APPLICATIONS IN FIXED POINT THEORY

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Banach's contraction principle is probably one of the most important theorems in fixed point theory. It has been used to develop much of the rest of fixed point theory. Another key result in the field is a theorem due to Browder, Göhde, and Kirk involving Hilbert spaces and nonexpansive mappings. Several applications of Banach's contraction principle are made. Some of these applications involve obtaining new metrics on a space, forcing a continuous map to have a fixed point, and using conditions on the boundary of a closed ball in a Banach space to obtain a fixed point. Finally, a development of the theorem due to Browder et al. is given with Hilbert spaces replaced by uniformly convex Banach spaces.

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CHAPTER 1

INTRODUCTION

Many of the theorems and lemmas quoted in chapters 2 and 3 are exercises from [1, p. 9-11, 18] that are stated but not proven.

In this introduction, we will introduce the spaces commonly used in fixed point theory as well as some of the theorems most often used with these spaces. We will also introduce the concept of contraction mappings and nonexpansive mappings.

Before we can talk about contraction and nonexpansive mappings, some preliminary definitions and theorems are required about the spaces we will be dealing with. We will first deal with metric spaces.

Definition 1.0.1 (Metric). Let X be a set. A metric is a function
$d : X \times X \to [0, +\infty)$ which satisfies the following properties:
$For\ all\ x, y, z\ in\ X$,

- $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.0.2 (Metric Space). A set X with a metric d defined on it is said to be a metric space.

Most times, however, having a metric space is not enough to guarantee having a fixed point for a contraction map or a nonexpansive map. What is most often required is at least having a complete metric space. The following definitions develop this notion.

Definition 1.0.3 (Convergence). A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to converge to a point x in X if and only if for all $\epsilon > 0$, there exists an N in $\mathbb{N}$ so that for every n in $\mathbb{N}$, $n \geq N$ implies $|x_n - x| < \epsilon$. This is also known as strong convergence.

Definition 1.0.4 (Cauchy Sequence). A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be Cauchy in X if and only if for all $\epsilon > 0$, there exists an N in $\mathbb{N}$ such that for every m, n in $\mathbb{N}$, if both $n \geq N$ and $m \geq N$, then $|x_n - x_m| < \epsilon$.

Definition 1.0.5 (Complete Metric Space). A metric space X is said to be complete if every Cauchy sequence in X converges in X.

Also, there will be times where we will rely on having a compact metric space. The following definition and two theorems give us the appropriate tools to work with.

Definition 1.0.6 (Compact Metric Space). A metric space X is said to be compact if every collection U of open sets V in X satisfies
$\bigcup U = X$ implies that there exists W a finite subset of U, also known as a finite subcollection of U, satisfying $\bigcup W = X$.

Theorem 1.0.7. *A metric space X is compact if and only if every sequence $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to some point x in X.*

Theorem 1.0.8 (Extreme Value Theorem). *Let X be a compact metric space and let $f : X \to \mathbb{R}$ be a continuous function. Then, f attains its extrema. Furthermore, $f[X]$ is a compact subset of X.*

Sometimes, even a compact or a complete metric space will not be enough to guarantee having a fixed point in a contraction map or in a nonexpansive map. But, in many of these circumstances having a Banach space will give us a fixed point for such mappings.

Definition 1.0.9 (Norm). Let X be a linear space. A norm is a function $\|\cdot\| : X \to [0, +\infty)$ which satisfies the following properties:

For every x, y in X,

- $\|x\| = 0$ in $\mathbb{R}$ if and only if $x = 0$ in X.
- For every α in $\mathbb{R}$, $\|\alpha x\| = |\alpha| \|x\|$.
- $\|x + y\| \leq \|x\| + \|y\|$.

Definition 1.0.10 (Normed Linear Space). A linear space X is called a normed linear space if there exists a norm on X.

Remark 1.0.11. A normed linear space X with norm $\|\cdot\|$ is also a metric space with metric d defined by $d(x, y) = \|x - y\|$.

Definition 1.0.12 (Banach Space). A normed linear space X is said to be a Banach space if X, as a metric space, is complete.

Before we can begin the fixed point theory discussion, we need the definitions of a non-expansive map and of a contraction map.

Definition 1.0.13 (Nonexpansive and Contraction Maps). Let X be a metric space. Let $f : X \to X$ be a function satisfying the property that there exists a λ (called the Lipschitzian constant) in $[0, 1]$ such that for every x, y in X, $d(f(x), f(y)) \leq \lambda d(x, y)$. The function f is said to be a nonexpansive map if $\lambda = 1$. If $\lambda < 1$, then f is said to be a contraction map.

Finally, we finish this chapter with Banach's Contraction Principle. This result is used in many of the proofs in the following chapters.

Theorem 1.0.14 (Banach's Contraction Principle). *Let (X, d) be a complete metric space and let $F : X \to X$ be a contraction with Lipschitzian constant L. Then, F has a unique fixed point u in X. Furthermore, for any x in X we have $\lim_{n\to\infty} F^n(x) = u$ with $d(F^n(x), u) \leq \frac{L^n}{1-L} d(x, F(x))$.*

Proof. [1, p. 1-2] We first show uniqueness. Suppose there exist x, y in X with $x = F(x)$ and $y = F(y)$. Then

$$d(x, y) = d(F(x), F(y)) \leq Ld(x, y),$$

therefore $d(x, y) = 0$.

To show existence, select an x in X. We first show that $\{F^n(x)\}$ is a Cauchy sequence. Notice for n in $\{0, 1, \ldots\}$ that

$$d\left(F^n(x), F^{n+1}(x)\right) \leq Ld\left(F^{n-1}(x), F^n(x)\right) \leq \cdots \leq L^n d(x, F(x)).$$

Thus for $m > n$ where n is in $\{0, 1, \ldots\}$,

$$\begin{aligned} d(F^n(x), F^m(x)) &\leq d\left(F^n(x), F^{n+1}(x)\right) + d\left(F^{n+1}(x), F^{n+2}(x)\right) \\ &\quad + \cdots + d\left(F^{m-1}(x), F^m(x)\right) \\ &\leq L^n d(x, F(x)) + \cdots + L^{m-1} d(x, F(x)) \\ &\leq L^n d(x, F(x))\left[1 + L + L^2 + \cdots\right] \\ &= \tfrac{L^n}{1-L} d(x, F(x)). \end{aligned}$$

That is for $m > n$, n in $\{0, 1, \ldots\}$,

$$d(F^n(x), F^m(x)) \leq \tfrac{L^n}{1-L} d(x, F(x)).$$

This shows that $\{F^n(x)\}$ is a Cauchy sequence and, since X is complete, there exists u in X with $\lim_{n\to\infty} F^n(x) = u$. Moreover the continuity of F yields

$$u = \lim_{n\to\infty} F^{n+1}(x) = \lim_{n\to\infty} F(F^n(x)) = F(u),$$

therefore u is a fixed point of F. Finally letting $m \to \infty$ in the inequality

$$d(F^n(x), F^m(x)) \leq \tfrac{L^n}{1-L} d(x, F(x))$$

yields

$$d(F^n(x), u) \leq \tfrac{L^n}{1-L} d(x, F(x)). \qquad \square$$

CHAPTER 2

COMPLETE METRIC SPACES

In this chapter, we will look into fixed point theorems involving metric spaces. These theorems appear as exercises in [1, p. 9-11].

We consider the following question: If we can obtain a fixed point for a contraction mapping in the scenario of a complete metric space, are we guaranteed a fixed point for a contraction mapping in an incomplete metric space?

Example 2.0.15. A contraction F from an incomplete metric space into itself need not have a fixed point.

Proof. Let $X = \mathbb{R} \setminus \{0\}$. Then, the map $f : X \to X$ defined by $f(x) = \frac{x}{2}$ is a contraction. However, as 0 is not in X, there is no fixed point of f. Furthermore, X with the standard metric from $\mathbb{R}$ restricted to X forms a metric space. Moreover, as $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence in X, with the aforementioned metric, that does not converge in X, X is not complete. $\square$

Hence, we will almost solely be dealing with spaces which are at least complete metric spaces.

Next, we look at a map F with the property that after n compostions of F with itself, we obtain a contraction mapping. The question: Does this guarantee that F has a fixed point?

Theorem 2.0.16. *Let (X, d) be a complete metric space and let $F : X \to X$ be such that $F^N : X \to X$ is a contraction for some positive integer N. Then F has a unique fixed point u in X and for each x in X, $\lim_{n\to\infty} F^n(x) = u$.*

Proof. As X is a complete metric space and F^N is a contraction on X, there exists an L in $[0,1)$ such that $d\left(F^N(x), F^N(y)\right) \leq Ld(x,y)$, for all x, y in X and, by Theorem 1.0.14, there exists a u in X that is a unique fixed point of F^N. Also, if we let x be in X, then $\lim_{m\to\infty} \left(F^N\right)^m(x) = u$. Furthermore, for every m in $\mathbb{N}$, $d\left(\left(F^N\right)^m(x), u\right) \leq \frac{L^m}{1-L} d\left(x, F^N(x)\right)$. Also, as u is in the domain of F^N, there exists x_0 in X such that $F(x_0) = u$, namely $x_0 = F^{N-1}(u)$. For each i in $\{0, 1, \ldots, N-2\}$, let $x_i = F^{(N-1)-i}(u)$. Then, $F^{i+1}(x_i) = u$ for all i in $\{0, 1, \ldots, N-2\}$. So,

$$\begin{aligned} d\left(\left(F^N\right)^m(x_i), u\right) &= d\left(\left(F^N\right)^m\left(F^{(N-1)-i}(u)\right), u\right) \\ &\leq \tfrac{L^m}{1-L} d\left(F^{(N-1)-i}(u), F^{(2N-1)-i}(u)\right) \\ &= \tfrac{L^m}{1-L} d\left(F^{(N-1)-i}(u), F^{(N-1)-i}\left(F^N(u)\right)\right) \\ &= \tfrac{L^m}{1-L} d\left(F^{(N-1)-i}(u), F^{(N-1)-i}(u)\right) \\ &= 0\; for\; all\; m\; in\, \mathbb{N}. \end{aligned}$$

Hence, $d\left(F^{mN}(x_i), F^{mN}(u)\right) = 0$ for all i in $\{0, \ldots, N-2\}$ and for all m in $\mathbb{N}$. Whence, $d\left(F^{(N-1)-i}(u), u\right) = 0$ for all i in $\{0, \ldots, N-2\}$. Therefore, $F^{(N-1)-i}(u) = u$ for all i in $\{0, \ldots, N-2\}$. Thus, $F(u) = u$. Furthermore, if there exists a v in X such that $F(v) = v$, then $F^N(v) = v$ and as u is the only fixed point of F^N, $F^N(v) = v = u = F^N(u)$. Whence, F has a unique fixed point u. Now, let x be in X. We already have $\lim_{m\to\infty} \left(F^N\right)^m(x) = u$. Let $\epsilon > 0$. Then, there exists an M_0 in $\mathbb{N}$ such that for every m in $\mathbb{N}$, $m \geq M$ implies $d\left(\left(F^N\right)^m(x), u\right) \leq \frac{L^m}{1-L} d\left(x, F^N(x)\right) < \epsilon$. As $u = F(u) = \cdots = F^{N-1}(u) = F^N(u)$, we can similarly obtain $M_1, \ldots, M_{N-1}$ and apply this reasoning to $F(x), \ldots, F^{N-1}(x)$, respectively. Let $\widehat{M} = \sup_{0\leq i\leq N-1} M_i$. Then, we can let $m \geq \widehat{M}$. So, $d(F^m(x), u) < \epsilon$. Therefore, $\lim_{m\to\infty} F^m(x) = u$. □

Thus, it is enough to have a contraction mapping appear after n self-compositions to guarantee that a map has a unique fixed point.

Now, we might explore what happens when a map starts in an arbitrary complete metric space X and maps to the nonnegative real numbers. Are there conditions on a map like this that would force a continuous map from X to itself to have a fixed point? The following is a lemma which helps answer this question.

Lemma 2.0.17. *Let (X, d) be a complete metric space and let $\phi : X \to [0, \infty)$ be a map (not necessarily continuous). Suppose* $\inf\{\phi(x) + \phi(y) : d(x, y) \geq \gamma\} = \mu(\gamma) > 0$ *for all* $\gamma > 0$. *Then each sequence $\{x_n\}$ in X, for which $\lim_{n\to\infty} \phi(x_n) = 0$, converges to one and only one point $u \in X$.*

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} \phi(x_n) = 0$. Then, let $\epsilon > 0$. Then, there exists γ such that $\inf\{\phi(x) + \phi(y) : d(x, y) \geq \gamma$ and x, y in $\{x_n\}\} = 2\epsilon > \mu(\gamma) > 0$. Also, there exists an N such that $n \geq N$ implies $\phi(x_n) < \epsilon$. Thus, for every $n, m \geq N$, $\phi(x_m) + \phi(x_n) < 2\epsilon$. Hence, for every $n, m \geq N$, $d(x_m, x_n) < \gamma$. Note that if we instead start out letting $\gamma > 0$ be arbitrary, $\mu(\gamma)$ can fullfil the role of 2ϵ. Hence, $\{x_n\}$ converges to some u in X. Now, let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ both converge to 0. A similar argument shows that there exists N_1, N_2 such that if $n \geq N_1$ and $m \geq N_2$, then $\phi(x_n) + \phi(y_m) < \mu(\gamma)$. Thus, taking $N = \max\{N_1, N_2\}$, we get $d(x_n, y_n) < \gamma$ for all $n \geq N$. Hence, $\{x_n\}$ and $\{y_n\}$ both converge to the same u in X. $\square$

Now, using a mapping with the property in the lemma above as well as a continous mapping from X to itself, do we obtain a fixed point? Is the fixed point unique?

Theorem 2.0.18. *Let (X, d) be a complete metric space and let $F : X \to X$ be continuous. Suppose $\phi(x) = d(x, F(x))$ satisfies* $\inf\{\phi(x) + \phi(y) : d(x, y) \geq \gamma\} = \mu(\gamma) > 0$ *for all* $\gamma > 0$, *and that* $\inf_{x \text{ in } X} d(x, F(x)) = 0$. *Then F has a unique fixed point.*

Proof. As $\inf_{x \text{ in } X} \phi(x) = 0$, by Lemma 2.0.17 above, we have that every sequence $\{x_n\}$, for which $\{\phi(x_n)\}$ converges to 0, converges to one and only one point u in X. We also have that such a sequence exists as the inf is 0. Furthermore, as $\phi(x)$ is defined as $d(x, F(x))$, this means F has the unique fixed point u. $\square$

To conclude the chapter, we ask if having a map from a metric space to itself that is bounded in the metric after n self-compositions gives us any useful results. Do we get fixed points? Do we get some other property, such as a new metric? Does the new metric bound the original function in the new metric? Note that we are not assuming this metric space is complete.

Theorem 2.0.19. *Let T be a map of the metric space (X, ρ) into itself such that, for a fixed positive integer n, $\rho(T^n x, T^n y) \leq \alpha^n \rho(x, y)$; here α is a positive real number. Then the function σ defined by $\sigma(x, y) := \rho(x, y) + \frac{1}{\alpha}\rho(Tx, Ty) + \cdots + \frac{1}{\alpha^{n-1}}\rho(T^{n-1}x, T^{n-1}y)$ is a metric on X and T satisfies $\sigma(Tx, Ty) \leq \alpha\sigma(x, y)$ for x, y in X.*

Proof. First, let us prove that σ is a metric on X.

Let x, y, z be in X.

- $\sigma(x, x) = \rho(x, x) + \frac{1}{\alpha}\rho(Tx, Tx) + \cdots + \frac{1}{\alpha^{n-1}}\rho(T^{n-1}x, T^{n-1}x) = 0.$
- $$\begin{aligned}\sigma(x, y) &= \rho(x, y) + \frac{1}{\alpha}\rho(Tx, Ty) + \cdots + \frac{1}{\alpha^{n-1}}\rho(T^{n-1}x, T^{n-1}y)\\ &\geq \rho(x, y) > 0 \; if \; x \neq y.\end{aligned}$$
- $$\begin{aligned}\sigma(x, y) &= \rho(x, y) + \frac{1}{\alpha}\rho(Tx, Ty) + \cdots + \frac{1}{\alpha^{n-1}}\rho(T^{n-1}x, T^{n-1}y)\\ &= \rho(y, x) + \frac{1}{\alpha}\rho(Ty, Tx) + \cdots + \frac{1}{\alpha^{n-1}}\rho(T^{n-1}y, T^{n-1}x)\\ &= \sigma(y, x).\end{aligned}$$
- $$\begin{aligned}\sigma(x, z) &= \rho(x, z) + \frac{1}{\alpha}\rho(Tx, Tz) + \cdots + \frac{1}{\alpha^{n-1}}\rho(T^{n-1}x, T^{n-1}z)\\ &\leq (\rho(x, y) + \rho(y, z)) + \frac{1}{\alpha}(\rho(Tx, Ty) + \rho(Ty, Tz))\\ &\quad + \cdots + \frac{1}{\alpha^{n-1}}(\rho(T^{n-1}x, T^{n-1}y) + \rho(T^{n-1}y, T^{n-1}z))\\ &= \rho(x, y) + \frac{1}{\alpha}\rho(Tx, Ty) + \cdots + \frac{1}{\alpha^{n-1}}\rho(T^{n-1}x, T^{n-1}y)\\ &\quad + \rho(y, z) + \frac{1}{\alpha}\rho(Ty, Tz) + \cdots + \frac{1}{\alpha^{n-1}}\rho(T^{n-1}y, T^{n-1}z)\\ &= \sigma(x, y) + \sigma(y, z).\end{aligned}$$

Thus, σ is a metric on X. Now, let us prove that T satisfies the claimed condition $\sigma(Tx, Ty) \leq \alpha\sigma(x, y)$ for x, y in X.

$$
\begin{aligned}
\sigma(Tx,Ty) &= \rho(Tx,Ty)+\tfrac{1}{\alpha}\rho(T^2x,T^2y)+\cdots+\tfrac{1}{\alpha^{n-1}}\rho(T^nx,T^ny)\\
&\leq \rho(Tx,Ty)+\tfrac{1}{\alpha}\rho(T^2x,T^2y)+\cdots+\alpha\rho(x,y)\\
&= \alpha\rho(x,y)+\rho(Tx,Ty)+\cdots+\tfrac{1}{\alpha^{n-2}}\rho(T^{n-1}x,T^{n-1}y)\\
&= \alpha\left(\rho(x,y)+\tfrac{1}{\alpha}\rho(Tx,Ty)+\cdots+\tfrac{1}{\alpha^{n-1}}\rho(T^{n-1}x,T^{n-1}y)\right)\\
&= \alpha\sigma(x,y)\,.
\end{aligned}
$$

$\square$

CHAPTER 3

BANACH SPACES

This chapter focuses on fixed point theorems relating to Banach spaces.

As several of the theorems and lemmas in this section use uniformly convex Banach spaces, we give several definitions and properties that are useful for these spaces.

Definition 3.0.20 (Uniformly Convex Banach Space [2, p. 51]). A Banach space X is said to be uniformly convex if for all $2 \geq \epsilon > 0$, there exists a $\delta > 0$ such that for every x, y in X, if $\|x\| \leq 1,\ \|y\| \leq 1$ and $\|x - y\| > \epsilon$, then $\left\|\frac{x+y}{2}\right\| \leq (1 - \delta)$.

Definition 3.0.21 (Weak Convergence). A sequence $\{x_n\}$ in a uniformly convex Banach space X is said to converge weakly to x in X if and only if $\lim_{n\to\infty} \langle x_n, x^* \rangle = \langle x, x^* \rangle$ for each x^* in X^*. Here, X^* is the dual space of X, defined by $X^* = \{T : X \to \mathbb{R} : T$ is a bounded linear functional $\}$.

The following property gives us a way to measure the convexity of our space.

Definition 3.0.22 (Modulus of Convexity [2, p. 52]). The modulus of convexity of a Banach space X is the function $\delta_X : [0, 2] \to [0, 1]$ defined by $\delta_X(\epsilon) = \inf\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}$.

Remark 3.0.23 ([2, p. 52]). Let $\epsilon > 0$. Then, $\delta_X(\epsilon) =$ $\sup\{A$ in $\mathbb{R}$: for x, y in X, if $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \epsilon$, then $\left\|\frac{x+y}{2}\right\| \leq 1 - A\}$.

Remark 3.0.24 ([2, p. 53]). Remark 3.0.23 is equivalent to the following:

For x, y, p in $X, R > 0$, and r in $[0, 2R]$, if $\|x - p\| \leq R$, $\|y - p\| \leq R$, and $\|x - y\| \geq r$, then $\left\|p - \frac{1}{2}(x + y)\right\| \leq \left(1 - \delta\left(\frac{r}{R}\right)\right) R$.

The following theorem is needed to prove Theorem 3.0.28.

Theorem 3.0.25. *Let $\overline{B_r}$ be the closed ball of radius $r > 0$, centered at zero, in a Banach space E with $F : \overline{B_r} \to E$ a contraction and $F\left(\partial \overline{B_r}\right) \subseteq \overline{B_r}$. Then F has a unique fixed point in $\overline{B_r}$.*

Proof. [1, p. 3-4] Consider

$$G(x) = \frac{x + F(x)}{2}.$$

We first show that $G : \overline{B_r} \to \overline{B_r}$. To see this let

$$x^{\star} = r\frac{x}{\|x\|} \text{ where } x \text{ is in } \overline{B_r} \text{ and } x \neq 0.$$

Now if x is in $\overline{B_r}$ and $x \neq 0$,

$$\|F(x) - F(x^{\star})\| \leq L\|x - x^{\star}\| = L(r - \|x\|),$$

since $x - x^{\star} = \frac{x}{\|x\|}(\|x\| - r)$, and as a result

$$\begin{aligned} \|F(x)\| &\leq \|F(x^{\star})\| + \|F(x) - F(x^{\star})\| \\ &\leq r + L(r - \|x\|) \\ &\leq 2r - \|x\|. \end{aligned}$$

Then for x in $\overline{B_r}$ and $x \neq 0$

$$\|G(x)\| = \left\|\frac{x + F(x)}{2}\right\| \leq \frac{\|x\| + \|F(x)\|}{2} \leq r.$$

In fact by continuity we also have

$$\|G(0)\| \leq r,$$

and consequently $G : \overline{B_r} \to \overline{B_r}$. Moreover $G : \overline{B_r} \to \overline{B_r}$ is a contraction since

$$\|G(x) - G(y)\| \leq \frac{\|x - y\| + L\|x - y\|}{2} = \frac{[1 + L]}{2}\|x - y\|.$$

Theorem 1.0.14 implies that G has a unique fixed point u in $\overline{B_r}$. Of course if $u = G(u)$ then $u = F(u)$. □

The following theorem is needed to give a result for use in Theorem 3.0.29. This result is nice in that it gives a midpoint property given two points in the space.

Theorem 3.0.26. *Suppose K is a nonempty, bounded, convex subset of a uniformly convex Banach space X and suppose $T : K \to X$ is nonexpansive. Then for $\{u_n\}, \{v_n\}$ in K and $z_n = \frac{1}{2}(u_n + v_n)$, if both $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$ and $\lim_{n\to\infty} \|v_n - Tv_n\| = 0$, then $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$.*

Proof. [2, p. 109] Suppose not. Then, there exist sequences $\{u_n\}, \{v_n\}$ in K with

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0$$

$$\lim_{n\to\infty} \|v_n - Tv_n\| = 0$$

and with $z_n = \frac{1}{2}(u_n + v_n)$ and $\epsilon > 0$ satisfying $\|z_n - Tz_n\| \geq \epsilon$ for all n in $\mathbb{N}$. Thus, we may pass to a subsequence and obtain that for some $r > 0$, $\lim_{n\to\infty} \|u_n - z_n\| = \lim_{n\to\infty} \|v_n - z_n\| = r$. Let $d = diam(K)$ and choose $t > 0$ so that $t < \epsilon/d$. Then clearly $t < \epsilon/\|u_n - z_n\|$ and so for n sufficiently large,

$$t < \epsilon/\left[\|u_n - Tu_n\| + \|u_n - z_n\|\right].$$

Also,

$$\|u_n - Tz_n\| \leq \|u_n - Tu_n\| + \|Tu_n - Tz_n\| \leq \|u_n - Tu_n\| + \|u_n - z_n\|.$$

Since the above inequalities hold if u_n is replaced by v_n, we have (for large n) by Remark 3.0.24:

$$\begin{aligned} \|u_n - v_n\| &\leq \left\|u_n - \tfrac{1}{2}(z_n - Tz_n)\right\| + \left\|v_n - \tfrac{1}{2}(z_n - Tz_n)\right\| \\ &\leq \left[(\|u_n - Tu_n\| + \|u_n - z_n\|)\right. \\ &\quad \left. + (\|v_n - Tv_n\| + \|v_n - z_n\|)\right](1 - \delta(t)). \end{aligned}$$

Letting $n \to \infty$ we obtain the contradiction

$$2r \leq 2r(1 - \delta(t)).$$

□

The following result is used in Theorem 3.0.29.

Theorem 3.0.27. *Suppose K is a nonempty, bounded, closed and convex subset of a uniformly convex Banach space X and suppose $T : K \to X$ is a nonexpansive mapping wich satisfies* $\inf\{\|x - Tx\| : x \text{ is in } K\} = 0$. *Then T has a fixed point in K.*

Proof. [2, p. 109-110] Let $B_r = B_r(0) = \{x \text{ in } X : \|x\| < r\}$. Let R denote the set of numbers $r > 0$ for which $B_r \cap K \neq \emptyset$ and $\inf\{\|x - Tx\| : x \text{ is in } B_r \cap K\} = 0$, and let $r_0 = \inf\ R$. Since K is bounded, $r_0 < \infty$, and if $r_0 = 0$, then 0 is in K and $T0 = 0$. So we may suppose $r_0 > 0$. It is clearly possible to select x_n in $B_{r_0+1/n} \cap K$ for each n so that $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. Since any strongly convergent subsequence of $\{x_n\}$ would have a fixed point of T as its limit, we may suppose there exists $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\left\|x_{n_k} - x_{n_{k+1}}\right\| \geq \epsilon, k = 1, 2, \ldots$. For each k, let $m_k = \frac{1}{2}\left(x_{n_k} - x_{n_{k+1}}\right)$. Then if $t > 0$ is any number smaller than $\epsilon/r_0, t \leq \epsilon/\|x_k\|$ for k sufficiently large and thus, by Remark 3.0.24,

$$\|m_k\| \leq (r_0 + (1/n_k))(1 - \delta(t)).$$

Thus $\limsup\limits_{k\to\infty} \|m_k\| \leq r_0(1 - \delta(t)) < r_0$. Since Theorem 3.0.26 implies $\lim\limits_{k\to\infty} \|m_k - Tm_k\| = 0$, this contradicts the definition of r_0. □

The following theorem, from [1, p. 9], is useful in obtaining a fixed point for a function on a closed ball centered at the origin. Basically, if the function behaves as an odd function on the boundary of the closed ball, it has a fixed point in the closed ball.

Theorem 3.0.28. *Let $\overline{B_r}$ be the closed ball of radius $r > 0$, centered at zero, in a Banach space E with $f : \overline{B_r} \to E$ a contraction and $F(-x) = -F(x)$ for x in $\partial\overline{B_r}$. Then F has a fixed point in $\overline{B_r}$.*

Proof. Let x be in $\partial\overline{B_r}$. Then, $2\|F(x)\| = \|F(x) - F(-x)\|$ $\leq L\|x - (-x)\| = 2L\|x\| = 2Lr < 2r$. Hence, $\|F(x)\| < r$. Thus, $F(x)$ is in $\overline{B_r}$. By Theorem 3.0.25, F has a unique fixed point u in $\overline{B_r}$. □

The next theorem, from [1, p. 18], gives a result as important as Theorem 1.0.14, but for uniformly convex Banach spaces and nonexpansive maps.

Theorem 3.0.29. *Let X be a uniformly convex Banach space and K a nonempty, closed, convex, bounded subset of X. Then every nonexpansive map $F : K \to K$ has a fixed point.*

Proof. Assume that 0_X is in K. (A modified argument from the one given below holds for any x_0 in K, therefore for simplicity we let $x_0 = 0_X$.) Also assume that $F(0_X) \neq 0_X$ (otherwise we are finished). For each $n = 2, 3, \ldots$, notice that $F_n := \left(1 - \frac{1}{n}\right) F : K \to K$ is a contraction. Now Theorem 1.0.14 guarantees that there exists a unique x_n in K with $x_n = F_n(x_n) = \left(1 - \frac{1}{n}\right) F(x_n)$. Thus $\|x_n - F(x_n)\| = \frac{1}{n}\|F(x_n)\| \leq \frac{1}{n}\delta(K)$, where $\delta(K)$ denotes the diameter of K. For each n in $\{2, 3, \ldots\}$, let $Q_n = \{x \text{ in } K : \|x - F(x)\| \leq \frac{1}{n}\delta(K)\}$. Now, $Q_2 \supseteq Q_3 \supseteq \cdots \supseteq Q_n \supseteq \cdots$ is a decreasing sequence of nonempty (given that for every $N \geq n$, x_N is in Q_n) closed sets, each of which is a subset of K. Let $d_n = \inf\{\|x\| : x \text{ is in } Q_n\}$ and since the $Q_n s$ are decreasing, we have $d_2 \leq d_3 \leq \cdots \leq d_n \leq \cdots$, with $d_i \leq \delta(K)$ for each i in $\{2, 3, \ldots\}$. Consequently, $d_n \uparrow d$ with $d \leq \delta(K)$. Next let $A_n = Q_{8n^2} \cap \overline{B(0_X, d + 1/n)}$, where $B(0_X, d + 1/n) = \{x \text{ in } X : \|x - 0_X\| < d + \frac{1}{n}\}$. Now A_n is a decreasing sequence of closed, nonempty sets, each of which is a subset of K. We now show that $\inf\{\|x - F(x)\| : x \text{ is in } K\} = 0$[1, p. 14-15]. To see this, let n be in $\mathbb{N}$ and let u_n be in A_n. Then, $\|u_n - 0_X\| \leq d + \frac{1}{n}$. Also, since u_n is in Q_{8n^2}, we have $\|u_n - F(u_n)\| \leq \frac{1}{8n^2}\delta(K)$. Thus, $\{u_n\}$ is a sequence in K such that $\lim_{n\to\infty} \|u_n - F(u_n)\| = 0$. So, we have that $\inf\{\|x - F(x)\| : x \text{ is in } K\} = 0$. Theorem 3.0.27 gives us that F has a fixed point in K, which is what we wanted to show. If 0_X is not in K, let x_0 be in K and replace all mention of 0_X in the above proof with x_0. □

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