [13].

Proposition 4.1. *If the linear transformation  $T$  on  $H$  to  $K$  is closed and if its domain is dense in  $H$ , the domain of  $T^t$  is dense in  $K$ ,  $(T^t)^t = T^{tt}$  exists, and  $T^{tt} = T$ .*

Theorem 4.2. *If the linear transformation  $T$  on  $H$  to  $K$  is closed and if its domain is dense in  $H$ , the transformations*

$$(16) \quad B = (I + T^t T)^{-1} \text{ and } C = T(I + T^t T)^{-1}$$

*are defined everywhere and bounded,*

$$(17) \quad \|B\| \leq 1 \text{ and } \|C\| \leq 1;$$

*moreover,  $B$  is symmetric and positive.*

Furthermore we have the following

Proposition 4.3. *Under the hypothesis of theorem 4.2, we have that for all  $x$  in the domain of  $T$*

$$(18) \quad T(I + T^t T)^{-1} = (I + T T^t)^{-1} T$$

Proof. Let  $x$  be in the domain of  $T$ , then we have

$$T x = T(I + T^t T)(I + T^t T)^{-1} x = (T + T T^t T)(I + T^t T)^{-1} x = (I + T T^t) T(I + T^t T)^{-1} x.$$

By using proposition 4.1 and theorem 4.2, we get that  $(I + T T^t)$  has an inverse with domain all of  $K$ . By applying  $(I + T T^t)^{-1}$  to the above we get  $(I + T T^t)^{-1}$  exists and

$$(I + T^t T)^{-1} T x = T(I + T^t T)^{-1} x$$

and we have the desired result. □

Regarding closed densely defined linear operators we have the following result due to von Neumann [17]

Theorem 4.4. *Suppose that each of  $H$  and  $K$  is a Hilbert space and  $T$  is a closed densely defined linear operator from  $H$  to  $K$ . Then the orthogonal projection of  $H \times K$  onto  $\left\{ \begin{pmatrix} x \\ T^t x \end{pmatrix} : x \in D(T) \right\}$  is given by the matrix*

$$P = \begin{pmatrix} (I + T^t T)^{-1} & T^t (I + T T^t)^{-1} \\ T (I + T^t T)^{-1} & I - (I + T T^t)^{-1} \end{pmatrix}$$

Thus in cases where  $F$ , the closed subspace, is the graph of closed densely defined linear operator, there is an explicit expression for  $P$ . In this section, I study properties of this projection. The two main results I obtain are:

- an extension of the formula given in theorem 4.4 for special cases.
- compactness results regarding the operators  $(I + T^t T)^{-1}$  and  $T^t (I + T T^t)^{-1}$  and the corresponding extensions.

In this section, let  $\Omega$  be a bounded region in the plane that satisfies the cone condition. Let  $K = L^2(\Omega)$ ,  $K^2 = K \times K$ , and  $K^3 = K \times K \times K$ . For  $p$  finite,  $L_p$  denotes  $L^p(\Omega)$ . For  $x \in H = H^{1,2}(\Omega)$ , define  $Wx = \nabla x$  where  $\nabla x$  is the list of the two partial derivatives of  $x$  taken in the Sobolev sense, see [2] for a discussion of Sobolev spaces. Then  $W$  is a closed and densely defined transformation on  $K$  to  $K^2$ , equivalently the graph of  $W$ ,  $G_W = \left\{ \vec{x} = \begin{pmatrix} x \\ \nabla x \end{pmatrix} : x \in H \right\}$  is a closed subspace of  $K^3$ . Von Neumann's formula gives that the projection,  $P$ , of  $K^3$  onto  $G_W$  is given by

$$\begin{pmatrix} (I + W^t W)^{-1} & W^t (I + W W^t)^{-1} \\ W (I + W^t W)^{-1} & I - (I + W W^t)^{-1} \end{pmatrix}$$

I want to extend the domain of the projection,  $P$ . Let  $1 < p < 2$ , define  $S_p$  to be the set of all  $\vec{y} = (f, g, h)$  so that  $f \in L_p$  and  $g, h \in K$ .  $S_p$  is a Banach space with the norm  $|\vec{y}|_{S_p} = |f|_{L_p} + |g|_K + |h|_K$ . I extend the domain of  $P$  to  $S_p$  and refer to the extension by  $P$  also. For the rest of this chapter, unless otherwise noted, assume that given  $1 < p < 2$ ,  $q$  is chosen to satisfy  $\frac{1}{p} + \frac{1}{q} = 1$ . I now define the following notation.

(i) For  $f \in L_p$  and  $x \in H$  define  $\langle f, x \rangle_K = \int_{\Omega} f x$ .

(ii) For  $\vec{y} \in S_p$  and  $\vec{x} \in G_W$  define  $\langle \vec{y}, \vec{x} \rangle_{K^3} = \int_{\Omega} f x + g x_1 + h x_2$ .

The integrals in 1) and 2) defined above are both finite since by theorem 2.10,  $H$  is continuously embedded in  $L^q(\Omega)$ . Thus if  $x \in H$ ,  $x \in L^q(\Omega)$  for  $q \geq 1$ , and there exists a constant  $c = c_q$  so that for all  $x \in H^{1,2}(\Omega)$ ,

$$\|x\|_{L^q(\Omega)} \leq c \|x\|_{H^{1,2}(\Omega)}.$$

Therefore we see that

$$|\langle f, x \rangle_K| \leq \int_{\Omega} |f x| \leq \left( \int_{\Omega} |f|^p \right)^{1/p} \left( \int_{\Omega} |x|^q \right)^{1/q} = \|f\|_{L_p} \|x\|_{L_q} \leq c \|f\|_{L_p} \|x\|_H < \infty.$$

Thus for  $f \in L_p$ ,  $\langle f, x \rangle_K$  is well defined and there is a constant  $m$  so that

$$(19) \quad |\langle f, x \rangle| \leq m \|x\|_H$$

for all  $x \in H$ . Similarly if  $\vec{y} \in S_p$ , then for  $\vec{x} \in G_W$ , by using Hölder's inequality, we have

$$\begin{aligned} |\langle \vec{x}, \vec{y} \rangle_{K^3}| &\leq \int_{\Omega} (|x f| + |x_1 g| + |x_2 h|) \leq \\ &\left( \int_{\Omega} |x|^q \right)^{1/q} \left( \int_{\Omega} |f|^p \right)^{1/p} + \left( \int_{\Omega} |x_1|^2 \right)^{1/2} \left( \int_{\Omega} |g|^2 \right)^{1/2} + \left( \int_{\Omega} |x_2|^2 \right)^{1/2} \left( \int_{\Omega} |h|^2 \right)^{1/2} \leq \\ &c \|x\|_H \|f\|_p + \|x\|_H \|g\|_K + \|x\|_H \|h\|_K. \end{aligned}$$

Thus  $\langle \vec{x}, \vec{y} \rangle_{K^3}$  as defined above is finite and if  $k$  is the max of  $c$  the embedding constant for the pair  $L_q$  and  $H$  and 1, then

$$(20) \quad |\langle \vec{x}, \vec{y} \rangle_{K^3}| \leq k \|\vec{y}\|_{S_p} \|x\|_H.$$

For  $\vec{y} = (f, g, h) \in S_p$  define

$$(21) \quad \alpha_{\vec{y}}(x) = \langle \vec{x}, \vec{y} \rangle_{K^3}$$

where  $x \in H$  and  $\vec{x} = \begin{pmatrix} x \\ \nabla x \end{pmatrix} \in G_W$ . Then

Lemma 4.5.  $\alpha_{\vec{y}}$  as defined in (21) is well defined and continuous from  $H$  to  $\mathbb{R}$ . Furthermore for each  $\vec{y} \in S_p$ , there exists a unique  $z = z_y$  so that  $\alpha_{\vec{y}}(x) = \langle x, z \rangle_H$  for all  $x \in H$ .

Proof. From equation (20) we have that,  $|\alpha_{\vec{y}}(x)| \leq k \|y\|_{S_p} \|x\|_H$  for all  $x \in H$ . Hence,  $\alpha_{\vec{y}}$  is continuous from  $H$  to  $\mathbb{R}$ , and by the Riesz representation theorem there is a unique element  $z \in H$  so that

$$(22) \quad \alpha_{\vec{y}}(x) = \langle x, z \rangle_H$$

for all  $x \in H$ . □

For  $\vec{y} \in S_p$ , define  $P\vec{y} = \vec{z}$ , where  $z \in H$  is defined in (22) and  $\vec{z} = \begin{pmatrix} z \\ \nabla z \end{pmatrix}$ . If  $\vec{y} \in G_W$  then  $\alpha_{\vec{y}}(x) = \langle x, y \rangle_H$  for all  $x \in H$ . Thus  $P\vec{y} = \vec{y}$ , so  $P$  stays fixed on  $G_W$ .  $P$  has the following properties.

Lemma 4.6. For all  $x \in H^{1,2}(\Omega)$  and  $\vec{y} \in S_p$ ,

- (i)  $\langle P\vec{x}, \vec{y} \rangle_{K^3} = \langle \vec{x}, P\vec{y} \rangle_{K^3}$  for all  $\vec{x} \in G_W$ .
- (ii)  $P(P(\vec{y})) = P\vec{y}$  for all  $\vec{y} \in S_p$ .
- (iii)  $\|P(\vec{y})\|_{K^3} \leq c_1 \|\vec{y}\|_{S_p}$  where  $c_1$  is the max of  $c$ , the Sobolev embedding constant for the pair  $L_q$  and  $H$ , and 1.

Proof. To show 1) observe the following. Let  $\vec{y} \in S_p$  and  $x \in G_W$ , then

$$\begin{aligned} \langle P\vec{x}, \vec{y} \rangle_{K^3} &= \langle \vec{x}, \vec{y} \rangle_{K^3} = \alpha_{\vec{y}}(x) = \\ &= \langle x, z \rangle_H = \langle \vec{x}, \vec{z} \rangle_{K^3} = \langle \vec{x}, P\vec{y} \rangle_{K^3}. \end{aligned}$$

To show 2) note that for  $\vec{y} \in S_p$  if we let  $P\vec{y} = \vec{z} \in G_W$ , then

$$P(P(\vec{y})) = P(\vec{z}) = \vec{z} = P\vec{y}$$

To show 3) note that

$$\|P(\vec{y})\|_{K^3}^2 = \langle P\vec{y}, P\vec{y} \rangle_{K^3} = \langle P^2\vec{y}, \vec{y} \rangle_{K^3} = \langle P\vec{y}, \vec{y} \rangle_{K^3}.$$

But using  $P\vec{y} = \vec{z}$  we have

$$\begin{aligned} \langle P\vec{y}, \vec{y} \rangle_{K^3} &= \langle \vec{z}, \vec{y} \rangle_{K^3} = \int_{\Omega} zf + z_1g + z_2h \leq \\ & \left( \int_{\Omega} |z|^q \right)^{1/q} \left( \int_{\Omega} |f|^p \right)^{1/p} + \left( \int_{\Omega} z_1^2 \right)^{1/2} \left( \int_{\Omega} g^2 \right)^{1/2} + \left( \int_{\Omega} z_2^2 \right)^{1/2} \left( \int_{\Omega} h^2 \right)^{1/2} \leq \\ & c \|z\|_H \|f\|_{L_p} + \|z\|_H \|g\|_K + \|z\|_H \|h\|_K \leq c_1 \|z\|_H \|\vec{y}\|_{S_p} \end{aligned}$$

Since  $\|z\|_H = \|P\vec{y}\|_{K^3}$ , we have that

$$\|P\vec{y}\|_{K^3} \leq c_1 \|\vec{y}\|_{S_p}.$$

□

So  $P$  extended to the Banach space  $S_p$  is bounded,  $P^2 = P$ , and the domain of  $P$  is extended to elements of the form  $\vec{y} = (f, g, h)$  where  $f \in L^p(\Omega)$  and  $g, h \in L^2(\Omega)$ . We see that some of the properties of  $P$  are preserved.

I now want to extend von Neumann's formula to  $S_p$ . In order to do this I define the following operator. I want to define  $M$  from  $K$  to  $H$  so that for  $y \in K$ ,

$$\langle x, y \rangle_K = \langle x, My \rangle_H \text{ for all } x \in H.$$

I give a precise definition for  $M$

Definition 4.7. Let  $y \in K$ , and define  $f_y$  from  $H$  to  $\mathbb{R}$  so that  $f_y(x) = \langle x, y \rangle_K$ . Since  $H$  is continuously embedded in  $K$ ,  $f_y$  is continuous from  $H$  to  $\mathbb{R}$ , thus there exists  $g_y \in H$  so that  $\langle x, y \rangle_K = \langle x, g_y \rangle_H$  for all  $x \in H$ . Define  $My = g_y$ .

In [7] we obtain the following results for  $M$ .

Theorem 4.8.  *$M$  as a transformation from  $K$  to  $H$  is continuous, and the operator norm of  $M$  is less than or equal to one.*

Theorem 4.9.  *$M$  as a transformation from  $K$  to  $K$  is compact iff  $M$  as a transformation from  $K$  to  $H$  is compact.*



I extend this operator as follows. For  $f \in L^p(\Omega)$ , define  $\beta_f : H \rightarrow \mathbb{R}$  by

$$(23) \quad \beta_f(x) = \langle x, f \rangle_K \text{ for all } x \in H.$$

Using equation (19), we get that  $\beta_f$  is well defined and continuous on  $H$ . Thus using the Riesz representation theorem, we get that there exists a unique member of  $H$ ,  $z_f$ , so that  $\beta_f(x) = \langle x, z_f \rangle_H$  for all  $x \in H$ . Define the operator  $M_p$  on  $L^p(\Omega)$  so that for  $f \in L_p$ ,  $M_p f = z_f$ . Thus we have that

$$(24) \quad \langle f, x \rangle_K = \langle M_p f, x \rangle_H$$

for  $f \in L_p$  and  $x \in H$ . From this we can see that  $M_p$  and  $M$  agree on  $K$ .

Theorem 4.10. *Let  $M_p$  be the operator defined above. Then we have the following:*

- (i)  $M_p$  is injective. I will show that if  $\langle f, x \rangle_K = \langle M_p f, x \rangle_H = 0$  for all  $x \in H$ , then  $f = 0$ .
- (ii)  $M_p \in L(X, Y)$  where  $X = H, L_p$  and  $Y = H, L_p$
- (iii)  $\langle x, M_p y \rangle_H = \langle M_p x, y \rangle_H \forall x, y \in H$
- (iv)  $M_p$  as an operator from  $H$  to  $H$  has a unique square root. If  $x \in H$ , then  $\|x\|_K = \|\sqrt{M_p}x\|_H$ .
- (v) If  $\{y_n\}_{n \geq 1}$  is a bounded sequence in  $L_p$ , then  $\{M_p y_n\}_{n \geq 1}$  has a subsequence which converges in  $L_p$  to  $u \in H$ .

Properties 1), 2), 3), and 5) are used in this work to obtain results for the Ginzburg-Landau energy functional. This will be discussed in detail in the later chapters. Property 4) is an analogue of a result we obtained in [7].

Proof. To show 1), suppose  $M_p f = 0$ , then

$$\langle f, x \rangle_K = \langle M_p f, x \rangle_H = 0$$

for all  $x \in H$ . I show that if  $\langle f, x \rangle_K = 0$  for all  $x \in H$  then  $\mu(f) = 0$  for all  $\mu \in L_p^*$ . Let  $\mu \in L_p^*$ , since  $L_p^*$  is isomorphic to  $L_q$  by the mapping  $v \in L_q \rightarrow l_v$ , where  $l_v(g) = \int_{\Omega} v g$  for

all  $g \in L_p$ , there exists  $v_\mu \in L_q$ , so that  $\mu(g) = \int_\Omega g v_\mu$  for all  $g \in L_p$ . Now since  $H$  is dense in  $L_q$  let  $\{v_n\}_{n \geq 1}$  be a sequence in  $H$  converging to  $v_\mu$  in  $L_q$ . Since

$$\int_\Omega |(v_n - v_\mu)f| \leq \|f\|_{L_p} \|v_n - v_\mu\|_{L_q}$$

we get that

$$\lim_n \int_\Omega f v_n = \int_\Omega f v_\mu.$$

Now

$$\begin{aligned} \mu(f) &= \int_\Omega f v_\mu = \lim_n \int_\Omega f v_n = \\ &= \lim_n \langle f, v_n \rangle_K = 0 \end{aligned}$$

as it was assumed that  $\langle f, x \rangle_K = 0$  for all  $x \in H$ . Thus  $\mu(f) = 0$  for all  $\mu \in L_p^*$ . This implies that  $f = 0$ .

To show 2) observe the following. Let  $c$  be the maximum of the Sobolev embedding constant,  $c_p$ , for the pair  $L_p$  and  $H$  and,  $c_q$ , the embedding constant for the pair  $L_q$  and  $H$ . Then for  $y \in L_p$

$$\begin{aligned} \|M_p y\|_{L_p} &\leq c \|M_p y\|_H = c \sup\{\langle M_p y, z \rangle_H : z \in H, \|z\|_H = 1\} = \\ &= c \sup\{\langle y, z \rangle_K : z \in H, \|z\|_H = 1\} \leq \\ &= c \sup\{\|y\|_{L_p} \|z\|_{L_q} : z \in H, \|z\|_H = 1\} \leq \\ &= c \|y\|_{L_p} \sup\{c \|z\|_H : z \in H, \|z\|_H = 1\} = c^2 \|y\|_{L_p}. \end{aligned}$$

If  $y \in H$  then we get that  $\|y\|_{L_p} \leq c \|y\|_H$ . From this it follows that  $M_p \in L(X, Y)$  where  $X = L_p, H$  and  $Y = L_p, H$ .

To show 3) observe that if  $x, y \in H$ , then

$$\langle M_p x, y \rangle_H = \langle x, y \rangle_K = \langle x, M_p y \rangle_H.$$

This also implies that for  $x \in H$ ,  $\langle M_p x, x \rangle_H = \|x\|_K^2$ , so  $M_p : H \rightarrow H$  is nonnegative. Refer to [13] for the following theorem,

Theorem 4.11. *Let  $F$  be a Hilbert space and  $A$  a nonnegative, symmetric, and bounded operator from  $F$  to itself. Then there exists a unique positive, symmetric, and bounded transformation  $B : F \rightarrow F$  so that  $B^2 = A$ . Also  $B$  is the strong limit of the sequence*

$$Y_0 = 0, Y_n = \frac{1}{2}((I - A) - Y_{n-1}^2), n = 1, 2, \dots$$

Using this theorem we have that  $M_p$  as an operator from  $H$  to  $H$  has a unique bounded, positive, and symmetric square root,  $\sqrt{M_p}$ . Furthermore if  $x \in H$ , then  $\|x\|_K^2 = \langle x, x \rangle_K = \langle x, M_p x \rangle_H = \langle \sqrt{M_p} x, \sqrt{M_p} x \rangle_H = \|\sqrt{M_p} x\|_H^2$ . Statement 4) follows from this.

To show 5) observe that condition 2) implies that if  $\{y_n\}_{n \geq 1}$  is bounded in  $L_p$ , then  $\{M_p y_n\}_{n \geq 1}$  is bounded in  $H$ . Hence using theorem 2.11, we get that there exists a subsequence  $\{M_p y_{n_k}\}_{k \geq 1}$  that converges in  $L_p$  to  $v \in L_p$ . But since  $\{M_p y_{n_k}\}_{k \geq 1}$  is bounded in  $H$ , there is a subsequence  $\{M_p y_{n_{k_j}}\}_{j \geq 1}$  so that  $\{M_p y_{n_{k_j}}\}_{j \geq 1}$  converges weakly to  $u \in H$ . Let  $a_j = M_p y_{n_{k_j}}$  and show  $u = v$ . Let  $x \in H$ , then

$$\begin{aligned} \lim_j \langle a_j, x \rangle_K &= \lim_j \langle a_j, M_p x \rangle_H = \\ &= \langle u, M_p x \rangle_H = \langle u, x \rangle_K. \end{aligned}$$

We also have that

$$|\langle a_j - v, x \rangle_K| \leq \|a_j - v\|_p \|x\|_q$$

so  $\lim_{j \rightarrow \infty} \langle a_j, x \rangle_K = \langle v, x \rangle_K$ . Thus we have that  $\langle u - v, x \rangle_K = 0$  for all  $x \in H$ . Using 1) we get that  $u = v$ . □

Theorem 4.12. *If  $\vec{y} = (f, g, h) \in S_p$ , then*

$$P\vec{y} = \begin{pmatrix} M_p & W^t(I + WW^t)^{-1} \\ WM_p & I - (I + WW^t)^{-1} \end{pmatrix} \begin{pmatrix} f \\ \begin{pmatrix} g \\ h \end{pmatrix} \end{pmatrix}$$

Proof. I show that if  $P\vec{y} = (z, z_1, z_2)$ , then

$$z = M_p f + W^t(I + WW^t)^{-1} \begin{pmatrix} g \\ h \end{pmatrix}.$$

We have for all  $\vec{x} \in G_W$ ,

$$\langle x, M_p f + W^t(I + WW^t)^{-1} \begin{pmatrix} g \\ h \end{pmatrix} \rangle_H = \langle x, M_p f \rangle_H + \langle x, W^t(I + WW^t)^{-1} \begin{pmatrix} g \\ h \end{pmatrix} \rangle_H =$$

$$\langle x, f \rangle_K + \langle x, W^t(I + WW^t)^{-1} \begin{pmatrix} g \\ h \end{pmatrix} \rangle_K + \langle Wx, WW^t(I + WW^t)^{-1} \begin{pmatrix} g \\ h \end{pmatrix} \rangle_{K^2} =$$

$$\langle x, f \rangle_K + \langle Wx, (I + WW^t)^{-1} \begin{pmatrix} g \\ h \end{pmatrix} \rangle_{K^2} + \langle Wx, (I - (I + WW^t)^{-1}) \begin{pmatrix} g \\ h \end{pmatrix} \rangle_{K^2} =$$

$$\langle x, f \rangle_K + \langle Wx, \begin{pmatrix} g \\ h \end{pmatrix} \rangle_{K^2} = \langle \vec{x}, \vec{y} \rangle_{K^3}$$

Since  $P\vec{y}$  is defined to be the unique element  $\vec{z}$  of  $G_W$  so that  $\langle \vec{x}, \vec{y} \rangle_{K^3} = \langle x, z \rangle_H$  for all  $x \in H$ , the above shows that

$$z = M_p f + W^t(I + WW^t)^{-1} \begin{pmatrix} g \\ h \end{pmatrix}.$$

This implies that

$$Wz = WM_p f + I - (I + WW^t)^{-1}(g, h)$$

since

$$WW^t(I + WW^t)^{-1} = I - (I + WW^t)^{-1}.$$

□

In theorem 4.10, we saw that the transformation  $M_p$  is compact in the sense that if  $\{y_n\}_{n \geq 1}$  is a bounded sequence in  $L_p$  then  $\{M_p y_n\}_{n \geq 1}$  has a subsequence that converges in the  $L_p$  norm to  $u \in H$ . In Chapter 6, I will use this result as well as well as the following

result to obtain convergence results for the steepest descent parameter. I now obtain the following result for the operator  $W^t(I + WW^t)^{-1}$ .

Theorem 4.13. *Suppose  $\{Y_n\}_{n \geq 1}$  is a bounded subset of  $K^2$ . Then*

- (i)  $\{W^t(I + WW^t)^{-1}Y_n\}_{n \geq 1}$  is bounded in  $H$ .
- (ii) There exists a subsequence  $\{W^t(I + WW^t)^{-1}Y_{n_k}\}_{k \geq 1}$  which converges in  $K$  to  $u \in H$ .

Proof. To show 1) observe that if  $x \in H$ , then  $\|x\|_H^2 = \|x\|_K^2 + \|Wx\|_K^2$ . Thus

$$\begin{aligned} \|W^t(I + WW^t)^{-1}Y_n\|_H^2 &= \langle W^t(I + WW^t)^{-1}Y_n, W^t(I + WW^t)^{-1}Y_n \rangle_H = \\ &\langle W^t(I + WW^t)^{-1}Y_n, W^t(I + WW^t)^{-1}Y_n \rangle_K + \\ &\langle WW^t(I + WW^t)^{-1}Y_n, WW^t(I + WW^t)^{-1}Y_n \rangle_{K^2} = \\ &\langle WW^t(I + WW^t)^{-1}Y_n, (I + WW^t)^{-1}Y_n \rangle_{K^2} + \\ &\langle WW^t(I + WW^t)^{-1}Y_n, WW^t(I + WW^t)^{-1}Y_n \rangle_{K^2} = \\ &\langle WW^t(I + WW^t)^{-1}Y_n, (I + WW^t)(I + WW^t)^{-1}Y_n \rangle_{K^2} = \\ &\langle WW^t(I + WW^t)^{-1}Y_n, Y_n \rangle_{K^2} = \langle I - (I + WW^t)^{-1}Y_n, Y_n \rangle_{K^2}. \end{aligned}$$

Since by theorem 4.2, the operator norm of  $(I + WW^t)^{-1}$  from  $K^2$  to  $K^2$  is less than or equal to one, we have

$$(25) \quad \|W^t(I + WW^t)^{-1}Y_n\|_H^2 \leq \|Y_n\|_{K^2}^2$$

for all  $n$ . Since by assumption  $\{Y_n\}_{n \geq 1}$  is bounded in  $K^2$  then  $\{W^t(I + WW^t)^{-1}Y_n\}_{n \geq 1}$  is bounded in  $H$ .

To show 2), recall that  $H^{1,2}(\Omega)$  is compactly embedded in  $L^2(\Omega)$ , when  $\Omega$  satisfies the cone condition and is bounded. If  $\{Y_n\}_{n \geq 1}$  is bounded in  $K$  then  $\{W^t(I + WW^t)^{-1}Y_n\}_{n \geq 1}$  is bounded in  $H$  by 1). Thus there is a subsequence  $\{W^t(I + WW^t)^{-1}Y_{n_k}\}_{k \geq 1}$  and  $v \in K$  so that  $\{W^t(I + WW^t)^{-1}Y_{n_k}\}_{k \geq 1}$  converges to  $v$  in  $K$ . We also have that there is  $u \in H$  and a subsequence  $\{W^t(I + WW^t)^{-1}Y_{n_{k_j}}\}_{j \geq 1}$  that converges to  $u$  weakly in  $H$ . I now show that  $u = v$ .

Let  $a_j = W^t(I + WW^t)^{-1}Y_{n_{k_j}}$ , then since  $\{a_j\}_{j \geq 1}$  converges strongly in  $K$  to  $v$ , it also converges weakly to  $v$  in  $K$ . But then for  $f \in K$ , we have

$$\begin{aligned} \langle f, v \rangle_K &= \lim_j \langle f, a_j \rangle_K = \\ \lim_j \langle Mf, a_j \rangle_H &= \langle Mf, u \rangle_H = \\ &= \langle f, u \rangle_K \end{aligned}$$

where  $M$  is as in Definition 4.7. Thus  $u = v$  and  $\{W^t(I + WW^t)^{-1}Y_{n_{k_j}}\}_{j \geq 1}$  converges strongly in  $K$  to  $u \in H$ . □

I now discuss the operator  $(I + WW^t)^{-1}$  from von Neumann's formula. Define the operator  $W^t$  so that the domain of  $W^t$  is all  $u \in K^2$  with

$$\langle Wx, u \rangle_{K^2} = \langle x, g \rangle_K$$

for all  $x \in H$ , in which case  $W^t u = g$ . If for  $u$  in the domain of  $W^t$ ,  $W^t u = g$  is in  $H$ , then  $(I + WW^t)(u) = u + WW^t u = u + Wg$ . As noted earlier  $W$  is closed and densely defined on  $K$  to  $K^2$ . I give the following characterization of  $M$  which will be used in Chapter 5.

*Proposition 4.14. Let  $M$  be the transformation defined in definition 4.7. Then  $(I + W^t W)^{-1}$  is the transformation  $M$ .*

Proof. I first show that if  $y \in K$  then  $My$  is in the domain of  $I + W^tW$  and  $(I + W^tW)(My) = y$ . Let  $y \in K$ , since  $My \in H$  then  $My$  is in the domain of  $W$ .  $My$  is in the domain of  $(I + W^tW)$  if there is  $z \in L$  so that

$$\langle Wx, WMy \rangle_{K^2} = \langle x, z \rangle_K$$

for all  $x \in H$ . But

$$\begin{aligned} \langle Wx, WMy \rangle_{K^2} &= \langle x, My \rangle_H - \langle x, My \rangle_K = \\ &= \langle x, y \rangle_K - \langle x, My \rangle_K = \langle x, y - My \rangle_K. \end{aligned}$$

Thus,  $z = W^tW(My) = y - My$  and

$$(I + W^tW)(My) = My + y - My = y$$

Now suppose  $y$  is in the domain of  $I + W^tW$ , I show that  $M(I + W^tW)y = y$ . Let  $x \in H$ , then

$$\langle x, (I + W^tW)y \rangle_K = \langle x, M(I + W^tW)y \rangle_H$$

But then,

$$\langle x, (I + W^tW)y \rangle_K = \langle x, y \rangle_K + \langle Wx, Wy \rangle_{K^2} = \langle x, y \rangle_H.$$

Thus,  $\langle x, M(I + W^tW)y \rangle_H = \langle x, y \rangle_H$  for all  $x \in H$ , which implies  $M(I + W^tW)y = y$ .  $\square$

I do not use the following result for the following applications. I do include it in this work because it gives another characterization of the operator  $M_p$  that might be significant. For  $1 < p < 2$ , define the operator  $T_p$  where the domain of  $T_p$  is all  $u \in H$ , such that there exists  $f \in L^p(\Omega)$  so that for all  $x \in H$ ,

$$\langle Wx, Wu \rangle_{K^2} = \langle x, f \rangle_K$$

in which case define  $T_p u = u + f$ . Then we have the following

Proposition 4.15.  $M_p$ , as defined above, is the inverse of  $T_p$ .

Proof. I first show that for all  $f$  in  $L_p$ , the domain of  $M_p$ ,  $M_p f$  is in the domain of  $T_p$  and  $T_p M_p f = f$ . For  $x \in H$

$$\langle Wx, WM_p f \rangle_{K^2} = \langle x, M_p f \rangle_H - \langle x, M_p f \rangle_K.$$

Since  $\langle x, M_p f \rangle_H = \langle x, f \rangle_K$  by definition we have that

$$\langle Wx, WM_p f \rangle_{K^2} = \langle x, f - M_p f \rangle_K.$$

Thus  $M_p f$  is in the domain of  $T_p$  and  $T_p M_p f = M_p f + f - M_p f = f$ .

Now suppose  $u$  is in the domain of  $T_p$ , then  $T_p u \in L^p(\Omega)$ . So  $T_p u = u + f$  is in the domain of  $M_p$ , where  $f$  is defined to be the element of  $L^p(\Omega)$  so that  $\langle Wx, Wu \rangle_{K^2} = \langle x, f \rangle_K$  for all  $x \in H$ . If  $x \in H$  then

$$\langle M_p T_p u - u, x \rangle_H = \langle T_p u, x \rangle_K - \langle u, x \rangle_H =$$

$$\langle u + f, x \rangle_K - \langle u, x \rangle_H =$$

$$\langle u, x \rangle_K + \langle Wu, Wx \rangle_{K^2} - \langle u, x \rangle_H = 0$$

This implies that  $M_p T_p u = u$  for all  $u$  in the domain of  $T_p$ . Hence  $M_p$  and  $T_p$  are inverses.  $\square$



## CHAPTER 5

### SIMPLIFIED GINZBURG-LANDAU

In this section I study the simplified Ginzburg-Landau functional as described below and present existence results for critical points of the functional.

Let  $\Omega$  be an open bounded set in  $\mathbb{R}^2$  that satisfies the cone property. I treat  $H^{1,2}(\Omega, \mathbb{C})$  as  $H^{1,2}(\Omega) \times H^{1,2}(\Omega) = H^2$ . To avoid any possible confusion note that that  $H^2$  is not  $H^{2,2}(\Omega)$ . Let  $K = L^2(\Omega)$ ,  $K^2$  is the cartesian product  $K \times K$ ,  $K^n$  is defined likewise for  $n = 4, 6$ .  $H$  denotes  $H^{1,2}(\Omega)$  as in the previous chapters. For  $u = \begin{pmatrix} r \\ s \end{pmatrix} \in H^2$ , let

$$\nabla u = \begin{pmatrix} r_1 \\ s_1 \\ r_2 \\ s_2 \end{pmatrix}$$

where  $x_i$  denotes the first partial derivative of  $x$  with respect to the  $i^{th}$  independent variable.

For  $u \in H^2$ , define

$$(26) \quad \phi(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{\kappa^2}{4} (|u|^2 - 1)^2.$$

This is the simplified Ginzburg-Landau functional.

Lemma 5.1. For  $u \in H^2$ ,  $\phi(u) < \infty$ .

Proof. For  $u \in H^2$ ,  $\|u\|_{H^2}^2 = \int_{\Omega} |u|^2 + |\nabla u|^2 < \infty$ . By theorem 2.10 we have that if  $1 \leq p < \infty$  there exists a constant  $c_p$  so that  $\|u\|_{L^p} \leq c_p \|u\|_{H^2}$ . Thus using inequality (11), we have

$$\int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{\kappa^2}{4} (|u|^2 - 1)^2 \leq$$

$$\begin{aligned}
& \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{\kappa^2}{2} ((r^2 + s^2)^4 + 1) \leq \\
& \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{\kappa^2}{2} 8(r^8 + s^8 + 1) \leq \\
& \frac{1}{2} \|u\|_{H^2}^2 + 4\kappa^2 (2c^8 \|u\|_{H^2}^8 + \text{vol}(\Omega))
\end{aligned}$$

where  $c$  is the embedding constant for the pair  $L^8(\Omega)$  and  $H$ . Thus we have that  $\phi(H^2) \subseteq [0, \infty)$ .  $\square$

We can write equation (26) as

$$\phi(u) = \int_{\Omega} F(Du)$$

where  $F$  is a function from  $\mathbb{R}^6$  to  $\mathbb{R}$  and is given by

$$F(x, y, z, a, b, c) = \kappa^2 \frac{(x^2 + a^2 - 1)^2}{4} + \frac{y^2}{2} + \frac{z^2}{2} + \frac{b^2}{2} + \frac{c^2}{2},$$

for

$$f \in H, Df = \begin{pmatrix} f \\ f_1 \\ f_2 \end{pmatrix}$$

and for

$$h = \begin{pmatrix} f \\ g \end{pmatrix} \in H^2, Dh = \begin{pmatrix} Df \\ Dg \end{pmatrix}.$$

I denote the list of partial derivatives of  $F$  by  $\nabla F$  and observe that

$$\nabla F(x, y, z, a, b, c) = \begin{pmatrix} Y1(x, y, z, a, b, c) \\ Y2(x, y, z, a, b, c) \end{pmatrix},$$

where

$$Y1(x, y, z, a, b, c) = \begin{pmatrix} \kappa^2(x^2 + a^2 - 1)x \\ y \\ z \end{pmatrix}$$

and

$$Y2(x, y, z, a, b, c) = \begin{pmatrix} \kappa^2(x^2 + a^2 - 1)a \\ b \\ c \end{pmatrix}.$$

I now show that  $\phi$  is Fréchet differentiable and that the Fréchet derivative of  $\phi$  can be written as

$$\phi'(u)h = \langle Dh, \nabla F(Du) \rangle_{K^6}.$$

Lemma 5.2.  $\phi$  is Fréchet differentiable from  $H^2$  to  $\mathbb{R}$  and  $\phi'(u)$  is a continuous linear functional for each  $u \in H^2$ . Furthermore we can write the Fréchet derivative of  $\phi$  as

$$(27) \quad \phi'(u)h = \langle Dh, \nabla F(Du) \rangle_{K^6}.$$

Proof. I use the following two calculations. Fix  $u = \begin{pmatrix} r \\ s \end{pmatrix} \in H^2$ , then if  $h = \begin{pmatrix} f \\ g \end{pmatrix} \in H^2$  we have,

(i)

$$\|\nabla(u + h)\|_{K^4}^2 - \|\nabla u\|_{K^4}^2 - 2\langle \nabla u, \nabla h \rangle_{K^4} = \|\nabla h\|_{K^4}^2$$

(ii)

$$\begin{aligned} \frac{\kappa^2}{4} \int_{\Omega} [(|u + h|^2 - 1)^2 - (|u|^2 - 1)^2] - \langle h, (|u|^2 - 1)u \rangle_{K^2} = \\ \frac{\kappa^2}{4} \int_{\Omega} p(r, s, f, g) \end{aligned}$$

where

$$\begin{aligned} p(r, s, f, g) = 6r^2f^2 + 4rf^3 + f^4 + 6s^2g^2 + 4sg^3 + g^4 + \\ 2(f^2s^2 + 4rsfg + 2f^2sg + f^2g^2 + 2rfg^2 + r^2g^2) \end{aligned}$$

Using inequality (12) we get that there exists a constant  $k_1$  so that

$$\int_{\Omega} 6r^2f^2 \leq k_1 \|r\|_H^2 \|f\|_H^2 \leq k_1 \|r\|_H^2 \|h\|_{H^2}^2.$$

Using a similar argument for the other terms of the polynomial  $p$  we get that there is a constant  $k_2$  so that

$$\int_{\Omega} p(r, s, f, g) \leq k_2 \|h\|_{H^2}^2.$$

Also

$$\|\nabla h\|_{K^4}^2 \leq \|h\|_{H^2}^2.$$

Now if we put calculations 1) and 2) together we get that there is a constant  $k_3$  so that

$$\frac{|\phi(u+h) - \phi(u) - \langle \nabla h, \nabla u \rangle_{K^4} - \kappa^2 \langle h, \kappa^2(|u|^2 - 1)u \rangle_{K^2}|}{\|h\|_{H^2}} \leq k_3 \|h\|_{H^2}$$

and thus we have that equation (27) holds. Now I show  $\phi'(u) \in H^*$

$$\begin{aligned} \phi'(u)h &= \langle \nabla h, \nabla u \rangle_{K^4} + \kappa^2 \langle h, (|u|^2 - 1)u \rangle_{K^2} \leq \\ &\|\nabla u\|_{K^4} \|\nabla h\|_{K^4} + \kappa^2 \|h\|_{K^2} \|( |u|^2 - 1)u \|_{K^2} \leq \\ &c' \|h\|_H, \end{aligned}$$

where  $c'$  is the max of  $\|\nabla u\|_{K^4}^2$  and  $\kappa^2 \|( |u|^2 - 1)u \|_{K^2}^2$ . □

I will also use the following result regarding coercivity of  $\phi$

Lemma 5.3.  $\phi$  as defined in (26) is coercive (i.e. If  $m \in \mathbb{R}$ , there exists a positive integer  $N$ , so that  $\|u\|_H \geq N$  implies that  $\phi(u) \geq m$ .)

Proof. Note that by using Hölder's inequality we have that

$$\left( \int_{\Omega} (|u|^2 - 1)^2 \right) \leq \text{vol}(\Omega) \int_{\Omega} (|u|^2 - 1)^2$$

Thus

$$\|u\|_{H^2}^2 \leq (\text{vol}(\Omega) \int_{\Omega} (|u|^2 - 1)^2)^{1/2} + \text{vol}(\Omega) + \int_{\Omega} |\nabla u|^2.$$

From this the desired result follows. □

Since  $\phi'(u)$  is a continuous linear functional from a Hilbert space to the real numbers, there exists a unique member of  $H^2$ , denoted by  $\nabla_H \phi(u)$  so that

$$\phi'(u)h = \langle h, \nabla_H \phi(u) \rangle_{H^2}$$

for all  $h = \begin{pmatrix} f \\ g \end{pmatrix} \in H^2$ . From lemma 5.2 we know that

$$\phi'(u)h = \langle Dh, \nabla F(Du) \rangle_{K^6}.$$

Note that  $\{Df : f \in H^{1,2}(\Omega)\}$  is closed in  $[L^2(\Omega)]^3$ . Hence there is an orthogonal projection  $P$  from  $[L^2(\Omega)]^3$  onto  $\{Df : f \in H^{1,2}(\Omega)\}$  so that  $P(Df) = Df$  for all  $f \in H^{1,2}(\Omega)$ . This implies that there is an orthogonal projection from  $[L^2(\Omega)]^6$  onto  $\{Dh = \begin{pmatrix} Df \\ Dg \end{pmatrix} : h = \begin{pmatrix} f \\ g \end{pmatrix} \in H^2\}$ . I denote this projection by  $P$  also and note that for

$$\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in [L^2(\Omega)]^3 \times [L^2(\Omega)]^3 = [L^2(\Omega)]^6,$$

we have

$$P \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} P\vec{x} \\ P\vec{y} \end{pmatrix}.$$

Now if  $u \in H^2$ , for any  $h \in H^2$  we have

$$\begin{aligned} \phi'(u)h &= \langle Dh, \nabla F(Du) \rangle_{K^6} = \langle P(Dh), \nabla F(Du) \rangle_{K^6} = \\ &= \langle Dh, P(\nabla F(Du)) \rangle_{K^6} = \langle h, \Pi P(\nabla F(Du)) \rangle_{H^2} \end{aligned}$$

where for

$$\begin{aligned} \nabla F(Du) &= \begin{pmatrix} Y_1(Du) \\ Y_2(Du) \end{pmatrix} \\ P\nabla F(Du) &= \begin{pmatrix} PY_1(Du) \\ PY_2(Du) \end{pmatrix} \end{aligned}$$

and

$$\Pi P(\nabla F(Du)) = \begin{pmatrix} \Pi PY_1(Du) \\ \Pi PY_2(Du) \end{pmatrix}.$$

$\Pi PY1(Du)$  is the first term of  $PY1(Du)$ .  $\Pi PY2(Du)$  is defined likewise. Thus using the definition of the Sobolev gradient, we have that

$$\nabla_H \phi(u) = \Pi P(\nabla F(Du)).$$

Von Neumann's formula tells us  $P$  as a transformation from  $L^2(\Omega)^3$  onto  $\{Df : f \in H\}$  is given by

$$\begin{pmatrix} (I + W^t W)^{-1} & W^t (I + W W^t)^{-1} \\ W (I + W^t W)^{-1} & I - (I + W W^t)^{-1} \end{pmatrix}$$

where for  $f \in H$ ,

$$Wf = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

and  $W^t$  is the transformation whose domain is all  $\begin{pmatrix} f \\ g \end{pmatrix} \in K^2$  so that there exists  $z \in K$  with

$$\langle Wg, \begin{pmatrix} f \\ g \end{pmatrix} \rangle_{K^2} = \langle g, z \rangle_K \text{ for all } g \in H,$$

in which case  $W^t \begin{pmatrix} f \\ g \end{pmatrix} = z$ . Using von Neumann's formula, our gradient becomes

$$\nabla_H \phi(u) = \begin{pmatrix} (I + W^t W)^{-1} \kappa^2 (r^2 + s^2 - 1)r + W^t (I + W W^t)^{-1} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \\ (I + W^t W)^{-1} \kappa^2 (r^2 + s^2 - 1)s + W^t (I + W W^t)^{-1} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \end{pmatrix}.$$

Now using Proposition 4.3,

$$\begin{aligned} & (I + W^t W)^{-1} \kappa^2 (r^2 + s^2 - 1)r + W^t (I + W W^t)^{-1} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \\ & (I + W^t W)^{-1} \kappa^2 (r^2 + s^2 - 1)r + W^t (I + W W^t)^{-1} W r = \\ & (I + W^t W)^{-1} (\kappa^2 (r^2 + s^2 - 1)r - r) + (I + W^t W)^{-1} r + W^t W (I + W^t W)^{-1} r = \\ & (I + W^t W)^{-1} (\kappa^2 (r^2 + s^2 - 1)r - r) + r. \end{aligned}$$

Similarly,

$$\begin{aligned} & (I + W^t W)^{-1} \kappa^2 (r^2 + s^2 - 1)s + W^t (I + W W^t)^{-1} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \\ & (I + W^t W)^{-1} (\kappa^2 (r^2 + s^2 - 1)s - s) + s. \end{aligned}$$

Let

$$\alpha = \kappa^2(r^2 + s^2 - 1)r - r$$

and

$$\beta = \kappa^2(r^2 + s^2 - 1)s - s$$

then

$$(28) \quad \nabla_H \phi(u) = \begin{pmatrix} (I + W^t W)^{-1} \alpha + r \\ (I + W^t W)^{-1} \beta + s \end{pmatrix}.$$

In order to define the descent parameter as given in theorem 3.2, I need to show that the Sobolev gradient is locally Lipschitzian. To do this I show that  $\phi$  as defined in equation (26) is continuously twice differentiable (i.e.  $C^2$ ) by giving a different formulation for  $\phi$ .

Let  $G$  be a function from  $\mathbb{R}^6$  to  $\mathbb{R}^5$  so that for  $u \in H^2$ ,

$$G(D(u)) = \begin{pmatrix} \frac{\kappa}{\sqrt{2}}(r^2 + s^2 - 1) \\ r_1 \\ r_2 \\ s_1 \\ s_2 \end{pmatrix}.$$

Then define  $J : H^2 \rightarrow K^5$  so that  $J(u) = G(D(u))$ . Note that for  $\phi$  as defined in equation (26)

$$\phi(u) = \|J(u)\|_{K^5}^2 / 2.$$

So to show that  $\phi$  is  $C^2$ , it suffices to show that  $J$  is  $C^2$ .

Proposition 5.4. *J as defined above is  $C^2$ .*

Proof. Suppose  $h = \begin{pmatrix} f \\ g \end{pmatrix} \in H^2$ . I claim that

$$J'(u)h = \begin{pmatrix} \sqrt{2}\kappa(rf + sg) \\ f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix}.$$

To minimize notation I refer to the above transformation as  $J'(u)$  even though I have yet to establish this. I use the result of the following calculations.

(i) For  $u = \begin{pmatrix} r \\ s \end{pmatrix}$  and  $h = \begin{pmatrix} f \\ g \end{pmatrix}$  in  $H^2$ ,

$$\|J(u+h) - J(u) - J'(u)h\|_{K^5}^2 = \frac{\kappa^2}{2} \|f^2 + g^2\|_K^2.$$

(ii) Using theorem 2.10, there is a constant  $k_1$  so that if  $h = \begin{pmatrix} f \\ g \end{pmatrix} \in H^2$

$$\|f^2 + g^2\|_K^2 \leq 2(\|f^2\|_K^2 + \|g^2\|_K^2) = 2(\|f\|_{L^4}^4 + \|g\|_{L^4}^4) \leq k_1(\|f\|_H^4 + \|g\|_H^4) \leq 2k_1\|h\|_{H^2}^4.$$

Thus using calculations 1) and 2) we get that

$$\frac{\|J(u+h) - J(u) - J'(u)h\|_{K^5}}{\|h\|_{H^2}} \leq 2k_1\|h\|_{H^2}.$$

Thus  $J$  is differentiable. Now I take a second derivative of  $J$ . Let  $u = \begin{pmatrix} r \\ s \end{pmatrix}$ ,  $h = \begin{pmatrix} f \\ g \end{pmatrix}$ , and  $j = \begin{pmatrix} p \\ q \end{pmatrix}$  then I claim that

$$J''(u)(h, j) = \begin{pmatrix} \sqrt{2}\kappa(fp + qg) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$



Again in our calculations I call the transformation above  $J''(u, A)$  even though I need to establish this. Observe that

$$\|J'(u+j)h - J'(u)h - J''(u)(h, j)\|_{K^5}^2 = 0.$$

Thus  $J$  is twice differentiable and  $J''$  is continuous as it is constant. □

By rearranging equation (28), we have that

$$(29) \quad u = \nabla_H \phi(u) - \begin{pmatrix} (I + W^t W)^{-1} \alpha \\ (I + W^t W)^{-1} \beta. \end{pmatrix}$$

For  $u_0$  any member of  $H^2$ , let the descent parameter  $z : [0, \infty) \rightarrow H^2$  be given by

$$z = \begin{pmatrix} r \\ s \end{pmatrix}.$$

Let

$$\alpha(t) = \kappa^2(r(t)^2 + s(t)^2 - 1)r(t) - r(t) \text{ and } \beta(t) = \kappa^2(r(t)^2 + s(t)^2 - 1)s(t) - s(t).$$

Then we have that

$$(30) \quad z(t) = \nabla_H \phi(z(t)) - \begin{pmatrix} (I + W^t W)^{-1} \alpha(t) \\ (I + W^t W)^{-1} \beta(t) \end{pmatrix}$$

for  $t \geq 0$  where  $z(0) = u_0$ . The rest of this chapter will focus on the following two results. I will conclude this chapter by putting the results together to obtain a critical point of functional (26) as a limit of the descent parameter. I will show

- The transformation  $(I + W^t W)^{-1}$  is the inverse of  $I - \Delta$  with Neumann boundary conditions.
- The functions  $\alpha, \beta : [0, \infty) \rightarrow K$  are both bounded.

Lemma 5.5.  *$\alpha$  and  $\beta$  as defined above are bounded.*

Proof. I just show  $\alpha$  is bounded as the same argument shows that  $\beta$  is bounded. From lemma 5.3, we know that  $\phi$  is coercive. This implies that  $z$  the descent parameter stays bounded as if not then  $\phi(z)$  must be unbounded. However, from equation (15) we know that  $\phi(z)$  is a decreasing nonnegative function which means that it must be bounded. Thus the range of  $z$  is bounded as a subset of  $H^2$ . For each  $t$ ,  $r(t)$  and  $s(t)$  are members of  $H$  thus using inequalities (11) and (12) and theorem 2.10, there is a constant  $k$  so that

$$\begin{aligned} \int_{\Omega} (r(t)^2 + s(t)^2 - 1)r(t)^2 &\leq 2 \int_{\Omega} [(r(t))^3 + (s(t))^2(r(t))]^2 + (r(t))^2 \leq \\ &4 \int_{\Omega} (r(t))^6 + (s(t))^4(r(t))^2 + 2 \int_{\Omega} (r(t))^2 \leq \\ &k(\|r(t)\|_H^6 + \|s(t)\|_H^4\|r(t)\|_H^2 + \|r(t)\|_H^2). \end{aligned}$$

Since  $z = \begin{pmatrix} r \\ s \end{pmatrix}$  is bounded in  $H^2$ , we have that  $r$  and  $s$  are bounded in  $H$ , thus  $(r^2 + s^2 - 1)r$  is bounded in  $H$ . From this it follows that  $\alpha$  is bounded.  $\square$

I want to use the result of proposition 4.14 and theorem 4.9 to show that  $M$  from  $K$  to  $H$  is compact. I show that the transformation  $I + W^tW$  is  $I - \Delta$  with Neumann boundary conditions on domains with Lipschitz continuous boundary. It is known that  $(I - \Delta)^{-1}$  with Neumann boundary conditions on domains with Lipschitz continuous boundary is compact as an operator on  $K$  to  $K$ . Using theorem 4.9, we get that  $M = (I + W^tW)^{-1}$  is compact from  $K$  to  $H$  iff it is compact from  $K$  to  $K$ .

Let  $\Omega$  be a region in  $\mathbb{R}^n$  with a bounded and Lipschitz continuous boundary  $\gamma$ . For all  $f \in H$ ,  $\vec{v} \in H^2$ , using Green's formula, we have

$$(31) \quad \langle \vec{v}, \nabla f \rangle_{K^2} + \langle \text{div} \vec{v}, f \rangle_K = \int_{\gamma} f \vec{v} \cdot n.$$

If  $f$  is in the domain of  $W^tW$ , for  $g \in H$ ,

$$\begin{aligned} \langle W^tWf, g \rangle_K &= \langle Wf, Wg \rangle_{K^2} = \\ &\langle \nabla f, \nabla g \rangle_{K^2} = \end{aligned}$$

$$\int_{\gamma} g \nabla f \cdot n - \langle \operatorname{div} \nabla f, g \rangle_K$$

Since

$$\langle W^t W f + \Delta f, g \rangle_K + \int_{\gamma} g \nabla f \cdot n = 0$$

for all  $g$ , we get that  $W^t W f = \Delta f$  and  $\nabla f \cdot n = 0$ . Thus if  $f$  is in the domain of  $W^t W$  then  $f$  is in the domain of the Neumann Laplacian and  $W^t W f = \Delta f$ . Similarly if  $f$  is in the domain of the Neumann Laplacian, then working backwards in the above sequence of equalities, we see that  $W^t W f = \Delta f$ .

I now give the statement and proof of the main result for this chapter.

*Theorem 5.6. Let  $\phi$  be as in equation (26). Since  $\phi$  is  $C^2$ , using theorem 3.2, given  $u_0 \in H^2$ , there exists a function  $z : [0, \infty) \rightarrow H^2$  so that  $z(0) = u_0$  and  $z'(t) = -\nabla_H \phi(z(t))$  for all  $t \in [0, \infty)$ . There exists an unbounded sequence of numbers  $\{t_n\}_{n \geq 1}$  so that the sequence  $\{z(t_n)\}_{n \geq 1}$  converges in  $H^2$  to  $u \in H^2$  and  $\nabla_H \phi(u) = 0$ .*

*Proof.* By theorem 3.3, there exists  $t_1, t_2, \dots$ , an unbounded sequence of real numbers, so that  $\|\nabla_H \phi(z(t_n))\|_{H^2} \rightarrow 0$ . Using lemma 5.5, the sequences

$$\alpha(t_n) = \{\kappa^2(r(t_n)^2 + s(t_n)^2 - 1)r(t_n) - r(t_n)\}_{n \geq 1}$$

$$\beta(t_n) = \{\kappa^2(r(t_n)^2 + s(t_n)^2 - 1)s(t_n) - s(t_n)\}_{n \geq 1}$$

each stay bounded in  $K$ . The transformation  $(I + W^t W)^{-1} = M$  (see Proposition 4.14) is compact as a transformation from  $K$  to  $H$  using theorem 4.9, thus we can find a subsequence of  $\{t_n\}_{n \geq 1}$ , which I denote by  $\{t_n\}_{n \geq 1}$  also, so that the sequence

$$\left\{ \begin{pmatrix} (I + W^t W)^{-1} \alpha(t_n) \\ (I + W^t W)^{-1} \beta(t_n) \end{pmatrix} \right\}_{n \geq 1}$$

converges in  $H^2$  to  $u \in H^2$ . From equation (30) we know that

$$z(t_n) = \nabla_H \phi(z(t_n)) - \begin{pmatrix} (I + W^t W)^{-1} \alpha(t_n) \\ (I + W^t W)^{-1} \beta(t_n) \end{pmatrix}.$$

Since  $\{\nabla_H(z(t_n))\}_{n \geq 1}$  converges in  $H^2$  to zero, there is  $u \in H$  so that  $z(t_n) \rightarrow u$  in  $H^2$ . By construction  $\nabla_H \phi(u) = 0$ .  $\square$

In [7] we give the following theorem along with a proof which generalizes this result. The ideas I use here follow the developments in [7] very closely.

*Theorem 5.7. Let  $H = L^2(\Omega)$  and  $H' = H^{1,2}(\Omega)$ . Assume that  $G$  is a real valued  $C^1$  function on  $\mathbb{R}$  so that  $G(u), G'(u) \in H$  for all  $u \in H'$ . For  $u \in H'$  let  $F(u) = \int_{\Omega} G(u)$ . Let*

$$(32) \quad \phi(u) = \|\nabla u\|_{H^n}^2/2 + F(u)$$

*Suppose also that  $\phi$  is coercive,  $F$  is a differentiable function on  $H'$ ,  $F'(u)h = \int_{\Omega} G'(u)h$  for all  $h \in H'$ , and that if  $S \subset H'$  is bounded, then  $G'(S) \subset H$  is bounded. Then there exists  $u \in H'$  so that  $\nabla_{H'} \phi(u) = 0$  and  $u$  is an  $\omega$ -limit point of  $z$ , where  $z$  is as in theorem 3.2.*

## CHAPTER 6

### THE GL ENERGY FUNCTIONAL WITH MAGNETIC FIELD ON A DOMAIN IN THE PLANE

Let  $\Omega$  be a bounded open set in the plane satisfying the cone condition. As in the previous chapter, I treat  $H^{1,2}(\Omega, \mathbb{C})$  as the cartesian product  $H^{1,2}(\Omega) \times H^{1,2}(\Omega)$ . Let  $H^2$  be the cartesian product  $H \times H$  where  $H = H^{1,2}(\Omega)$ .  $H^3$  and  $H^4$  are defined likewise.  $K = L^2(\Omega)$ ,  $L_p = L^p(\Omega)$  for  $p \neq 2$ , and  $K^i$  denotes the cartesian product for  $i = 2, 3, 4$ .

For

$$(u, A) = \begin{pmatrix} r \\ s \\ a \\ b \end{pmatrix}$$

in  $H^4$  define

$$D(u, A) = \begin{pmatrix} \vec{r} \\ \vec{s} \\ \vec{a} \\ \vec{b} \end{pmatrix}.$$

As in the previous chapters, for  $x \in H$ ,  $\vec{x}$  denotes

$$\begin{pmatrix} x \\ x_1 \\ x_2 \end{pmatrix}$$

where  $x_i$  denotes the first partial derivative of  $x$  with respect to the  $i^{th}$  independent variable.

For  $(u, A) \in H^4$  define

$$(33) \quad \phi(u, A) = \int_{\Omega} F(D(u, A))$$

where  $F : \mathbb{R}^{12} \rightarrow \mathbb{R}$  is constructed so that  $F(D(u, A))$  is given by

$$(34) \quad \frac{1}{2}(|r_1 + as|^2 + |s_1 - ar|^2 + |r_2 + bs|^2 + |s_2 - br|^2 + |\nabla \times A - H_0|^2) + \frac{\kappa^2}{4}(|u|^2 - 1)^2.$$

$\kappa$  is the material constant discussed in Chapter 1 and  $H_0$ , the applied magnetic field, is treated as a constant also.  $\nabla \times A$  is the third component of  $curl(A)$  (i.e. if  $A = \begin{pmatrix} a \\ b \end{pmatrix}$  then  $\nabla \times A = b_1 - a_2$ ).

Using theorem 2.10,  $H^4$  is embedded in  $[L^p(\Omega)]^4$  for all finite  $p \geq 1$ . I now show that  $\phi(H^4) \subseteq [0, \infty)$ .

Lemma 6.1.  $\phi$  as defined in equation (33) is finite valued.

Proof. We have that  $\phi(u, A) =$

$$\frac{1}{2} \int_{\Omega} |r_1 + as|^2 + |s_1 - ar|^2 + |r_2 + bs|^2 + |s_2 - br|^2 + |b_1 - a_2 - H_0|^2 + \int_{\Omega} \frac{\kappa^2(r^2 + s^2 - 1)^2}{4}.$$

By using equation (11), for any real numbers  $x, y$ ,  $(x + y)^2 \leq 2(x^2 + y^2)$ . Using this and Hölder's inequality, we get that

$$\begin{aligned} \int_{\Omega} |f_i + gh|^2 &\leq 2 \int_{\Omega} (|f_i|^2 + |gh|^2) \leq \\ 2 \left( \int_{\Omega} |f_i|^2 + \left( \int_{\Omega} |g|^4 \right)^{1/2} \left( \int_{\Omega} |h|^4 \right)^{1/2} \right) &\leq 2(\|f\|_H^2 + c^2 \|g\|_H^2 \|h\|_H^2) < \infty \end{aligned}$$

for  $i = 1, 2$  and  $f, g, h \in H$ . Here  $c$  is the embedding constant for the pair  $L^4(\Omega), H$ . I showed that

$$\int_{\Omega} \frac{\kappa^2(r^2 + s^2 - 1)^2}{4}$$

is finite for  $r, s \in H$  in lemma 5.1. Also

$$\int_{\Omega} |b_1 - a_2 - H_0|^2 \leq 2 \int_{\Omega} b_1^2 + (a_2 + H_0)^2 < \infty$$

since  $a, b \in H$ . It should now be clear that  $\phi$  as defined in equation (33) is well defined and finite valued on  $H^4$ . □

I now show that  $\phi$  is Fréchet differentiable on  $H^4$  using the following two lemmas. The expression I obtain for the derivative will be used in obtaining an expression for the Sobolev gradient.

Lemma 6.2. *Let  $\phi$  be the functional in equation (33), then*

$$(35) \quad \frac{\phi((u, A) + (v, B)) - \phi(u, A) - \int_{\Omega} D(v, B) \cdot \nabla F(D(u, A))}{\|(v, B)\|_{H^4}}$$

*converges to zero as  $\|(v, B)\|_{H^4}$  converges to zeros.*

If  $F : \mathbb{R}^{12} \rightarrow \mathbb{R}$  is chosen so that  $F(D(u, A))$  satisfies (34), then by  $\nabla F(D(u, A))$  I denote the composition of the 12 partial derivatives of  $F$  and  $D(u, A)$  as defined above in the order given below.

$$\nabla F(D(u, A)) = \begin{pmatrix} Y1(D(u, A)) \\ Y2(D(u, A)) \\ Y3(D(u, A)) \\ Y4(D(u, A)) \end{pmatrix}$$

where

$$Y1(D(u, A)) = \begin{pmatrix} \kappa^2(r^2 + s^2 - 1)r - a(s_1 - ar) - b(s_2 - br) \\ r_1 + as \\ r_2 + bs \end{pmatrix},$$

$$Y2(D(u, A)) = \begin{pmatrix} \kappa^2(r^2 + s^2 - 1)s + a(r_1 + as) + b(r_2 + bs) \\ s_1 - ar \\ s_2 - br \end{pmatrix},$$

$$Y3(D(u, A)) = \begin{pmatrix} s(r_1 + as) - r(s_1 - ar) \\ 0 \\ a_2 - b_1 + H_0 \end{pmatrix},$$

and

$$Y4(D(u, A)) = \begin{pmatrix} s(r_2 + bs) - r(s_2 - br) \\ b_1 - a_2 - H_0 \\ 0 \end{pmatrix}.$$

For  $(v, B) = (f, g, p, q)$  in  $H^4$ , note that  $D(v, B) \cdot \nabla F(D(u, A))$  is

$$\vec{f} \cdot Y1(D(u, A)) + \vec{g} \cdot Y2(D(u, A)) + \vec{p} \cdot Y3(D(u, A)) + \vec{q} \cdot Y4(D(u, A)).$$

In order to minimize notation, I refer to the transformation

$$(v, B) \rightarrow \int_{\Omega} D(v, B) \cdot \nabla F(D(u, A))$$

as  $\phi'(u, A)$  in the following two lemmas.

Proof. I prove the lemma by showing the following

(i) For  $u = \begin{pmatrix} r \\ s \end{pmatrix} \in H^2$ ,

$$\frac{1}{\delta} \left| \int_{\Omega} (|u + \vec{h}|^2 - 1)^2 - (|u|^2 - 1)^2 - 4(|u|^2 - 1)u \cdot \vec{h} \right|$$

converges to zero as  $\|\vec{h}\|_{H^2} \rightarrow 0$ . Here  $\vec{h} = \begin{pmatrix} f \\ g \end{pmatrix} \in H^2$  and  $\delta = \|\vec{h}\|_{H^2}$ .

(ii) For  $r, s, a, b \in H$ ,

$$\frac{|\int_{\Omega} |(r + f)_1 + (a + p)(s + g)|^2 - |r_1 + as|^2 - 2(f_1 + ps + ag)(r_1 + as)|}{\delta}$$



converges to zero as  $\|\vec{h}\|_{H^4} \rightarrow 0$ . Here  $\vec{h} = (f, g, p, q) \in H^4$  and  $\delta = \|\vec{h}\|_{H^4}$ . A similar argument shows that

$$\frac{|\int_{\Omega} |(s+g)_1 - (a+p)(r+f)|^2 - |s_1 - ar|^2 - 2(g_1 - (af+pr))(s_1 - ar)|}{\delta}$$

converges to zero as  $\delta = \|\vec{h}\|_{H^4} \rightarrow 0$ ,

$$\frac{|\int_{\Omega} |(r+f)_2 + (b+q)(s+g)|^2 - |r_2 + bs|^2 - 2(f_2 + qs + bg)(r_2 + bs)|}{\delta}$$

converges to zero as  $\delta = \|\vec{h}\|_{H^4} \rightarrow 0$ , and

$$\frac{|\int_{\Omega} |(s+g)_2 - (b+q)(r+f)|^2 - |s_2 - br|^2 - 2(g_2 - (bf+qr))(s_2 - br)|}{\delta}$$

converges to zero as  $\delta = \|\vec{h}\|_{H^4} \rightarrow 0$ .

(iii) For  $p, q \in H$ ,

$$\frac{|\int_{\Omega} |(b+q)_1 - (a+p)_2 - H_0|^2 - |b_1 - a_2 - H_0|^2 - 2(b_1 - a_2 - H_0)(q_1 - p_2)|}{\delta}$$

converges to zero as  $\delta = \|\vec{h}\|_{H^2} \rightarrow 0$ . Here  $\vec{h} = \begin{pmatrix} p \\ q \end{pmatrix}$ .

If the above statements are true, then for  $(u, A) = (r, s, a, b)$  and  $(v, B) = (f, g, p, q)$ ,

$$\begin{aligned} \phi'(u, A)(v, B) &= \int_{\Omega} (f_1 + ps + ag)(r_1 + as) + (g_1 - (af + pr))(s_1 - ar) + \\ &\quad (f_2 + qs + bg)(r_2 + bs) + (g_2 - (bf + qr))(s_2 - br) \\ &\quad + (b_1 - a_2 - H_0)(q_1 - p_2) + \kappa^2(|u|^2 - 1)(rf + sg). \end{aligned}$$

By rearranging this expression we see that if 1), 2), and 3) are satisfied then

$$\phi'(u, A)(v, B) = \int_{\Omega} D(v, B) \cdot \nabla F(D(u, A)).$$

I proved 1) in lemma 5.2.

To show 2), let  $r, s, a, b \in H$ , then for  $\vec{h} = (f, g, p, q) \in H^4$  we have that

$$\frac{1}{\delta} \int_{\Omega} |(r+f)_1 + (s+g)(a+p)|^2 - |r_1 + as|^2 - 2(f_1 + ps + ag)(r_1 + as) =$$

$$\frac{1}{\delta} \int_{\Omega} O((u, A), (v, B)).$$

where

$$\begin{aligned} O((u, A), (v, B)) = & f_1^2 + 2(r_1(pg + ps) + f_1(ag + pg + ps)) + \\ & p^2(g + s)^2 + (ag)^2 + 2pg^2a + 4pgas \end{aligned}$$

Using Hölder's inequality and the Sobolev embedding theorem observe that for any  $x, y, z, w \in H$ ,

$$\int_{\Omega} |x_1 y z| \leq \left( \int_{\Omega} x_1^2 \right)^{1/2} \left( \int_{\Omega} y^4 \right)^{1/4} \left( \int_{\Omega} z^4 \right)^{1/4}$$

also

$$\int_{\Omega} |x y z w| \leq \left( \int_{\Omega} x^4 \right)^{1/4} \left( \int_{\Omega} y^4 \right)^{1/4} \left( \int_{\Omega} z^4 \right)^{1/4} \left( \int_{\Omega} w^4 \right)^{1/4}.$$

From theorem 2.10 we get that there is a constant  $k$  so that

$$\int_{\Omega} O((u, A)(v, B)) \leq k\delta^2$$

for  $\delta \leq 1$ . Statement 2) follows from this. To show 3) note that the computation

$$|(b + q)_1 - (a + p)_2 - H_0|^2 - |b_1 - a_2 - H_0|^2 - 2(b_1 - a_2 - H_0)(q_1 - p_2)|$$

simplifies to

$$(p_2 - q_1)^2.$$

Thus

$$\begin{aligned} \left| \int_{\Omega} |(b + q)_1 - (a + p)_2 - H_0|^2 - |b_1 - a_2 - H_0|^2 - 2(b_1 - a_2 - H_0)(q_1 - p_2) \right| = \\ \int_{\Omega} (p_2 - q_1)^2 \leq 2\|h\|^2 = 2\delta^2. \end{aligned}$$

□

Lemma 6.3. For  $(u, A) \in H^4$ , there exists a constant  $k$  so that

$$\phi'(u, A)(v, B) \leq k\|(v, B)\|_{H^4}$$

for all  $(v, B) \in H^4$ .

Proof. Using equation (35), we have that

$$\phi'(u, A)(v, B) = \int_{\Omega} \vec{f} \cdot Y1(D(u, A)) + \vec{g} \cdot Y2(D(u, A)) + \vec{p} \cdot Y3(D(u, A)) + \vec{q} \cdot Y4(D(u, A))$$

The following computation uses the Sobolev embedding theorem and Hölder's inequality to show that there is a constant  $k_1$  so that

$$(36) \quad \int_{\Omega} \vec{f} \cdot Y1(D(u, A)) \leq k_1 \|f\|_H.$$

$$\begin{aligned} \int_{\Omega} \vec{f} \cdot Y1(D(u, A)) &= \\ \int_{\Omega} (\kappa^2(r^2 + s^2 - 1)r - a(s_1 - ar) - b(s_2 - br))f + \\ &\quad \int_{\Omega} (r_1 + as)f_1 + (r_2 + bs)f_2. \end{aligned}$$

Using inequality (12), we have that  $(r^2 + s^2 - 1)r \in K$ . Thus using the Cauchy-Schwarz inequality we have

$$\int_{\Omega} |(r^2 + s^2 - 1)rf| \leq \|(r^2 + s^2 - 1)r\|_K \|f\|_K.$$

Also, using inequality (12) we know that  $af \in K$  and  $(s_1 - ar) \in K$  thus by using the Cauchy-Schwarz inequality and inequality (12), we have that

$$\int_{\Omega} |a(s_1 - ar)f| \leq \|af\|_K \|s_1 - ar\|_K \leq \|a\|_{L^4} \|s_1 - ar\|_K \|f\|_{L^4}$$

Finally we have that

$$\int_{\Omega} |(r_1 + as)f_1| \leq \|r_1 + as\|_K \|f_1\|_K.$$

Using theorem 2.10 we can find a constant  $c_1$  so that  $\|f\|_K, \|f\|_{L^4} \leq c_1\|f\|_H$ . I use this fact and a similar argument for the other terms in equation (36) and get the desired result.

Similarly there are constants  $k_2, k_3,$  and  $k_4$  so that

$$\int_{\Omega} \vec{g} \cdot Y_2(D(u, A)) \leq k_2\|g\|_H,$$

$$\int_{\Omega} \vec{p} \cdot Y_3(D(u, A)) \leq k_3\|p\|_H,$$

and

$$\int_{\Omega} \vec{q} \cdot Y_4(D(u, A)) \leq k_4\|q\|_H.$$

Let  $k$  be the maximum of  $k_1, k_2, k_3,$  and  $k_4$ , then we see that

$$\phi'(u, A)(v, B) \leq 4k\|(v, B)\|_{H^4}.$$

□

The above lemma shows that  $\phi'(u, A) \in (H^4)^*$  so by using the Riesz representation theorem we can define  $\nabla_H\phi(u, A)$  to be the unique member of  $H^4$  so that

$$(37) \quad \phi'(u, A)(v, B) = \langle (v, B), \nabla_H\phi(u, A) \rangle_{H^4}$$

for all  $(v, B) = (f, g, p, q) \in H^4$ .

I wish to obtain an explicit expression for  $\nabla_H\phi(u, A)$  using the projections of Chapter 4. Using equation (35) we know that

$$\phi'(u, A)(v, B) = \langle D(v, B), \nabla F(D(u, A)) \rangle_{(L^2(\Omega))^{12}}.$$

for all  $(v, B) \in H^4$ .

I first show that

Lemma 6.4.  $Y_1(D(u, A)), Y_2(D(u, A)), Y_3(D(u, A)),$  and  $Y_4(D(u, A))$  are subsets of  $S$  where for  $1 < p < 2$ , define  $S = S(p) = L^p(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ . As in chapter 4, denote the members of  $S$  by  $\vec{y}$  where for  $f \in L_p, g, h \in K, \vec{y} = (f, g, h)$ .

Proof. If  $1 < p < 2$  choose  $n$  so that  $pn = 2$  then choose  $m$  so that  $\frac{1}{m} + \frac{1}{n} = 1$ . Note that since  $n = 2/p$  then

$$m = \frac{n}{n-1} = \frac{2}{2-p}.$$

Thus

$$mp = \frac{2p}{2-p} > 1$$

and if  $f, g, c, d \in H$  and  $i = 1, 2$ , using Hölder's inequality, we have

$$\begin{aligned} \int_{\Omega} |f(g_i - cd)|^p &= \int_{\Omega} |f|^p |(g_i - cd)|^p \leq \\ & \left( \int_{\Omega} |f|^{pm} \right)^{1/m} \left( \int_{\Omega} |(g_i - cd)|^{pn} \right)^{1/n} = \\ & \left( \int_{\Omega} |f|^{pm} \right)^{1/m} \left( \int_{\Omega} |(g_i - cd)|^2 \right)^{1/n}. \end{aligned}$$

Since  $H$  is embedded in  $L^q(\Omega)$  for all finite  $q \geq 1$  we get that

$$\int_{\Omega} |f|^{pm}$$

is finite. Also since  $g, c, d \in H$ ,  $g_i - cd \in L^2(\Omega)$  thus

$$\int_{\Omega} |(g_i - cd)|^2$$

is finite. Also for  $f, g \in H$ , using theorem 2.10 and inequality (12), we have that

$$\int_{\Omega} ((f^2 + g^2 - 1)f)^2$$

is finite hence  $(f^2 + g^2 - 1)f$  is in  $L^p(\Omega)$  for  $1 < p < 2$ . This shows that each first term of  $Yi(D(u, A))$ , is in  $L^p(\Omega)$  for  $i=1, 2, 3$ , or  $4$ .

If  $f, g, h \in H$ , then

$$\int_{\Omega} (f_i + gh)^2 \leq 2(\|f_i\|_{L^2}^2 + \|gh\|_{L^2}^2) < \infty.$$

Thus  $f_i + gh \in L^2(\Omega)$  and each second and third term of  $Yi(D(u, A))$ , is in  $L^2(\Omega)$  for  $i=1, 2, 3$ , or  $4$ . Hence  $Yi(D(u, A)) \in S$  for  $i= 1, 2, 3$ , or  $4$ .  $\square$

In the lemma 4.6, I showed that for  $1 < p < 2$  there exists a bounded operator  $P$  with domain  $S$  and range  $\left\{ \begin{pmatrix} x \\ \nabla x \end{pmatrix} : x \in H \right\}$  so that for all  $x \in H^{1,2}(\Omega)$  and  $\vec{y} \in S$ ,

$$(38) \quad \langle P\vec{x}, \vec{y} \rangle_{K^3} = \langle \vec{x}, P\vec{y} \rangle_{K^3}$$

and  $P \begin{pmatrix} x \\ \nabla x \end{pmatrix} = \begin{pmatrix} x \\ \nabla x \end{pmatrix}$  for all  $x \in H$ .

For

$$(\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4) \in S^4,$$

the quadruple cartesian product of  $S$ , let

$$P(\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4) = (Py_1, Py_2, Py_3, Py_4).$$

Thus we have that

$$\begin{aligned} \phi'(u, A)(v, B) &= \\ \langle D(v, B), \nabla F(D(u, A)) \rangle_{L^2(\Omega)^{12}} &= \\ \langle P(D(v, B)), \nabla F(D(u, A)) \rangle_{L^2(\Omega)^{12}} &= \\ \langle D(v, B), P\nabla F(D(u, A)) \rangle_{L^2(\Omega)^{12}} &= \\ \langle (v, B), \Pi P(\nabla F(D(u, A))) \rangle_{H^4} \end{aligned}$$

where if

$$P\nabla F(D(u, A)) = \left( \begin{pmatrix} y_1 \\ \nabla y_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ \nabla y_2 \end{pmatrix}, \begin{pmatrix} y_3 \\ \nabla y_3 \end{pmatrix}, \begin{pmatrix} y_4 \\ \nabla y_4 \end{pmatrix} \right),$$

then

$$\Pi P(\nabla F(D(u, A))) = (y_1, y_2, y_3, y_4) \in H^4.$$

So using equation (37), we get that

$$\nabla_H \phi(u, A) = \Pi P(\nabla F(D(u, A))).$$

From theorem 4.12, we also know that  $P$ , the mapping of  $L^p(\Omega) \times L^2(\Omega) \times L^2(\Omega)$  onto  $\left\{ \begin{pmatrix} x \\ \nabla x \end{pmatrix} : x \in H \right\}$  is given by

$$\begin{pmatrix} M_p & W^t(I + WW^t)^{-1} \\ WM_p & I - (I + WW^t)^{-1} \end{pmatrix}$$

where for  $f \in H$ ,  $W(f) = \nabla f$ . Recall that if for  $\begin{pmatrix} f \\ g \end{pmatrix} \in [L^2(\Omega)]^2$  there is  $z \in L^2(\Omega)$  so that

$$\langle Wh, \begin{pmatrix} f \\ g \end{pmatrix} \rangle_{[L^2(\Omega)]^2} = \langle h, z \rangle_{L^2(\Omega)}$$

for all  $h \in H$ . Then I define  $W^t \begin{pmatrix} f \\ g \end{pmatrix} = z$ . See equation (24) for a definition of  $M_p$ .

Using the above formula we get that

$$(39) \quad \nabla_H \phi(u, A) = (y_1, y_2, y_3, y_4)$$

where

$$y_1 = y_1(D(u, A)) = M_p(\alpha_1) + W^t(I + WW^t)^{-1} \begin{pmatrix} r_1 + as \\ r_2 + bs \end{pmatrix},$$

$$y_2 = y_2(D(u, A)) = M_p(\alpha_2) + W^t(I + WW^t)^{-1} \begin{pmatrix} s_1 - ar \\ s_2 - br \end{pmatrix},$$

$$y_3 = y_3(D(u, A)) = M_p(\alpha_3) + W^t(I + WW^t)^{-1} \begin{pmatrix} 0 \\ -(b_1 - a_2 - H_0) \end{pmatrix},$$

and

$$y_4 = y_4(D(u, A)) = M_p(\alpha_4) + W^t(I + WW^t)^{-1} \begin{pmatrix} b_1 - a_2 - H_0 \\ 0 \end{pmatrix}.$$

Here

$$\alpha_1 = \alpha_1(D(u, A)) = \kappa^2(r^2 + s^2 - 1)r - a(s_1 - ar) - b(s_2 - br),$$

$$\alpha_2 = \alpha_2(D(u, A)) = \kappa^2(r^2 + s^2 - 1)s + a(r_1 + as) + b(r_2 + bs),$$

$$\alpha_3 = \alpha_3(D(u, A)) = s(r_1 + as) - r(s_1 - ar),$$

and

$$\alpha_4 = \alpha_4(D(u, A)) = s(r_2 + bs) - r(s_2 - br).$$

In proposition 4.3, I showed that if  $T$  is a closed and densely defined operator on a Hilbert space  $H$  to a Hilbert space  $K$  then for  $x$  in the domain of  $T$ ,

$$(I + TT^t)^{-1}Tx = T(I + T^tT)^{-1}x.$$

Applying this result to the transformation  $W : H \subset K \rightarrow K^2$  we get that

$$W^t(I + WW^t)^{-1}Wx = W^tW(I + W^tW)^{-1}x.$$

for  $x \in H$ . In proposition 4.14, I showed that

$$(I + W^tW)^{-1} = M$$

where  $M$  is defined in Definition 4.7. Let  $\Gamma = W^t(I + WW^t)^{-1}$  then

$$\begin{aligned} y_1 &= M_p(\alpha_1) + W^t(I + WW^t)^{-1}(Wr + \begin{pmatrix} as \\ bs \end{pmatrix}) = \\ &= M_p(\alpha_1) + W^tW(I + W^tW)^{-1}r + \Gamma \begin{pmatrix} as \\ bs \end{pmatrix} = \\ &= M_p(\alpha_1) - Mr + Mr + W^tWMr + \Gamma \begin{pmatrix} as \\ bs \end{pmatrix}. \end{aligned}$$

From equation (24) we know that  $M_p$  and  $M$  agree on  $L^2(\Omega)$  thus  $Mr = M_p r$ . Also

$$Mr + W^tWMr = (I + W^tW)Mr = r,$$

thus

$$(40) \quad y_1 = M_p(\alpha_1 - r) + r + \Gamma \begin{pmatrix} as \\ bs \end{pmatrix}.$$

Similarly we can show that

$$(41) \quad y_2 = M_p(\alpha_2 - s) + s - \Gamma \begin{pmatrix} ar \\ br \end{pmatrix},$$

$$(42) \quad y_3 = M_p(\alpha_3 - a) + a + \Gamma \begin{pmatrix} -a_1 \\ -b_1 + H_0 \end{pmatrix}, \text{ and}$$

$$(43) \quad y_4 = M_p(\alpha_4 - b) + b - \Gamma \begin{pmatrix} a_2 + H_0 \\ b_2 \end{pmatrix}.$$



To define the steepest descent parameter, we need to check that the Sobolev gradient as defined above is locally Lipschitzian. As in Chapter 5, I show that  $\phi$  as defined in (33) is continuously twice differentiable. To do this I give a different formulation of  $\phi$  than the one I used above. This formulation will also allow us to give a different characterization of the Sobolev gradient.

Let  $G$  be a function from  $\mathbb{R}^{12}$  to  $\mathbb{R}^6$  so that

$$(44) \quad G(D(u, A)) = \begin{pmatrix} r_1 + as \\ s_1 - ar \\ r_2 + bs \\ s_2 - br \\ b_1 - a_2 - H_0 \\ \frac{\kappa}{\sqrt{2}}(r^2 + s^2 - 1) \end{pmatrix}.$$

Define  $J : H^4 \rightarrow K^6$  by  $J(u, A) = G(D(u, A))$ . Then we have that

$$(45) \quad \phi(u, A) = \frac{1}{2} \|J(u, A)\|_{K^6}^2.$$

Proposition 6.5.  $\phi$  as defined in (33) is continuously twice differentiable.

Proof. I need to show that  $J$  is continuously twice differentiable. I claim that if

$$(u, A) = \left( \begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right) \text{ and } (v, B) = \left( \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right)$$

then

$$J'(u, A)(v, B) = \begin{pmatrix} f_1 + ag + sp \\ -af + g_1 - rp \\ f_2 + bg + sq \\ g_2 - bf - rq \\ q_1 - p_2 \\ \sqrt{2}\kappa(rf + sq) \end{pmatrix}.$$

In order to minimize notation, in the statement below I refer to the above transformation as  $J'(u, A)$  even though I have not yet established this.

$$\begin{aligned} & \|J(u + v, A + B) - J(u, A) - J'(u, A)(v, B)\|_{K^6}^2 = \\ & \|pg\|_K^2 + \|pf\|_K^2 + \|qg\|_K^2 + \|qf\|_K^2 + \frac{\kappa^2}{2}\|f^2 + g^2\|_K^2. \end{aligned}$$

Now using Hölder's inequality and theorem 2.10 there exists a constant  $k_1$  so that

$$\|pg\|_K^2 \leq \|p\|_{L^4}^2 \|g\|_{L^4}^2 \leq k_1 \|p\|_H^2 \|g\|_H^2 \leq k_1 \|(v, B)\|_{H^4}^4.$$

Similarly each of

$$\|pf\|_K^2, \|qg\|_K^2, \|qf\|_K^2$$

is less than or equal to  $k_1 \|(v, B)\|_{H^4}^4$ . As in the proof of proposition 5.4, there exists a constant  $k_2$  so that

$$\|f^2 + g^2\|_K^2 \leq k_2 \|(v, B)\|_{H^4}^4.$$

Thus there is a constant  $k_3$  so that

$$\|J(u + v, A + B) - J(u, A) - J'(u, A)(v, B)\|_{K^6}^2 \leq k_3 \|(v, B)\|_{H^4}^4$$

hence

$$\frac{\|J(u + v, A + B) - J(u, A) - J'(u, A)(v, B)\|_{K^6}}{\|(v, B)\|_{H^4}} \leq k_3 \|(v, B)\|_{H^4}.$$

This implies that  $J'(u, A)$  as defined above is the Fréchet derivative of  $J$ . Now I show that  $J$  is twice differentiable and that the second derivative of  $J$  is constant. Let  $(v, B) \in H^4$

be as above, then I claim that for  $(w, C) = \left(\begin{pmatrix} h \\ l \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right)$

$$J''(u, A)((v, B), (w, C)) = \begin{pmatrix} pl + gc \\ -fc - ph \\ gd + ql \\ -fd - qd \\ 0 \\ \sqrt{2}\kappa(fh + gl) \end{pmatrix}.$$

Since

$$\|J'(u + w, A + C)(v, B) - J'(u, A)(v, B) - J''(u, A)(v, B)\|_{K^6} = 0,$$

$J''(u, A)$  as defined above is the second derivative of  $J$  which is constant. □

I can use the formulation of the Ginzburg-Landau functional given in equation (45) to give a different characterization of the Sobolev gradient. When we differentiate  $\phi$  as given in equation (45) we get that

$$\phi'(u, A)(v, B) = \langle J'(u, A)(v, B), J(u, A) \rangle_{K^6}.$$

Let  $(J'(u, A))^*$  denote the adjoint of the linear transformation  $J'(u, A)$  from  $H^4$  to  $K^6$ . Since  $J'(u, A)$  is continuous,  $(J'(u, A))^*$  is everywhere defined and thus

$$\phi'(u, A)(v, B) = \langle (v, B), (J'(u, A))^* J(u, A) \rangle_{H^4}$$

for all  $(v, B) \in H^4$ . Thus by definition  $(J'(u, A))^* J(u, A)$  is the Sobolev gradient of  $\phi$  at the point  $(u, A)$ . One might ask how this expression for the Sobolev gradient relates to the one

I got in equations (40), (41), (42), and (43). Note that we can write  $J'(u, A)(v, B)$  as

$$\begin{pmatrix} 0 & a & s & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & 0 & -r & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & s & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b & 0 & 0 & -r & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ k_1 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (v, B) \\ \nabla(v, B) \end{pmatrix}$$

where  $k_1 = \sqrt{2}\kappa r$ ,  $k_2 = \sqrt{2}\kappa s$ , and

$$(v, B) = \begin{pmatrix} f \\ g \\ p \\ q \end{pmatrix}$$

and

$$\nabla(v, B) = \begin{pmatrix} \nabla f \\ \nabla g \\ \nabla p \\ \nabla q \end{pmatrix}.$$

I can rewrite equations (40) to (43) as

$$M_p \left( \begin{pmatrix} 0 & -a & 0 & -b & 0 & \sqrt{2}\kappa r \\ a & 0 & b & 0 & 0 & \sqrt{2}\kappa s \\ s & -r & 0 & 0 & 0 & 0 \\ 0 & 0 & s & -r & 0 & 0 \end{pmatrix} J(u, A) \right) + \Gamma \left( \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} J(u, A) \right)$$

where  $M_p(x, y, z, w) = (M_px, M_py, M_pz, M_pw)$  and  $\Gamma(x_1, \dots, x_8) = (\Gamma(x_1, x_2), \dots, \Gamma(x_7, x_8))$ .

Thus in a certain sense

$$\nabla_H \phi(u, A) = \pi P(J'(u, A)^t J(u, A)).$$

I now give the main result of this work regarding convergence of the steepest descent parameter.

*Theorem 6.6. Let  $\phi$  be the functional given in 33, fix  $1 < p < 2$ , and let  $(u_0, A_0) \in H^4$ . From proposition 6.5 we have that  $\phi$  is continuously twice differentiable from  $H^4$  to  $\mathbb{R}$ . Using theorem 3.2 we can define  $z : [0, \infty)$  to  $H^4$  so that  $z(0) = (u_0, A_0)$  and  $z'(t) = -\nabla_H \phi(z(t))$  for all  $t \geq 0$ . If the range of  $z$  is bounded in  $H^4$  then the following is true:*

*There exists an unbounded sequence of real numbers  $\{t_n\}_{n \geq 1}$  so that  $z(t_n)$  converges to  $(u, A) \in H^4$  in the  $[L^p(\Omega)]^4$  norm and  $(u, A)$  is a critical point of  $\phi$ .*

Proof. Let  $z(t) = (u(t), A(t))$  and

$$\nabla_H \phi(z) = \begin{pmatrix} y_1(D(u, A)) \\ y_2(D(u, A)) \\ y_3(D(u, A)) \\ y_4(D(u, A)) \end{pmatrix}$$

where  $u(t) = \begin{pmatrix} r(t) \\ s(t) \end{pmatrix}$  and  $A(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ . By theorem 3.3, we know that there exists an unbounded sequence of numbers  $\{t_n\}_{n \geq 1}$  so that  $\nabla_H \phi(z(t_n)) \rightarrow 0$ . By rearranging equations (40), (41), (42), (43) we get that

$$(46) \quad r = y_1 - M_p(\alpha_1 - r) - \Gamma \begin{pmatrix} as \\ bs \end{pmatrix},$$

$$(47) \quad s = y_2 - M_p(\alpha_2 - s) + \Gamma \begin{pmatrix} ar \\ br \end{pmatrix},$$

$$(48) \quad a = y_3 - M_p(\alpha_3 - a) - \Gamma \begin{pmatrix} a_1 \\ b_1 + H_0 \end{pmatrix},$$

and

$$(49) \quad b = y_4 - M_p(\alpha_4 - b) + \Gamma \begin{pmatrix} a_2 + H_0 \\ b_2 \end{pmatrix}.$$

Since it was assumed that the range of  $z$  stays bounded as a subset of  $H^4$ , we get that each of

$$\alpha_1 - r = \kappa^2(r^2 + s^2 - 1)r - a(s_1 - ar) - b(s_2 - br) - r$$

$$\alpha_2 - s = \kappa^2(r^2 + s^2 - 1)s + a(r_1 + as) + b(r_2 + bs) - s$$

$$\alpha_3 - a = s(r_1 + as) - r(s_1 - ar) - a$$

$$\alpha_4 - b = s(r_2 + bs) - r(s_2 - br) - b$$

stays bounded as functions from  $[0, \infty)$  to  $L^p(\Omega)$ . We can see this if we use inequalities (12) and the inequalities we used in lemma 6.4. Similarly  $as, bs, ar, br$  stay bounded as functions from  $[0, \infty)$  to  $L^2(\Omega)$ . Using theorems 4.10 and 4.13, we get that there is a subsequence of  $\{t_n\}_{n \geq 1}$ , denoted by  $\{t_n\}_{n \geq 1}$  also, so that each of the sequences

$$\{M_p[(\alpha_1 - r)(t_n)] - \Gamma \begin{pmatrix} as \\ bs \end{pmatrix}(t_n)\}_{n \geq 1}$$

$$\{M_p[(\alpha_2 - s)(t_n)] + \Gamma \begin{pmatrix} ar \\ br \end{pmatrix}(t_n)\}_{n \geq 1}$$

$$\begin{aligned} & \{M_p[(\alpha 3 - a)(t_n)] - \Gamma\left(\begin{matrix} a_1 \\ b_1 + H_0 \end{matrix}\right)(t_n)\}_{n \geq 1} \\ & \{M_p[(\alpha 4 - b)(t_n)] + \Gamma\left(\begin{matrix} a_2 + H_0 \\ b_2 \end{matrix}\right)(t_n)\}_{n \geq 1} \end{aligned}$$

converge in  $L^p(\Omega)$  to an element of  $H$ . But now using equations (46), (47), (48), and (49) and the fact that  $\nabla_H \phi(z(t_n))$  converges to zero in  $H$  we get that each of  $\{r(t_n)\}_{n \geq 1}$ ,  $\{s(t_n)\}_{n \geq 1}$ ,  $\{a(t_n)\}_{n \geq 1}$ , and  $\{b(t_n)\}_{n \geq 1}$  converges in  $L^p(\Omega)$  to an element of  $H$ . We call this element of  $H^4$ ,  $(u, A)$  and note that

$$\nabla_H \phi(u, A) = 0$$

by construction of the sequence  $\{t_n\}_{n \geq 1}$ . □

I end this chapter by giving an informal discussion regarding the assumption in the above theorem that  $z$ , the steepest descent parameter, stays bounded. I do not have a proof verifying this assumption yet, however I do have evidence that suggests this assumption is valid.

- (i) I am interested in studying this problem using continuous steepest descent and developing a numerical analogue. the numerical results are discussed in Chapter 7 in detail. However I mention here that the numerical evidence suggests that that discrete descent parameter  $z$  stays bounded.
- (ii) Another way to show that the range of  $z$  is bounded in  $H^4$  is to show that  $\phi$  as defined in (33) is coercive. This implies that if  $z$  is unbounded then  $\phi(z)$  is unbounded which is a contradiction as  $\phi(z)$  is a nonnegative decreasing function. However,  $\phi$  may not be coercive from  $H^4$  to  $\mathbb{R}$ . To remedy this I consider the subspace

$$(50) \quad H' = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in H^2 : f_1 + g_2 = 0 \text{ in } \Omega \text{ and } \begin{pmatrix} f \\ g \end{pmatrix} \cdot n = 0 \text{ on } \partial\Omega \right\}$$

where  $n$  is the unit outward normal and  $\partial\Omega$  denotes the boundary of  $\Omega$ . Then

$$(51) \quad \left( \int_{\Omega} (g_1 - f_2)^2 \right)^{1/2}$$

defines a norm on  $H'$  that is equivalent to the  $H^2$  norm [11], [4]. Now consider the subspace  $H^2 \times H'$  instead of  $H^4$ ,  $\phi$  will be coercive from  $H^2 \times H'$  to  $\mathbb{R}$ . I also make use of the following definition and result which can be found in [11], [15].

(a) *Gauge Equivalence*

Let  $(u, A)$  and  $(v, B)$  be elements of  $H^{1,2}(\Omega, \mathbb{C}) \times H^2$ .  $(u, A)$  and  $(v, B)$  are said to be gauge equivalent if there exists  $\alpha \in H^{2,2}(\Omega)$  so that

$$v = ue^{i\alpha} \text{ and } B = A + \nabla\alpha$$

(b) *Gauge Invariant Property of Functional (33)*

If  $(u, A)$  and  $(v, B)$  are gauge equivalent, then  $\phi(u, A) = \phi(v, B)$ .

For  $(u, A) \in H^4$ ,  $\phi'(u, A)$  is a continuous linear functional from  $H^4$  to  $\mathbb{R}$ . Using the equivalence of norms,  $\phi'(u, A)$  is a continuous linear functional from  $H^2 \times H'$  to  $\mathbb{R}$ . Thus there exists a unique element of  $H^2 \times H'$  denoted by  $\nabla_{H'}\phi(u, A)$  so that

$$\phi'(u, A)(v, B) = \langle (v, B), \nabla_{H'}\phi(u, A) \rangle_{H \times H'}.$$

The problem now lies in determining a form for  $\nabla_{H'}\phi(u, A)$ . Ideally such a form would be determined by a projection as developed in Chapter 4 and would have a numeric analogue. This issue seems very interesting and will be pursued in a separate work.

- (iii) The approach above (using coercivity) does treat the hypothesis that  $z$  is bounded. However, it might be advantageous to treat the problem directly without using properties such as coercivity. One reason why one might want to consider this is that if we are interested in finding critical point of the Ginzburg-Landau functional in a higher order Sobolev space such as  $[H^{2,2}(\Omega)]^4$ , coercivity will almost certainly not be satisfied, however the descent parameter might stay bounded anyway.



## CHAPTER 7

### NUMERICAL RESULTS

In this chapter I discuss the numerical study of the Ginzburg-Landau energy functional with magnetic field. As discussed in Chapter 1, it is known that superconductors can be classified as type I or type II. The distinguishing factor is that in type I superconductors  $\kappa$ , the Ginzburg-Landau parameter, is less than the critical value of  $1/\sqrt{2}$  and for type II superconductors  $\kappa$  is greater than  $1/\sqrt{2}$ . Type II superconductors can be in three states: normal state where the modulus of the order parameter is zero everywhere, superconducting state where the modulus of the order parameter is one, and mixed state where tubes of magnetic flux called vortices form in the interior of the superconducting medium. For fixed  $\kappa$  these states are determined by the value of the external magnetic field.

In this work I try to numerically simulate the phase transition from normal state to mixed state to superconducting state for type II superconductors. The motivation here is numerical experimenting. Both  $H_0$ , the external magnetic field, and  $\kappa$  are treated as constants. I will first give a description of the numerical method I am using and describe some numerical difficulties I encountered.

#### 7.1. Method

I follow the developments in [8] closely. For all of the simulations  $\Omega$  is a rectangle. I discretize  $\Omega$  into  $N + 1$  by  $M + 1$  grid points and let  $\Omega_G$  be the set of all  $c = (N + 1)(M + 1)$  grid points and  $\Omega'_G$  the set of all  $c' = NM$  cell centers. Let  $h_x = l_x/M$  and  $h_y = l_y/N$  where  $l_x$  is the horizontal length of  $\Omega$  and  $l_y$  is the vertical length of  $\Omega$ . Let  $H$  be the set of all  $\mathbb{R}$  valued functions with domain  $\Omega_G$  and  $K$  the set of all  $\mathbb{R}$  valued functions on  $\Omega'_G$ .  $H$  is

analogous to  $H^{1,2}(\Omega)$  and  $K$  is analogous to  $L^2(\Omega)$ . I define  $W_G : H \rightarrow K^2$  so that for  $f \in H$ ,

$$W_G f = \begin{pmatrix} D_1 f \\ D_2 f \end{pmatrix}$$

where if  $e$  is a cell center with corners  $x_1, x_2, x_3, x_4$  ordered counterclockwise starting with the top left corner then

$$(D_1 f)(e) = \frac{f(x_4) - f(x_1) + f(x_3) - f(x_2)}{2h_x}$$

and

$$(D_2 f)(e) = \frac{f(x_1) - f(x_2) + f(x_4) - f(x_3)}{2h_y}.$$

$D_1$  is the discretized partial derivative operator in the first independent variable, and  $D_2$  is the discretized partial derivative operator in the second independent variable. Note that there is more than one way to define  $D_1$  and  $D_2$ . I have not experimented with any other definition.

I also define  $\hat{I} : H \rightarrow K$  and  $I : H \rightarrow H$  so that

$$(\hat{I}_G f)(e) = \frac{f(x_1) + f(x_2) + f(x_3) + f(x_4)}{4}$$

where  $e, x_1, x_2, x_3, x_4$  are as above. For  $x$  a grid point

$$(I_G f)(x) = x.$$

For  $(u, A) = (r, s, a, b) \in H^4$ , the quadruple cartesian product of  $H$ , define

$$\hat{I}(u, A) = (\hat{I}_G r, \hat{I}_G s, \hat{I}_G a, \hat{I}_G b)$$

and

$$W(u, A) = (W_G r, W_G s, W_G a, W_G b).$$

I define the following inner products. For  $f, g \in H$  define

$$\langle f, g \rangle_H = \sum_{a \in \Omega_G} f(a)g(a).$$

For  $f, g \in K$

$$\langle f, g \rangle_K = \sum_{e \in \Omega'_G} f(e)g(e).$$

For  $f, g \in H$  let

$$\langle f, g \rangle_S = \langle f, g \rangle_H + \langle D_1 f, D_1 g \rangle_K + \langle D_2 f, D_2 g \rangle_K.$$

Since  $D_1$  and  $D_2$  are transformations from  $H$  to  $K$  then  $D_1^t$  and  $D_2^t$  are transformations from  $K$  to  $H$ . Thus  $D_1^t D_1$  and  $D_2^t D_2$  are transformations from  $H$  to  $H$ . Thus note that we can write the  $S$  inner product as

$$\langle f, g \rangle_S = \langle (I_G + D_1^t D_1 + D_2^t D_2) f, g \rangle_H.$$

I now discretize the full Ginzburg-Landau functional as follows. Let  $F$  be a function from  $\mathbb{R}^{16}$  to  $\mathbb{R}$  so that if  $e \in \Omega'_G$  then

$$\begin{aligned} F \left( \left( \left( \left( \hat{I} \right) \right) \right) \left( \left( \left( \hat{W} \right) \right) \right) (u, A) \right) (e) &= |(D_1 r + \hat{I} s \hat{I} a)(e)|^2 + |(D_1 s - \hat{I} r \hat{I} a)(e)|^2 + \\ &| (D_2 r + \hat{I} s \hat{I} b)(e)|^2 + |(D_2 s - \hat{I} r \hat{I} b)(e)|^2 + \\ &| (D_1 b - D_2 a)(e) - H_0|^2 + \frac{\kappa^2}{2} |(\hat{I} r(e))^2 + (\hat{I} s(e))^2 - 1|^2. \end{aligned}$$

Define  $\phi_G : \mathbb{R}^{4c} \rightarrow \mathbb{R}$  so that

$$(52) \quad \phi_G(u, A) = \frac{h_x h_y}{2} \sum_{e \in \Omega'_G} F \left( \left( \left( \left( \hat{I} \right) \right) \right) \left( \left( \left( \hat{W} \right) \right) \right) (u, A) \right) (e)$$

$\phi'_G(u, A)$  is a continuous linear functional from  $\mathbb{R}^{4c}$  to  $\mathbb{R}$  thus we can represent it as

$$\phi'_G(u, A) h = \langle h, \nabla \phi_G(u, A) \rangle_{H^4}$$

where  $h = (v, B) \in H^4$  and  $\nabla \phi_G(u, A)$  is the list of all  $4c$  partial derivatives of  $\phi_G$ . Note that  $\phi'_G(u, A)$  also has another representation using the  $S$  inner product.

$$\begin{aligned} \phi'_G(u, A) h &= \langle h, \nabla_S \phi_G(u, A) \rangle_{S^4} = \\ &\langle h, (I + D_1^t D_1 + D_2^t D_2) \nabla_S \phi_G(u, A) \rangle_{H^4}. \end{aligned}$$

Thus we have that

$$\nabla_S \phi_G(u, A) = (I + D_1^t D_1 + D_2^t D_2)^{-1} \nabla \phi_G(u, A).$$

Now let  $(u_0, A_0) \in H^4$  and

$$z_0 = (u_0, A_0).$$

Then define

$$(53) \quad z_n = z_{n-1} - \alpha_n \nabla_S \phi_G(z_{n-1})$$

for  $n \geq 1$  where  $\alpha_n$  is the minimum of the function

$$t \rightarrow \phi_G(z_{n-1} - t \nabla_S \phi_G(z_{n-1})).$$

$z_n$  as defined in equation (53) is the approximation to a critical point of the functional (33) for  $n$  large enough to satisfy the measure of convergence. I use two measures of convergence.

(i) The relative change in the functional has to be less than the error tolerance of  $err_\phi$ .

In other words if

$$\frac{\phi_G(z_{n-1}) - \phi_G(z_n)}{\phi_G(z_{n-1})} < err_\phi$$

then the first measure of convergence is satisfied.

(ii) A critical point of the functional is defined to be a point  $z_n \in H^4$  at which  $\nabla_S \phi_G(z_n) = 0$ . Thus for the second measure of convergence I check that this is the case. In other words if for the error tolerance of  $err_{\nabla_S \phi}$

$$\|\nabla_S \phi_G(z_n)\|_S < err_{\nabla_S \phi}$$

then the second measure of convergence is satisfied.

For all experiments  $err_{\nabla_S \phi} = err_\phi = 10^{-7}$ .

## 7.2. Experiments

I present the results of two of the experiments here. For the first experiment the aim is to numerically verify the phase diagram presented in [15] for a fixed value of  $\kappa$ . I take  $\kappa$  to be 50 and increase  $H_0$ , the applied magnetic field, by units of 25. The phase diagram predicts that for  $\Omega = \mathbb{R}^2$ , if  $\frac{H_0}{\kappa}$  is less than  $\frac{|\ln \kappa|}{2}$  then the medium is in superconducting state. If  $\frac{H_0}{\kappa}$  is greater than  $\frac{|\ln \kappa|}{2}$  but less than  $\kappa$ , the medium is in mixed state. Thus for  $\kappa = 50$  and  $\Omega = \mathbb{R}^2$ , it is predicted that if  $H_0$  is less than 98 then the medium is in superconducting state. If  $98 < H_0 < 2500$ , the medium is in mixed state. In the simulations  $\Omega$  is the unit square partitioned into a 100 by 100 grid. For  $\kappa = 50$  and  $H_0 = 100$  I did not observe any vortices. For  $\kappa = 50$  and  $H_0 = 125$  I observed four.

I produce surface plots for  $\|u\|^2$ . The images in figures 7.1 to 7.6 show a contour plot for  $\kappa = 50$  and  $H_0$  ranging from 125 to 275. Vortices correspond to regions where  $\|u\|^2$  is zero in the region. They correspond to dark blue regions in the plots. The red region indicates that  $\|u\|^2$  is approximately one. Thus the red regions correspond to superconducting state and the blue regions correspond to normal state. It is predicted that at a critical point  $\|u\|^2 \leq 1$ . In all of the simulations this condition was satisfied without being enforced. The axis are labeled with grid cell counts. Since the domain is the unit square for this experiment, the axis label 100 corresponds to a length of 1. I also give the winding number around the boundary,  $d$ , and the energy.

For the second experiment I fix  $\kappa$  to be 4 and  $H_0$ , the external magnetic field, to be 6. I start the simulation in superconducting state (i.e.  $\|u\| = 1$  and  $\nabla \times A = 0$ ) then I produce a plot of  $\|u\|^2$  every 20 or so iterations to describe how vortices form under steepest descent. Here  $n$  gives the descent iteration for which the picture was produced. The domain is  $[0, 5] \times [0, 5]$  and the mesh size is .05. Again the axis are labeled with grid cell counts. I give the contour plots for  $\|u\|^2$  in figures 7.7 to 7.13. We see the first set of vortices forming then stabilizing in figures 7.7 to 7.9. They form on the boundary and move toward the center. In

figures 7.10 to 7.12 we see the next set of eight vortices forming then stabilizing. Figure 7.13 corresponds to the contour plot of  $\|u\|^2$  for  $n$  large enough to meet the convergence criteria mentioned above. Figure 7.14 is the energy plot for  $\kappa = 4$  and  $H_0 = 6$ . I plot energy as a function of  $n$  where  $n$  gives the number of descent iterations. We see that for  $75 < n < 225$ , the maximum rate of change in the functional is around  $n = 170$ . From figure 7.11 we see that when  $n = 160$ , the second set of vortices are forming.

Figure 7.15 is the energy plot for  $k = 4$  and  $H_0 = 7$ . I used the domain  $[0, 5] \times [0, 5]$  and a mesh size of .05. Starting the simulation in superconducting state I produced a critical point with 16 vortices. These vortices formed in three stages. A set of four vortices formed on the boundary and moved toward the center. Once the vortices stabilized, a set of eight vortices formed on the boundary and stabilized. Finally the last set of four vortices formed and stabilized. When energy is plotted against the number of iteration we observe that the relative change in the functional is small between  $n = 100$  and  $n = 300$ , between  $n = 600$  and  $n = 1350$ , and for  $n > 2000$ . I make a few comments about this observation.

- (i) In almost all of the experiments I observed that the relative change of the energy functional is small when the vortices are stabilizing and large when they are forming. Thus the energy plot is similar for other choices of  $H_0$  and  $\kappa$ , provided that  $H_0$  is significantly larger than  $H_{c1}$  and  $\kappa$  is significantly larger than  $1/\sqrt{2}$ .
- (ii) One explanation may be that a metastable state is reached when the relative change in the functional is small. I do not have enough evidence or an explanation to fully support this claim. It is an interesting topic for future study.
- (iii) I observed that in all of our experiments when steepest descent had converged to a critical point  $u$ ,  $\|u\|^2$  was approximately one around the boundary. However at the intermediate states when the relative change in the functional is small,  $\|u\|^2$  is

approximately zero around the boundary. Thus I do not believe that these intermediate states correspond to local minima as the behavior around the boundary is very different from a critical point.

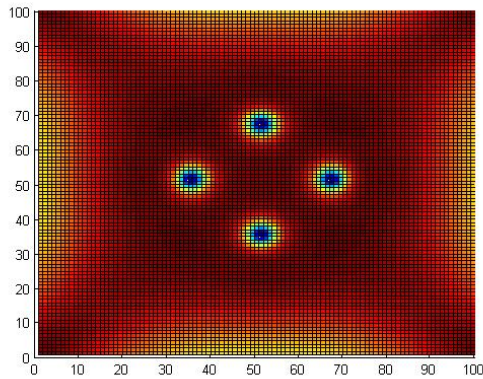


Figure 7.1. Plot of  $\|u\|^2$  for  $\kappa = 50$  and  $H_0 = 125$ .  $d = 4$  and energy is 124.75.

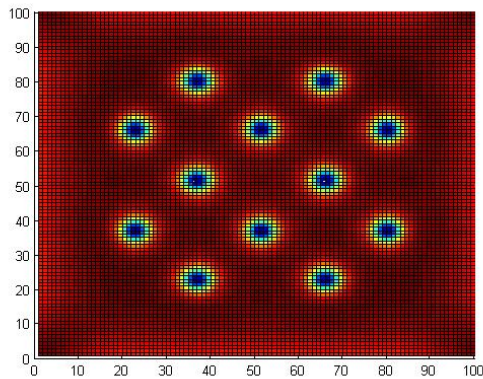


Figure 7.2. Plot of  $\|u\|^2$  for  $\kappa = 50$  and  $H_0 = 150$ .  $d = 12$  and energy=91.96.

### 7.3. Discussion

I now discuss some difficulties I encountered and some ideas on how one may improve our scheme. During the experiments I observed a few things that might be cause for caution. As discussed in the second experiment, I observed in the experiments that vortices form on the boundary and move toward the center. When the vortices are forming the relative

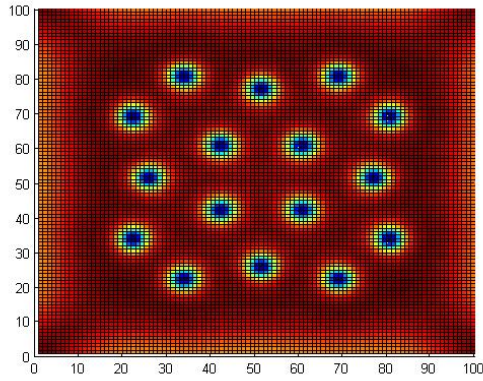


Figure 7.3. Plot of  $\|u\|^2$  for  $\kappa = 50$  and  $H_0 = 200$ .  $d = 16$  and energy=128.36.

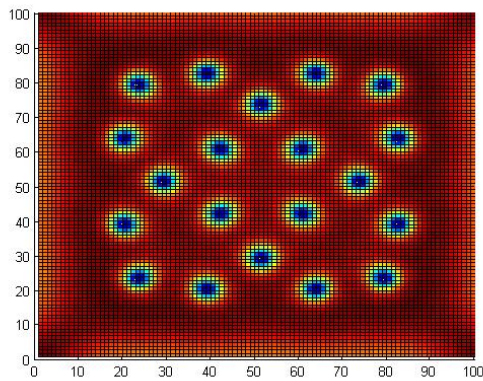


Figure 7.4. Plot of  $\|u\|^2$  for  $\kappa = 50$  and  $H_0 = 225$ .  $d = 20$  and energy=137.33.

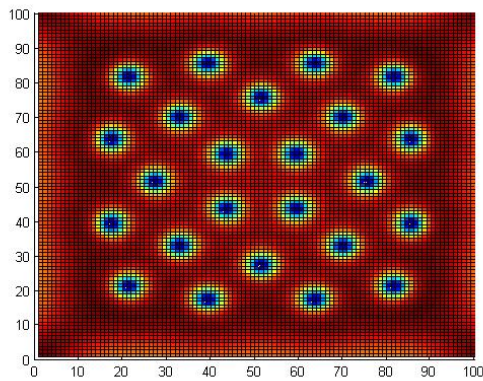


Figure 7.5. Plot of  $\|u\|^2$  for  $\kappa = 50$  and  $H_0 = 250$ .  $d = 24$  and energy=145.13.



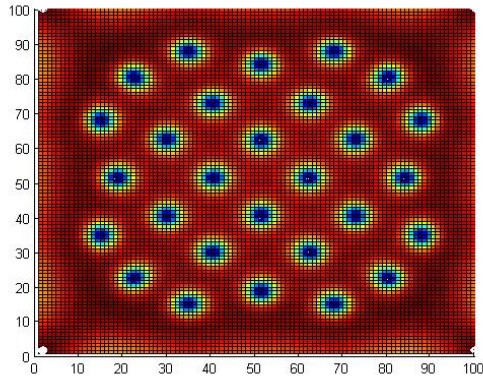


Figure 7.6. Plot of  $\|u\|^2$  for  $\kappa = 50$  and  $H_0 = 275$ .  $d = 28$  and energy=157.02.

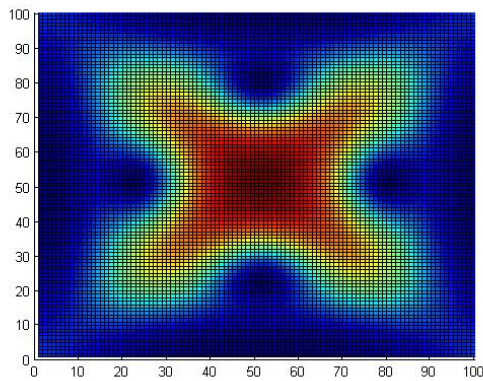


Figure 7.7. Plot of  $\|u\|^2$  for  $\kappa = 4$ ,  $H_0 = 6$ , and  $n = 40$ .

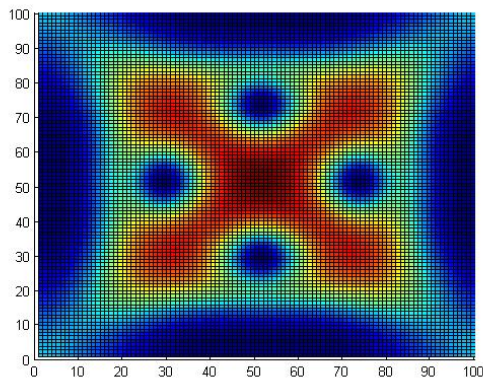


Figure 7.8. Plot of  $\|u\|^2$  for  $\kappa = 4$ ,  $H_0 = 6$ , and  $n = 60$ .

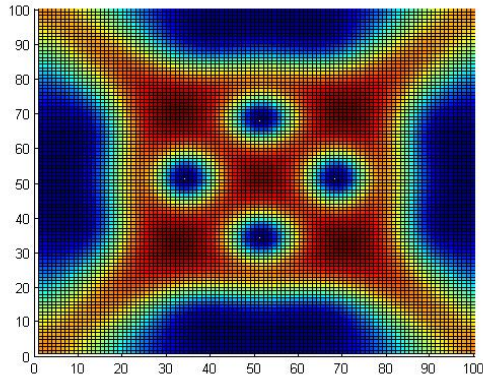


Figure 7.9. Plot of  $\|u\|^2$  for  $\kappa = 4$ ,  $H_0 = 6$ , and  $n = 100$ .

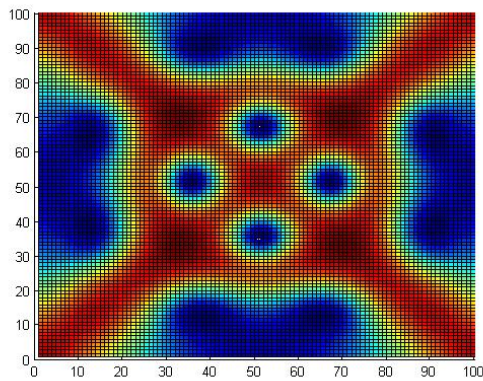


Figure 7.10. Plot of  $\|u\|^2$  for  $\kappa = 4$ ,  $H_0 = 6$ , and  $n = 140$ .

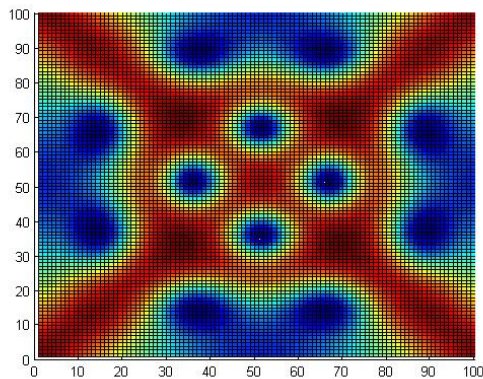


Figure 7.11. Plot of  $\|u\|^2$  for  $\kappa = 4$ ,  $H_0 = 6$ , and  $n = 160$ .

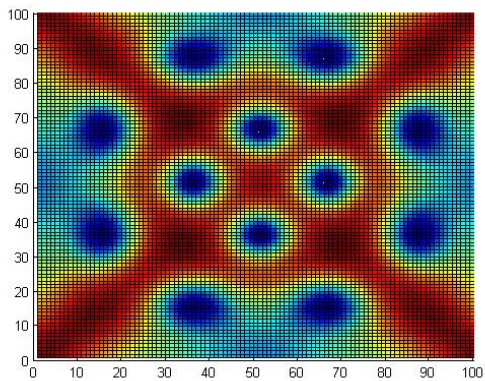


Figure 7.12. Plot of  $\|u\|^2$  for  $\kappa = 4$ ,  $H_0 = 6$ , and  $n = 180$ .

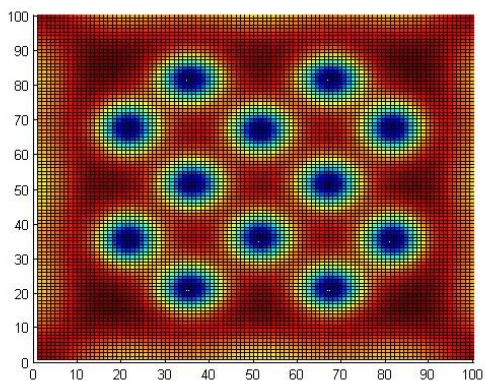


Figure 7.13. Plot of  $\|u\|^2$  for  $\kappa = 4$ ,  $H_0 = 6$ ,  $n = 837$ , and  $energy = 55.92$ .

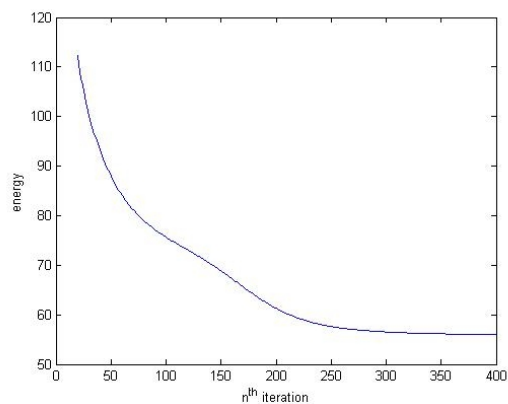


Figure 7.14. Plot of energy vs. iteration for  $\kappa = 4$  and  $H_0 = 6$ .

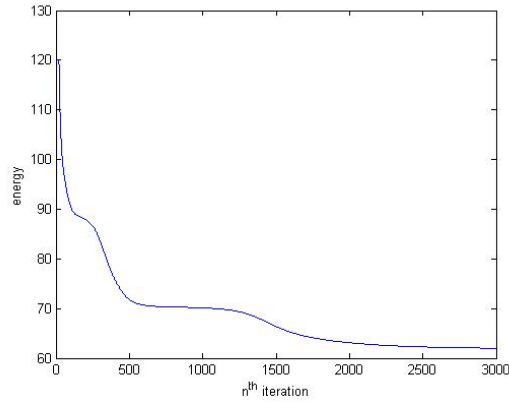


Figure 7.15. Plot of energy vs. iteration for  $\kappa = 4$  and  $H_0 = 7$ .

change in the functional is large compared to the error tolerance, but when the vortices are stabilizing, both the relative change in the functional and  $\|\nabla_S \phi_G\|_S$  can be very small causing a premature stop before all the vortices have formed. To address the immediate issue I decreased the error tolerances I was using and observed a more regular pattern for vortex formation. This observation brings up the question of what is the right test for convergence?

The next concern is computation time. It seems that computation time grows not only with domain size, the number of grid points but also with the experiment parameters  $\kappa$  and  $H_0$ . Thus for high values of  $\kappa$  the computation time might be days when the simulation is run on a standard PC. This is of some concern because superconductor with high values of  $\kappa$  are important from a scientific standpoint. One might use an alternate descent method with the Sobolev gradient. The possible candidates are the Barzilai-Borwein gradient method for large scale unconstrained minimization [12] and the Polak-Ribière conjugate gradient method [6]. By using a different descent method I hope to optimize computation time by improving the algorithm for finding step size or by reducing the number of steps required to meet our convergence criteria. For large values of  $\kappa$  and  $H_0$ , it might be advantageous to start the simulation from a previous critical point. In other words one could produce a critical point for a lower value of  $\kappa$  and  $H_0$  and use the critical point as the initial estimate to the descent

parameter for larger values of  $\kappa$  and  $H_0$ . I used this method in some experiments and observed that computation time did improve.

The third point was brought to my attention by Walter Richardson [18]. He experimented with minimizing the time dependent Ginzburg-Landau functional and found that the number and position of vortices depends on the mesh size. Although I did not observe this, it is cause for some concern and will be addressed in a separate work.

Finally I point out that this method for finding minimizers of the Ginzburg-Landau energy functional finds critical points, but does not distinguish between local minima and global minima. Also the critical points I obtain depend on the initial estimate to the descent parameter. Thus for each value of  $H_0$  and  $\kappa$  I can find several critical points by varying the initial estimate. This brings up several questions. The first question is whether there are only finitely many critical points for each value of  $H_0$  and  $\kappa$ ? The second question is whether it is possible to find the global minimizer by varying the initial estimate? If the answer to this question is yes, then is there an organized way to go about this task?

## CHAPTER 8

### CONCLUSION

Superconductivity was discovered nearly 100 years ago. Since that time, many individuals have contributed ideas toward developing a theory that correctly describes superconductivity. Several of these individuals have been awarded the Nobel prize for their contributions. Like many other discoveries in physics, the theory of superconductivity has evolved with time and technology. For low temperature superconductors the London brothers, Bardeen, Cooper, Schrieffer, Ginzburg, Landau, Abrikosov, and Górkov all made contribution that led to the current theory of superconductivity. In 1987 Bednerz and Müller were awarded the Nobel prize in physics for their discovery in high temperature superconductors. Due to the industrial applications of high temperature superconductors, their work brought back attention both from the scientific community and the mathematical community. Among the many open problems in superconductivity is the challenge of developing a theory that incorporates high temperature superconductors.

The contribution of the present work is to present a constructive method for finding critical points of the Ginzburg-Landau energy functional. The term constructive is open to interpretation. The solution I present is constructive in two ways. First the critical point is obtained as a limit of the descent parameter  $z$  as described in equations (30) and equations (46) to (49). I gave a precise statement of these results in Theorem 5.6 and Theorem 6.6. The second reason that the method is constructive is that it provides a numerical analogue which I use to find the critical points. Von Neumann's formula and its generalization play a key role in obtaining the expression for  $z$  and in obtaining convergence results. I certainly hope that the approach I present here will adapt itself to the future of superconductivity.

In this work I brought up several questions which at this point remain unanswered to me. I will not state all of the issues I brought up in this work, however the three prominent ones are as follows.

- In this work I obtained the existence of a sequence of numbers  $t_1, t_2, \dots$  so that the sequence  $\{z(t_n)\}_{n \geq 1}$  was convergent for  $z$  as given in equation (30) or equations (46) to (49). See Theorem 5.6 and Theorem 6.6 for a precise statement. Is it true that  $\lim_{t \rightarrow \infty} z(t)$  exists where  $z$  is as in (30) or equations (46) to (49)?
- In treating the full Ginzburg-Landau energy functional (33), I used the quadruple cartesian product of  $H^{1,2}(\Omega)$ . It would be an interesting problem to consider the minimization using the technique using a different space. A possible candidate is  $H^{1,2}(\Omega, \mathbb{C}) \times H'$  where  $H'$  is the divergence free subspace defined in chapter 6.
- Regarding numerics, how can I optimize running time for the simulator? I would also like to design and run more experiments that will add to the knowledge of superconductivity.

I hope that through correspondence with knowledgeable individuals, I will be able to answer all of these questions in the near future.

## BIBLIOGRAPHY

- [1] A. A. Abrikosov, *On magnetic properties of superconductors of the second group*, Soviet Physics JETP 5 (1957), no. 6, 1174–1182.
- [2] R. Adams, *Sobolev spaces*, Academic, New York, 1975.
- [3] V.L. Ginzburg and L.D. Landau, *Journal of Experimental and Theoretical Physics* 20 (1950), no. 12.
- [4] V. Girault and P.-A. Raviart, *Finite element methods for navier-stokes equations*, Springer, Berlin, 1986.
- [5] L.P. Gor'kov, *Microscopic derivation of the ginzburg-landau equations in the theory of superconductivity*, Sov. Phys.-JETP 9 (1959), 1364–1367.
- [6] L. Grippo and S. Lucidi, *A globally convergent version of the polak-ribière conjugate gradient method*, *Mathematical Programming* 78 (1997), 375–391.
- [7] Parimah Kazemi and J. W. Neuberger, *Potential theory and applications to a constructive method for finding critical points of ginzburg-landau type equations*, *Nonlinear Analysis: Theory, Methods and Applications* (2008), DOI: doi:10.1016/j.na.2008.02.074.
- [8] J. W. Neuberger and R. J. Renka, *Critical points of the ginzburg-landau functional on multiply-connected domains*, *Experimental Mathematics* 9 (2000), no. 4, 523–533.
- [9] ———, *Numerical determination of vortices in superconductors: Simulation of cooling*, *Supercond. Sci. Technol.* 16 (2003), 1413–1416.
- [10] J.W. Neuberger, *Sobolev gradients and differential equations*, *Lecture notes in mathematics*, vol. 1670, Springer-Verlag, 1997.
- [11] Max D. Gunzburger Qiang Du and Janet S. Peterson, *Analysis and approximation of the ginzburg-landau model of superconductivity*, *SIAM Review* 34 (1992), no. 1, 54–81.



- [12] Marcos Raydan, *The barzilai and borwein gradient method for the large scale unconstrained minimization problem*, Saim J. Optim. 7 (1997), no. 1, 26–33.
- [13] Riesz and Sz. Nagy, *Functional analysis*, Dover Publications Inc., 1990.
- [14] A. C. Rose-Innes and E.H. Rhoderick, *Introduction to superconductivity*, Pergamon Press, 1978.
- [15] Etienne Sandier and Sylvia Serfaty, *Vortices in the magnetic ginzburg-landau model*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, 2007.
- [16] Michael Tinkham, *Introduction to superconductivity*, McGraw-Hill Book Company, 1975.
- [17] J. VonNeumann, *Functional operators ii*, Annls. Math. Stud., vol. 22, Princeton University Press, 1950.
- [18] Graham F. Carey Walter B. Richardson, Anand L. Pardhanani and Alexandre Ardelea, *Numerical effects in the simulation of ginzburglandau models for superconductivity*, Int. J. Numer. Meth. Engng 59 (2004), 12511272.