MAXWELL'S EQUATIONS FROM ELECTROSTATICS AND EINSTEIN'S GRAVITATIONAL FIELD EQUATION FROM NEWTON'S UNIVERSAL LAW OF GRAVITATION USING TENSORS

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Maxwell's equations are obtained from Coulomb's Law using special relativity. For the derivation, tensor analysis is used, charge is assumed to be a conserved scalar, the Lorentz force is assumed to be a pure force, and the principle of superposition is assumed to hold.

Einstein's gravitational field equation is obtained from Newton's universal law of gravitation. In order to proceed, the principle of least action for gravity is shown to be equivalent to the maximization of proper time along a geodesic. The conservation of energy and momentum is assumed, which, through the use of the Bianchi identity, results in Einstein's field equation.

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CHAPTER 1

INTRODUCTION

Thousands of years ago, the more advanced ancient societies were aware of the forces of electricity and magnetism.[1] Several hundreds of years ago, these forces were considered to be different manifestations of the same kind of force. Ironically, this unification was dismissed for centuries as the scientific community pursued a more enlightened philosophy. It wasn't until the time of James Clerk Maxwell that the unification of these two forces was again accepted. Of course, this did not play well with the mechanics proposed by Isaac Newton.[2] The two theories, electromagnetism and classical mechanics, required different treatments when transformed from one coordinate system to another. This indicated that at least one of the two theories needed an adjustment. The first idea was to adjust Maxwell's equations to accommodate classical mechanics, but Albert Einstein saw things the other way around.[2]

Einstein used symmetry to show algebraically that length contraction and time dilation are consequences of the invariance of the speed of light.[3] To solidify the principle of relativity, he developed a geometrical picture in which tensors that transform according to Lorentz transformation matrices represent all physical objects in flat space-time.[2] In this regard, the magnetic field is a consequence of the application of special relativity to the electric field under a Lorentz transformation. Ultimately, tensor analysis can be used to derive Maxwell's equations from Coulomb's Law and the assumptions that charge is a conserved scalar and that the principle of superposition holds.[2]

After completing his theory of special relativity, Einstein realized that the next great accomplishment would be to make gravitation consistent with the special theory of relativity.[2] Mass, the source of gravitation, is very different than charge, and the analogy between gravity and electrostatics breaks down.[2] Einstein showed that the gravitational field is actually a distortion of space-time itself.[2] For general relativity, tensors had to be used so that the form of the equations of physical laws remained invariant under general coordinate transformations. This is called the principle of general covariance.[4]

Applying the principle of general covariance and the equivalence principle to Newton's universal law of gravitation gives Einstein's gravitational field equation, if energy is conserved. Poisson's equation for gravity can be generalized to Einstein's gravitational field equation, a second rank tensor equation.[2] In the limit that the sources are stationary and the fields they produce are weak, Einstein's equation reduces to Poisson's equation for gravity.[2] Therefore, Newton's law for gravity, which he presented[5] in 1686, is still a valid way to describe "weak gravitational interactions" in which the interacting bodies have constant mass and have negligible velocity compared to the speed of light. That Newton's law for gravity is still valid today in the appropriate limit demonstrates the smooth progression of classical physics over the last few centuries.

CHAPTER 2

SPECIAL RELATIVITY: DERIVATION OF MAXWELL'S EQUATIONS

2.1. Introduction

Magnetism was most likely first recognized thousands of years ago by the Chinese as a force, though most thought it to be of magical origin.[1] The source of this magic was known as lodestone. Almost as long ago, the Greeks first recognized that a force emanated from amber. This was the first observation of the electrostatic force.[1] A millennium later, in the sixteenth century during the renaissance, interest in science grew, and sparked an explosion of scientific discoveries. Charles Coulomb experimentally demonstrated in 1785 that the electrostatic force obeys an inverse square law.[1] In 1873, James Clerk Maxwell presented his *Treatise on Electricity and Magnetism* which showed a unification of the contemporary theories of electricity and magnetism into an elegant set of four equations known as Maxwell's equations.[6] Along with the elegance of the equations came the realization that electromagnetic waves should propagate away from a source at the speed of light (a phenomenon that Heinrich Hertz verified experimentally in a series of experiments conducted between 1879 and 1889).[6] In 1890, Hendrik A. Lorentz proposed transformations that maintained the form of Maxwell's equations for frames of reference moving at different velocities with respect to the source of the electromagnetic field.[7] A transformation of this type is known as a Lorentz transformation. Einstein showed in 1905 that the Lorentz transformations could be derived algebraically with the assumption that the speed of light is the same for any observer in any inertial frame of reference,[3] and so modified classical mechanics to be consistent with electromagnetism.[2]

There have been many attempts to obtain Maxwell's equations from Coulomb's Law using special relativity, though the approaches and assumptions vary. R. P. Feynman uses the scalar and vector potentials.[8] E. Krefetz indicates that a multitude of assumptions must be made in order to carry out the derivation.[9] W. Rindler gives a very similar approach, as well as using potentials to justify certain steps.[10] D. H. Frisch and L. Wilets offer a very detailed and rigorous approach.[11] One of the most significant approachs to the one given in this chapter was presented by D. H. Kobe.[2] However, there are some considerations that are given more attention in this chapter.

Tensors are discussed in general in Sec. 3. In Sec. 4 Gauss' Law for electrostatics is generalized to Gauss' Law for time varying fields and to the Ampere-Maxwell Law. In Sec. 5 the conservative nature of the electrostatic field is generalized to Faraday's Law and the law of no magnetic monopoles. The proper time introduced in Sec. 3 and the Faraday tensor developed in Secs. 4 and 5 are used to generalize Newton's second law for a charged particle to the Lorentz force in Sec. 6. Sec. 7 gives the conclusion for this chapter.

2.2. Lorentz Transformations

Einstein postulated that the speed of light should be the same for both of two observers, regardless of their relative motion (as long as neither experiences an acceleration). Using this postulate, the Lorentz transformations can be obtained. Consider two observers, A and B, moving along the x-axis. They both pass through the origin at time $t = 0$. Let A observe B to move to the right (in the $+x$ direction) at a constant speed, v. Now, assume that A sends out a light pulse isotropically from the origin at $t = 0$. Of course, observer A will consider this as a spherical wavefront, centered at herself, the radius of which increases at a rate, c. The "strange" consequence of Einstein's postulate is that B will also consider this as a spherical wavefront, centered at himself, the radius of which also increases at a rate, c. This is strange in terms of Galilean relativity. While B should consider a spherical wavefront, he should consider the center of the sphere to move to the left (in the $-x$ direction). However, according to Einstein's special theory of relativity, the center of the sphere will *not* move at all with respect to B; B will also remain at the center of the sphere. Therefore, some aspect of the common sense of Galilean relativity must change. Einstein's relativity abandons the notions of absolute time and absolute space, and it replaces them with the invariance of the speed of light in all inertial frames. The Lorentz transformations, instead of Galilean transformations, determine Cartesian coordinate transformations from one inertial frame to another.

To proceed with the derivation, it is convenient to introduce two coordinate systems, K and K', and restrict the discussion to 1 spatial dimension. Let K be defined as the coordinate system in which A is at rest and B is moving to the right at a constant velocity, v. Let K' be defined as the coordinate system in which B is at rest. Since B is moving to the right relative to A at a speed, v, then A moves to the left relative to B at a speed, v. Therefore, in K', A is moving to the left at a constant velocity, v. This has a straightforward representation in Cartesian coordinates.

In coordinate system K, let x_A be the value of the x coordinate of A's position, and let x_B be the value of the x coordinate of B's position. Then[a]

$$x_A = 0, \tag{1}$$

[a] Note: the coordinates are, in general, functions of time.

and

$$x_B = vt\,. \tag{2}$$

In coordinate system K', let x'_A be the value of the x coordinate of A's position, and let x'_B be the value of the x coordinate of B's position.[b] Then

$$x'_A = -vt'\,, \tag{3}$$

and

$$x'_B = 0\,. \tag{4}$$

Since the speed of light is invariant, the light pulse has the same coordinate representation in both coordinate systems. Along the x -axis

$$x_{c\pm} = \pm ct\,, \tag{5}$$

and

$$x'_{c\pm} = \pm ct'\,. \tag{6}$$

Equations (5) and (6) show Einstein's postulate in mathematical form. The (+) and (-) signs in equations (5) and (6) indicate a rightward and leftward traveling light pulse, respectively. Equations (1) through (6) suggest an ostensible contradiction. The right side of the light pulse relative to B in coordinate system K seems to be traveling more slowly than c relative to B, and the left side seems to be exceeding the speed c. To resolve the contradiction, one must realize that $x' = x - vt$, and that $t' \neq t$ (in

[b] Notice the prime on the time parameter in equation (3). Time is not necessarily the same from one coordinate system to another. This is how the paradox is resolved.

contradiction to the Galilean transformation that says $t' = t$, *always*). The next step[c] is to rewrite equations (5) and (6) as

$x_{c+} - ct = 0$ and $x'_{c+} - ct' = 0$

for the rightward traveling light pulse, and

$x_{c-} + ct = 0$ and $x'_{c-} + ct' = 0$

for the leftward traveling light pulse. The assumption that an affine transformation (linear in this particular example) exists between the coordinates in K and the coordinates in K' leads to[3]

$$(x_{c+} - ct) = a(x'_{c+} - ct'), \quad (7)$$

and

$$(x_{c-} + ct) = b(x'_{c-} + ct'), \quad (8)$$

where a and b are arbitrary constants. This affine transformation is assumed to apply to all events in space-time, not just those on the light cone, since a light cone can always be defined through any given event. Generalizing x_{c+} and x_{c-} to x and x'_{c+} and x'_{c-} to x', equations (7) and (8) can be rearranged to give the primed coordinates in terms of the unprimed coordinates[d]

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[c] This step is not at all obvious. The motivation is based on the desire to find a transformation for all points, not just those on the light cone.

[d] This results in two equations, four variables, and two arbitrary constants (to be determined subsequently). So, given two of the variables, the equations may be solved for the other two, up to the arbitrary constants.

$$x' = \frac{(a+b)}{2ab}x + \frac{(a-b)}{(a+b)}\frac{(a+b)}{2ab}ct$$

and

$$t' = \frac{(a-b)}{(a+b)}\frac{(a+b)}{2ab}\frac{x}{c} + \frac{(a+b)}{2ab}t\ .$$

The role of the two arbitrary constants, a and b, can be fulfilled by two different arbitrary constants, β and γ, defined as[e]

$$-\beta = \frac{(a-b)}{(a+b)} \quad \text{and} \quad \gamma = \frac{(a+b)}{2ab}$$

so that

$$x' = \gamma\, x - \beta\gamma\, ct\ , \tag{9}$$

and

$$t' = -\beta\gamma\,\frac{x}{c} + \gamma\, t\ . \tag{10}$$

The arbitrary constant, β, can be found by using the primed and unprimed coordinates to describe the position of observer B. Combining equation (9) with equations (2) and (4) gives

$$\beta = \frac{v}{c}\ . \tag{11}$$

To find the arbitrary constant, γ, symmetry must be used. Equations (9) and (10) give the coordinates of an event as seen by observer B, in terms of the coordinates as seen

[e] This does not change the generality, since the number of arbitrary constants does not change. These particular symbols are chosen in anticipation of the result and recognition of the standard usage.

by observer A, given that B is moving with a velocity, v, in the x direction as seen by observer A. The symmetry suggests that there should also be two such equations to tell B where an event will appear in A's perspective. The coordinate values should also match between the two sets of equations if they are to be meaningful. Equation (11) is used to replace β in equations (9) and (10).

$$x' = \gamma x - v\gamma t \,, \tag{12}$$

and

$$t' = -v\gamma \frac{x}{c^2} + \gamma t \,. \tag{13}$$

Solving equations (12) and (13) for x and t gives

$$x = \frac{c^2}{\gamma (c^2 - v^2)} x' + v \frac{c^2}{\gamma (c^2 - v^2)} t' \,, \tag{14}$$

and

$$t = v \frac{c^2}{\gamma (c^2 - v^2)} x' + \frac{c^2}{\gamma (c^2 - v^2)} t' \,. \tag{15}$$

Since the velocity of A relative to K' is the opposite of the velocity of B relative to K, the v's must be multiplied by -1 in equations (14) and (15) in order to compare the form to equations (12) and (13). In doing this, it is seen that all four factors agree so that γ must satisfy

$$\gamma = \frac{c^2}{\gamma (c^2 - v^2)} \,.$$

This gives

$$\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}} \,. \tag{16}$$

Finally, note that no transformation of the y or z coordinates occur, so that

$$\begin{aligned} y' &= y \\ z' &= z \end{aligned}. \tag{17}$$

Equations (12), (13), and (17), together with the definitions (11) and (16), give the primed coordinates of an event in space-time in terms of the unprimed coordinates.[f] In matrix notation, this can be written as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}. \tag{18}$$

The inverse of the matrix in equation (18) represents a boost in the opposite direction, which is accomplished by changing the (-) signs to (+) signs.

2.3. Tensors

The previous section emphasized an algebraic derivation of the Lorentz boost. However, there is a much more geometrical interpretation of the Lorentz boost. The benefit of a geometrical interpretation is that it provides an intuitive understanding of the effects of a Lorentz boost on the coordinates (length contraction and time dilation). To develop an understanding of the representation of physical objects by tensors, consider three immediate examples of three different tensor ranks: the speed of light as a scalar,

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[f] The primed coordinates are the coordinates of the event as seen by an observer moving with a speed v in the x direction as seen with respect to the unprimed coordinate system. Since the orientations of the coordinate systems are arbitrary, they can always be rotated so that their x axes coincide with the direction of the boost.

an event (or, more appropriately, a displacement) in space-time as a 4-vector, and the metric as a second rank tensor.

2.3a. Scalar Fields (0^{th} Rank Tensors)

The speed of light is a constant scalar field in special relativity. Scalar fields are tensors of rank 0, and vice versa. Being a field means that the object is defined over a connected region of space-time. Being a constant means that the object has the same value at every point in the region of interest.

Scalar fields are invariant to Lorentz transformations. Consider an arbitrary point in space-time, (also called an "event"), represented in shorthand notation by, x^{μ}. Let an arbitrary scalar field, ϕ, take the value, $\phi\left(x^{\mu}\right)$, at the point x^{μ}. Let primes indicate transformed values under some Lorentz transformation. Since ϕ is a scalar field, it satisfies the relationship

$$\phi'\left(x'^{\alpha}\right) = \phi\left(x^{\mu}\right). \tag{19}$$

Equation (19) defines ϕ as a tensor of rank 0. This means that the value of the scalar field at an arbitrary point in space-time transforms to the same value at the transformed point in space-time under a Lorentz transformation. In other words, changing the space-time perspective (coordinate system) does not change the numerical value of the scalar field at any given point in space-time.

2.3b. Vector Fields (1^{st} Rank Tensors)

Increasing the rank of the tensor increases the level of complexity. Equation (18) can be written in component form as

$$x'^{\mu} = \sum_{\nu=0}^{3} a^{\mu}{}_{\nu} x^{\nu} , \tag{20}$$

where x'^{μ} is the μ^{th} component of the four-component column vector on the L.H.S. of equation (18), x^{ν} is the ν^{th} component of the four-component column vector on the R.H.S. of equation (18), and $a^{\mu}{}_{\nu}$ is the component in the μ^{th} row and ν^{th} column of the matrix in equation (18).[g] The scripts run from 0 to 3.[h] Einstein's summation convention provides a shorthand notation for equation (20). A summation from 0 to 3 over any Greek-letter index that appears once as a superscript and once as a subscript in a given term is implied in Einstein's summation convention.[12] Using this convention, equation (20) can be written as[12] [i]

$$x'^{\mu} = a^{\mu}{}_{\nu} x^{\nu} . \tag{21}$$

This kind of summation is called "contraction." An event in space-time can be represented by a position 4-vector, which can in turn be represented by one temporal component (ct), and three spatial components (x, y, and z). The first component of

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[g] In other words, equation (20) is the operation of a matrix on a vector in linear algebra written out explicitly.

[h] Starting the index at 0 is merely a convention. This convention will be used in this paper since it is ubiquitous in the literature.

[i] There is no physical significance to what specific Greek letter is used for any given index. The letters μ and ν happen to be popular choices. What is physically significant is whether or not the specific letter is repeated and therefore involved in a summation.

the position 4-vector,[j] x^0, is identified with the product of the speed of light with the time of the event, ct. The other three components of the position 4-vector, x^1, x^2, and x^3, are arbitrarily identified with the Cartesian components of the position 3-vector that represents the point in space at which the event occurs, namely, (x, y, z). This refers explicitly to the x^ν on the R.H.S. of equation (21). The x'^μ on the L.H.S. is treated similarly, but with respect to components in the primed coordinate system.

It is more appropriate to express equation (21) in terms of differentials.[k]

$$dx'^\mu = a^\mu{}_\nu dx^\nu, \tag{22}$$

where the coordinates refer to some trajectory represented as a curve in 4-dimensional space-time.[l] Equation (22) establishes a paradigm[2] for a tensor of rank 1 in the context of special relativity (Minkowski space-time)[12] in terms of its components (the components in equation (22) are the contravariant components). 4-vectors are tensors of rank 1, and vice versa.[2] The components, $a^\mu{}_\nu$, are the corresponding components of

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[j] Do not confuse these upper indices with exponents. Exponentiation will be indicated by first putting the variable in parenthesis, unless the context makes the use as an exponent obvious.

[k] The reason behind this, though not extremely complicated, is deferred to the next chapter in the discussion of curved manifolds. As a brief justification, it involves the more appropriate concept of a tangent vector from differential geometry.

[l] It is chosen as a scalar so that it remains unaffected by the Lorentz transformation matrix and is thus a global parameter.

the Jacobian transformation matrix. In other words, equation (22) just shows the application of the chain rule in multivariable calculus such that

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}, \qquad (23)$$

where Einstein's summation convention is being used. Equation (23) shows the geometrical meaning of the Lorentz transformations.

A tensor of rank 1 is similar to a vector in linear algebra in the sense that the matrix multiplication of a vector has the form of equation (23). However, there is a stronger requirement on 4-vectors in the geometrical context of tensors. The contravariant components of a tensor of rank 1 (4-vector) must transform according to equation (22) under a Lorentz transformation. That is, no extra terms should result from a Lorentz transformation than those that appear in equation (22).

The components of a vector field transform under a Lorentz transformation. Consider an arbitrary point in space-time represented in shorthand notation by, x^{ν}. Let the contravariant components of an arbitrary vector field, A, take the values, $A^{\mu}\left(x^{\nu}\right)$, at the point x^{ν}. Let primes indicate transformed values under some particular Lorentz transformation. Since A^{μ} are the contravariant components of a vector field, they satisfy the relationship

$$A'^{\alpha}\left(x'^{\beta}\right) = a^{\alpha}{}_{\mu} A^{\mu}\left(x^{\nu}\right). \qquad (24)$$

Equation (24) defines A as a tensor of rank 1. This means that the value of the contravariant components of a vector field at an arbitrary point in space-time transform into a contraction with components of the Lorentz transformation matrix at the transformed point in space-time under a Lorentz transformation. In a sense, the value of the tensor itself does not change. However, the need to represent the tensor as a set of 4 numbers requires a transformation of these four numbers according to equation (24), much like rotating the coordinate axes changes the individual components of a vector without actually changing the vector itself.

The components of a tensor can also take the covariant form. The paradigm for this form of component is the partial derivative operator, which can be written using the shorthand

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}. \tag{25}$$

An important difference in the notation for covariant components is that the index is written as a subscript, rather than a superscript. The transformation rule for such components is similar to equation (24), except that the components have subscripts instead of superscripts and the indices on the components of the transformation matrix are switched:

$$A'_{\alpha}\left(x'^{\beta}\right) = a^{\mu}{}_{\alpha} A_{\mu}\left(x^{\nu}\right). \tag{26}$$

The two different forms of components are discussed in more detail in the next two subsections.

2.3c. Second Rank Tensors

In order to have a sense of scale ("closeness"), a scalar product[m] is defined using a tensor of rank 2 called the metric tensor. In special relativity there are four dimensions, so a tensor of rank 2 will have 4^2 = 16 components, in general. These

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[m] The scalar product is a generalization of the dot product to spaces that are not Euclidean.

tensors seem quite similar to 4x4 matrices in linear algebra, but there are several differences that make this comparison dangerously confusing. One confusing similarity between second rank tensors and 4x4 matrices is the compact manner of listing the components in four rows and four columns. This makes a second rank tensor look much like a matrix, but, keep in mind that the tensor is an object independent of the particular arrangement of components. However, out of convenience, the components of the metric tensor in special relativity are usually given as[12]

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{27}$$

The scalar product is written in component form as

$$\mathbf{u} \cdot \mathbf{v} = \eta_{\mu\nu} u^{\mu} v^{\nu}, \tag{28}$$

where $\mathbf{u}$ and $\mathbf{v}$ are arbitrary 4-vectors, and u^{μ} and v^{ν} are their contravariant components.[n] The scalar product is a true scalar in the sense that it is a tensor of rank 0. As an example, consider the squared distance from the origin to a point in space. In 3-dimensional Euclidean space, this can be found from the Cartesian components as the sum of the squares of the components. This is actually the scalar (dot) product of the position vector with itself. In flat space-time (special relativity), the distance is generalized to something called the proper distance, but is fundamentally the same: it is the scalar product of the position 4-vector with itself. However, in space-time this is

[n] Don't forget that, since the Greek indices appear once as a subscript and once as a superscript, they are summed over, from 0 to 3. This summation is called contraction when both factors in the summation are components of tensors.

not exactly the sum of the squares. Using equation (28) with the definition from equation (27), the square of the differential proper distance is given by

$$\mathbf{dx} \cdot \mathbf{dx} = \eta_{\mu\nu} dx^{\mu} dx^{\nu}$$

$$= \left(dx^0\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 =$$

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 = ds^2 \,, \tag{29}$$

where ds is the differential proper distance. Notice that the surface, $ds = 0$, gives a 4-dimensional cone. This is called the light cone,[o] because it represents the set of all events in space-time through which a ray of light could pass, given that it passes through the origin of 3-dimensional space at time $t = 0$. If $ds \neq 0$, then equation (29) gives a 4-dimensional hyperboloid.[p]

Derived from the differential proper distance is the fundamentally important scalar called the differential proper time, which is literally just the differential proper distance divided by the speed of light.

$$d\tau = \frac{1}{c}\sqrt{c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right)}, \tag{30}$$

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[o] It is called a "cone," but don't forget that this is a 4-D cone. Picture a sphere that shrinks down to a point and then immediately expands again. If that point is the origin, and the radius of the sphere shrinks and expands at the constant rate, c, then this shrinking and expanding sphere is the light cone. If space were two dimensional, then this could be represented as the familiar version of a cone.

[p] That is why the hyperbolic functions are a very straightforward way to express the Lorentz transformations.

where $d\tau$ is the differential proper time. This gives the time experienced by an observer who undergoes a given displacement in space-time. This is the parameter used to generalize the time derivative in many physical laws.

The usefulness of second rank tensors has been demonstrated by the Minkowski metric tensor. In general, the components of a second rank tensor field follow a certain transformation rule. Consider an arbitrary point in space-time represented in shorthand notation by, x^{κ}. Let the contravariant components of an arbitrary second rank tensor field, B, take the values, $B^{\mu\nu}\left(x^{\kappa}\right)$, at the point x^{κ}. Let primes indicate transformed values under some particular Lorentz transformation. Since $B^{\mu\nu}$ are the contravariant components of a second rank tensor field, they satisfy the relationship

$$B'^{\alpha\beta}\left(x'^{\lambda}\right) = a^{\alpha}{}_{\mu} a^{\beta}{}_{\nu} B^{\mu\nu}\left(x^{\kappa}\right). \qquad (31)$$

Equation (31) defines B as a tensor of rank 2. The contravariant components of a second rank tensor field at an arbitrary point in space-time transform into a contraction of each index with components of the Lorentz transformation matrix at the transformed point in space-time under a Lorentz transformation. Again, the tensor itself does not change, but the contravariant components change according to equation (31).

At this point one should notice that there is one Lorentz transformation factor for every rank of the tensor. As in the case for the covariant components of a first rank tensor, in the transformation rule for the covariant components of a second rank tensor, the indices on the components of the Lorentz transformation matrices are switched. A more detailed discussion of the distinction between contravariant form and covariant form follows in the next subsection.

2.3d. Superscripts vs. Subscripts

There is a shorthand notation for the contraction of the metric tensor with another tensor,[q] called raising and lowering an index, or "index gymnastics."[2] Since the metric tensor plays a fundamental roll in the analysis, it frequently appears in the formulae. Rather than explicitly write it out, it is more convenient to use this shorthand. If the metric tensor is contracted with the superscript of another tensor, then the superscript is written as a subscript and the metric tensor is omitted. As an example, the scalar product can be simplified using the shorthand as

$$\eta_{\mu\nu} dx^{\mu} dx^{\nu} = dx_{\nu} dx^{\nu} = dx^{\mu} dx_{\mu} \,. \tag{32}$$

This introduces a new kind of component, which is indexed with a subscript, called covariant. Intuitively, the contravariant components (with the superscript) refer to the coordinates in terms of the constant coordinate hyperplanes, while the covariant components (with the subscript) refer to the coordinates as measured along the axes (which are the intersections of all other hyperplanes). This gives the scalar product the intuitive meaning of the number of hyperplanes through which a line segment passes.[13] The partial derivative is the paradigm for the covariant component,[2] since it holds all other variables fixed, and thus "moves along the coordinate axis." In Cartesian coordinates, the distinction is trivial, since the coordinate system is orthogonal. In special relativity there is a non-trivial distinction, in that, changing between contravariant and covariant form changes the sign on the 1st, 2nd, and 3rd components. This can be

[q] This is actually more than just a convenient shorthand; it shows that the contraction of the components of the metric tensor with the contravariant components of a 4-vector gives the covariant components of the 4-vector.

seen by comparing equation (29) with equation (32). Since the partial derivative is the paradigm for the covariant component, it is given a special symbol.

$$\frac{\partial f}{\partial x^{\mu}} \equiv \partial_{\mu} f \,, \tag{33}$$

where f is a scalar function of the position in space-time. Notice that the symbol takes a subscript, indicating that it is covariant. So, the differential is the paradigm for the contravariant form of a vector, and the partial derivative is the paradigm for the covariant form of a vector.

The components of the metric tensor also have a contravariant form that raises the index of covariant components to make them contravariant. The different types of components of the metric tensor satisfy the relationship:[12]

$$\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^{\mu}{}_{\nu} \,, \tag{34}$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker Delta.[r] Notice that the two forms of the components of the metric tensor are contracted on one index, leaving one free index[s] from each tensor. In general, the contraction takes away one rank from each tensor, so, the components of two 2nd rank tensors contracted on one of their indices leaves 2 - 1 + 2 -1 = 2 indices, and therefore results in components of a 2nd rank tensor. The contraction of the components of two 1st rank tensors (4-vectors) results in a tensor of rank 0 (scalar). Also notice that the result preserves the placement of the remaining indices as a subscript or superscript. It turns out that the covariant components of the Minkowski

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[r] The Kronecker Delta can be thought of as the mixed components of the metric tensor of space-time.

[s] A free index is an index that is not involved in the contraction.

metric tensor are the same as the contravariant components. So, equation (23) also represents the covariant components.

There is another 2nd rank tensor, called the Faraday tensor, which encapsulates the electric and magnetic field into one physical object. It is developed from first principles of electrostatics in the next three sections.

2.4. Generalizing Electrostatics to the Time Dependent Gauss' Law and the Ampere Maxwell Law

Here, a brief review of electrostatics is given. The goal is to propose the minimum number of physical postulates in order to generalize to a relativistic version of the electromagnetic field. This generalization results in a second rank tensor, called the electromagnetic field strength tensor, or Faraday tensor. Maxwell's equations follow as a natural consequence of this generalization.

Coulomb's law states that the force between two charges is proportional to the product of the charges and inversely proportional to the square of the distance between them. For two charges, q and q', located at points in space represented by the position vectors, $\vec{x}$ and $\vec{x}'$, respectively, Coulomb's law gives the force, $\vec{F}$, experienced by the charge, q, as

$$\vec{F} = \frac{qq'}{4\pi\varepsilon_0}\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^3}, \tag{35}$$

where ε_0 is the permittivity of free space. The electric field, $\vec{E}$, at the point in space represented by the position vector, $\vec{x}$, due to charge, q', is defined as

$$\vec{E} = \frac{\vec{F}}{q}. \tag{36}$$

If the source charge, q', is generalized to the charge density at a point in space, $\vec{x}'$, as a function of the position vector that represents that point, $dq' = d^3x'\,\rho(\vec{x}')$, where d^3x' is a small volume element, then, using the *principle of superposition*, the electric field, $\vec{E}(\vec{x})$, is given as an integral over all of space.

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} d^3x'\,\rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}. \tag{37}$$

The divergence and curl of the electric field can be found by taking the divergence and curl of the integral in equation (37). As a result[t]

$$\nabla\cdot\vec{E}(\vec{x}) = \frac{1}{\varepsilon_0}\rho(\vec{x}), \tag{38}$$

and,

$$\nabla\times\vec{E}(\vec{x}) = 0. \tag{39}$$

Equations (38) and (39) are the equations of electrostatics that follow from Coulomb's Law. The electric charge density, $\rho(\vec{x})$, is the 0th contravariant component of a 4-vector.[u,14] This 4-vector is called the charge-current density 4-vector.[14] The contravariant components of the charge-current density 4-vector are:[14] $j^0 = \rho c$, $j^1 = j_x$, $j^2 = j_y$, and $j^3 = j_z$. Notice the similarity to the assignment of the components in the position 4-vector. This indicates that the electric charge density transforms in the same way that time transforms under a Lorentz transformation.

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[t] The derivation for equations (38) and (39) can be found in Appendix A.

[u] The justification for the static charge being the 0th component of a 4-vector is shown in Appendix B.

Consider the contraction of the partial derivative 4-vector with the charge-current density 4-vector, $\partial_\mu j^\mu$. This is called the 4-divergence of the charge-current density. From Section 3.d, this contraction can be recognized as a scalar field. Consider a static distribution of charge in its own rest frame, so that $j^i = 0$. In this case the scalar field is just the time derivative of the charge density. Since charge is a conserved quantity, then the time rate of change of the static charge distribution is

$$\frac{\partial\rho}{\partial t} = \frac{\partial(\rho c)}{\partial(ct)} = \frac{\partial j^0}{\partial x^0} = 0 \tag{40}$$

in its own rest frame. Since the contraction is a scalar field, then in any coordinate system

$$\partial_\mu j^\mu = 0 . \tag{41}$$

Equation (41) is the continuity equation,

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\vec{j} , \tag{42}$$

where $\vec{j}$ is the 3-dimensional electric current density vector.[v]

Replacing the charge density, ρ, in equation (38) with the 0^{th} component of the charge-current density 4-vector,

$$\nabla\cdot\vec{E}(\vec{x}) = \frac{1}{\varepsilon_0 c} j^0(\vec{x}). \tag{43}$$

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[v] Physically, equation (42) says that, if there is an increase or decrease of charge at a point in space, then there is a net current flow into or out of that region of space that contains the charge.

Since the R.H.S. of equation (43) transforms as the 0th component of a 4-vector, so must the L.H.S. The divergence of the electric field generalizes to a contraction of the partial derivatives with the contravariant components of a tensor. This, along with the 4-vector transformation property of the L.H.S., requires that the components of the electric field be generalized to components of a 2nd rank tensor.[14]

$$\begin{aligned} E_x &= F^{10} \\ E_y &= F^{20} \\ E_z &= F^{30} \end{aligned} \tag{44}$$

so that

$$\nabla \cdot \vec{E} = \partial_\mu F^{\mu 0} - \partial_0 F^{00} . \tag{45}$$

These new components that have been introduced are components of the Faraday tensor. Again, since equation (45) must transform as the 0th component of a 4-vector, the last term, $-\partial_0 F^{00}$, must vanish.[w] Equation (43) then becomes

$$\partial_\mu F^{\mu 0} = \frac{1}{\varepsilon_0 c} j^0 . \tag{46}$$

Substituting the definitions that have been used so far in this section, equation (46) is a generalization of equation (38) to a time-dependent[x] electric field and current

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[w]The last term in equation (45) must vanish because it transforms as the 000 mixed component of a 3rd rank tensor, and therefore does not transform as the 0th component of a 4-vector.

[x] The time dependence is a consequence of the requirement that the physical objects must be tensor fields, which must be defined as functions of space-*time*, not just functions of space, in order to be Lorentz invariant.

density. In order to have form invariance under Lorentz transformations, the other three components of the 4-vector must be included. Therefore, equation (46) generalizes to[14]

$$\partial_\mu F^{\mu\nu} = \frac{1}{\varepsilon_0 c} j^\nu . \tag{47}$$

There is only one subtle difference in the notation between equation (46) and (47), namely, that the 0 superscript has become a Greek letter ν. Physically, however, there is a very profound difference, namely, that equation (47) accounts for all components of the Faraday tensor whereas equation (46) does not. A consequence of this natural generalization is that equation (47) gives another one of Maxwell's laws, the Ampere-Maxwell law.[14]

To find the other components of the Faraday tensor, the divergence of equation (47) gives[14]

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0 , \tag{48}$$

by equation (41). Equation (48) suggests that the Faraday tensor is antisymmetric.[y,14] Assuming that the Faraday tensor *is* antisymmetric reveals 7 more of its components by symmetry: $F^{\mu\mu} = 0$, $F^{01} = -E_x$, $F^{02} = -E_y$, and $F^{03} = -E_z$. To find the remaining 6 components, equation (47) is examined. As previously mentioned, the ν = 0 component of equation (47) returns Gauss' Law for a time-dependent electric field and charge density.[z]

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[y] For a discussion of the antisymmetry of the Faraday tensor, refer to Appendix C.

[z] The result of Gauss' Law with time dependence should not be at all surprising considering the development up to equation (46).

$$\nabla \cdot \vec{E}(\vec{x},t) = \frac{1}{\varepsilon_0} \rho(\vec{x},t), \qquad (49)$$

where the argument represents a dependence on space as well as time. For ν = 1, equation (47) gives[aa]

$$c^2 \left[\frac{\partial}{\partial y} \frac{F^{21}}{c} + \frac{\partial}{\partial z} \frac{F^{31}}{c} \right] = \frac{1}{\varepsilon_0} j_x + \frac{\partial E_x}{\partial t}, \qquad (50)$$

and similarly for the ν = 2 and 3 terms. Equation (50) gives the x-component of the Ampere-Maxwell Law,[14] given certain assignments for the components of the Faraday tensor that appear, namely $F^{23} = -cB_x$, $F^{31} = -cB_y$, $F^{12} = -cB_z$, $F^{32} = cB_x$, $F^{13} = cB_y$, and $F^{21} = cB_z$. Therefore, the Faraday tensor may now be sumarized in a 4x4 matrix in terms of the electric and magnetic fields as

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix}, \qquad (51)$$

where the first index gives the row, starting with zero from the top, and the second index gives the column, starting with zero from the left. If these assignments are made, then equation (50) gives the Ampere-Maxwell Law

$$\nabla \times \vec{B}(\vec{x},t) = \mu_0 \vec{j}(\vec{x},t) + \frac{1}{c^2} \frac{\partial \vec{E}(\vec{x},t)}{\partial t}, \qquad (52)$$

where μ_0 is the permeability of free space.

This accounts for all of the components of the Faraday tensor. Finding the last six is a bit speculative. Reference to the Ampere-Maxwell Law seems to break the

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[aa] For the details of the derivation of equation (50), refer to Appendix D.

promise that only electrostatics would be needed. The resolution is that the naming of the components is arbitrary. The oblique components of the Faraday tensor were recognized to satisfy the Ampere-Maxwell equation from equation (50), but the tensor had already been defined physically to satisfy equation (47), which describes the fundamental response of the Faraday tensor to the charge-current density. In other words, equation (47) shows that these oblique components satisfy the Ampere-Maxwell equation, and therefore they are associated with the magnetic field components. The naming convention should not be obscured as a physical indication; equation (47) gives the Ampere-Maxwell equation, not vice versa. For the reader who is still not convinced, ultimately, the definition for these oblique components is verified operationally by their appearance in the Lorentz force law in Section 6.

2.5. Generalizing Electrostatics to Faraday's Law and the Law of No Magnetic Monopoles

Faraday's Law, equation (39), for electrostatics was obtained in Section 4 from the basic principles already mentioned. Written in component form

$$\sum_{k=1}^{3}\sum_{j=1}^{3}\varepsilon_{ijk}\frac{\partial E_k}{\partial x_j}=0\,, \tag{53}$$

where ε_{ijk} is the Levi-Civita symbol defined as[14]

$$\varepsilon_{ijk}=\begin{cases}+1 & if\ \ ijk\ is\ an\ even\ permutation\ of\ 1,2,3\\ -1 & if\ \ ijk\ is\ an\ odd\ permutation\ of\ 1,2,3\\ 0 & if\ \ ijk\ is\ not\ a\ permutation\ of\ 1,2,3\end{cases}\ . \tag{54}$$

Using the notation from section 3 and equation (44), equation (53) becomes

$$\sum_{k=1}^{3}\sum_{j=1}^{3}\varepsilon_{ijk}\partial_j F^{k0}=0\,. \tag{55}$$

The Levi-Civita symbol can be generalized to a 4th rank tensor called the Levi-Civita tensor,[14] by using a definition similar to equation (54) for the contravariant components of the tensor.

$$\varepsilon^{\kappa\lambda\mu\nu} = \begin{cases} +1 & \text{if } \kappa\lambda\mu\nu \text{ is an even permutation of } 0,1,2,3 \\ -1 & \text{if } \kappa\lambda\mu\nu \text{ is an odd permutation of } 0,1,2,3 \\ 0 & \text{if } \kappa\lambda\mu\nu \text{ is not a permutation of } 0,1,2,3 \end{cases} . \tag{56}$$

Replacing the Levi-Civita symbol in equation (55) with the appropriate contravariant components of the Levi-Civita tensor, and lowering the indices on the components of the Faraday tensor,

$$\sum_{k=1}^{3}\sum_{j=1}^{3}\varepsilon^{ijk0}\partial_j F_{k0} = 0 , \tag{57}$$

where[14] $\varepsilon^{ijk0} = -\varepsilon_{ijk}$, and $F_{k0} = \eta_{k\mu}\eta_{0\nu}F^{\mu\nu} = -F^{k0}$. Recognizing that the component of the Levi-Civita tensor vanishes if any of the other indices = 0, equation (57) is identical to

$$\varepsilon^{\mu\nu\alpha 0}\partial_\mu F_{\alpha 0} = 0 . \tag{58}$$

To generalize this to a tensor equation, the 0 must be generalized to an index that ranges from 0 to 3. Since the components of the Levi-Civita tensor are constant, they can be moved between the partial derivative to directly multiply the components of the Faraday tensor. Equation (58) can also be multiplied by 1/2. This gives

$$\partial_\mu\left(\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\right) = 0 . \tag{59}$$

The quantity in parenthesis is defined as ${}^*F^{\mu\nu}$, the dual of the $F^{\mu\nu}$ component of the Faraday tensor.[14] These components can be expressed as a matrix similar to equation (51).

$$^*F^{\mu\nu} = \begin{bmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{bmatrix}. \qquad (60)$$

Replacing the quantity in parenthesis using this definition, equation (59) becomes:

$$\partial_\mu {}^*F^{\mu\nu} = 0 . \qquad (61)$$

"A generalization of Helmholtz's theorem states that an antisymmetric second-rank tensor is completely determined by specifying its divergence and the divergence of its dual."[15] These specifications have been made by equations (47) and (61) in terms of the charge-current density 4-vector.

For ν = 0, using the definition of the dual, and the suggested antisymmetry of the Faraday tensor from Section 4, equation (61) gives

$$\frac{1}{c}\frac{\partial F^{32}}{\partial x} + \frac{1}{c}\frac{\partial F^{13}}{\partial y} + \frac{1}{c}\frac{\partial F^{21}}{\partial z} = 0 , \qquad (62)$$

where the factor of $1/c$ has been introduced arbitrarily without changing the validity of the equation. Equation (62) gives the law of no magnetic monopoles using the component definitions in equation (51).

$$\nabla \cdot \vec{B} = 0 . \qquad (63)$$

This supports the discussion in the previous section concerning the relationship of these components to the magnetic field.

For ν = 1, equation (61) gives:

$$\partial_0 {}^*F^{01} + \partial_1 {}^*F^{11} + \partial_2 {}^*F^{21} + \partial_3 {}^*F^{31} = 0 , \qquad (64)$$

and similarly for ν = 2 and 3. Rearranging equation (64) and making the assignments to the components of the Faraday tensor according to equation (51) gives Faraday's Law for time-dependent electric and magnetic fields.

$$\nabla \times \vec{E}\left(\vec{x},t \right) = -\frac{\partial \vec{B}\left(\vec{x},t \right)}{\partial t} . \tag{65}$$

This further supports the discussion in the previous section concerning the relationship of these components to the magnetic field.

2.6. The Lorentz Force

The previous sections developed the Faraday tensor from a few reasonable assumptions. As a result, two similar equations were found, (47) and (61), that determine the Faraday tensor in terms of a given charge-current density 4-vector. In this section, the response of a point-charge is found to a given Faraday tensor, which completes the mathematical description of the interaction between charge and field. To begin the development, Newton's second law is considered in the context of the electric field (equation (36)).

$$\frac{d\vec{p}}{dt} = q\vec{E} , \tag{66}$$

where $\vec{p}$ is the 3-dimensional momentum vector of the charged point-particle of charge q, and $\vec{E}$ is the electrostatic field at the charge. Equation (66) is a first order, linear, ordinary differential equation. Applying the usual technique of separating the dependent variables from the independent variables, an implicit equation can be obtained in the derivatives of $\vec{p}$ and t with respect to the proper time τ .

$$\frac{d\vec{p}}{d\tau} = \frac{dt}{d\tau} q \vec{E} . \tag{67}$$

As in the previous 2 sections, the components of the electric field are identified with components of the Faraday tensor $E_x \to F^{10}$, etc., according to equation (44). Applying this direct substitution to equation (67) gives

$$\frac{dp^i}{d\tau} = \frac{dt}{d\tau} q F^{i0}, \tag{68}$$

where p^i is the i^{th} component of the momentum vector $\vec{p}$. Multiplying the R.H.S. of equation (68) by $\frac{c}{c}$ gives

$$\frac{dp^i}{d\tau} = \frac{q}{c}\left(\frac{c\,dt}{d\tau}\right) F^{i0} = \frac{q}{c}\left(\frac{dx_0}{d\tau}\right) F^{i0} = \frac{q}{c} u_0 F^{i0}, \tag{69}$$

where u_0 is the 0th covariant component of the proper velocity u_ν defined as the proper time derivative of the position 4-vector.

$$u_\nu \equiv \frac{dx_\nu}{d\tau}. \tag{70}$$

The L.H.S. of equation (69) can be generalized to a 1st rank tensor in a straightforward manner by changing the Latin superscript i to a Greek letter μ, thus including all four components. This will also change the i on the R.H.S. of the equation to a μ. Since the L.H.S. is a 1st rank tensor, so must be the R.H.S. This requires that the 0 subscript and superscript be generalized to contracted indices rather than free indices.

$$\frac{dp^\mu}{d\tau} = \frac{q}{c} u_\nu F^{\mu\nu}. \tag{71}$$

Equation (71) is the relativistic equation of motion for a charged particle. It includes every component of the momentum 4-vector p^μ (and velocity 4-vector u_ν) of the charged particle and every component of the Faraday tensor $F^{\mu\nu}$. Consider, as an example, $\mu = 1$, which represents a typical spatial component (the x-component). This gives, directly from equation (71),

$$\frac{dp^1}{d\tau} = \frac{q}{c}u_0 F^{10} + \frac{q}{c}u_2 F^{12} + \frac{q}{c}u_3 F^{13}. \qquad (72)$$

Using the chain rule and equation (70), and substituting $p^1 = p_x$,

$$\frac{dt}{d\tau}\frac{dp_x}{dt} = \frac{q}{c}\frac{dx_0}{d\tau}F^{10} + \frac{q}{c}\frac{dx_2}{d\tau}F^{12} + \frac{q}{c}\frac{dx_3}{d\tau}F^{13}. \qquad (73)$$

The components of the position 4-vector can be identified according to section 2.3b, and the F^{10} component of the Faraday tensor is the x-component of the electric field. The chain rule can also be applied to the R.H.S. of the equation.

$$\frac{dt}{d\tau}\frac{dp_x}{dt} = \frac{q}{c}\frac{dt}{d\tau}\frac{c\,dt}{dt}E_x + \frac{q}{c}\frac{dt}{d\tau}\left[-\frac{dy}{dt}F^{12} - \frac{dz}{dt}F^{13}\right]. \qquad (74)$$

Simplifying,

$$\frac{dp_x}{dt} = q\,E_x + \frac{q}{c}\left[-v_y\,F^{12} - v_z\,F^{13}\right], \qquad (75)$$

where $v_y = \frac{dy}{dt}$ and $v_z = \frac{dz}{dt}$. Letting $F^{12} = -c\,B_z$ and $F^{13} = c\,B_y$ in accordance with the previous 2 sections, equation (75) gives the x-component of the force experienced by a point particle of charge q moving with velocity $\vec{v}$ through an electric field $\vec{E}$ and magnetic field $\vec{B}$.

$$\frac{dp_x}{dt} = q\,E_x + q\left[\vec{v}\times\vec{B}\right]_x. \qquad (76)$$

The y and z-components follow similarly.

The $\mu = 0$ component of equation (71) yields a well-known feature of electromagnetism, and does so naturally from the principle of relativity.

$$\frac{dp^0}{dt} = \frac{q}{c}\frac{d\tau}{dt}\left(\frac{dx_1}{d\tau}F^{01} + \frac{dx_2}{d\tau}F^{02} + \frac{dx_3}{d\tau}F^{03} \right) = \frac{q}{c}\left(\frac{dx}{dt}E_x + \frac{dy}{dt}E_y + \frac{dz}{dt}E_z \right), \tag{77}$$

which can be rearranged to give

$$\frac{d\left(p^0 c\right)}{dt} \equiv \frac{d\left(energy\right)}{dt} = q\left(\vec{v}\cdot\vec{E}\right). \tag{78}$$

Equation (78) gives the power delivered to the charge by the electric field, and shows that the magnetic field does no work.

2.7. Conclusion

Assuming that Coulomb's Law and the principle of superposition are valid for a static electric field and static charge distribution, the electrostatic field is found to be conservative and Gauss' Law obtained. By assuming that charge is a conserved scalar, the equation of continuity was found for the charge density. This allowed Gauss' Law for the electrostatic field to be generalized, using the principle special relativity, to Gauss' Law for electrodynamics and the Ampere-Maxwell equation. The curl of the electrostatic field was also generalized to give the law of no magnetic monopoles and Faraday's Law. Finally, the force of the electrostatic field was generalized to the Lorentz force and an equation for the rate of energy equal to the power delivered to a point charge in an electric field.

All of this information can be written in terms of three tensor equations in special relativity, (47), (61), and (68), that maintain their form under all Lorentz transformations. The first two equations determine the response of the field to charged matter, which is represented by the response of the Faraday tensor to the charge-current density 4-

vector. These equations are Maxwell's equations. The third equation determines the response of charged matter to the field, which is represented by the response of a point charge to the Faraday tensor. This equation is Newton's second law of motion with the Lorentz force. The mutual interaction between the field and charge upholds Newton's third law of motion.

CHAPTER 3

GENERAL RELATIVITY: EINSTEIN'S GRAVITATIONAL FIELD EQUATION FROM NEWTON'S LAW OF UNIVERSAL GRAVIATION

3.1. Introduction

After completing his theory of special relativity, Albert Einstein realized that the next great challenge would be to make gravitation consistent with the special theory of relativity.[2] Isaac Newton himself found fault in his universal law of gravitation because it required a "spooky action at a distance."[16] In order to have an instantaneous action at a distance, an interaction must travel at an infinite speed. This becomes a problem for causality in special relativity, because, what is simultaneous in one frame of reference is not simultaneous in a boosted frame. In other words, an action in one frame of reference could inconsistently be a *re*-action in another frame of reference. The alternative to instantaneous action at a distance is an interaction that propagates at the speed of light, which is accomplished in general relativity. Another problem with Newton's universal law of gravitation is the source term, namely the mass. Obviously, what is a static mass distribution in one frame of reference induces a mass current in a boosted frame. So, the source of gravitation must demonstrate this in general relativity as a tensor. It turns out that the source of gravitation is matter and energy, partly as a consequence of Einstein's equivalence principle, so a second rank tensor is used, in contrast to the source of the electromagnetic field, which is a first rank tensor.[2] Another consequence of the equivalence principle, and perhaps the most profound (certainly the most popular) feature of general relativity, is that the gravitational field is actually a distortion of space-time itself.[2]

There are two basic approaches to demonstrate the correspondence between Einstein's gravitational field equation and Newton's universal law of gravitation. One of these is to show that Einstein's gravitational field equation reduces to Poisson's equation for gravity in the limit of static, weak fields. This is mathematically straightforward, and involves neglecting certain terms in the equation at the appropriate stages. Newton's universal law of gravitation has been demonstrated as a limiting form of Einstein's equation by several authors.[17,18,19] The other approach is to show that Einstein's gravitational field equation is an inevitable consequence of Newton's universal law of gravitation when certain assumptions are made concerning the physical universe. The latter approach is taken in this chapter.

Many other authors have shown Einstein's gravitational field equation as a generalization of Newton's universal law of gravitation. Ohanian shows that Einstein's gravitational field equation is the result of a generalization of Poisson's equation, with the assumption that the gravitational potential is the 00 component of the space-time metric tensor.[20] Misner, *et. al.* gives six "reasonable axiomatic structures" that lead to Einstein's equation, with the underlying assertion that the generalization *must* lead to Einstein's equation.[21] Kobe shows a heuristic treatment of the speed of light to establish a relationship between the gravitational potential and the metric tensor, similar to Einstein's first theory using the speed of light as a scalar field.[22,23] Costa de Beauregard introduces a "generalized Poisson potential" that is not closely related to the gravitational potential in Newton's universal law of gravitation.[24] Moore derives Schwarzschild's equation from Poisson's equation, but does not use tensors (general covariance) nor develops the general form of Einstein's equation.[25] Mannheim mentions the relationship between Einstein's gravitational field equation and Poisson's equation, but does not give the derivation.[26] Rindler proposes a canonical form for the metric of curved space-time, compares Poisson's equation to the geodesic equation for curved space-time and then relates the gravitational potential to the Christoffel symbol.[27] Bondi also assumes a canonical metric for a spherically symmetric mass distribution and does not give detailed justification for the use of the stress-energy tensor or Einstein tensor, but does give an excellent intuitive illustration of non-Euclidean space.[28] In a lecture on general relativity and astrophysics delivered at the DPG School, a linear relationship is assumed between the stress-energy tensor, Ricci tensor, and curvature scalar, in order to reflect the linear relationship of the tidal force to the mass density in Newton's theory.[29] Weinberg gives similar points in the development that is found in this paper.[30] Perhaps the most significant approach to the one presented in this paper was by Chandrasekhar, who gives more detail in some parts of the generalization and far less detail in others, and uses a different set of assumptions.[31]

In this chapter, the concept of general covariance (tensor analysis in the context of general coordinate transformations, in contrast to the linear transformations in special relativity) is applied to the classical Poisson's equation for gravity, which is based on Newton's universal law of gravitation and the principle of superposition. Poisson's equation for gravity can then be written as the 00 component (time-like component) of a second rank tensor equation.[2] Einstein's equation is obtained by relating the mass density (times c^2) to the 00 component (being the energy component) and trace of the energy-stress tensor.[2] Poisson's equation, which is valid for slowly moving particles in static and weak gravitational fields, is generalized to give Einstein's gravitational field equation, which is valid for particles at any speed in an arbitrary gravitational field.[2]

A brief development of Poisson's equation for Newtonian gravity is presented in Sec. 2. Section 3 gives a discussion of space-time curvature and introduces some important tensors. The principle of least action for gravity is shown to be equivalent to the definition of a geodesic in Sec. 4. In Sec. 5, the mass density (times c^2) is generalized to the stress-energy tensor, and the Bianchi identity is used to give Einstein's equation. Section 6 gives the conclusion.

3.2. Poisson's Equation for Gravity

In 1686, Isaac Newton presented his law of universal gravitation to the world.[5] The key features of this law are that the force is proportional to the product of the point masses, and that the force is inversely proportional to the square of the distance between the point masses. Given two masses, m and m', the force $\vec{F}(\vec{x})$ on mass m at $\vec{x}$ by the mass m' at the origin is given by Newton's universal law of gravitation:

$$\vec{F}(\vec{x}) = -Gmm'\frac{\vec{x}}{|\vec{x}|^3}, \tag{79}$$

where G is Newton's universal gravitational constant.[5] Equation (79) can be generalized so that m' is at some point $\vec{x}'$:

$$\vec{F}(\vec{x}) = -Gmm'\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}. \tag{80}$$

The forces on m at $\vec{x}$ due to masses m_i at $\vec{x}_i$ ($i = 1, 2, \ldots N$) obey the principle of superposition, so that the total force on mass m at $\vec{x}$ is:

$$\vec{F}(\vec{x}) = -Gm\sum_i m_i \frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3}. \tag{81}$$

If the constellation of masses in equation (81) is generalized to a continuous distribution of mass with a mass density $\rho(\vec{x}')$ at the position $\vec{x}'$, then the force on mass m at $\vec{x}$ by the continuous mass distribution is given by:

$$\vec{F}(\vec{x}) = -Gm\int d^3x'\,\rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}, \tag{82}$$

where the integral is a volume integral on $\vec{x}'$ over all space.

A gravitational field $\vec{g}(\vec{x})$ at the displacement $\vec{x}$ can be defined as:

$$\vec{g}(\vec{x}) \equiv \frac{\vec{F}(\vec{x})}{m}. \tag{83}$$

Combining equations (82) and (83), the gravitational field at the point $\vec{x}$ due to a continuous distribution of mass with a mass density $\rho(\vec{x}')$ as a function of the position $\vec{x}'$ is given as:

$$\vec{g}(\vec{x}) = -G\int d^3x'\,\rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}. \tag{84}$$

The gravitational field is conservative because $\nabla\times\vec{g}(\vec{x}) = 0$.[bb] Therefore a gravitational potential function $\Phi(\vec{x})$ for the gravitational field $\vec{g}(\vec{x})$ can be found, which is given by:[cc]

$$\Phi(\vec{x}) = -G\int d^3x'\,\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{85}$$

[bb] This is shown in Appendix E

[cc] The derivations of equation (85) from equation (84) and equation (86) from equation (85) can be found in Appendix E.

Taking the Laplacian of equation (85) gives Poisson's equation for the Newtonian gravitational potential:

$$\nabla^2 \Phi(\vec{x}) = 4\pi G \rho(\vec{x}). \tag{86}$$

3.3. Space-time Curvature

The Minkowski space-time of special relativity is often referred to as "flat" space-time. The space-time of general relativity is often referred to as "curved" space-time.[2] This section demonstrates the significance of this curvature and provides a mechanism to quantify it.

Euclidean geometry has several features, called axioms, such as the existence of parallel lines and an open set of points, that serve the same purpose as intuitive geometric notions. The advent of Riemannian geometry demonstrated that the axioms of Euclidean geometry are nontrivial. The most obviously nontrivial notion is the Euclidean construct of parallel lines, which require a more careful definition in Riemannian geometry. Riemannian geometry is the generalization of Euclidean geometry to include the notion of curvature.

In this section the relevant distinctions between Euclidean space, Minkowski space-time, and pseudo-Riemannian[dd,32] space-time are demonstrated, and the tensors used to describe this curvature in the context of space-time (4-D) are developed.

[dd] The geometry of general relativity is not truly Riemannian because the geometry of special relativity is not truly Euclidean. This subtly is not important for the development in this paper, but, to be rigorous, the prefix "pseudo-" is added to indicate this distinction.

A key feature of Euclidean geometry is Pythagoras' theorem that essentially defines the distance between two points to be the distance that one would measure if one were to put a physical ruler across them. This implies that the metric tensor of Euclidean geometry is the unit second rank tensor, the Kronecker Delta, δ_{mn}, in Cartesian coordinates. In special relativity, the idea of distance is generalized to proper distance or proper time. This implies that the metric tensor[ee] used in the geometry of special relativity is the Minkowski metric tensor,[12] $\eta_{\mu\nu}$, in Cartesian coordinates. The Minkowski metric tensor is generalized to the metric tensor for curved space-time,[33] $g_{\mu\nu}$, in general relativity. The metric tensor itself is a dynamical variable in general relativity.[34] It interacts with mass and energy and reacts accordingly. This profound consequence of general relativity has evidence in Einstein's equivalence principle in which he proposed gravitational and inertial mass to be equivalent.

The notion of parallel is more appropriately applied to vectors than to lines in general relativity. Two vectors at the same point in space-time are parallel to each other if and only if one is a scalar multiple of the other. In order to determine whether vectors at two different points in space-time are parallel, a connection must be established. The connection used in general relativity is parallel transport. Parallel transport is the transporting of a tangent vector from one point to another along a piecewise geodesic path while maintaining the orientation of the vector to each

[ee] A discussion of tensors can be found in Chapter 2, Section 3.

geodesic in each piece.[35] A geodesic is a path that extremizes the proper distance or proper time between two given points.[ff,32]

Parallel transport can have a peculiar consequence in a curved geometry. The vector that is parallel transported along one path can disagree with the same vector transported along a different path between the same two points. The simplest example of this disagreement occurs on the surface of the Earth. Consider an arrow pointing westward on the equator. Parallel transporting this arrow along the equator to the opposite side of the Earth will result in a westward pointing arrow on the opposite side of the Earth. Parallel transporting this arrow along a line of longitude to the same point will result in an *eastward* pointing arrow at that point. This disagreement is a direct result of the curvature of the geometry, and it can be used to quantify the curvature.

First, note that parallel transportation along a single geodesic is closely related to partial differentiation.[gg] Next, note that the tensor product of the partial derivative tensor with a 4-vector results in an object that is *not* a tensor.[hh] A new kind of derivative can

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[ff] Actually, more generally, a geodesic is a curve of *stationary* proper length in pseudo-Riemannian geometry. However, it is of *maximum* proper length for all *massive particles*. A geodesic represents the "straightest" path connecting two points in curved space.

[gg] Recall that a partial derivative is a change in a function with respect to a change along a coordinate axis.

[hh] Appendix F shows why the tensor product of the partial derivative with a 4-vector results in an object that is not a tensor.

be defined that is related to the partial derivative, traditionally called the covariant derivative, that is a tensor.

Some new notation is introduced to simplify the expression for a partial derivative. A comma before a lower index will be used to indicate a partial derivative with respect to the coordinate that takes that index:

$$A^{\mu}{}_{,\beta} \equiv \partial_{\beta} A^{\mu}, \tag{87}$$

where $\partial_{\beta} \equiv \partial / \partial x^{\beta}$ according to equation (25) or (33) in Section 3 of Chapter 2.

For an arbitrary 4-vector A^{μ}, the covariant derivative with respect to x^{β} is denoted by a semicolon before a lower index and is defined as:[36]

$$A^{\mu}{}_{;\beta} \equiv \partial_{\beta} A^{\mu} + \Gamma^{\mu}_{\alpha\beta} A^{\alpha}, \tag{88}$$

where $\Gamma^{\mu}_{\alpha\beta}$ is known as a connection coefficient. In general relativity, the connection coefficient is called the Christoffel symbol. The Christoffel symbol can be expressed in terms of the metric tensor as:[37]

$$\Gamma^{\mu}_{\alpha\beta} = g^{\mu\nu} \left(g_{\nu\alpha,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right). \tag{89}$$

The term on the left-hand side of equation (88) is a second rank mixed tensor, whereas the object in equation (87) is not. Though the covariant derivative is a tensor, two consecutive covariant derivatives of a 4-vector do not commute, in general. This property can be used to quantify the curvature of space-time. A fourth rank tensor, known as the Riemann curvature tensor $R^{\alpha}{}_{\beta\mu\nu}$, is defined in terms of the Christoffel symbols as:[35]

$$R^{\alpha}{}_{\beta\mu\nu} \equiv -\Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\beta\nu,\mu} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\alpha}_{\sigma\mu} - \Gamma^{\sigma}_{\beta\mu}\Gamma^{\alpha}_{\sigma\nu}. \tag{90}$$

The Riemann tensor is essentially the commutator of two covariant derivatives acting on a 4-vector.[ii] It indicates the path dependence of differentiation in curved space. If the space is flat, then the Riemann tensor vanishes.

Two other important tensors come directly from the Riemann curvature tensor. The Ricci tensor $R_{\mu\nu}$ is defined as the Riemann curvature tensor contracted on its first and last indices:[38]

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\nu\alpha} \,. \tag{91}$$

The Riemann curvature scalar R is defined by the contraction of the two remaining indices:[38]

$$R \equiv R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\alpha}{}_{\mu\nu\alpha} \,. \tag{92}$$

The Ricci tensor and Riemann curvature scalar together satisfy the Bianchi identity:[38]

$$\left(R^{\mu}{}_{\nu} - \tfrac{1}{2} \delta^{\mu}{}_{\nu} R \right)_{;\mu} = 0 \,. \tag{93}$$

Equation (93) is valid geometrically, meaning it is independent of the details of the physical situation.

3.4. Principle of Least Action

Points in 3-D space extrude into 4-D space-time as worldlines. In Minkowski space-time, the notion of a straight line is intuitively the same as that in familiar Euclidean geometry. That is, the worldlines of free particles "look" straight in Minkowski space-time. More formally, a geodesic is the generalization of a straight line to curved

[ii] Note that the left-hand side of equation (90) is a tensor, even though every term on the right-hand side is not a tensor in itself.

space, and is defined as the path that maximizes the proper time experienced by a particle that travels between two given points.[39]

$$\int_{geodesic} d\tau = \max . \tag{94}$$

A geodesic is a path over which a particle will not experience any force. Any deviation from this path will result in an experienced acceleration and a decrease in the proper time experienced by the particle.[jj] Since a deviation from this path results in a force, it should not be surprising that this requirement is related to the principle of least action for the gravitational potential energy.

$$\int_{actual\ path} L\, dt = \min , \tag{95}$$

where L is the Lagrangian function for a massive particle in a gravitational potential. Using the definition of the Lagrangian as

$$L \equiv T - V , \tag{96}$$

equation (95) can be rewritten as

$$\int_{actual\ path} \left(\tfrac{1}{2} mv^2 - m\Phi \right) dt = \min . \tag{97}$$

In the limit

$$g_{\mu\nu} \to \eta_{\mu\nu} , \tag{98}$$

equation (94) can be rewritten as[kk]

[jj] The fact that acceleration reduces the experienced proper time is valid in special relativity. This is one way to address the infamous twins paradox.

[kk] See Appendix F for a derivation of equation (99).

$$\int_{geodesic} \left(\tfrac{1}{2} m v^2 - m\left(\tfrac{1}{2} c^2 g_{00} + \tfrac{1}{2} c^2 \right) \right) dt = \min , \qquad (99)$$

In equation (97), gravity is treated as a force for which there exists a potential function. This is the notion of gravity given by Newton. In equation (99), the influence of geometry on the trajectory of an otherwise free particle is demonstrated. Solely under the influence of gravity, the actual path of a physical particle *is* a geodesic. Comparing equations (97) and (99), the gravitational potential can be expressed in terms of the metric tensor as[40]

$$\Phi = \tfrac{1}{2} c^2 g_{00} + \tfrac{1}{2} c^2 . \qquad (100)$$

Equation (100) gives the relationship between the Newtonian gravitational potential Φ and the 00 component of the metric tensor g_{00}.

3.5. Einstein's Equation

The mass density, ρ, is related to the 00 mixed component of the stress-energy tensor $T^0{}_0$ in the cloud of dust model[41,||] which Einstein used in his paper, *The Meaning of Relativity*.[42] First, consider the trace of the stress-energy tensor.

$$T \equiv T^\mu{}_\mu = \tilde{\rho} u^\mu u_\mu = \tilde{\rho} c^2 , \qquad (101)$$

where $\tilde{\rho}$ is the proper mass density scalar field. In the static limit, the space-like components of the stress-energy tensor (in the cloud of dust model) vanish.

$$T^k{}_k = 0 + 0 + 0 = 0 . \qquad (102)$$

[||] See Appendix H for a description of the stress-energy tensor.

This implies (numerically) that

$$T = T^0{}_0 = \tilde{\rho} c^2 = \rho c^2 \tag{103}$$

in the static limit.

According to equation (103), the Newtonian mass density (times c^2) can be written as either the 00 mixed component or the trace of the stress-energy tensor. More generally, it can be written as a linear combination of the two.

$$\rho c^2 = aT^0{}_0 + (1-a)T\delta^0{}_0, \tag{104}$$

where a is a constant to be determined and $\delta^0{}_0$ is the 00 component of the Kronecker Delta tensor (and therefore equal to unity). Though seemingly superfluous at this point, the Kronecker Delta tensor is necessary for subsequent generalization in order to maintain the rank and form of the tensor components. Substituting equation (100) and (104) into Poisson's equation (86) gives

$$\nabla^2 g_{00} = \frac{8\pi G}{c^4}\left(aT^0{}_0 + (1-a)T\delta^0{}_0\right). \tag{105}$$

The L.H.S. of equation (105) can be related to the 00 component of the Ricci tensor.[2,mm]

$$\nabla^2 g_{00} = -2R^0{}_0. \tag{106}$$

Combining equations (105) and (106) gives

$$R^0{}_0 = -\frac{4\pi G}{c^4}\left(aT^0{}_0 + (1-a)T\delta^0{}_0\right). \tag{107}$$

Both the L.H.S. and R.H.S. of equation (107) are 00 mixed components of second rank tensors. Generalization of equation (107) to be form invariant under

[mm] See Appendix I for the generalization of $\nabla^2 g_{00}$ to $-2R^0{}_0$.

general coordinate transformations is simply a matter of recognizing that equation (107) is the 00 component of a tensor equation, and that all sixteen mixed components of the second rank tensor equation must exist in general.[2]

$$R^{\mu}{}_{\nu} = -\frac{4\pi G}{c^4}\left(aT^{\mu}{}_{\nu} + (1-a)T\,\delta^{\mu}{}_{\nu} \right) \tag{108}$$

for $\mu,\nu = 0, 1, 2, 3$.

The Riemann curvature scalar can be related to the trace of the stress-energy tensor by taking the trace of equation (108).

$$R = -\frac{4\pi G}{c^4}\left(aT + (1-a)T\left(4\right)\right). \tag{109}$$

Solving this equation for *T*, substituting this back into equation (108) and rearranging gives

$$R^{\mu}{}_{\nu} - \frac{(1-a)}{(4-3a)}R\,\delta^{\mu}{}_{\nu} = -\frac{4\pi G\,a}{c^4}T^{\mu}{}_{\nu}\,. \tag{110}$$

Energy and momentum are conserved if the 4-divergence of the stress-energy tensor vanishes as[20]

$$T^{\mu}{}_{\nu;\mu} = 0\,. \tag{111}$$

On physical grounds, it is assumed that energy and momentum are conserved in the theory. Taking the 4-divergence of equation (110) gives

$$\left(R^{\mu}{}_{\nu} - \frac{(1-a)}{(4-3a)}R\,\delta^{\mu}{}_{\nu} \right)_{;\mu} = 0\,. \tag{112}$$

Comparing equation (112) with the Bianchi identity equation (93) requires

$$\frac{(1-a)}{(4-3a)} = \frac{1}{2}\,. \tag{113}$$

Therefore the value of a is 2. Substituting $a = 2$ into equation (110) gives

$$R^{\mu}{}_{\nu} - \tfrac{1}{2} R\,\delta^{\mu}{}_{\nu} = -\frac{8\pi G}{c^4} T^{\mu}{}_{\nu}\,. \tag{114}$$

Equation (114) is Einstein's equation for the gravitational field in terms of the Ricci tensor, Riemann curvature scalar, and the stress-energy tensor.[20] Another form of Einstein's equation is obtained by raising the lower index in equation (114).

$$R^{\mu\nu} - \tfrac{1}{2} R\,g^{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu}\,. \tag{115}$$

3.6. Conclusion

Poisson's equation for gravity is a direct consequence of Newton's law of gravity if superposition, continuity, and conservation of energy are assumed. This assumption is valid in static, weak gravitational fields. However, in general relativity, the gravitational field is related to the curvature of space-time. Therefore, the metric tensor of special relativity must be generalized to the metric tensor of curved space-time. Using the principle of least action to relate the gravitational potential to the 00 component of the metric tensor, Einstein's equation follows as a natural consequence of tensor analysis, the Bianchi identity and the conservation of energy and momentum.

CHAPTER 4

CONCLUSION

Classical physical theory effectively began when Isaac Newton proposed his classical dynamics and universal law of gravitation. This sparked a classical world view of physical phenomena to which science remains loyal even today, in the appropriate limits of consideration, usually referred to as "everyday experience."[23] Classical mechanics was not successfully contested until Albert Einstein introduced his special theory of relativity. However, Einstein further extended the classical view by discovering the appropriate way in which to generalize Newtonian gravitation.[2] In this way, Einstein's general relativity marks the completion of the classical deterministic physical theory of the universe.[2]

APPENDIX A

Derivation of Equations (38) and (39) from Coulombs Law and the Principle of Superposition

1. Derivation of Equation (38)

Equation (37) was derived from Coulomb's Law and the principle of superposition:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}. \tag{37}$$

Taking the divergence of this integral equation with respect to the unprimed coordinates (coordinates of the observation point where the electric field is being defined), recognize that the divergence operator does not act on the primed coordinates:

$$\nabla\cdot\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(\vec{x}')\left(\nabla\cdot\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right), \tag{A1}$$

where ∇ is the gradient with respect to the unprimed coordinates. The divergence in parenthesis must be determined from an auxiliary integral.

By the divergence theorem:

$$\int_{V:|\vec{x}-\vec{x}'|\le R} d^3x\left(\nabla\cdot\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right) = \oint_{S:|\vec{x}-\vec{x}'|=R} d\vec{s}\cdot\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}. \tag{A2}$$

Making the substitution $\vec{x}-\vec{x}'=\vec{r}$, the surface integral may easily be evaluated:

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} R^2\sin\theta\,d\theta\,d\phi\,\hat{n}\cdot\frac{\hat{n}R}{R^3} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} R^2\sin\theta\,d\theta\,d\phi\,\hat{n}\cdot\frac{\hat{n}R}{R^3} = 4\pi\ , \tag{A3}$$

where $\hat{n}$ represents the unit normal vector to the spherical surface of integration. This result is valid for all values of R > 0. This condition is met for all observation points $\vec{x}\ne\vec{x}'$. Examining the value of the divergence at $\vec{x}$ directly using the same substitution:

$$\nabla\cdot\frac{\vec{r}}{r^3} = \nabla\cdot\frac{\hat{r}}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0. \tag{A4}$$

Under the same condition that $\vec{x} \neq \vec{x}'$. Therefore, this divergence must satisfy two properties, namely:

$$\nabla \cdot \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = 0 \text{, if } \vec{x} \neq \vec{x}', \tag{A5}$$

and

$$\int_V d^3x \left(\nabla \cdot \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \right) = 4\pi \text{, if V contains } \vec{x}'. \tag{A6}$$

This gives, by definition, the 3-dimensional Dirac-Delta function:

$$\frac{1}{4\pi} \nabla \cdot \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \delta(\vec{x}-\vec{x}'). \tag{A7}$$

Therefore:

$$\nabla \cdot \vec{E}(\vec{x}) = \frac{1}{\varepsilon_0} \int d^3x' \rho(\vec{x}')(\delta(\vec{x}-\vec{x}')) = \frac{1}{\varepsilon_0} \rho(\vec{x}), \tag{38}$$

which is Gauss' Law for electrostatics.

2. Derivation of Equation (39)

Taking the curl of equation (37):

$$\nabla \times \vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \rho(\vec{x}') \left(\nabla \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \right). \tag{A8}$$

Examining the curl in parenthesis directly by arbitrarily considering the x component:

$$\left[\nabla \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \right]_x = \frac{\partial}{\partial y} \frac{z-z'}{r^3} - \frac{\partial}{\partial z} \frac{y-y'}{r^3}$$

$$= (z-z') \frac{\partial}{\partial y} \frac{1}{r^3} - (y-y') \frac{\partial}{\partial z} \frac{1}{r^3}$$

$$= (z - z')\frac{-3}{r^4}\frac{\partial r}{\partial y} - (y - y')\frac{-3}{r^4}\frac{\partial r}{\partial z}$$

$$= (y - y')\frac{3}{r^4}\frac{z - z'}{r} - (z - z')\frac{3}{r^4}\frac{y - y'}{r} = 0\,. \tag{A9}$$

The other two components of the curl are similarly found to vanish. This is not a conditional result; therefore, the curl in parenthesis is identically 0. This gives equation (39),

$$\nabla \times \vec{E}(\vec{x}) = 0\,, \tag{39}$$

which says that the electrostatic field is a conservative vector field.

APPENDIX B

Demonstration of Charge Density as the 0-component of a 4-vector

Charge is assumed to be a conserved scalar. A differential element of charge, dq, is related to a static charge density, ρ, by the relationship:

$$dq = \rho\, d^3x\,, \tag{B1}$$

where d^3x is a differential element of 3-dimensional spatial volume. Since the charge is a scalar, the R.H.S. of this equation must be a scalar, but the differential volume element of 3-dimensional space is not a scalar:

$$d^3x \equiv dx^1 dx^2 dx^3\,. \tag{B2}$$

There is an element of 4-dimensional space-time volume that is a scalar:[43]

$$d^4x \equiv dx^0 dx^1 dx^2 dx^3 = dx^0 d^3x\,. \tag{B3}$$

To show that equation (B3) represents a scalar (it is not directly obvious from the discussions in Chapter 2), consider the element of volume under a coordinate transformation.

$$d^4x' = dx'^0 dx'^1 dx'^2 dx'^3\,. \tag{B4}$$

From multivariable calculus, this relates to the unprimed element of volume as

$$d^4x' = |\, J\,|\, d^4x\,, \tag{B5}$$

where is the determinant of the coordinate transformation matrix (a.k.a. the Jacobian).[44] For the purposes of Chapter 2, the transformation matrix is the Lorentz transformation matrix. Without loss of generality, the x^1-axis may be aligned with the Lorentz boost. Then, using equation (18), the Jacobian of such a boost is

$$|\, J\,| = \begin{vmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \gamma^2 - \beta^2\gamma^2 = \frac{1-\beta^2}{1-\beta^2} = 1\,. \tag{B6}$$

Inserting this result into equation (B5) gives

$$d^4x' = d^4x . \tag{B7}$$

Upon comparison to equation (19), equation (B7) shows that the element of space-time volume is a scalar. Rearranging equation (B3) gives

$$\frac{d^4x}{d^3x} = dx^0 , \tag{B8}$$

so this scalar density is the zero component of a 4-vector. This can be generalized to

$$\frac{df\left(x^{\mu}\right)}{d^3x} = A^0\left(x^{\mu}\right), \tag{B9}$$

where $df\left(x^{\mu}\right)$ is an arbitrary scalar field and

$$A^0\left(x^{\mu}\right) = g\left(x^{\mu}\right)dx^0 = \frac{df\left(x^{\mu}\right)}{d^4x}dx^0 \tag{B10}$$

is the 0-component of an arbitrary 4-vector field. This shows that the division of an arbitrary scalar field by a differential element of 3-dimensional spatial volume, which is a scalar density, results in the 0-component of a 4-vector field. Applying this to electric charge:

$$\frac{c\,dq}{d^3x} = \rho c = j^0 . \tag{B11}$$

This shows that the electric charge density is the 0-component of a 4-vector field. This 4-vector field is conventionally given the symbol j^{μ} and called the charge-current density 4-vector. The other three components of this 4-vector are the components of the electric current density 3-vector.

APPENDIX C

Antisymmetry of the Faraday Tensor

1. Primary Approach

By definition, the components, $S^{\mu\nu}$, of a second rank tensor are symmetric if

$$S^{\mu\nu} = S^{\nu\mu} \tag{C1}$$

and the components, $A^{\mu\nu}$, of a second rank tensor are antisymmetric if

$$A^{\mu\nu} = -A^{\nu\mu} \tag{C2}$$

This definition also holds for covariant and mixed components.

According to equation (48):

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0 . \tag{48}$$

In special relativity, the twice-repeated partial derivatives are symmetric covariant components of a second rank tensor. They are symmetric because the result of the differentiation is the same under an exchange of subscripts. That is:

$$\partial_\nu \partial_\mu [\cdot] = \partial_\mu \partial_\nu [\cdot]. \tag{C3}$$

As a consequence of the construction of tensors, the contraction on both indices of the symmetric covariant components of a second rank tensor with the antisymmetric contravariant components of an second rank tensor vanishes. To show this, let $S_{\mu\nu}$ be the symmetric covariant components of a second rank tensor, and let $A^{\mu\nu}$ be the antisymmetric contravariant components of a second rank tensor. Contracting on both indices gives

$$S_{\mu\nu} A^{\mu\nu} = \left(S_{\nu\mu}\right)\left(-A^{\nu\mu}\right) = -S_{\nu\mu} A^{\nu\mu} , \tag{C4}$$

where equations (C1) and (C2) have been used. Recognizing that both of the indices that appear in equation (C4) are dummy indices, they may be substituted with any arbitrary Greek letter without changing the validity of the equation in any way, and without using any identity or definition. Choosing the mutual substitution $\mu \leftrightarrow \nu$ gives

$$S_{\mu\nu} A^{\mu\nu} = -S_{\mu\nu} A^{\mu\nu} . \tag{C5}$$

The only way that equation (C5) can be true is that the contraction definitively vanishes.

A second rank tensor has 16 independent components, in general, whereas there are only 6 independent antisymmetric components. Since the electric and magnetic field vectors contribute six independent components, the Faraday tensor is intuitively antisymmetric, because any extra independent components would bring superfluous information.[9]

2. An Alternative Approach

Any second rank tensor may be written as the sum of a symmetric second rank tensor and an antisymmetric second rank tensor, much like a matrix in linear algebra can be broken into a symmetric and antisymmetric matrix. So, let the Faraday tensor be:

$$F^{\mu\nu} = A^{\mu\nu} + S^{\mu\nu} , \tag{C6}$$

where $A^{\mu\nu}$ is the antisymmetric part and $S^{\mu\nu}$ is the symmetric part. Inserting this into equation (48):

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu \partial_\mu \left[A^{\mu\nu} + S^{\mu\nu} \right] = \partial_\nu \partial_\mu A^{\mu\nu} + \partial_\nu \partial_\mu S^{\mu\nu} = 0 . \tag{C7}$$

The term containing the antisymmetric part $A^{\mu\nu}$ is identically zero. Therefore, the symmetric part $S^{\mu\nu}$ must satisfy:

$$\partial_\nu \partial_\mu S^{\mu\nu} = 0 . \tag{C8}$$

The simplest solution to this second order partial differential tensor equation is:[9 23]

$$S^{\mu\nu} = 0 . \tag{C9}$$

3. Another Consideration

If the Lorentz force is assumed to be a "pure force," that is, a force that does not affect the rest mass m of the particle on which it acts, then[45]

$$u^{\mu} \frac{dp_{\mu}}{d\tau} = \frac{1}{m} p^{\mu} \frac{dp_{\mu}}{d\tau}$$

$$= \frac{1}{2m}\left(\frac{dp^{\mu}}{d\tau} p_{\mu} + p^{\mu} \frac{dp_{\mu}}{d\tau} \right)$$

$$= \frac{1}{2m} \frac{d}{d\tau}\left(p^{\mu} p_{\mu} \right)$$

$$= \frac{1}{2m} \frac{d}{d\tau}\left(\left(mc \right)^2 \right) = 0 ,$$

where u^{μ} is the proper velocity, or 4-velocity of the particle, and the invariance of $p^{\mu} p_{\mu} = \left(mc \right)^2$ has been used. Given that the Lorentz force is of the form:

$$\frac{dp_{\mu}}{d\tau} = kF_{\mu\nu} u^{\nu} ,$$

it follows that:

$$F_{\mu\nu} u^{\nu} u^{\mu} = 0 .$$

It should be obvious that the tensor product of the 4-velocity with itself is a symmetric second rank tensor (because the order of the tensor product does not change the value of the components). Therefore, from the result in the first section of this appendix it can clearly be seen that the Faraday tensor must be antisymmetric to satisfy the above relationship (that is, for the Lorentz force to be a pure force).

APPENDIX D

Explicit Calculation of Equation (50)

The $\nu = 1$ component of equation (47) is

$$\partial_{\mu} F^{\mu 1} = \frac{1}{\varepsilon_0 c} j^1 . \tag{D1}$$

Writing out all of the terms in equation (D1) explicitly gives

$$\frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} = \frac{1}{\varepsilon_0 c} j^1 . \tag{D2}$$

Using the definitions from Chapter 2 for x^{μ}, j^{μ}, and $F^{\mu\nu}$, equation (D2) may be written as

$$\frac{\partial(-E_x)}{\partial(ct)} + \frac{\partial(0)}{\partial x} + \frac{\partial F^{21}}{\partial y} + \frac{\partial F^{31}}{\partial z} = \frac{1}{\varepsilon_0 c} j_x . \tag{D3}$$

Rearranging equation (D3) gives

$$-\frac{\partial E_x}{\partial t} + c\frac{\partial F^{21}}{\partial y} + c\frac{\partial F^{31}}{\partial z} = \frac{1}{\varepsilon_0} j_x . \tag{D4}$$

A further rearrangement of equation (D4) gives

$$c^2 \frac{\partial\left(F^{21}/c\right)}{\partial y} + c^2 \frac{\partial\left(F^{31}/c\right)}{\partial z} = \frac{1}{\varepsilon_0} j_x + \frac{\partial E_x}{\partial t} , \tag{D5}$$

and, finally,

$$c^2\left[\frac{\partial}{\partial y}\frac{F^{21}}{c} + \frac{\partial}{\partial z}\frac{F^{31}}{c}\right] = \frac{1}{\varepsilon_0} j_x + \frac{\partial E_x}{\partial t} , \tag{50}$$

which is equation (50).

APPENDIX E

Demonstration of Equation (85) and Derivation of Poisson's Equation (86) from Equation (84)

1. Demonstration of Equation (85)

Equation (84) gives a 3-dimensional vector field $\vec{g}(\vec{x})$ according to a mass density distribution $\rho(\vec{x}')$. This is derived from the superposition of a central field, and is therefore a conservative field. Since it is conservative, it has a potential function $\Phi(\vec{x})$ that is related by

$$\vec{g}(\vec{x}) = -\nabla\Phi(\vec{x}), \tag{E1}$$

where the gradient operates on the unprimed coordinates.

Since the gradient does not operate on the primed coordinates, it can be taken inside the integration in equation (85) to operate on only the unprimed coordinates:

$$-\nabla\Phi(\vec{x}) = G\int d^3x'\,\rho(\vec{x}')\nabla\left[\frac{1}{|\vec{x}-\vec{x}'|}\right] \tag{E2}$$

$$= G\int d^3x'\,\rho(\vec{x}')\nabla\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right]$$

$$= G\int d^3x'\,\rho(\vec{x}')\left[-\hat{x}\frac{(x-x')}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{3/2}}+\cdots\right],$$

where addition of the $\hat{y}$ and $\hat{z}$ vector terms are implied,

$$= -G\int d^3x'\,\rho(\vec{x}')\left[\frac{\hat{x}(x-x')+\hat{y}(y-y')+\hat{z}(z-z')}{|\vec{x}-\vec{x}'|^3}\right]$$

$$= -G\int d^3x'\,\rho(\vec{x}')\left[\frac{(\hat{x}x+\hat{y}y+\hat{z}z)-(\hat{x}x'+\hat{y}y'+\hat{z}z')}{|\vec{x}-\vec{x}'|^3}\right]$$

$$= -G\int d^3x'\,\rho(\vec{x}')\left[\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right], \tag{84}$$

which is the R.H.S. of equation (84).

2. Derivation of Equation (86)

To get equation (86), the demonstration is analogous to that in Appendix A for deriving equation (38) from equation (37). The differences are the following two explicit direct replacements:

$$\vec{E} \to -\nabla\Phi\left(\vec{r}\,\right), \tag{E3}$$

$$\frac{1}{\varepsilon_0} \to -4\pi G\,, \tag{E4}$$

and to replace the charge density from Appendix A with a mass density in this case. The negative sign before the constant of proportionality is due to the fact that gravitational force is always attractive between mass (a mass at the origin exerts a force on another mass in the $-\hat{r}$ direction).

APPENDIX F

Demonstration of the Partial Derivative of a 4-vector

Consider the operation of the partial derivative on an arbitrary 4-vector:

$$A^{\mu}{}_{,\nu} = \frac{\partial}{\partial x^{\nu}}\left[A^{\mu}(x)\right]. \tag{F1}$$

Now consider an arbitrary coordinate transformation to primed coordinates:

$$\frac{\partial}{\partial x'^{\beta}}\left[A'^{\alpha}(x')\right] = \frac{\partial x^{\nu}}{\partial x'^{\beta}}\frac{\partial}{\partial x^{\nu}}\left[\frac{\partial x'^{\alpha}}{\partial x^{\mu}}A^{\mu}(x)\right]. \tag{F2}$$

If the space-time is flat, and the transformation is from one Cartesian coordinate system to another (a Lorentz transformation), then the partial derivatives of the coordinates are constant, which allows them to be pulled out of the partial derivative of the 4-vector:

$$\frac{\partial x^{\nu}}{\partial x'^{\beta}}\frac{\partial}{\partial x^{\nu}}\left[\frac{\partial x'^{\alpha}}{\partial x^{\mu}}A^{\mu}(x)\right] = a_{\beta}{}^{\nu}\frac{\partial}{\partial x^{\nu}}\left[a^{\alpha}{}_{\mu}A^{\mu}(x)\right] = a^{\alpha}{}_{\mu}a_{\beta}{}^{\nu}A^{\mu}{}_{,\nu}\,. \tag{F3}$$

Therefore:

$$A'^{\alpha}{}_{,\beta} = a^{\alpha}{}_{\mu}a_{\beta}{}^{\nu}A^{\mu}{}_{,\nu}\,. \tag{F4}$$

This has the form of equation (31), and therefore the operation of the partial derivative on a four vector is a tensor product that results in a second rank tensor if there is no contraction, or a scalar if there is contraction. This idea was used more than once in Chapter 2.

If, however, the space-time is not flat, then the partial derivatives of the coordinates are not constant. According to the product rule for differentiation:

$$\frac{\partial x^{\nu}}{\partial x'^{\beta}}\frac{\partial}{\partial x^{\nu}}\left[\frac{\partial x'^{\alpha}}{\partial x^{\mu}}A^{\mu}(x)\right] = \frac{\partial x^{\nu}}{\partial x'^{\beta}}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}\left[A^{\mu}(x)\right] + \frac{\partial x^{\nu}}{\partial x'^{\beta}}\frac{\partial^{2} x'^{\alpha}}{\partial x^{\nu}\partial x^{\mu}}A^{\mu}(x)$$

$$= \frac{\partial x^{\nu}}{\partial x'^{\beta}}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}A^{\mu}{}_{,\nu} + \frac{\partial x^{\nu}}{\partial x'^{\beta}}\frac{\partial^{2} x'^{\alpha}}{\partial x^{\nu}\partial x^{\mu}}A^{\mu}\,. \tag{F5}$$

It is this second term that appears on the R.H.S. that prevents the ordinary partial derivative operation from resulting in a tensor in curved spacetime.

This demonstration is shown by Ohanian and most other references that use tensor analysis to discuss space-time curvature.[33]

To show how the Christoffel symbol subtracts this problem from the transformation, consider the transformation of the covariant derivative using equation (88) and (89).

$$A'^{\alpha}{}_{;\beta} = \partial'_{\beta} A'^{\alpha} + \Gamma'^{\alpha}_{\sigma\beta} A'^{\sigma} . \tag{F6}$$

The form of the first term on the R.H.S. of equation (F6) has already been shown in equation (F5). To find the form of the second term, note the transformation property of the Christoffel symbol[37]

$$\Gamma'^{\alpha}_{\sigma\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\lambda}}{\partial x'^{\sigma}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \Gamma^{\mu}_{\lambda\nu} + \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\sigma} \partial x'^{\beta}} , \tag{F7}$$

which is a consequence of applying a coordinate transformation explicitly to the R.H.S. of equation (89) and then simplifying. Equation (F6) becomes

$$A'^{\alpha}{}_{;\beta} = \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} A^{\mu}{}_{,\nu} + \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\nu} \partial x^{\mu}} A^{\mu}$$

$$+ \left(\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\lambda}}{\partial x'^{\sigma}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \Gamma^{\mu}_{\lambda\nu} + \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\sigma} \partial x'^{\beta}} \right) \frac{\partial x'^{\sigma}}{\partial x^{\kappa}} A^{\kappa} . \tag{F8}$$

Rearranging

$$A'^{\alpha}{}_{;\beta} = \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} A^{\mu}{}_{,\nu} + \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \Gamma^{\mu}_{\lambda\nu} \frac{\partial x^{\lambda}}{\partial x'^{\sigma}} \frac{\partial x'^{\sigma}}{\partial x^{\kappa}} A^{\kappa}$$

$$+ \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\nu} \partial x^{\mu}} A^{\mu} + \frac{\partial x'^{\sigma}}{\partial x^{\kappa}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\sigma} \partial x'^{\beta}} A^{\kappa} . \tag{F9}$$

Recognizing the product of the transformation matrix with its own inverse as the Kronecker delta tensor in the second term, equation (F9) becomes

$$A'^{\alpha}{}_{;\beta} = \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} A^{\mu}{}_{,\nu} + \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \Gamma^{\mu}_{\lambda\nu} A^{\lambda}$$

$$+ \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\nu} \partial x^{\mu}} A^{\mu} + \frac{\partial x'^{\sigma}}{\partial x^{\kappa}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\sigma} \partial x'^{\beta}} A^{\kappa} . \tag{F10}$$

Consider the partial derivative of the Kronecker delta tensor, which vanishes in any coordinate system.

$$\frac{\partial}{\partial x'^{\sigma}} \delta^{\alpha}{}_{\beta} = 0 . \tag{F11}$$

The Kronecker delta tensor can be replaced by the product of the coordinate transformation matrix with its own inverse.

$$\frac{\partial}{\partial x'^{\sigma}} \left(\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x'^{\beta}} \right) = 0 = \frac{\partial^2 x'^{\alpha}}{\partial x'^{\sigma} \partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x'^{\beta}} + \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\sigma} \partial x'^{\beta}} . \tag{F12}$$

Using the chain rule

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\sigma} \partial x'^{\beta}} = - \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial x^{\lambda}}{\partial x'^{\sigma}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\lambda} \partial x^{\mu}} . \tag{F13}$$

Substituting equation (F13) into equation (F9) gives

$$A'^{\alpha}{}_{;\beta} = \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} A^{\mu}{}_{,\nu} + \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \Gamma^{\mu}_{\lambda\nu} A^{\lambda}$$

$$+ \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\nu} \partial x^{\mu}} A^{\mu} - \frac{\partial x'^{\sigma}}{\partial x^{\kappa}} \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial x^{\lambda}}{\partial x'^{\sigma}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\lambda} \partial x^{\mu}} A^{\kappa}$$

$$= \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} A^{\mu}{}_{,\nu} + \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \Gamma^{\mu}_{\lambda\nu} A^{\lambda}$$

$$+ \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\nu} \partial x^{\mu}} A^{\mu} - \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\lambda} \partial x^{\mu}} \delta^{\lambda}{}_{\kappa} A^{\kappa}$$

$$= \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} A^{\mu}{}_{,\nu} + \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \Gamma^{\mu}_{\lambda\nu} A^{\lambda} + \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\nu} \partial x^{\mu}} A^{\mu} - \frac{\partial x^{\mu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\lambda} \partial x^{\mu}} A^{\lambda}. \qquad \text{(F14)}$$

Utilizing the freedom to reassign contracted indices and the fact the repeated partial derivatives commute gives

$$A'^{\alpha}{}_{;\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \left(A^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\lambda\nu} A^{\lambda} \right) + \left(\frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\nu} \partial x^{\mu}} - \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\nu} \partial x^{\mu}} \right) A^{\mu}$$

$$= \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} A^{\mu}{}_{;\nu}. \qquad \text{(F15)}$$

Therefore, according to the rules set forth in Section 2.3, the form of equation (F15) shows that the covariant derivative of a 4-vector is a tensor of second rank. By induction (though not formally), the covariant derivative operator itself is a covariant 4-vector, or a 4-covector.

$$A^{\mu}{}_{;\nu} = D_{\nu} A^{\mu}. \qquad \text{(F16)}$$

Combining equations (F15) and (F16) gives

$$D'_{\beta} = \frac{\partial x^{\nu}}{\partial x'^{\beta}} D_{\nu}. \qquad \text{(F15)}$$

APPENDIX G

Derivation of Equation (99) from Equation (94)

Equation (94) says that a geodesic maximizes the proper time:

$$\int_{geodesic} d\tau = \max . \tag{G1}$$

From Chapter 1 Sec 3 the definition of proper time leads to:

$$\int d\tau = \int \frac{1}{c}\sqrt{g_{\mu\nu} dx^{\mu} dx^{\nu}} = \int \frac{1}{c}\sqrt{g_{00}\left(c\,dt\right)^2 + 2\,g_{0k}\,dx^k\,c\,dt + g_{kl}\,dx^k dx^l}$$

$$= \int dt \sqrt{g_{00} + 2\,g_{0k}\frac{dx^k}{c\,dt} + g_{kl}\frac{dx^k}{c\,dt}\frac{dx^l}{c\,dt}} = \int dt \sqrt{g_{00} + 2\,g_{0k}\,\beta^k + g_{kl}\,\beta^k\,\beta^l}\,, \tag{G2}$$

where $\beta^k = \dfrac{dx^k}{c\,dt}$, similar to the definition in equation (11). If this integration is carried out along a geodesic, then

$$\int_{geodesic} \sqrt{g_{00} + 2\,g_{0k}\,\beta^k + g_{kl}\,\beta^k\,\beta^l}\;dt = \max . \tag{G3}$$

Multiplying[2] by $-mc^2$ gives

$$-\int_{geodesic} mc^2\sqrt{g_{00} + 2\,g_{0k}\,\beta^k + g_{kl}\,\beta^k\,\beta^l}\;dt = \min . \tag{G4}$$

Notice that the integral is now minimized due to the negative sign. Assuming that the metric tensor deviates by a small amount from the Minkowski Metric tensor, and that the velocities are slow compared to the speed of light, the square root can be approximated.[2]

$$\int_{geodesic} d\tau = -\int_{geodesic} mc^2\sqrt{1-\left(1-g_{00} - 2\,g_{0k}\,\beta^k - g_{kl}\,\beta^k\,\beta^l\right)}dt$$

$$\approx -\int_{geodesic} mc^2\left(1-\tfrac{1}{2}\left(1-g_{00} - 2\,g_{0k}\,\beta^k - g_{kl}\,\beta^k\,\beta^l\right)\right)dt$$

$$\approx -\int_{geodesic} mc^2\left(1-\tfrac{1}{2}\left(1-g_{00} - \beta^k\beta_k\right)\right)dt\,. \tag{G5}$$

where the l superscript in the last term has been lowered by the metric tensor and the raised and lowered Latin index k indicates a summation from 1 to 3. The radical has been expanded assuming that the parenthetical is $<<1$, and, aside from the g_{00} component, the metric tensor is approximated as the Minkowski metric tensor.[2]

Simplifying, and making use of equation (11), equation (G5) becomes

$$\int_{geodesic} d\tau = -\int_{geodesic} \left(mc^2 - \tfrac{1}{2}mc^2 + \tfrac{1}{2}mc^2\, g_{00} - \tfrac{1}{2}mc^2\, \beta^2 \right) dt$$

$$= \int_{geodesic} \left(-mc^2 + \tfrac{1}{2}mc^2 - \tfrac{1}{2}mc^2\, g_{00} + \tfrac{1}{2}mv^2 \right) dt$$

$$= \int_{geodesic} \left(\tfrac{1}{2}mv^2 - m\left(\tfrac{1}{2}c^2 g_{00} + \tfrac{1}{2}c^2 \right) \right) dt = \min\,, \qquad (99)$$

upon rearranging, which is equation (99).

APPENDIX H

The Stress-energy Tensor from the "Cloud of Dust" Model

Imagine an extremely dense (in the continuum limit) cloud of identically massive dust particles (point masses) of rest mass m, but that they do not interact with each other. This is the popular "cloud of dust" model used to demonstrate the stress-energy tensor.[41] In some small element of spatial volume ΔV, there is some number of particles n, each with a rest mass m. For simplicity, consider all of the particles to be co-moving. Then, the momentum in this element of volume is

$$p^{\mu} = n m u^{\mu} . \tag{H1}$$

The energy of this element of the dust cloud is given as

$$\Delta E = p^{0} c = n m u^{0} c . \tag{H2}$$

Therefore, the energy density in this volume is

$$\frac{\Delta E}{\Delta V} = \frac{n m}{\Delta V} u^{0} c . \tag{H3}$$

In the limit that $\Delta V \to 0$, equation (H3) gives

$$\frac{d(n m)}{d^{3}x} u^{0} c = \frac{1}{c} j^{0} u^{0} c = \tilde{\rho} u^{0} u^{0} \equiv T^{00} , \tag{H4}$$

where $\tilde{\rho}$ is the rest frame mass density of the dust cloud and T^{00} is defined as the energy density of the matter field (i.e. the density of $m c^{2}$). The appearance of j^{0} is by a similar argument to that found in Appendix B, except that here it indicates the 0-component of the mass current density 4-vector.

In order to form a tensor, all that must be done is to generalize the indices from 0 to μ and ν. Then, equation (H4) is still identically valid in the rest frame, and furthermore, validity is maintained under a Lorentz boost.

$$T^{\mu\nu} \equiv \tilde{\rho} u^{\mu} u^{\nu} . \tag{H5}$$

Lowering the second index of equation (H5) for the 00 component gives

$$T^{0}{}_{0} = \tilde{\rho}\, g_{\alpha 0}\, u^{0}\, u^{\alpha} . \tag{H6}$$

In the static weak field limit, $g_{\alpha 0} \to \eta_{\alpha 0}$, and $u^{0} \to c$. Therefore equation (H6) gives

$$T^{0}{}_{0} \to \tilde{\rho}\, \eta_{\alpha 0}\, u^{0}\, u^{\alpha} = \tilde{\rho}\, u^{0}\, u^{0} \to \tilde{\rho}\, c^{2} . \tag{H7}$$

For an interpretation of this new tensor, consider three sets of components, T^{00}, T^{k0}, and T^{kl}.

T^{00} is the energy density,[41] as before. It is the amount of energy per unit volume by virtue of the presence of rest mass, whether or not the mass is in motion.

$T^{k0} = T^{0k}$ is the momentum density, or energy flux density.[41] This is quite similar to current density. It is due to a stream of massive point particles flowing in the k direction (or, at least, the flow has a projection in the k direction).

$T^{kl} = T^{lk}$ is the momentum flux density of k directed momentum transferred in the l direction.[41] This is similar to a transverse wave in the case that $k \neq l$, and a longitudinal wave in the case that $k = l$.

As a simple example, consider a constant, continuous, and static distribution of non-interacting mass. The stress energy tensor has components $T^{00} = \tilde{\rho}\, c^{2}$, and $T^{\mu\nu} = 0$ if either $\mu \neq 0$ or $\nu \neq 0$. A Lorentz boosted version of this stress energy tensor in the x direction is essentially the same as a uniform distribution of mass moving as a whole in the x direction (resulting in a Lorentz factor due to relativistic velocity addition) with an increased mass density (due to length contraction and resulting in another Lorentz factor). This gives $T'^{00} = \gamma^{2}\, \tilde{\rho}\, c^{2}$, $T'^{10} = T'^{01} = \gamma^{2}\, c\, \tilde{\rho}\, v$, and $T'^{11} = \gamma^{2}\, \tilde{\rho}\, v^{2}$. Therefore, the 00 component is still the familiar rest energy density with the relativistic correction factor of γ^2, the 10 component is the classical density of the momentum in the x direction with the relativistic correction factor of $\gamma^2 c$, and the 11 component is the classical kinetic energy density with the relativistic correction factor γ^2.

APPENDIX I

Generalization of $\nabla^2 g_{00}$ to $-2R^0{}_0$

According to the notation of Chapter 2, the Laplacian of g_{00} can be written as

$$\nabla^2 g_{00} = -\partial^k \partial_k g_{00} = -\eta_{kl}\, g_{00}{}^{,k,l}\,, \tag{I1}$$

where the summations on k and l are from 1 to 3. The use of Latin indices is allowed by the diagonal nature of the Minkowski metric tensor, since including the 0 index terms would not contribute anything to the summation.

The quantity in equation (I1) must be related to a tensor in order to be a physical object. As it stands, the object in equation (I1) has two free indices, both 0. So, it is reasonable to generalize to the 00 component of a second rank tensor. The most obvious first attempt is to simply generalize the ordinary partial derivatives summed from 1 to 3 to covariant derivatives summed from 0 to 3. This, by definition, would make the object the 00 component of a tensor directly in a single step.

$$g_{00}{}^{,k}{}_{,k} \rightarrow -g_{00}{}^{;\mu}{}_{;\mu}\,. \tag{I2}$$

Unfortunately, this is not an acceptable candidate, since this object, also by definition, vanishes when g_{00} is interpreted as a component of the general metric tensor.

$$g_{00;\mu} = 0 \tag{I3}$$

The next attempt (not at all obvious, and not nearly as direct) is to transpose the indices in order to account for all of the components of the metric tensor. Consider the explicit summation in equation (I1).

$$\nabla^2 g_{00} = -\sum_{k=1}^{3}\sum_{l=1}^{3}\eta_{kl}\, g_{00}{}^{,k,l} \tag{I4}$$

Transposing the indices on $g_{00}{}^{,k,l}$ gives

$$\nabla^2 g_{00} = -\sum_{k=1}^{3}\sum_{l=1}^{3}\eta_{kl}\left(g_{00}{}^{,k,l} + g_{k0}{}^{,0,l} + g_{kl}{}^{,0,0} + g_{0l}{}^{,k,0}\right). \tag{I5}$$

The equality remains in equation (I5) in the static limit since all three additional terms contain a derivative with respect to time and therefore vanish. This accounts for all of the components of the general metric tensor. The operation of raising an lowering can be done implicitly by accounting for a change of sign when the operation involves a Latin index. This gives

$$\nabla^2 g_{00} = -\sum_{k=1}^{3}\sum_{l=1}^{3}(-1)(-1)\eta^{kl}_{\uparrow\uparrow}\left((-1)(-1)g_{00,\overset{\downarrow}{k},\overset{\downarrow}{l}} + (-1)g_{k0,0,l}^{\;\downarrow\downarrow} + g_{kl,0,0}^{\;\downarrow\downarrow} + (-1)g_{0l,k,0}^{\;\downarrow\downarrow}\right)$$

$$= -\sum_{k=1}^{3}\sum_{l=1}^{3}\eta^{kl}\left(g_{00,k,l} - g_{k0,0,l} + g_{kl,0,0} - g_{0l,k,0}\right) \tag{I6}$$

The Latin indices can be generalized to Greek indices also without changing the validity of equation (I6) in the static limit, since such a generalization will only introduce time derivatives which vanish in the static limit. The explicit summation is now removed in favor of notational convenience over functional explicitness.

$$\nabla^2 g_{00} = -\eta^{\mu\nu}\left(g_{00,\mu,\nu} - g_{\mu0,0,\nu} + g_{\mu\nu,0,0} - g_{0\nu,\mu,0}\right). \tag{I7}$$

Two mutually canceling terms may be added to equation (I7), namely $-g_{\nu0,0,\mu}$ and $g_{\nu0,0,\mu}$. Inserting these two terms and rearranging gives

$$\nabla^2 g_{00} = -\eta^{\mu\nu}\left(-g_{\nu0,0,\mu} - g_{0\nu,\mu,0} + g_{00,\mu,\nu} + g_{\nu0,0,\mu} + g_{\mu\nu,0,0} - g_{\mu0,0,\nu}\right). \tag{I8}$$

Using the symmetry of the metric tensor and the commutability of the ordinary partial derivatives, equation (I8) may be rewritten.

$$\nabla^2 g_{00} = -\eta^{\mu\nu}\left(-g_{\nu0,0,\mu} - g_{0\nu,0,\mu} + g_{00,\nu,\mu} + g_{\nu0,\mu,0} + g_{\mu\nu,0,0} - g_{0\mu,\nu,0}\right). \tag{I9}$$

Grouping the first three terms together and the last three terms together and recognizing the Minkowski metric tensor as a constant gives

$$\nabla^2 g_{00} = \partial_\mu\left[\eta^{\mu\nu}\left(g_{\nu0,0} + g_{0\nu,0} - g_{00,\nu}\right)\right] - \partial_0\left[\eta^{\mu\nu}\left(g_{\nu0,\mu} + g_{\mu\nu,0} - g_{0\mu,\nu}\right)\right]. \tag{I10}$$

In local geodesic coordinates $g^{\mu\nu} = \eta^{\mu\nu}$ at the pole of these coordinates. In such coordinates, equation (I10) may be written as

$$\nabla^2 g_{00} = \partial_\mu \left[g^{\mu\nu} \left(g_{\nu 0,0} + g_{0\nu,0} - g_{00,\nu} \right) \right] - \partial_0 \left[g^{\mu\nu} \left(g_{\nu 0,\mu} + g_{\mu\nu,0} - g_{0\mu,\nu} \right) \right]. \tag{I11}$$

The quantities in square brackets are recognized as constant scalar multiples of the Christoffel symbol according to equation (89).

$$\nabla^2 g_{00} = \partial_\mu \left[2\Gamma^\mu_{00} \right] - \partial_0 \left[2\Gamma^\mu_{0\mu} \right] = -2\left(-\Gamma^\mu_{00,\mu} + \Gamma^\mu_{0\mu,0} \right). \tag{I12}$$

In geodesic coordinates, the quantity in parenthesis of equation (I12) is recognized as exactly the R_{00} component of the Ricci tensor according to equations (90) and (91).

$$\nabla^2 g_{00} = -2 R_{00} . \tag{I13}$$

The first subscript of this component of the Ricci tensor may be raised with the metric tensor. In geodesic coordinates $g^{\alpha 0} \to \eta^{\alpha 0}$, therefore, raising and lowering a 0 index does not change the value of the tensor component. This gives (numerically)

$$\nabla^2 g_{00} = -2 R^0{}_0 . \tag{I14}$$

The justification for using geodesic coordinates is in the weak field limit. Geodesic coordinates are coordinates for which the metric tensor is locally flat, and therefore can be replaced by the Minkowski metric tensor. This is representative of the metric that would be observed, for instance, inside an elevator that is freely falling in the Earth's gravitational field. Essentially, a freely falling Cartesian coordinate system is a geodesic coordinate system in a weak gravitational field for some small amount of time.

At this point, it should be emphasized that the equality in equation (I10) is exact in the static limit, and that, by choosing a geodesic coordinate system, the generalization from (I10) to (I11) is locally exact. A transformation from the R.H.S. of equation (I11) to a general coordinate system will generate two extra terms that are products of Christoffel symbols. These two terms match those found in the definition of the Riemann tensor so that the R.H.S. of equation (I11) is found to transform as a tensor in general coordinates.

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