AROUND THE FIBONACCI NUMERATION SYSTEM

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Let 1, 2, 3, 5, 8, … denote the Fibonacci sequence beginning with 1 and 2, and then setting each subsequent number to the sum of the two previous ones. Every positive integer *n* can be expressed as a sum of distinct Fibonacci numbers in one or more ways. Setting *R(n)* to be the number of ways *n* can be written as a sum of distinct Fibonacci numbers, we exhibit certain regularity properties of *R(n)*, one of which is connected to the Euler φ–function. In addition, using a theorem of Fine and Wilf, we give a formula for *R(n)* in terms of binomial coefficients modulo two.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. Introduction of the Fibonacci Numbers and Their Generalizations

In 1202, an Italian mathematician, Leonardo Pisano, who was called Fibonacci, published a book called *Liber Abaci*, the earliest Western publication that mentions the Fibonacci sequence. He posed the following question. How many rabbits can be produced in a year from a single pair of rabbits if each pair produces a new pair each month, starting in its second month, with the additional assumption that rabbits never die? The answer is given by the Fibonacci numbers, $1, 2, 3, 5, 8, 13, 21, 34, \ldots$, which are defined by $f_0 = f_1 = 1$, and $f_k = f_{k-1} + f_{k-2}$, for all $k \geq 2$. It turns out that in month k, there are f_k-many rabbits. Even though this is the most well-known example using the Fibonacci numbers, this sequence in fact turns up in the studies of Indian scholars at least by 200 B.C.

In the sixth century A.D., the Indian mathematician Virahanka studied, in detail, Sanskrit vowel sounds. In the twelfth century, the Indian philosopher Hemachondra wrote a text which revisited the work of Virahanka. Sanskrit vowel sounds can be characterized as long and short. The question at the time was how many combinations of vowel sounds there were of a given length assuming that the long vowel sound was twice the length of the short one. The answer here is also given by the Fibonacci numbers. For example,

$$
\begin{aligned}
&\text{Length } 1: && S\\
&\text{Length } 2: && SS, L\\
&\text{Length } 3: && SSS, SL, LS\\
&\text{Length } 4: && SSSS, SSL, LSS, SLS, LL\\
&\text{Length } 5: && SSSSS, SSSL, SSLSS, SLSS, LSSS, SLL, LSL, LLS
\end{aligned}
$$

Fibonacci visited India before writing the *Liber Abaci* and studied the numeration system used there. In addition, he also studied these vowel sounds and mentions them specifically in his text.

One way to calculate the nth Fibonacci number in terms of elementary functions involves the well-known Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$. The Golden Ratio ϕ is the positive solution of the equation $x^2 - x - 1 = 0$. In 1753, R. Simson, a Scottish mathematician, showed that the ratios of consecutive Fibonacci numbers, $\frac{f_n}{f_{n-1}}$, converge to ϕ. Binet's formula, so named in honor of the French mathematician Binet but actually attributed to Euler, gives a closed formula for the nth Fibonacci number in terms of elementary functions; $f(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$. There is also a closed formula in terms of matrices whose eigenvalues are related to ϕ.

The Fibonacci numbers appear in various academic fields as well as nature. The number of petals on flowers, arrangements of seeds on flower heads, and the spirals on pine cones can all involve Fibonacci numbers [37], [41]. There have been numerous analyses of well-known musical compositions [37], [29], such as Beethoven's *Fifth*, in order to detect any adherence to the Fibonacci numbers or ϕ. In addition, Fibonacci numbers occur in Pascal's triangle and Lozanic's triangle, in cryptography in relation to pseudorandom number generators [53], [39], in theoretical physics in relation to quasicrystals [4], in the run-time analysis of the Euclidean algorithm [53], and in numeration systems in [6], [12], [17], [18], [20], [36], [37], [38], [40], [54] for example. The list is long and the above-mentioned occurrences do not come close to exhausting it.

There are many generalizations of the Fibonacci numbers and the Fibonacci recurrence relation. A first example is the Lucas numbers, which keep the Fibonacci recurrence relation but use different initial conditions, $L_0 = 2$ and $L_1 = 1$. So, the Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, \ldots$. Alternative generalizations of the Fibonacci numbers are obtained by changing the linear recurrence relation. For example, the Pell numbers are defined as follows: $P_0 = 0, P_1 = 1$, and for $n \geq 2, P_n = 2P_{n-1} + P_{n-2}$. Even other processes for generalization have been used, such as adding two consecutive numbers further back in the sequence or extending the Fibonacci sequence and its generalizations to negative numbers. In this text, we will consider a natural generalization of the Fibonacci recurrence relation leading to the m-bonacci numbers. Instead of adding the two most recent numbers to get the next one, m-bonacci requires adding the m most recent. For each $m \geq 2$, we define the m-bonacci numbers by $F_k = 2^k$ for $0 \leq k \leq m-1$ and $F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m}$ for $k \geq m$. Notice that when $m = 2$, these are the Fibonacci numbers. When $m = 3$, we obtain the Tribonacci numbers, $1, 2, 4, 7, 13, 24, 44, 81, \ldots$.

1.2. The Infinite Fibonacci Word

There is an infinite word ω associated with the Fibonacci sequence. It is called the infinite Fibonacci word and is the fixed point of the morphism

$$\begin{aligned} \tau : 0 &\mapsto 01 \\ 1 &\mapsto 0. \end{aligned}$$

To generate ω, we iterate τ, beginning with 0. Using the rule that every 0 gets replaced by 01 and every 1 gets replaced by 0, we get $\tau^0(0) = 0, \tau^1(0) = 01, \tau^2(0) = 010, \tau^3(0) = 01001, \tau^4(0) = 01001010, \ldots$. The infinite Fibonacci word ω is equal to

$$\lim_{n\to\infty} \tau^n(0) = 0100101001001010010100\ldots.$$

It is the fixed point of τ since $\tau(\omega) = \omega$, meaning if every 0 is replaced by 01 and every 1 by 0 in ω, we still have ω. Note that 11 and 000 can never occur in ω. The word 11 can never appear because there is no way to concatenate 01 and 0 to get it. The word 00 never occurs because it would have come from an iteration of τ applied to 11. A couple of observations relating the Fibonacci numbers to ω are the following.

- The lengths of the iterations of τ are the Fibonacci numbers.
- The finite word obtained by the nth iteration of τ contains f_n-many 0's and f_{n-1}-many 1's. For example, the fourth iteration of τ produces 01001010, which contains $f_4 = 5$0's and $f_3 = 3$1's.

The Fibonacci word ω is probably the most well-known example of a *Sturmian word*, an infinite word having exactly $n+1$ factors of length n. See [7]. For example, the four factors of length 3 of the Fibonacci word are $\{100, 010, 001, 101\}$. The words $011, 110, 111$, and 000, cannot occur since there are no 000's and 11's. A *random* binary sequence has 2^n-many factors of length n, and a *periodic* sequence has less than $n+1$-many factors of length n. Periodic sequences eventually repeat and random sequences are anything but predictable. A Sturmian word is as close to periodic as one can get without actually being periodic.

The study of Sturmian words dates back to the astronomer J. Bernoulli III (1772) in [5] and there is also early work by Christoffel and Markoff in [13], [44], but the first comprehensive study of Sturmian words was given by G.A. Hedlund and M. Morse in 1940 in [30]. In this 1940 paper, Hedlund and Morse discuss Sturmian words in terms of symbolic dynamics. The term *Sturmian* is introduced in this paper to honor the mathematician Jacques Charles

François Sturm. Sturmian words can be characterized by an arithmetic formulation which provides a link from combinatorics to number theory. They also appear in ergodic theory [46], computer graphics [11], and crystallography [43] and have been additionally characterized in terms of the continued fraction expansions of irrational numbers, palindromic closures, return words, left- and right-special factors, and suffix replication. See [16], [47], and [51] for example.

A generalization of Sturmian words are Arnoux-Rauzy sequences, introduced in a 1991 paper [3] by P. Arnoux and G. Rauzy, which are infinite words on three symbols having exactly $2n+1$ factors of length n. The most well-known example of an Arnoux-Rauzy sequence is the Tribonacci sequence which is the fixed point of the morphism

$$\begin{aligned} \tau : 0 &\mapsto 01 \\ 1 &\mapsto 02 \\ 2 &\mapsto 0. \end{aligned}$$

The Tribonacci word $\nu = 0102010010201\ldots$, and has a similar relationship to the Tribonacci numbers as ω does with the Fibonacci numbers. The lengths of the iterations of τ are the Tribonacci numbers, $(T_n)_{n\geq 0}$. $\tau^0(0) = 0, \tau^1(0) = 01, \tau^2(0) = 0102, \tau^3(0) = 0102010, \tau^4(0) = 0102010010201, \ldots$. For $n \geq 3$, in the nth iteration of τ, the number of 0's is T_{n-1}, the number of 1's is T_{n-2}, and the number of 2's is T_{n-3}. The Tribonacci sequence and Tribonacci word can be used together to obtain what is called the *Rauzy fractal.* The Rauzy fractal can be used to tile the plane in two different ways, one of which is actually periodic. See [8] and [9].

1.3. Numeration Systems and the Fibonacci Numbers

A *numeration system* is a base or a sequence of numbers together with an alphabet that allows us to represent numbers. In this sense, the notion of a numeration system is very basic and natural. Typically, when one thinks of a numeration system, the *base* β is a natural number that signifies intent to write another natural number in terms of powers of β, where the number of times we can use a given power of β is $0, \ldots \beta-1$. In this case, we call β together with the alphabet $A = \{0, \ldots, \beta - 1\}$ a *standard numeration system.* The *β-representation* of an integer $n \geq 0$ is a finite word $d_k \ldots d_0$ over the alphabet $A = \{0, \ldots, \beta - 1\}$ where $n = \sum_{i=0}^{k} d_i \beta^i$. Such a representation is unique if $d_k \neq 0$. For example, when $\beta = 10$ and $A = \{0, \ldots, 9\}$, we have our usual base-ten numeration system. Another common numeration system (base two) is given by $\beta = 2$ and $A = \{0, 1\}$. The base-ten representation of 6752 is just 6752 but since, as a sum of powers of two, $6572 = 4096+2048+256+128+32+8+4$, its base-two representation is 1100110101100.

An extension of this idea, still using a natural number for the base β and alphabet $A = \{0, \ldots, \beta - 1\}$, allows us to represent a non-negative real number x as an infinite sequence $(x_i)_{i \leq k}$, where $x = \sum_{i \leq k}(x_i \beta^i)$. Further, we can use a real number $\beta > 1$ as a base to represent real numbers in the interval $[0, 1]$ as an infinite sequence. Though many extensions of the idea of a standard numeration system have been studied, we shall be concerned with the numeration system that arises by replacing the base with a strictly increasing sequence of integers. General information about numeration systems can be found in [24].

Let $U = (u_n)_{n \geq 0}$ be a strictly increasing sequence of integers with $u_0 = 1$. A representation $d_k \ldots d_0$ of a non-negative integer n is called the normal U-representation of n, if it is produced by the greedy algorithm which is defined as follows. If $u_k \leq n < u_{k+1}$, then $d_k = \lfloor \frac{n}{u_k} \rfloor$, $d_i = \lfloor \frac{n-(d_k u_k + \ldots + d_{i+1} u_{i+1})}{u_i} \rfloor$, for $i = 0, \ldots, k-1$. Then $n = d_k u_k + \ldots + d_0 u_0$ and the normal representation of n, obtained by the greedy algorithm, is $d_k \ldots d_0$.

The Fibonacci, and more generally the m-bonacci, sequence together with the alphabet $A = \{0, 1\}$ determines a numeration system [24]. For example, $41 = 34 + 5 + 2$ and so a representation of 41 in the Fibonacci base is 10001010. However, $41 = 21 + 13 + 5 + 2$ so that another representation is 1101010. In the Fibonacci numeration system, to get other representations, a 100 can be replaced by a 011 and *vice versa*, because instead of using f_k, for some k, we can use f_{k-1} and f_{k-2} if these numbers are not already in use (and *vice versa*). For m-bonacci, 10^m can be replaced by 01^m and *vice versa.* This implies that in the Fibonacci (m-bonacci) numeration system, representations are not necessarily unique, unlike base 10 and base 2. However, for every positive integer n, there is a canonical representation of n obtained from the greedy algorithm. In our Fibonacci (m-bonacci) case, we shall call this the Zeckendorff (m-Zeckendorff) representation and denote it $Z(n)$, or $Z_m(n)$ in the case of m-bonacci.

The m-Zeckendorff representation has several notable properties; we shall list a few here.

- Because $Z_m(n)$ is obtained from the greedy algorithm, it has no occurrences of 1^m. In particular, in the case of Fibonacci ($m = 2$), there are no occurrences of 11.
- Because all other representations of n are obtained from $Z_m(n)$ by substituting 01^m for 10^m, $Z_m(n)$ is the lexicographically largest representation.

- An integer n has a unique representation in the m-bonacci base if and only if $Z_m(n)$ contains no occurrences of 0^m. In the case of Fibonacci, n has a unique representation if and only if Z(n) contains no occurrence of 00, which occurs if and only if $Z(n)$ is a prefix of $(10)^\infty$, which occurs if and only if $n = f_k - 1$ for some Fibonacci number f_k.

There in fact is a connection in the Fibonacci case between the Zeckendorff representations of all integers and the infinite Fibonacci word. We will begin with 0 and say that $Z(0) = 0$. Reading the last entry in each representation beginning from 0, we recover the infinite Fibonacci word. The following is $Z(0), Z(1), Z(2), \ldots$.

$\mathbf{0}, \mathbf{1}, 1\mathbf{0}, 10\mathbf{0}, 10\mathbf{1}, 100\mathbf{0}, 100\mathbf{1}, 101\mathbf{0}, 1000\mathbf{0}, 1000\mathbf{1}, 1001\mathbf{0}, 1010\mathbf{0}, 1010\mathbf{1}, 10000\mathbf{0}, \ldots$

1.4. The Function $R(n)$

Since a representation of n in Fibonacci (m-bonacci) is not necessarily unique, unlike base ten or two, it is a natural question to ask how many representations n has. We set $R(n)$, or $R_m(n)$ in the case of m-bonacci, to be the number of representations n has in the Fibonacci (m-bonacci) base.

Example 1.1. Let $n = 50$. We see that $Z(50) = 10100100$. All other representations are obtained by replacing 100 by 011. We then have the following 6 representations (arranged in decreasing lexicographic order) so that $R(50) = 6$:

10100100
10100011
10011100
10011011
1111100
1111011

Many mathematicians have studied the function R and representations of integers in the Fibonacci base. C. G. Lekkerkerker was the first (in 1952) to publish a work [40] on the Fibonacci numeration system. In this work, he showed that the Zeckendorff representation of an integer is unique. In a 1965 paper [20], H. H. Ferns gives the number of integers N in the interval $f_n \leq N < f_{n+1}$ that have m-many 1's in their Zeckendorff representations. D. Klarner, in a 1966 paper [36], gives a formula for $R(f_n - 1)$ and bounds on $R(N)$ for

$f_n \leq N \leq f_{n+1} - 1$. In a 1968 paper [12], L. Carlitz gives a formula for $R(n)$ in very special cases, when $Z(n)$ contains one, two, or three 1's. Carlitz gives explicit formulae in these special cases but states, "While explicit formulas are obtained for $r = 1, 2, 3$ in a canonical representation, the general case is very complicated." E. Zeckendorff (of the so-called Zeckendorff representation) did not publish a paper on the Fibonacci numeration system until 1972 [54] but claims that in 1939 he had a proof that the so-called Zeckendorff representation is unique. In 2001, J. Berstel published a paper [6] expressing $R(n)$ as a product of 2×2 matrices. In this paper, he states, " As already mentioned, the behavior of $R(n)$ is rather irregular as a function of $n.''$ Earlier in the paper, Berstel had said that after some experimentation, the function R seems rather erratic. In 2005 [38], P. Kocábová, Z. Masácová, and E. Pelantová, extended Berstel's result to m-bonacci numbers using material that appears in [17] and gave maxima for $R_m(n)$.

1.5. Main Results of the Thesis

Chapters 2 and 3: Contrary to Berstel's remark that the behavior of R seems rather erratic, the sequence $R(n)$ exhibits quite a rigid structure when represented as an array of rows where the kth row has length f_{k-1}. We study regularity properties of R in Chapters 2, 3, and 4. In Chapter 2, we discuss a palindromic property of the sequence as well as different recursive methods for generating $R(n)$. A first method for generating $R(n)$ can be found in section 2.1 and a second method in section 2.3. The palindrome structure of R(n) is discussed in section 2.2 with the main result being Proposition 2.3. The representation of R as an array gave rise to a new algorithm, for computing $R(n)$ using Fibonacci towers (to be defined in Chapter 3), which is closely related to the Euclidean algorithm. This new algorithm allows us to see structure in $\{R^{-1}(n)\}$. Given m, the set $R^{-1}(m) = \{n : R(n) = m\}$ is infinite but it is finitely generated in the sense that there are a finite number of words, called m-basis words, so that any word w having m representations is either an m-basis word or can be written as $w = uv$, where u is an m-basis word and v is a prefix of $(10)^{\infty}$. A relation between the number of these m-basis words and the Euler ϕ-function is given in Theorem 3.10.

Chapter 4: A theorem of Fine and Wilf, in [21], allows a definition of special words associated with these towers. In chapter 4, using a recursive definition for $R(n)$ and the structure of these *Fine and Wilf words*, we obtain a formula for $R(n)$ as a product of sums of binomial coefficients modulo 2. This formula, given in Theorem 4.1, provides a resolution to the work started by Carlitz.

Chapter 5: At a conference on the Fibonacci infinite word, in Turku, Finland, in October, 2006, Jean Berstel presented a never-ending open problem. Choosing any identity involving the Fibonacci numbers, interpret this identity combinatorially in terms of the Fibonacci word. He mentioned two specifically that he had attempted and had not yet found a combinatorial interpretation. One of these,

$$f_{2n} = \sum_{i=0}^{n} \binom{n}{i} f_{n-i} \quad \text{and} \quad f_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} f_{n-i+1},$$

is addressed in the last chapter. The main result can be found in Theorem 5.15.

The material in Chapters 2, 3, and 4 is joint work with my advisor, Luca Zamboni, and has been published in papers [17] and [18]. The material in Chapter 5 will soon be submitted for publication.

A general reference for combinatorics on words, which includes Sturmian words and numeration systems, is [42].

CHAPTER 2

ALGORITHMS FOR GENERATING $R(n)$.

2.1. Introduction

In this chapter, we demonstrate structure in the sequence $R(n)$ that seems to have been previously unnoticed. J. Berstel's remark in [6] that $R(n)$ is rather erratic as a function of n was an inspiration to search for regularity properties of the sequence. This structure is more apparent when the sequence is represented as a two-dimensional array (shown in Figure 2.1) having row lengths equal to Fibonacci numbers. The array consists of an infinite number of rows $L_1, L_2, L_3, \ldots$ where the kth row L_k contains f_{k-1} terms of the sequence [1].

$$
\begin{array}{lcl}
L_1 & : & 1 \\
L_2 & : & 1 \\
L_3 & : & 21 \\
L_4 & : & 221 \\
L_5 & : & 32231 \\
L_6 & : & 33242331 \\
L_7 & : & 4335244253341 \\
L_8 & : & 443635526446255363441 \\
L_9 & : & 5447366385572664846627558366374451 \\
L_{10} & : & 5548477396693885T5772866T4884T6682775T5883966937748455 1 \\
\vdots & \vdots & \vdots
\end{array}
$$

FIGURE 2.1. Schematic Representation of the Sequence $(R(n))_{n\geq 1}$

The sequence $(R(n))_{n\geq 1}$ is simply the concatenation of the blocks L_k so

$$(R(n))_{n\geq 1} = 1121221322313324233143352442533414436355264462553634 41\cdots$$

We begin, in Section 2.2, with a decomposition of $R(n)$ and a recursive combinatorial construction for the above 2-dimensional representation of $R(n)$. That is, we give a recipe for generating the kth row L_k from earlier rows L_j with $j < k$. In Section 2.3, we explain

[1]In L_{10} the entry T denotes the value 10.

the underlying palindrome structure of each row L_k mentioned above. Finally, in Section 2.4, we consider a different decomposition of $R(n)$ and an alternate recursive algorithm for generating $R(n)$. With respect to this second decomposition of $R(n)$, it is natural to consider another sequence, which we denote $(d(n))_{n\geq 0}$, that gives the coefficients in the power series expansion of the infinite product

$$F(x) = \prod_{k=1}^{\infty}(1-x^{f_k}) = (1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^8)(1-x^{13})\cdots.$$

See [2]. A different algorithm for generating the coefficients $d(n)$ was originally discovered by Tamura [49] but remains unpublished.

2.2. The First Algorithm

Denote, by $\{0,1\}^*$, the set of all finite words $w = w_1w_2\cdots w_k$ with $w_i \in \{0,1\}$. To each $w = w_1w_2\cdots w_k \in \{0,1\}^*$ with $w_1 = 1$, we associate a positive integer n_w defined by

$$n_w = w_1f_k + w_2f_{k-1} + \cdots + w_{k-1}f_2 + w_kf_1.$$

We say that w is a representation of n_w in the Fibonacci base.

Given $n \in \mathbb{Z}^{>0}$, set

$$\Omega(n) = \{w = w_1w_2\cdots w_k \in \{0,1\}^* \,|w_1 = 1 \text{ and } n_w = n\}$$

and put $R(n) = \#\Omega(n)$. For $w \in \Omega(n)$, we write $R(w)$ for $R(n_w)$. For each $w \in \Omega(n)$, we have $|Z(n)| - |w| \in \{0,1\}$. Thus, we define

$$\Omega^+(n) = \{w \in \Omega(n) \,|\, |w| = |Z(n)|\}$$

and

$$\Omega^-(n) = \{w \in \Omega(n) \,|\, |w| = |Z(n)| - 1\},$$

and put $R^+(n) = \#\Omega^+(n)$ and $R^-(n) = \#\Omega^-(n)$. Then for each n, we have $R^+(n) \geq 1$, $R^-(n) \geq 0$, and $R(n) = R^+(n) + R^-(n)$. For example, it follows from Example 1.1 that $R^+(50) = 4$ and $R^-(50) = 2$. For $w \in \Omega(n)$ we write $R^+(w)$ and $R^-(w)$ for $R^+(n_w)$ and $R^-(n_w)$ respectively.

We now give a lemma that will be used repeatedly.

LEMMA 2.1. *Let w be a Zeckendorff word. Then:*

(i) $R^+(f_r) = R^+(10^{r-1}) = 1$ *for each* $r \geq 1$.

(ii) $R^+(10^\ell w) = R(w)$ *whenever* $\ell \geq 1$.

(iii) $R^-(10^\ell w) = R(10^{\ell-2}w)$ *whenever* $\ell \geq 3$.

(iv) $R^-(100w) = R(w)$.

(v) $R^-(10w) = R^-(w)$.

(vi) $R(100w) = 2R(w)$.

(vii) *If* $w = vu$ *with* $v \in \{0,1\}^*$ *and* u *a prefix of* $(10)^\infty$, *then* $R^-(w) = R^-(v)$ *and* $R^+(w) = R^+(v)$.

PROOF. As for items (i) and $(ii), R^+(n)$ counts the number of representations of n where the left-most 1 remains unchanged. For $(iii), R^-(10^\ell w) = R(0110^{\ell-2}w) = R(10^{\ell-2}w)$. For $(iv), R^-(100w) = R(011w) = R(w)$. For (v), to calculate $R^-(10w)$, we count the number of representations of $10w$ where the left-most one is replaced by a 0. For this to happen, the left-most 1 of w also must be replaced by a 0. Thus, $R^-(10w) = R^-(w)$. For $(vi), R(100w) = R^+(100w) + R^-(100w) = R(w) + R(w) = 2R(w)$. Finally, (vii) holds because a prefix of $(10)^\infty$ contributes nothing for new representations. □

We next give a recursive combinatorial construction for the sequence $R(n)$. We begin with a recursive construction of a sequence $(x_n)_{n\geq 1} \in \mathbb{N}^\infty$, which we shall see coincides with the sequence $R^-(n)$. In order to define the sequence (x_n), we arrange it schematically as shown below in a 2-dimensional array consisting of an infinite collection of rows so that each row $k \geq 0$ contains f_k entries of the sequence (x_n) (see Figure 2.2.)

$$
\begin{array}{lcllllllllllllll}
\text{row } 0 & : & x_1 \\
\text{row } 1 & : & x_2 \\
\text{row } 2 & : & x_3 & x_4 \\
\text{row } 3 & : & x_5 & x_6 & x_7 \\
\text{row } 4 & : & x_8 & x_9 & x_{10} & x_{11} & x_{12} \\
\text{row } 5 & : & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} & x_{20} \\
\text{row } 6 & : & x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} & x_{29} & x_{30} & x_{31} & x_{32} & x_{33} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

FIGURE 2.2. Schematic Representation of the Sequence (x_n)

The following statements all follow from the lengths of the rows of Figure 2.2.

- The first entry in row k is $x_{f_{k+1}}$. Hence x_n is in row k, if and only if $|Z(n)| = k+1$.
- If x_n is in row k, then the entry below x_n in row $k+1$ is x_{n+f_k}.
- For $k \geq 3$, each of the first f_{k-3} entries of row k has three or more entries above it in the same column. We denote this section of row k by A_k. Each of the next f_{k-4} entries of row k has exactly two entries above it in the same column (we put $f_{-1} = 0$). We denote this section of row k by B_k. Each of the next f_{k-3} entries of row k has exactly one entry above it in the same column. We denote this section of row k by C_k. Each of the remaining f_{k-2} entries of row k has no entry above it in the same column. We denote this section of row k by D_k.
- $x_n \in A_{k+1}$ if and only if $f_{k+2} \leq n < f_{k+2} + f_{k-2}$. $x_n \in B_{k+1}$ if and only if $f_{k+2} + f_{k-2} \leq n < f_{k+2} + f_{k-1}$. $x_n \in C_{k+1}$ if and only if $f_{k+2} + f_{k-1} \leq n < f_{k+2} + f_k$. $x_n \in D_{k+1}$ if and only if $f_{k+2} + f_k \leq n < f_{k+3}$.
- An entry $x_n \in A_k$ if and only if $Z(n)$ begins in 10000, $x_n \in B_k$ if and only if $Z(n)$ begins in 10001, $x_n \in C_k$ if and only if $Z(n)$ begins in 1001, and $x_n \in D_k$ if and only if $Z(n)$ begins in 101.
- x_n belongs to row k if and only if $x_{n+f_{k+1}}$ belongs to $C_{k+1} \cup D_{k+1}$.

We put $x_1 = x_2 = x_4 = 0$ and $x_3 = 1$. This defines rows $0, 1$ and 2. Having defined rows $0, 1, 2, \ldots, k$ for $k \geq 2$, we now describe how to obtain row $k+1$ from prior rows. We consider three cases:

Case 1. If x_n belongs to A_{k+1}, then the column containing x_n has at least three entries above x_n:

$$\begin{array}{lll} & & \vdots \\ \text{row } k-2 & \cdots & x_{n-f_k-f_{k-1}-f_{k-2}} \\ \text{row } k-1 & \cdots & x_{n-f_k-f_{k-1}} \\ \text{row } k & \cdots & x_{n-f_k} \\ \text{row } k+1 & \cdots & x_n \end{array}$$

and we set

$$x_n = x_{n-f_k} + x_{n-f_k-f_{k-1}} - x_{n-f_k-f_{k-1}-f_{k-2}}. \tag{1}$$

Case 2. If x_n belongs to B_{k+1} then the column containing x_n has two entries above x_n:

$$\begin{array}{lll} \text{row } k-1 & \cdots & x_{n-f_k-f_{k-1}} \\ \text{row } k & \cdots & x_{n-f_k} \\ \text{row } k+1 & \cdots & x_n \end{array}$$

and we set

$$x_n = x_{n-f_k} + x_{n-f_k-f_{k-1}}. \tag{2}$$

Case 3. If x_n belongs to $C_{k+1} \cup D_{k+1}$ then $x_{n-f_{k+1}}$ belongs to row k and we set

$$x_n = x_{n-f_{k+1}}, \tag{3}$$

in other words $C_{k+1} \cup D_{k+1}$ is obtained by simply copying row k.

These recursive rules define the sequence (x_n) shown in Figure 2.2 below.

$$\begin{array}{lcl} \text{row } 0 & : & 0 \\ \text{row } 1 & : & 0 \\ \text{row } 2 & : & 10 \\ \text{row } 3 & : & 110 \\ \text{row } 4 & : & 21110 \\ \text{row } 5 & : & 22121110 \\ \text{row } 6 & : & 3223122121110 \\ \text{row } 7 & : & 3324233132231\ 22121110 \\ \text{row } 8 & : & 4335244253341\ 33242331\ 3223122121110 \\ \text{row } 9 & : & \underbrace{4436355264462}_{A_9}\ \underbrace{55363441}_{B_9}\ \underbrace{4335244253341}_{C_9}\ \underbrace{332423313223122121110}_{D_9} \\ \vdots & \vdots & \vdots \end{array}$$

FIGURE 2.3. Schematic Representation of the Sequence $(R^{-}(n))_{n\geq 1}$

PROPOSITION 2.2. *$R^{-}(n) = x_n$ for each $n \geq 1$.*

PROOF. Clearly $R^{-}(1) = R^{-}(2) = R^{-}(4) = 0$, and $R^{-}(3) = 1$. To show that $R^{-}(n) = x_n$ for $n \geq 4$, it suffices to show that $R^{-}(n)$ satisfies the same recursive conditions which defined (x_n) in Cases 1-3 above. In doing so we will make repeated use of Lemma 2.1.

Case 1. Suppose x_n belongs to A_{k+1}. Then we can write $Z(n) = 10^\ell w$ for some $\ell \geq 4$ and for some Zeckendorff word w. Then

$$\begin{aligned}
R^-(n) &= R^-(10^\ell w) \\
&= R(10^{\ell-2}w) \\
&= R^-(10^{\ell-2}w) + R^+(10^{\ell-2}w) \\
&= R^-(10^{\ell-2}w) + R^+(10^{\ell-3}w) \\
&= R^-(10^{\ell-2}w) + R^+(10^{\ell-3}w) + R^-(10^{\ell-3}w) - R^-(10^{\ell-3}w) \\
&= R^-(10^{\ell-2}w) + R(10^{\ell-3}w) - R^-(10^{\ell-3}w) \\
&= R^-(10^{\ell-2}w) + R^-(10^{\ell-1}w) - R^-(10^{\ell-3}w) \\
&= R^-(n - f_{k+2} + f_k) + R^-(n - f_{k+2} + f_{k+1}) - R^-(n - f_{k+2} + f_{k-1}) \\
&= R^-(n - f_k - f_{k-1}) + R^-(n - f_k) - R^-(n - f_k - f_{k-1} - f_{k-2})
\end{aligned}$$

as required by (1).

Case 2. Suppose next x_n belongs to B_{k+1}. Then we can write $Z(n) = 1000w$ for some Zeckendorff word w. This gives us that

$$\begin{aligned}
R^-(n) &= R^-(1000w) \\
&= R(10w) \\
&= R^-(10w) + R^+(10w) \\
&= R^-(10w) + R(w) \\
&= R^-(10w) + R^-(100w) \\
&= R^-(n - f_{k+2} + f_k) + R^-(n - f_{k+2} + f_{k+1}) \\
&= R^-(n - f_k - f_{k-1}) + R^-(n - f_k)
\end{aligned}$$

as required by (2).

Case 3. Suppose x_n belongs to $C_{k+1} \cup D_{k+1}$. In this case according to (3) we must show that $R^-(n) = R^-(n - f_{k+1})$. We proceed by induction on k. The result is readily verified for $k \leq 2$.

For $k \leq 2$, the result is shown explicitly.

$$\begin{array}{lllllll}
R^-(1) & & & = & 0 & = & x_1 \\
R^-(2) & = & R^-(10) & = & 0 & = & x_2 \\
R^-(3) & = & R^-(100) & = & 1 & = & x_3 \\
R^-(4) & = & R^-(101) & = & 0 & = & x_4 \\
R^-(5) & = & R^-(1000) & = & 1 & = & x_5 \\
R^-(6) & = & R^-(1001) & = & 1 & = & x_6 \\
R^-(7) & = & R^-(1010) & = & 0 & = & x_7
\end{array}$$

We consider two subcases. The first (which does not require induction hypothesis) is when $Z(n) = 10010w$ for some $\{0,1\}$-word w. In this case we have

$$\begin{aligned}
R^-(n) &= R^-(10010w) \\
&= R(10w) \\
&= R^-(1000w) \\
&= R^-(n - f_{k+2} + f_{k+1} - f_{k-1}) \\
&= R^-(n - f_{k+1} - f_k - f_{k-1} + f_k + f_{k-1} \\
&= R^-(n - f_{k+1}).
\end{aligned}$$

The second sub-case is when $Z(n) = 10w$ for some Zeckendorff word w. In this case

$$\begin{aligned}
R^-(n) &= R^-(10w) \\
&= R^-(w) \\
&= R^-(n - f_{k+2}) \\
&= R^-(n - f_{k+2} + f_k) \quad \text{(by inductive hypothesis)} \\
&= R^-(n - f_{k+1}).
\end{aligned}$$

The inductive hypothesis applies to the next-to-last equality for the following reasons. Since x_n is in D_{n+1}, $f_k \leq n - f_{k+2} < f_{k+1}$. This implies that $f_{k+1} + f_{k-2} = 2f_k \leq n - f_{k+2} + f_k <$

$f_{k+1} + f_k = f_{k+2}$. Thus $x_n - f_{k+2} + f_k$ is in $C_k \cup D_k$. So by the inductive hypothesis, $R^-(x_n - f_{k+2} + f_k) = R^-(x_n - f_{k+2} + f_k - f_k) = R^-(x_n - f_{k+2})$.

$\square$

Having constructed the sequence $R^-(n)$ we use Lemma 2.1 to compute the sequence $R(n)$. We have that $R(n) = R(Z(n)) = R^-(100Z(n))$ so that $R(n)$ is obtained from the previous chart by concatenating the C_k for $k \geq 3$. Putting $L_k = C_{k+2}$, we obtain the schematic representation of $(R(n))$ given in Figure 2.1.

We end this section by making a few observations; the first three are immediate, the fourth is discussed in the next section of this chapter, and the last one will be discussed in Chapter 3.

- L_k has f_{k-1} entries.
- The first entry of L_k is $R(f_k)$, and hence the last entry of L_k is $R(f_{k+1} - 1) = 1$.
- $R(n)$ is in level L_k if and only if $|Z(n)| = k$.
- For $k \geq 3$, level L_k can be written in the form $L_k = W_k1$ where W_k is a palindrome of length $f_{k-1} - 1$.
- For each integer $m \geq 1$, there exists a positive integer $\mathrm{rk}(m)$ such that for each $k \geq 2m$, the integer m occurs exactly $\mathrm{rk}(m)$ times in level L_k. For instance, in each row L_k for $k \geq 6$, the value 3 is assumed exactly 4 times, so that $\mathrm{rk}(3) = 4$. Similarly, in each row L_k for $k \geq 8$, the value 4 is assumed exactly 6 times, so that $\mathrm{rk}(4) = 6$.

2.3. The Underlying Palindrome Structure of $R(n)$

PROPOSITION 2.3. *The sequence $(R(n))_{n\geq 1}$ can be factored in the form*

$$(R(n))_{n\geq 1} = 11W_31W_41W_51W_61\cdots$$

where W_k is a palindrome of length $f_{k-1} - 1$ for each $k \geq 3$.

PROOF. We saw in the previous section that $(R(n))_{n\geq 1}$ factors as $(R(n))_{n\geq 1} = 11L_3L_4L_5L_6\cdots$ where L_k has length f_{k-1} and the first and last entries of L_k are $R(f_k)$ and $R(f_{k+1} - 1) = 1$ respectively. Thus writing $L_k = W_k1$ we see that $(R(n))_{n\geq 1} = 11W_31W_41W_51W_61\cdots$ and that the length of W_k is $f_{k-1} - 1$. To see that each W_k is a palindrome, we consider an entry $R(n)$ in W_k. Hence, $|Z(n)| = k$ and $n \neq f_{k+1} - 1$. Let $\overline{Z(n)}$ denote the $\{0,1\}$-word obtained from $Z(n)$ by exchanging 0s and 1s. Since $Z(n)$ begins in 10, it follows that $\overline{Z(n)}$ begins in

01, and thus deleting the first 0 in $\overline{Z(n)}$ we obtain that $0^{-1}\overline{Z(n)} \in \Omega(\hat{n})$ for some positive integer $\hat{n}$.

To show that W_k is a palindrome, it suffices to show

- $R(n) = R(\hat{n})$
- $n - f_k = f_{k+1} - 2 - \hat{n}$; in other words, the entries $R(n)$ and $R(\hat{n})$ are located the same distance away from the 'center' of W_k.

The first point is clear since the number of elements congruent to $z(n)$ under the relation $011 \equiv 100$ (which is $R(n)$) is equal to the number of elements congruent to $\overline{Z(n)}$ under the same relation (which is $R(\hat{n})$). As for the second point, we show $n+\hat{n}+2 = f_{k+1}+f_k = f_{k+2}$. Now $n + \hat{n}$ can be represented as $\underbrace{11\cdots1}_{k}$ in the Fibonacci base. Hence,

$$Z(n+\hat{n}) = \begin{cases} (10)^{\ell}01 & \text{if } k = 2\ell+1 \\ (10)^{\ell}0 & \text{if } k = 2\ell \end{cases}$$

which implies $Z(n+\hat{n}+2) = \underbrace{1000\cdots0}_{k+2}$. Since $\underbrace{1000\cdots0}_{k+2}$ is a representation of f_{k+2}, $n+\hat{n}+2 = f_{k+2}$, as needed.

□

2.4. A Second Algorithm for Generating $R(n)$

In this section, we present a second recursive construction for generating $R(n)$ based on an alternate decomposition of $R(n)$. We begin by defining $\Omega_{\text{odd}}(n)$ to be the set of all $w \in \Omega(n)$ having an odd number of 1's and $\Omega_{\text{ev}}(n)$ to be the set of all $w \in \Omega(n)$ having an even number of 1's, and set $R_{\text{odd}}(n) = \#\Omega_{\text{odd}}(n)$ and $R_{\text{ev}}(n) = \#\Omega_{\text{ev}}(n)$. This gives rise to the decomposition

$$R(n) = R_{\text{odd}}(n) + R_{\text{ev}}(n).$$

In this context it is also natural to consider the difference

$$d(n) = R_{\text{odd}}(n) - R_{\text{ev}}(n).$$

The sequence $d(n)$ may also be defined as the coefficients in the power series expansion of the infinite product

$$F(x) = \prod_{k=1}^{\infty}(1 - x^{f_k}) = (1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^8)(1-x^{13})\cdots$$

More precisely:

$$F(x) = \sum_{n=0}^{\infty} -d(n)x^n.$$

In an unpublished paper, Tamura gives an ingenious recursive construction for generating the sequence of coefficients $(d(n))_{n\geq 0}$. In what follows, we will present a different algorithm for constructing $d(n)$. Once again our approach involves arranging the sequence $(d(n))_{n\geq 0}$ in a 2-dimensional array as shown in Figure 2.4 below:

$$\begin{array}{lcllllllll}
\text{row } 0 & : & d(0) \\
\text{row } 1 & : & d(1) \\
\text{row } 2 & : & d(2) \\
\text{row } 3 & : & d(3) & d(4) \\
\text{row } 4 & : & d(5) & d(6) & d(7) \\
\text{row } 5 & : & d(8) & d(9) & d(10) & d(11) & d(12) \\
\text{row } 6 & : & d(13) & d(14) & d(15) & d(16) & d(17) & d(18) & d(19) & d(20) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

FIGURE 2.4. Schematic Representation of the Sequence $(d(n))_{n\geq 0}$

We observe that for $k \geq 1$, row k contains f_{k-1} entries of the sequence $d(n)$ beginning with $d(f_k)$ and ending with $d(f_{k+1} - 1)$.

Figure 2.4 illustrates the first 11 rows of this 2-dimensional representation of the sequence $d(n)$. Here $\bar{1}$ denotes the value -1. A first observation is that the sequence assumes only the values $\{\bar{1}, 0, 1\}$. Also, as is the case with the sequence $R(n)$, the sequence $d(n)$ exhibits certain regularity properties when represented in this fashion. In fact, we observe that row $2k$ is of the form $U_{2k}(\bar{1})^{k+1}$ where U_{2k} is a palindrome. And, if for a $\{\bar{1}, 0, 1\}$-word $U = u_1u_2\dots u_n$, we put $U^* = \bar{u_n}\dots\bar{u_2}\bar{u_1}$ (U^* is obtained from U by first reflecting the word, then exchanging 1's and $\bar{1}$'s) then row $2k + 1$ is of the form $U_{2k+1}(\bar{1})^k$ where U_{2k+1} satisfies $U^*_{2k+1} = U_{2k+1}$.

We now state the key lemma which will be the basis for the algorithms for generating $R(n)$ and $d(n)$.

LEMMA 2.4. *Let $k \geq 3$. Then:*

row 0	:	$\bar{1}$
row 1	:	1
row 2	:	1
row 3	:	$0\bar{1}$
row 4	:	$00\bar{1}$
row 5	:	$100\bar{1}1$
row 6	:	$1\bar{1}000\bar{1}11$
row 7	:	$0\bar{1}\bar{1}10000\bar{1}110\bar{1}$
row 8	:	$00\bar{1}011\bar{1}000000\bar{1}110\bar{1}00\bar{1}$
row 9	:	$100\bar{1}10010\bar{1}\bar{1}1000000000\bar{1}110\bar{1}00\bar{1}100\bar{1}1$
row 10	:	$1\bar{1}000\bar{1}11\bar{1}001\bar{1}00\bar{1}011\bar{1}00000000000000\bar{1}110\bar{1}00\bar{1}100\bar{1}11\bar{1}000\bar{1}11$
$\vdots$	$\vdots$	$\vdots$

FIGURE 2.5. 2-Dimensional Representation of the Sequence $(d(n))_{n\geq 0}$

(i) *For* $f_{k+1} \leq n \leq f_{k+1} + f_{k-2} - 1$, *we have*

$$R_{ev}(n) = R_{odd}(n - f_{k+1} + f_{k-1}) + R_{odd}(n - f_{k+1})$$

and

$$R_{odd}(n) = R_{ev}(n - f_{k+1} + f_{k-1}) + R_{ev}(n - f_{k+1}).$$

(ii) *For* $f_{k+1} + f_{k-2} \leq n \leq f_{k+1} + f_{k-2} + f_{k-3} - 1$, *we have*

$$R_{ev}(n) = R_{odd}(n - f_{k+1}) + R_{ev}(n - f_{k+1})$$

and

$$R_{odd}(n) = R_{odd}(n - f_{k+1}) + R_{ev}(n - f_{k+1}).$$

(iii) *For* $f_{k+1} + f_{k-1} \leq n \leq f_{k+1} + f_{k-1} + f_{k-2} - 2$, *put* $\hat{n} = f_{k+1} + f_{k+2} - n - 2$. *Then we have*

$$R_{ev}(n) = R_{ev}(\hat{n})$$

and

$$R_{odd}(n) = R_{odd}(\hat{n})$$

whenever $k+1$ *is even, while*

$$R_{ev}(n) = R_{odd}(\hat{n})$$

and

$$R_{odd}(n) = R_{ev}(\hat{n})$$

whenever $k+1$ *is odd.*

(iv) *For* $n = f_{k+2} - 1$ *we have*

$$R_{ev}(n) = 0 \ \ \textit{and} \ \ R_{odd}(n) = 1$$

whenever $k \equiv 0, 1 \mod 4$ *and*

$$R_{ev}(n) = 1 \ \ \textit{and} \ \ R_{odd}(n) = 0$$

whenever $k \equiv 2, 3 \mod 4$.

Proof. We first note that case (1) is equivalent to the Zeckendorff representation of n beginning in 1000, case (2) to the Zeckendorff representation of n beginning in 1001, case (3) to the Zeckendorff representation of n beginning in 101 which is not a prefix of $(10)^\infty$, and case (4) to the Zeckendorff representation of n being equal to the prefix of $(10)^\infty$ of length $k+1$.

In order to verify (1), we observe that:

$$\#\left(\Omega_{\text{odd}}(n) \cap \Omega_1(n)\right) = R_{\text{ev}}(n - f_{k+1}),$$

$$\#\left(\Omega_{\text{odd}}(n) \cap \Omega_0(n)\right) = R_{\text{ev}}(n - f_{k+1} + f_{k-1}),$$

$$\#\left(\Omega_{\text{ev}}(n) \cap \Omega_1(n)\right) = R_{\text{odd}}(n - f_{k+1}),$$

$$\#\left(\Omega_{\text{ev}}(n) \cap \Omega_0(n)\right) = R_{\text{odd}}(n - f_{k+1} + f_{k-1}).$$

Thus

$$R_{\text{odd}}(n) = R_{\text{ev}}(n - f_{k+1}) + R_{\text{ev}}(n - f_{k+1} + f_{k-1}),$$

and

$$R_{\text{ev}}(n) = R_{\text{odd}}(n - f_{k+1}) + R_{\text{odd}}(n - f_{k+1} + f_{k-1}).$$

In case (2), we consider the representations of n of the form $100w$ and $011w$, for some $\{0,1\}$-word w, and we note that $w = n - f_{k+1}$. The following observations will complete case (2).

- w has odd many 1's if and only if $100w$ has even many 1's.
- w has even many 1's if and only if $011w$ has even many 1's.
- w has even many 1's if and only if $100w$ has odd many 1's.
- w has odd many 1's if and only if $011w$ has odd many 1's.

The first two items above imply that

$$R_{\text{odd}}(n) = R_{\text{ev}}(n - f_{k+1}) + R_{\text{ev}}(n - f_{k+1} + f_{k-1}),$$

and the second two items imply that

$$R_{\text{ev}}(n) = R_{\text{odd}}(n - f_{k+1}) + R_{\text{odd}}(n - f_{k+1} + f_{k-1}).$$

In case (3), note that if n is on row $k+1$, then so is $\hat{n}$. Since $\Omega(\hat{n})$ is obtained from $\Omega(n)$ by exchanging 0's and 1's, we have the following.

- If $k+1$ is even then $R_{\text{ev}}(n) = R_{\text{ev}}(\hat{n})$ and $R_{\text{odd}}(n) = R_{\text{odd}}(\hat{n})$.
- If $k+1$ is odd then $R_{\text{ev}}(n) = R_{\text{odd}}(\hat{n})$ and $R_{\text{odd}}(n) = R_{\text{ev}}(\hat{n})$.

Finally in case (4), we have that $Z(n)$ is the only representation of n and that if $k \equiv 0, 1 \mod 4$ then $Z(n)$ contains an odd number of 1's, while if $k \equiv 2, 3 \mod 4$ then $Z(n)$ contains an even number of 1's. □

The following is an immediate consequence of the above lemma:

COROLLARY 2.5. *Let $k \geq 3$. Then:*

(i) *For $f_{k+1} \leq n \leq f_{k+1} + f_{k-2} - 1$, we have*

$$R(n) = R(n - f_{k+1} + f_{k-1}) + R(n - f_{k+1})$$

and

$$d(n) = -\left(d(n - f_{k+1} + f_{k-1}) + d(n - f_{k+1})\right).$$

(ii) *For $f_{k+1} + f_{k-2} \leq n \leq f_{k+1} + f_{k-2} + f_{k-3} - 1$, we have*

$$R(n) = 2R(n - f_{k+1})$$

and

$$d(n) = 0.$$

(iii) *For* $f_{k+1} + f_{k-1} \leq n \leq f_{k+1} + f_{k-1} + f_{k-2} - 2$, *we have*

$$R(n) = R(\hat{n})$$

and

$$d(n) = d(\hat{n})$$

whenever $k+1$ *is even, while*

$$R(n) = R(\hat{n})$$

and

$$d(n) = -d(\hat{n})$$

whenever $k+1$ *is odd.*

(iv) *For* $n = f_{k+2} - 1$ *we have*

$$R(n) = 1 \text{ and } d(n) = 1$$

whenever $k \equiv 0, 1 \mod 4$ *and*

$$R(n) = 1 \text{ and } d(n) = -1$$

whenever $k \equiv 2, 3 \mod 4$.

The above corollary provides a recursive algorithm for computing the $k+1$st row of the 2-dimensional representations of $R(n)$ and $d(n)$. First, we discuss the algorithm in the case of $d(n)$.

Computing directly the values for rows $0, 1, 2$, and 3 of Figure 2.4, we get

$$\begin{aligned}
d(0) &= R_{\text{odd}}(0) - R_{\text{ev}}(0) &&= -1 \\
d(1) &= R_{\text{odd}}(1) - R_{\text{ev}}(1) &&= 1 \\
d(2) &= R_{\text{odd}}(10) - R_{\text{ev}}(10) &&= 1 \\
d(3) &= R_{\text{odd}}(100) - R_{\text{ev}}(100) &&= 0 \\
d(4) &= R_{\text{odd}}(101) - R_{\text{ev}}(101) &&= -1
\end{aligned}$$

Now let $k \geq 3$, and suppose we have computed rows $0, 1, 2, \ldots, k$ of Figure 2.4. We now compute the entries in row $k+1$.

• The first f_{k-2} entries of row $k+1$ are computed as follows: We form an array of 3 rows each consisting of f_{k-2} columns: The top row consists of the first f_{k-2} entries of $d(n)$; the middle row is given by row $k-1$ of Figure 2.4; the bottom row (to be computed) consists of the first f_{k-2} entries of row $k+1$. This array is shown in Figure 2.6.

$$\begin{array}{llllll}
d(0) & d(1) & d(2) & d(3) & \cdots & d(f_{k-2}-1) \\
d(f_{k-1}) & d(f_{k-1}+1) & d(f_{k-1}+2) & d(f_{k-1}+3) & \cdots & d(f_{k-1}+f_{k-2}-1) \\
d(f_{k+1}) & d(f_{k+1}+1) & d(f_{k+1}+2) & d(f_{k+1}+3) & \cdots & d(f_{k+1}+f_{k-2}-1)
\end{array}$$

FIGURE 2.6. The Computation of Row $k+1$ of the 2-Dimensional Representation of $d(n)$

It follows from Corollary 2.5 that the entries in the bottom row are computed by:

$$d(n) = -\left(d(n - f_{k+1} + f_{k-1}) + d(n - f_{k+1})\right). \tag{4}$$

In other words, each entry in the bottom row is the negated sum of the two entries above it.

• The next f_{k-3} entries are all equal to 0.

• Following this block of 0's, the next $f_{k-2}-1$ entries of row $k+1$ are the reflections (or mirror images) of the first $f_{k-2}-1$ entries (of row $k+1$) in case $k+1$ is even; otherwise, if $k+1$ is odd we must, after reflecting, exchange 1's and $\bar{1}$'s.

• Finally the last entry is 1 if $k \equiv 0, 1 \mod 4$ and $\bar{1}$ otherwise.

• We conclude this discussion of $(d(n))_{\geq 0}$ by illustrating the algorithm with an example.

EXAMPLE 2.6. Taking $k = 7$, we compute row 8 of Figure 2.4. The first $f_5 = 8$ entries of row 8 are obtained by writing the first 8 terms of the sequence $d(n)$ over row 6 and negating the sum of the terms in each of the 8 columns as shown below:

$$\begin{array}{cccccccc}
\bar{1} & 1 & 1 & 0 & \bar{1} & 0 & 0 & \bar{1} \\
1 & \bar{1} & 0 & 0 & 0 & \bar{1} & 1 & 1 \\
0 & 0 & \bar{1} & 0 & 1 & 1 & \bar{1} & 0
\end{array}$$

In other words the first 8 entries are $00\bar{1}011\bar{1}0$. We next adjoin a block of $f_4 = 5$ many 0's to obtain $00\bar{1}011\bar{1}000000$. Next, since 8 is even, we adjoin the reflection of the initial 7 entries generated thus far to obtain the palindrome $00\bar{1}011\bar{1}000000\bar{1}110\bar{1}00$. Finally, since $k = 7 \equiv 3 \mod 4$, we adjoin $\bar{1}$ so that

$$\text{row } 8 = 00\bar{1}011\bar{1}000000\bar{1}110\bar{1}00\bar{1}.$$

We end this chapter by giving the recursive construction of the sequence $R(n)$ which is based on its even/odd decomposition. Since this construction relies heavily on Corollary 2.5, it is very similar to the construction given for the sequence $d(n)$. First, define R(0)=1 and suppose we have computed rows $1, 2, \ldots, k$ of Figure 2.1. We now compute the entries in row $k + 1$.

• For the first f_{k-2} entries of row $k+1$, we have a similar array as for $d(n)$. We form an array of 3 rows each consisting of f_{k-2} columns: The top row consists of the first f_{k-2} entries of $R(n)$, using $R(0) = 1$; the middle row is given by row $k-1$ of Figure 2.1; the bottom row (to be computed) consists of the first f_{k-2} entries of row $k+1$. This array is shown in Figure 2.7.

$$\begin{array}{llllll}
R(0) & R(1) & R(2) & R(3) & \cdots & R(f_{k-2}-1) \\
R(f_{k-1}) & R(f_{k-1}+1) & R(f_{k-1}+2) & R(f_{k-1}+3) & \cdots & R(f_{k-1}+f_{k-2}-1) \\
R(f_{k+1}) & R(f_{k+1}+1) & R(f_{k+1}+2) & R(f_{k+1}+3) & \cdots & R(f_{k+1}+f_{k-2}-1)
\end{array}$$

FIGURE 2.7. The Computation of Row $k+1$ of the 2-Dimensional Representation of $R(n)$

It follows from Corollary 2.5 that the entries in the bottom row are computed by:

$$R(n) = R(n - f_{k+1} + f_{k-1}) + R(n - f_{k+1}). \tag{5}$$

So each entry in the bottom row is the sum of the two entries above it.

• For the next f_{k-3} entries: By Corollary 2.5, $R(n) = 2R(n - f_{k+1})$. Since $f_{k+1} + f_{k-2} \leq n \leq f_{k+1} + f_{k-2} + f_{k-3} - 1$, then $f_{k-2} \leq n - f_{k+1} \leq f_{k-1} - 1$. This implies these entries are simply twice row $k-2$.

• The next $f_{k-2} - 1$ entries of row $k+1$ are the reflections (or mirror images) of the first $f_{k-2} - 1$ entries of row $k+1$.

•Finally the last entry is 1.

CHAPTER 3

A PARTICULAR REGULARITY PROPERTY OF R(n)

3.1. Basis Words and Rank

The next two sections are devoted to studying the structure of all Zeckendorff words $Z(n)$ such that $R(n) = m$, for $m \geq 1$. We begin by giving a matrix formulation of Lemma 2.1 which will in turn allow us to recover a recent result of Berstel in [6] for computing $R(n)$ from the Zeckendorff representation of n.

Lemma 3.1. *Let w be a Zeckendorff word. Then*

(i) $$\begin{pmatrix} R_0(10^\ell w) \\ R_1(10^\ell w) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_0(10^{\ell-2} w) \\ R_1(10^{\ell-2} w) \end{pmatrix} \qquad \textit{for } l \geq 3$$

(ii) $$\begin{pmatrix} R_0(100w) \\ R_1(100w) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} R_0(w) \\ R_1(w) \end{pmatrix}$$

(iii) $$\begin{pmatrix} R_0(10w) \\ R_1(10w) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} R_0(w) \\ R_1(w) \end{pmatrix}$$

Using the identities

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{d-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} d & d \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{d} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} d+1 & d \\ 1 & 1 \end{pmatrix}$$

we deduce that for any Zeckendorff word w

$$\begin{pmatrix} R_0(10^r w) \\ R_1(10^r w) \end{pmatrix} = \begin{pmatrix} \lceil \frac{r}{2} \rceil & \lfloor \frac{r}{2} \rfloor \\ 1 & 1 \end{pmatrix} \begin{pmatrix} R_0(w) \\ R_1(w) \end{pmatrix} \qquad \text{for } r \geq 1.$$

This yields the following result due to Berstel[1]:

[1]The matrices occurring in Berstel's formulation of the same result differ slightly from ours as a consequence of notational differences.

Proposition 3.2. *[Berstel [6]] Let* $w = 10^{r_1}10^{r_2}\cdots 10^{r_k}$ *with* $r_j \geq 1$. *Then*

$$R(w) = R(w1) = R_0(w1) + R_1(w1)$$
$$= (1\ 1)\left(\prod_{j=1}^{k}\begin{pmatrix} \lceil \frac{r_j}{2} \rceil & \lfloor \frac{r_j}{2} \rfloor \\ 1 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let m be a positive integer. Let $R^{-1}(m) = \{n \geq 1 \,|\, R(n) = m\}$. We also define for $m \geq 2$

$$\mathcal{B}(m) = \{w = 10^{r_1}10^{r_2}\cdots 10^{r_k} \,|\, r_j \geq 1, r_k \geq 2, \text{ and } R(w) = m\}$$

and $\mathcal{B}(1) = \{\varepsilon\}$ where ε denotes the empty word. Then it follows that $n \in R^{-1}(m)$ if and only if $Z(n) = wu$ where $w \in \mathcal{B}(m)$ and u is a (possibly empty) prefix of $(10)^\infty$. We call an element $w = 10^{r_1}10^{r_2}\cdots 10^{r_k} \in \mathcal{B}(m)$ a m- *basis word*[2] While for each m the set $R^{-1}(m)$ is infinite, the set $\mathcal{B}(m)$ is finite. The cardinality of $\mathcal{B}(m)$ will be denoted by

$$\mathrm{rk}(m) = \#\mathcal{B}(m)$$

and called the *m-rank*.

Lemma 3.3. *For each* $m \geq 2$, *the longest* m*-basis word has length* $2m$.

Proof. Let $m \geq 2$, and consider the word 10^{2m-1}. Then it follows from Berstel's result that

$$R(10^{2m-1}) = (1\ 1)\begin{pmatrix} \lceil \frac{2m-1}{2} \rceil & \lfloor \frac{2m-1}{2} \rfloor \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1\ 1)\begin{pmatrix} m & m-1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = m.$$

Hence 10^{2m-1} is a m-basis word of length $2m$.

We next claim that any m-basis word w is of length less than or equal to $2m$. The proof is by induction on m. In case $m = 2$, then the only basis words are 100 and 1000. Next suppose that for all $2 \leq m' < m$, each m'-basis word is of length less than or equal to $2m'$, and let w be a m-basis word. If w is of the form $w = 10^k$, then it is easy to see that $k \leq 2m - 1$, since otherwise $R(w) > m$. So we can suppose w is of the form $w = 10^k w'$ for some Zeckendorff word w'. We now use Lemma 3.1 and consider three cases: a) $k = 1$, b) $k = 2$, and c) $k \geq 3$. Since the arguments in each case are essentially identical, we consider only the first case. In

[2] In [25] such words are called relational words, a term stemming from earlier terminology introduced in [1].

this case we see that:

$$\begin{pmatrix} R_0(w) \\ R_1(w) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} R_0(w') \\ R_1(w') \end{pmatrix}.$$

Putting $m' = R(w')$, it follows that $m' = R(w') = R_1(w) < R(w) = m$. Since w' is an m'-basis word, by induction hypothesis we have that the length of w' is less than or equal to $2m'$ which in turn is less than or equal to $2m - 2$. Hence the length of w is less than or equal to $2m$. □

PROPOSITION 3.4. *Let $m \geq 1$, and let $k \geq 2m$. Then level L_k of Figure 1 contains exactly rk(m) occurrences of m.*

PROOF. Consider the $\mathrm{rk}(m)$ infinite words $\{w(10)^\infty \,|w \in \mathcal{B}(m)\}$. Then by taking the prefix of length k of each of the $\mathrm{rk}(m)$ infinite words, we obtain $\mathrm{rk}(m)$ distinct Zeckendorff words of length k each having precisely m different representations in the Fibonacci base. Hence, this gives rise to $\mathrm{rk}(m)$ occurrences of the integer m in level L_k of Figure 1. But since each occurrence of m in L_k must arise in this way, we see that L_k has exactly $\mathrm{rk}(m)$ occurrences of m. □

Recall that each L_k may be written as $L_k = W_k1$ where W_k is a palindrome of length $f_{k-1} - 1$. In particular, for infinitely many k, the length of W_k is even. Hence:

COROLLARY 3.5. *The constant rk(m) is an even number for each $m \geq 2$.*

Let $m \geq 2$, and let $w = 10^{r_1}10^{r_2}\cdots 10^{r_k} \in \mathcal{B}(m)$. It follows from Proposition 3.2 that if r_k is even, then $10^{r_1}10^{r_2}\cdots 10^{r_k+1} \in \mathcal{B}(m)$, while if r_k is odd (and hence $r_k \geq 3$), then $10^{r_1}10^{r_2}\cdots 10^{r_k-1} \in \mathcal{B}(m)$. In fact if r_k is even, then

$$\begin{pmatrix} \lceil \frac{r_k+1}{2} \rceil & \lfloor \frac{r_k+1}{2} \rfloor \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lceil \frac{r_k}{2} \rceil & \lfloor \frac{r_k}{2} \rfloor \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

while if r_k is odd, then

$$\begin{pmatrix} \lceil \frac{r_k-1}{2} \rceil & \lfloor \frac{r_k-1}{2} \rfloor \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lceil \frac{r_k}{2} \rceil & \lfloor \frac{r_k}{2} \rfloor \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence for each m-basis word ending in an even number of 0's, there is a m-basis word ending in an odd number of 0's and vice versa. This gives an alternate argument to the fact that $\mathrm{rk}(m)$ is even. For $m \geq 2$, denote by $\mathcal{B}_0(m)$ (respectively $\mathcal{B}_1(m)$) the set of all

m-basis words ending in an even (respectively odd) number of 0's and set $\mathrm{rk}_i(m) = \#\mathcal{B}_i(m)$ for $i \in \{0, 1\}$.

3.2. Fibonacci Towers

Associated to each of the matrices in Lemma 3.1 are mappings $f_{00}, f_{01}, f_{001} : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ defined as follows:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow f_{00} : (a, b) \mapsto (a + b, b)$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \longrightarrow f_{01} : (a, b) \mapsto (a, a + b)$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow f_{001} : (a, b) \mapsto (a + b, a + b)$$

Note that f_{00} and f_{01} are each one-to-one, while f_{001} is generally many to one.

Let $m \geq 2$. We now consider all arrays of the form

$$\begin{array}{c} (x_k, y_k) \\ (x_{k-1}, y_{k-1}) \\ (x_{k-2}, y_{k-2}) \\ \vdots \\ (x_1, y_1) \end{array}$$

such that $x_k + y_k = m$, $(x_1, y_1) = (1, 1)$, and such that for each $1 \leq j \leq k - 1$, we have

$$(x_{j+1}, y_{j+1}) = f_{u_j}(x_j, y_j)$$

for some $u_j \in \{00, 01, 001\}$.

We call such an array a *m-Fibonacci tower* of height k. The mappings $f_{u_1}, f_{u_2}, \ldots, f_{u_{k-1}}$ between the various levels will be called *stair maps* and the words $u_1, u_2, \ldots, u_{k-1}$ *stair indeces*. Clearly, each Fibonacci tower is uniquely determined from its associated stair maps, and hence stair indeces. We let

$$\mathcal{T}(m) = \{m\text{-Fibonacci towers}\}$$

and put $\tau(m) = \#\mathcal{T}(m)$.

Let $T \in \mathcal{T}(m)$ be a m-Fibonacci tower of height k with associated stair indeces $u_1, u_2, \ldots, u_{k-1}$. We define $\psi(T)$ to be the Zeckendorff word

$$\psi(T) = 1u_{k-1}u_{k-2}\cdots u_1 00.$$

Then it follows from Proposition 3.2 that $R(\psi(T)) = m$. Moreover, since $\psi(T)$ ends in an even number of 0's, we deduce that $\psi(T)$ is an even m-basis word, i.e., $\psi(T) \in \mathcal{B}_0(m)$. Hence we obtain a mapping $\psi : \mathcal{T}(m) \to \mathcal{B}_0(m)$.

PROPOSITION 3.6. *The mapping $\psi : \mathcal{T}(m) \to \mathcal{B}_0(m)$ is a bijection. Hence $rk(m) = 2\tau(m)$.*

PROOF. Consider an even m-basis word $w \in \mathcal{B}_0(m)$. Then it is easy to see that w can be written in the form $w = 1u_{k-1}u_{k-2}\cdots u_1 00$ for some choice of k and words $u_i \in \{00, 01, 001\}$. Moreover, this representation of w is unique. Let T be the (unique) Fibonacci tower whose associated stair indeces are $u_{k-1}, u_{k-2}, \ldots, u_1$. Then $T \in \mathcal{T}(m)$ and we have $\psi(T) = w$. Hence ψ is both one-to-one and onto. $\square$

Let $T \in \mathcal{T}(m)$ be a m-Fibonacci tower with stair indeces $u_{k-1}, u_{k-2}, \ldots, u_1$. Let T' denote the m-Fibonacci tower of height k with stair indeces $u'_{k-1}, u'_{k-2}, \ldots, u'_1$ where $(01)' = 00$, $(00)' = 01$, and $(001)' = 001$. Then for all $1 \le j \le k$, we have $(x'_j, y'_j) = (y_j, x_j)$. In other words, the tower T' is obtained from T by reversing the coordinates at each level $1 \le j \le k$. Then:

LEMMA 3.7. $\psi(T') = Z(\overline{\psi(T)})$.

PROOF. Recall that $\overline{\psi(T)}$ is the word obtained from $\psi(T)$ by exchanging 0's and 1's. Here $Z(\overline{\psi(T)})$ denotes the unique Zeckendorff word in the equivalence class of $\overline{\psi(T)}$.

In order to prove the lemma, we must show that

$$Z(\overline{1u_{k-1}u_{k-2}\cdots u_1 00}) = 1u'_{k-1}u'_{k-2}\cdots u'_1 00.$$

We proceed by induction on k. For $k = 1$, we must show that $Z(\overline{1u_1 00}) = 1u'_1 00$ where $u_1 \in \{01, 00, 001\}$. In case $u_1 = 01$, we have

$$Z(\overline{10100}) = Z(01011) = 10000 = 1(01)'00.$$

If $u_1 = 00$, we have

$$Z(\overline{10000}) = Z(01111) = 10100 = 1(00)'00.$$

If $u_1 = 001$, we have

$$Z(\overline{100100}) = Z(011011) = 100100 = 1(001)'00.$$

We next suppose that

$$Z(\overline{1u_{k-1}u_{k-2}\cdots u_1 00}) = 1u'_{k-1}u'_{k-2}\cdots u'_1 00$$

holds and we show that

$$Z(\overline{1u_k u_{k-1}u_{k-2}\cdots u_1 00}) = 1u'_k u'_{k-1}u'_{k-2}\cdots u'_1 00$$

for $u_k \in \{01, 00, 001\}$. We consider only the first case of $u_k = 01$, as the arguments in the remaining two cases are essentially identical. We have

$$\begin{aligned} Z(\overline{101u_{k-1}u_{k-2}\cdots u_1 00}) &= Z(01\overline{1u_{k-1}\cdots u_1 00}) \\ &= Z(011u'_{k-1}\cdots u'_1 00) \\ &= 100u'_{k-1}\cdots u'_1 00 \\ &= 1(01)'u'_{k-1}\cdots u'_1 00 \end{aligned}$$

as required. $\square$

We observe that for $T \in \mathcal{T}(m)$, we have $T' \neq T$ unless all stair indeces for T are 001 in which case $m = 2^n$ for some n. Thus we deduce that

LEMMA 3.8. *$\tau(m)$ is even if and only if m is not of the form 2^n for some n. Hence $rk(m)$ is divisible by 4 whenever m is not of the form 2^n for some n.*

EXAMPLE 3.9. We consider the case $m = 7$. There are six 7-Fibonacci towers $T_1, T_2, \ldots, T_6$. They are

T_1					$T_6 = T'_1$
$(1,6)$	T_2	T_3	$T_4 = T'_3$	$T_5 = T'_2$	$(6,1)$
$(1,5)$	$(2,5)$	$(3,4)$	$(4,3)$	$(5,2)$	$(5,1)$
$(1,4)$	$(2,3)$	$(3,1)$	$(1,3)$	$(3,2)$	$(4,1)$
$(1,3)$	$(2,1)$	$(2,1)$	$(1,2)$	$(1,2)$	$(3,1)$
$(1,2)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(2,1)$
$(1,1)$					$(1,1)$

The associated stair indeces are $01, 01, 01, 01, 01$ for T_1, $00, 01, 01$ for T_2, $00, 00, 01$ for T_3, $01, 01, 00$ for T_4, $01, 00, 00$ for T_5 and $00, 00, 00, 00, 00$ for T_6. Hence

$$\mathcal{B}_0(7) = \{1010101010100, 100010100, 100000100, 101010000, 101000000, 1000000000000\}$$

and

$$\mathcal{B}_1(7) = \{10101010101000, 1000101000, 1000001000, 1010100000, 1010000000, 10000000000000\}$$

Hence $\text{rk}(7) = 12$.

We note that in the previous example the mapping f_{001} never occurs. This is because the mapping f_{001} only occurs in a Fibonacci tower between level j and level $j+1$ whenever $x_{j+1} = y_{j+1}$. Because 7 is a prime number, the only level j in which $x_j = y_j$ is when $j = 1$. In fact, starting with any pair of positive integers (a, b) whose sum is m, there exists one or more m-Fibonacci towers whose top level is given by the pair (a, b). If $a < b$, then the next level down is $(a, b - a)$. If $a > b$, then the next level down is $(a - b, b)$. On the other hand if $a = b$, then the next level down consists of any pair of positive integers (a', b') whose sum $a' + b' = a = b$. It is well known that by iterating this process of leaving the smaller coordinate the same while subtracting it from the larger coordinate eventually yields the pair (d, d) where $d = \gcd(a, b)$. Thus if $\gcd(a, b) = 1$, then there is one and only one Fibonacci tower whose top level is (a, b). In particular, if m is prime, then for any pair of positive integers (a, b) whose sum is m, there is a unique m-Fibonacci tower whose top level is (a, b).

If m is not prime, then for each positive integer $1 \leq d < m$ dividing m, there are $\phi(m/d)$ distinct pairs (a, b) whose sum is m and whose greatest common divisor is d, where ϕ denotes the Euler ϕ- function. Each such pair (a, b) will generate in a unique way (from the top down) the upper levels of a m-Fibonacci tower (with top level (a, b)) until the level is reached consisting of the pair (d, d). At which point there are $\tau(d)$ different ways of continuing the tower downward. (In case $d = 1$, we adopt the convention $\tau(1) = 1$). In other words, for each pair of positive integers (a, b) whose sum is m and whose greatest common divisor is d, there are $\tau(d)$ many m-Fibonacci towers whose upper level is (a, b).

In summary we have:

THEOREM 3.10. *Let $m \geq 2$ be a positive integer. Then the number of m-Fibonacci towers is given by:*

$$\tau(m) = \sum_{1 \leq d < m;\ d|m} \phi(m/d)\tau(d).$$

Hence

$$rk(m) = 2\left(\sum_{1 \leq d < m;\ d|m} \phi(m/d)\tau(d)\right).$$

In particular if m is prime we have $rk(m) = 2\phi(m) = 2(m-1)$.

For $m \geq 2$, let

$$\mathrm{Fac}(m) = \{(n_1, n_2, \ldots, n_k) |\, n_j \geq 2 \text{ and } \prod_{j=1}^{k} n_j = m\}.$$

For instance, $\mathrm{Fac}(16) = \{(16), (2,8), (8,2), (4,4), (2,2,4), (2,4,2), (4,2,2), (2,2,2,2)\}$.

Then using the above formula together with a straightforward induction argument (on m) we obtain:

COROLLARY 3.11. *For each $m \geq 2$ we have*

$$\tau(m) = \sum_{(n_1,n_2,\ldots,n_k) \in Fac(m)} \phi(n_1)\phi(n_2)\cdots\phi(n_k)$$

and

$$\phi(m) = \sum_{(n_1,n_2,\ldots,n_k) \in Fac(m)} (-1)^{k+1}\tau(n_1)\tau(n_2)\cdots\tau(n_k).$$

For instance, $\tau(16) = \phi(16) + 2\phi(2)\phi(8) + \phi(4)^2 + 3\phi(2)^2\phi(4) + \phi(2)^4$ while $\phi(16) = \tau(16) - 2\tau(2)\tau(8) - \tau(4)^2 + 3\tau(2)^2\tau(4) - \tau(2)^4$.

CHAPTER 4

A FORMULA FOR $R_m(n)$

4.1. Introduction and Preliminaries

For each $m \geq 2$, we define the m-bonacci numbers by $F_k = 2^k$ for $0 \leq k \leq m-1$ and $F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m}$ for $k \geq m$. When $m = 2$, these are the usual Fibonacci numbers. Each positive integer n may be expressed as a sum of distinct m-bonacci numbers in one or more different ways. That is, we can write $n = \sum_{i=1}^{k} w_i F_{k-i}$ where $w_i \in \{0,1\}$ and $w_1 = 1$. Call the associated $\{0,1\}$-word $w_1 w_2 \cdots w_k$ a *representation* of n. One way of obtaining such a representation is by applying the "greedy algorithm." Recall, that this representation obtained by the greedy algorithm is the m-Zeckendorff representation, which does not contain m consecutive 1's, and is denoted by $Z_m(n)$ [54]. For example, taking $m = 2$ and applying the greedy algorithm to $n = 50$, we obtain $50 = 34 + 13 + 3 = F_7 + F_5 + F_2$ which gives rise to the representation $Z_2(50) = 10100100$. A $\{0,1\}$-word w beginning in 1 and having no occurrences of 1^m will be called a *m-Zeckendorff word.*

Other representations arise from the fact that an occurrence of 10^m in a given representation of n may be replaced by 01^m to obtain another representation of n, and *vice versa.* Thus a number n has a unique representation in the m-bonacci base if and only if $Z_m(n)$ does not contain any occurrences of 0^m. For example, again taking $m = 2$ and $n = 50$ we obtain the following 6 representations (arranged in decreasing lexicographic order):

10100100

10100011

10011100

10011011

1111100

1111011

We are interested in the sequence $R_m(n)$ which counts the number of distinct partitions of n in the m-bonacci base. More precisely, given $n \in \mathbb{Z}^{>0}$, we set

$$\Omega_m(n) = \{w = w_1 w_2 \cdots w_k \in \{0,1\}^* \,|w_1 = 1 \text{ and } n = \sum_{i=1}^{k} w_i F_{k-i}\}$$

and put $R_m(n) = \#\Omega_m(n)$. For $w \in \Omega_m(n)$, we will sometimes write $R_m(w)$ for $R_m(n)$. Also, we let $R_m^{\leq}(w)$ denote the number of representations of n which are less than or equal to w in the lexicographic order. As $Z_m(n)$ is the largest representation of n with respect to the lexicographic order, it follows that $R_m(n) = R_m^{\leq}(Z_m(n))$.

In this chapter, a formula for $R_m(n)$ is given that involves sums of binomial coefficients modulo 2. The proof makes use of the well-known Fine and Wilf Theorem [21]. In order to state the main result, we first consider a special factorization of $Z_m(n)$: Either $Z_m(n)$ contains no occurrences of 0^m (in which case $R_m(n) = 1$), or $Z_m(n)$ can be factored uniquely in the form

$$Z_m(n) = V_1 U_1 V_2 U_2 \cdots V_N U_N W$$

where

- $V_1, V_2, \ldots, V_N$ and W do not contain any occurrences of 0^m.
- 0^{m-1} is not a suffix of $V_1, V_2, \ldots, V_N$.
- Each U_i is of the form

  $$U_i = 10^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^{m-1} x_0 0^m$$

  with $x_i \in \{0, 1\}$.

We shall refer to this factorization as the *principal factorization* of $Z_m(n)$ and call the U_i *indecomposable factors*. We observe that in the special case of $m = 2$, the factors V_i are empty. Each indecomposable factor U_i may be coded by a positive integer r_i whose base 2 expansion is $1 x_k x_{k-1} \cdots x_0$; in other words $r_i = 1 \cdot 2^{k+1} + x_k \cdot 2^k + \cdots x_1 \cdot 2 + x_0$.
Given a positive integer r whose base 2 expansion is $1 x_k x_{k-1} \cdots x_0$, we set

$$[r] = 10^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^{m-1} x_0 0^m.$$

We now state our main result:

THEOREM 4.1. *Let $m \geq 2$. Given a positive integer n, let $Z_m(n) = V_1 U_1 V_2 U_2 \cdots V_N U_N W$ be the principal factorization of the m-Zeckendorff representation of n as defined above. Then the number of distinct partitions of n as a sum of distinct m-bonacci numbers is given by*

$$R_m(n) = \prod_{i=1}^{N} \sum_{j=0}^{r_i} \binom{2r_i - j}{j} \pmod 2$$

where $[r_i] = U_i$ for each $1 \leq i \leq N$.

4.2. Proof of Theorem 4.1

Let $Z_m(n) = V_1U_1V_2U_2 \cdots V_NU_NW$ be the principal factorization of $Z_m(n)$ described above. Then the number of partitions of n is simply the product of the number of partitions of each indecomposable factor:

$$(\dagger) \qquad R_m(n) = \prod_{i=1}^{N} R_m(U_i).$$

In fact, any representation of n as a sum of distinct m-bonacci numbers may be factored in the form

$$V_1U_1'V_2U_2' \cdots V_NU_N'W$$

where for each $1 \le i \le N$, U_i' is an equivalent representation of U_i. To see this, we first observe that since the V_i and W contain no 0^m, we have $R_m(V_i) = R_m(W) = 1$. So the only way that V_i or W could change in an alternate representation of n would be as a result of a neighboring indecomposable factor. If V_i contains an occurrence of 1, then since V_i does not end in 0^{m-1}, the last occurrence of 1 in V_i can never be followed by 0^m. In other words the last 1 in V_i can never move into the U_i that follows. If V_i contains no occurrences of 1, then $V_i = 0^r$ with $r < m-1$. Since the indecomposable factor U_{i-1} preceding V_i ends in Km many consecutive 0's (for some $K \ge 1$), any equivalent representation of U_{i-1} either ends in 0^m or in 1^m, and since V_i does not begin in 0^m, any representation of U_{i-1} terminating in 1^m will never be followed by 0^m. In other words, no 1 in U_{i-1} can ever move into V_i or in the following U_i. A similar argument applies to the indecomposable factor U_N preceding W.

Thus in view of ($\dagger$) above, in order to prove Theorem 4.1, it remains to show that for each positive integer $r = 1 \cdot 2^{k+1} + x_k \cdot 2^k + \cdots x_1 \cdot 2 + x_0$, we have

$$(\dagger\dagger) \qquad R_m([r]) = \sum_{j=0}^{r} \binom{2r-j}{j} \pmod 2.$$

For each positive integer n, there is a natural decomposition of the set $\Omega_m(n)$ of all partitions of n in the m-bonacci base. Let F be the largest m-bonacci number less than or equal to n. We denote by $\Omega_m^+(n)$ the set of all partitions of n involving F and $\Omega_m^-(n)$ the set of all partitions of n not involving F, and set $R_m^+(n) = \#\Omega_m^+(n)$ and $R_m^-(n) = \#\Omega_m^-(n)$. Clearly

$$R_m(n) = R_m^+(n) + R_m^-(n).$$

We will make use of the following recursive relations:

LEMMA 4.2. *Let* $U = 10^{m-1}x_k 0^{m-1}x_{k-1}\cdots 0^{m-1}x_0 0^m$ *with* $x_i \in \{0,1\}$. *Then*

$$\begin{aligned}
R_m^+(10^{m-1}10^{m-1}x_k 0^{m-1}x_{k-1}\cdots 0^{m-1}x_0 0^m) &= R_m(U) &&= R_m^+(U) + R_m^-(U)\\
R_m^-(10^{m-1}10^{m-1}x_k 0^{m-1}x_{k-1}\cdots 0^{m-1}x_0 0^m) &= R_m^-(U) &&\\
R_m^+(10^{m-1}00^{m-1}x_k 0^{m-1}x_{k-1}\cdots 0^{m-1}x_0 0^m) &= R_m^+(U) &&\\
R_m^-(10^{m-1}00^{m-1}x_k 0^{m-1}x_{k-1}\cdots 0^{m-1}x_0 0^m) &= R_m(U) &&= R_m^+(U) + R_m^-(U)
\end{aligned}$$

PROOF. It is easy to see that $w \in \Omega_m^+(10^{m-1}U)$ if and only if w is of the form $w = 10^{m-1}w'$ for some $w' \in \Omega_m(U)$. Whence $R_m^+(10^{m-1}U) = R_m(U)$. Similarly, $w \in \Omega_m^-(10^{m-1}U)$ if and only if w is of the form $w = 01^m w'$ for some $w' \in \Omega_m^-(U)$. Whence $R_m^-(10^{m-1}U) = R_m^-(U)$. A similar argument applies to the remaining two identities. $\square$

Fix a positive integer $r = 1\cdot 2^{k+1} + x_k \cdot 2^k + \cdots x_1 \cdot 2 + x_0$. The above lemma can be used to compute $R_m([r])$ as follows: We construct a tower of $k+2$ levels $L_0, L_1, \cdots, L_{k+1}$, where each level L_i consists of an ordered pair (a,b) of positive integers. We start with level 0 by setting $L_0 = (1,1)$. Then L_{i+1} is obtained from L_i according to the value of x_i. If $L_i = (a,b)$, then $L_{i+1} = (a, a+b)$ if $x_i = 0$, and $L_{i+1} = (a+b, b)$ if $x_i = 1$. It follows from Lemma 5.8 that $L_{k+1} = (R_m^+([r]), R_m^-([r]))$. Hence $R_m([r])$ is the sum of the entries of level L_{k+1}.

By the well known Fine and Wilf Theorem [21], given a pair of relatively prime numbers (p,q), there exists a $\{0,1\}$-word w of length $p+q-2$ (unique up to isomorphism) having periods p and q, and if p and q are both greater than 1, then this word contains both 0's and 1'; in other words, $1 = \gcd(p,q)$ is not a period. We call such a word a *Fine and Wilf word* relative to (p,q). Moreover it can be shown (see [50] for example) that if both p and q are greater than 1, then the suffixes of w of lengths p and q begin in different symbols. We denote by $FW(p,q)$ the unique Fine and Wilf word relative to (p,q) with the property that its suffix of length p begins in 0 and its suffix of length q begins in 1.

We now apply this to the ordered pair $(p,q) = (R_m^+([r]), R_m^-([r]))$. It is well known that $FW(R_m^+([r]), R_m^-([r]))01$ is given explicitly by the following composition of morphisms:

$$FW(R_m^+([r]), R_m^-([r]))01 = \tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)$$

where

$$\begin{aligned}
\tau_0(0) &= 0 \qquad & \tau_0(1) &= 01\\
\tau_1(0) &= 10 \qquad & \tau_1(1) &= 1
\end{aligned}$$

(see for instance [31, 50]).

Let

$$\alpha(r) = |FW(R_m^+([r]), R_m^-([r]))01|_1$$

and

$$\beta(r) = |FW(R_m^+([r]), R_m^-([r]))01|_0$$

. In other words, $\alpha(r)$ is the number of occurrences of 1 in $FW(R_m^+([r]), R_m^-([r]))01$ and $\beta(r)$ the number of 0's in $FW(R_m^+([r]), R_m^-([r]))01$.

In summary:

$$\begin{aligned}
R_m([r]) &= R_m^+([r]) + R_m^-([r]) \\
&= R_m^+([r]) + R_m^-([r]) - 2 + 2 \\
&= |FW(R_m^+([r]), R_m^-([r]))| + 2 \\
&= |FW(R_m^+([r]), R_m^-([r]))01| \\
&= |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)| \\
&= |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|_1 + |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|_0 \\
&= \alpha(r) + \beta(r) \\
&= |\tau_1 \circ \tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|_1 \\
&= \alpha(2r+1).
\end{aligned}$$

The key step in the proof of Theorem 4.1 is to replace the sum of the periods $R_m^+([r])+R_m^-([r])$ of the Fine and Wilf word $FW(R_m^+([r]), R_m^-([r]))$ by the sum of the number of occurrences of 0's and 1's in $FW(R_m^+([r]), R_m^-([r]))01$.

The following basic identities are readily verified:

- $\alpha(1) = \beta(1) = 1$.
- $\alpha(2r) = \alpha(r)$.
- $\beta(2r) = \alpha(r) + \beta(r)$.
- $\alpha(2r+1) = \alpha(r) + \beta(r)$.
- $\beta(2r+1) = \beta(r)$.
- $\beta(r) = \alpha(r+1)$.

Summarizing we have

Proposition 4.3. *Let $U = 10^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_00^m$ with $x_i \in \{0,1\}$. Let r be the number whose base 2 expansion is given by $1x_kx_{k-1}\cdots x_0$. Then $R_m(U) = \alpha(2r+1)$ where the sequence $\alpha(r)$ is defined recursively by:*

- $\alpha(1) = 1$
- $\alpha(2r) = \alpha(r)$
- $\alpha(2r+1) = \alpha(r) + \alpha(r+1)$.

We now consider a new function $\psi(r)$ defined by $\psi(1) = 1$, and for $r \geq 1$

$$\psi(r+1) = \sum_{j=0}^{2j\leq r} \binom{r-j}{j} \pmod 2.$$

We will show that $\psi(r)$ and $\alpha(r)$ satisfy the same recursive relations, namely $\psi(2r) = \psi(r)$ and $\psi(2r+1) = \psi(r) + \psi(r+1)$. Thus, $\alpha(r) = \psi(r)$ for each r, thereby establishing formula (††).

We shall make use of the following lemma:

Lemma 4.4. $\binom{n}{k} \pmod 2 = \binom{2n+1}{2k} \pmod 2 + \binom{2n}{2k+1} \pmod 2$.

Proof. This follows immediately from the so-called Lucas' identities: $\binom{2n}{2k+1} = 0 \pmod 2$ for $0 \leq k \leq n-1$, and $\binom{n}{k} = \binom{2n+1}{2k} \pmod 2$ for $0 \leq k \leq n$. $\square$

Proposition 4.5. *For $r \geq 0$ we have $\psi(2r+2) = \psi(r+1)$ and for $r \geq 1$ we have $\psi(2r+1) = \psi(r) + \psi(r+1)$.*

Proof. By Lemma 4.4 we have

$$\begin{aligned}
\psi(r+1) &= \sum_{j=0}^{2j\leq r} \binom{r-j}{j} \pmod 2 \\
&= \sum_{j=0}^{2j\leq r} \left(\binom{2r-2j+1}{2j} \pmod 2 + \binom{2r-2j}{2j+1} \pmod 2\right) \\
&= \sum_{i=0}^{r} \binom{2r+1-i}{i} \pmod 2 \\
&= \psi(2r+2).
\end{aligned}$$

As for the second recursive relation we have

$$\begin{aligned}
\psi(2r+1) &= \sum_{j=0}^{r} \binom{2r-j}{j} \pmod 2 \\
&= \sum_{i=0}^{2i\le r} \binom{2r-2i}{2i} \pmod 2 + \sum_{i=0}^{2i\le r-1} \binom{2r-2i-1}{2i+1} \pmod 2.
\end{aligned}$$

But

$$\begin{aligned}
\binom{2r-2i}{2i} \pmod 2 &= \frac{(2r-2i)!}{(2i)!(2r-4i)!} \pmod 2 \\
&= \frac{(2r-2i+1)!}{(2i)!(2r-4i+1)!} \pmod 2 \\
&= \binom{2r-2i+1}{2i} \pmod 2 \\
&= \binom{r-i}{i} \pmod 2 \quad \text{by Lemma 4.4.}
\end{aligned}$$

Hence

$$\sum_{i=0}^{2i\le r} \binom{2r-2i}{2i} \pmod 2 = \sum_{i=0}^{2i\le r} \binom{r-i}{i} \pmod 2 = \psi(r+1).$$

Similarly

$$\begin{aligned}
\binom{2r-2i-1}{2i+1} \pmod 2 &= \frac{(2r-2i-1)!}{(2i+1)!(2r-4i-2)!} \pmod 2 \\
&= \frac{(2r-2i-1)!}{(2i)!(2r-4i-1)!} \pmod 2 \\
&= \binom{2r-2i-1}{2i} \pmod 2 \\
&= \binom{r-1-i}{i} \pmod 2 \quad \text{by Lemma 4.4.}
\end{aligned}$$

Hence

$$\sum_{i=0}^{2i\le r-1}\binom{2r-2i-1}{2i+1} \pmod 2 = \sum_{i=0}^{2i\le r-1}\binom{r-1-i}{i} \pmod 2 = \psi(r).$$

It follows that $\psi(2r+1)=\psi(r)+\psi(r+1)$. $\square$

Having established that $\alpha(r)=\psi(r)$ for each $r\ge 1$, we deduce that:

Corollary 4.6. *Let $U=10^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_00^m$ with $x_i\in\{0,1\}$. Let r be the number whose base 2 expansion is given by $1x_kx_{k-1}\cdots x_0$. Then $R_m(U)=\sum_{j=0}^{r}\binom{2r-j}{j} \pmod 2$.* This concludes the proof of Theorem 4.1.

4.3. Concluding Remarks

4.3.1. A formula for $R_m^{\le}(w)$

The proof applies more generally to give a formula for $R_m^{\le}(w)$ for each representation w of n. In other words, given $w\in\Omega_m(n)$, then either w does not contain any occurrences of 0^m (in which case $R_m^{\le}(w)=1$) or w may be factored in the form

$$w=V_1U_1V_2U_2\cdots V_NU_NW$$

where the V_i and W do not contain any occurrences of 0^m and the V_i do not end in 0^{m-1}, and where the U_i are of the form

$$U_i=10^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_00^m$$

with $x_i\in\{0,1\}$. Each factor U_i is coded by a positive integer r_i whose base 2 expansion is $1x_kx_{k-1}\cdots x_0$. It is easy to see that any representation of n less than or equal to w may be factored in the form

$$V_1U_1'V_2U_2'\cdots V_NU_N'W$$

where for each $1\le i\le N$, U_i' is an equivalent representation of U_i. Hence $R_m^{\le}(w)=\prod_{i=1}^{N}R_m(U_i)$, from which it follows that

$$R_m^{\le}(w)=\prod_{i=1}^{N}\sum_{j=0}^{r_i}\binom{2r_i-j}{j} \pmod 2.$$

4.3.2. Episturmian numeration systems

Let A be a finite non-empty set. Associated to an infinite word $\omega = \omega_1\omega_2\omega_3 \ldots \in A^{\mathbb{N}}$ is a non- decreasing sequence of positive integers $\mathcal{E}(\omega) = E_1, E_2, E_3, \ldots$ defined recursively as follows: $E_1 = 1$, and for $k \geq 1$, the quantity E_{k+1} is defined by the following rule. If $\omega_{k+1} \neq \omega_j$ for each $1 \leq j \leq k$, then set

$$E_{k+1} = 1 + \sum_{j=1}^{k} E_j.$$

Otherwise let $\ell \leq k$ be the largest integer such that $\omega_{k+1} = \omega_\ell$, and set

$$E_{k+1} = \sum_{j=\ell}^{k} E_j.$$

In particular, note that $E_{k+1} = E_k$ if and only if $\omega_{k+1} = \omega_k$.

Set $\mathcal{N}(\omega) = \{E_k | k \geq 1\}$. For $E \in \mathcal{N}(\omega)$ let $k \geq 1$ be such that $E = E_k$. We define $\sigma(E) = \omega_k$ and say that E is *based* at $\omega_k \in A$. We also define the quantity $\rho(E)$, which we call the *multiplicity* of E, by $\rho(E) = \#\{i \geq 1 | E = E_i\}$. We can write $\mathcal{N}(\omega) = \{x_1, x_2, x_3, \ldots\}$, where for each $i \geq 1$, we have $x_i < x_{i+1}$. Thus we have that $\omega = \sigma(x_1)^{\rho(x_1)}\sigma(x_2)^{\rho(x_2)} \ldots$.

It can be verified that the set $\mathcal{N}(\omega)$ defines a numeration system (see [34]). More precisely, each positive integer n may be written as a sum of the form

$$(*) \qquad n = m_k x_k + m_{k-1} x_{k-1} + \cdots + m_1 x_1$$

where for each $1 \leq i \leq k$ we have $0 \leq m_i \leq \rho(x_i)$ and $m_k \geq 1$. While such a representation of n is not necessarily unique, one way of obtaining such a representation is to use the "greedy algorithm". In this case we call the resulting representation the Zeckendorff representation of n and denote it $Z_\omega(n)$. We call the above numeration system a *generalized Ostrowski system* or an *Episturmian numeration system*. In fact, the quantities E_i are closely linked to the lengths of the palindromic prefixes of the characteristic Episturmian word associated to the directive sequence ω (see [32, 33, 34, 35]). In the case where $\#A = 2$, this is known as the Ostrowski numeration system (see [6, 10, 45]). In the case where $A = \{1, 2, \ldots, m\}$ and ω is the periodic sequence $\omega = (1, 2, 3, \ldots, m,)^\infty$, then the resulting numeration system is the m-bonacci system defined earlier.

Given an infinite word $\omega = \omega_1\omega_2\omega_3 \ldots \in A^{\mathbb{N}}$, we are interested in the number of distinct ways of writing each positive integer n as a sum of the form (*). More precisely, denoting by $\hat{A}$ the set $\{\hat{a}|a \in A\}$, we set $R_\omega(n) = \#\Omega_\omega(n)$ where $\Omega_\omega(n)$ is the set of all expressions of the form

$$(**) \qquad \widehat{\sigma(x_k)}^{m_k}\sigma(x_k)^{\rho(x_k)-m_k}\widehat{\sigma(x_{k-1})}^{m_{k-1}}\sigma(x_{k-1})^{\rho(x_{k-1})-m_{k-1}}\cdots\widehat{\sigma(x_1)}^{m_1}\sigma(x_1)^{\rho(x_1)-m_1}$$

in $(A \cup \hat{A})^*$, such that $n = m_kx_k + m_{k-1}x_{k-1} + \cdots + m_1x_1$ where $\mathcal{N}(\omega) = \{x_1, x_2, x_3, \ldots | 1 = x_1 < x_2 < x_3 \ldots\}$ and where $0 \leq m_i \leq \rho(x_i)$ and $m_k \geq 1$.[1] For $w \in \Omega_\omega(n)$ we sometimes write $R_\omega(w)$ for $R_\omega(n)$.

Just as in the previous section, we begin with a unique special factorization of the Zeckendorff representation of n. In this case, this factorization was originally defined by Justin and Pirillo (see Theorem 2.7 in [34]). We factor $Z_\omega(n)$ as

$$Z_\omega(n) = V_1U_1V_2U_2\cdots V_NU_NW$$

where for each $1 \leq i \leq N$, we have that U_i is a a_i- *based maximal semigood multiblock* for some $a_i \in A$. Moreover, any other representation of n may be factored in the form

$$Z_\omega(n) = V_1U_1'V_2U_2'\cdots V_NU_N'W$$

where U_i' is an equivalent representation of U_i (see Theorem 2.7 in [34]). Thus, as before (see (†)) we have

$$R_\omega(n) = \prod_{i=1}^{N} R_\omega(U_i).$$

For each $1 \leq i \leq N$, the factor U_i corresponds to a sum of the form

$$m_Kx_K + m_{K-1}x_{K-1} + \cdots + m_kx_k$$

for some $K > k$ with $m_K \neq 0$. In addition, for each $K \geq j \geq k$, we have that if $m_j \neq 0$, then $\sigma(x_j) = a_i$ [34]. In other words the only "accented" symbol occurring in U_i is a_i, i.e., $U_i \in (A \cup \{\hat{a_i}\})^*$.

Associated to U_i is a $\{0,1\}$-word $\nu(U_i) = \nu_K\nu_{K-1}\ldots\nu_k$, where $\nu_K = 10$ and

- $\nu_j = \varepsilon$ (the empty word) if $\sigma(x_j) \neq a_i$,
- $\nu_j = 10$ if $\sigma(x_j) = a_i$ and $m_j = \rho(x_j)$,
- $\nu_j = 010$ if $\sigma(x_j) = a_i$ and $0 < m_j < \rho(x_j)$, and

[1]Our notation here differs somewhat from that of Justin and Pirillo in [34]. For instance, in [34] the authors use the notation $\bar{a}$ for in lieu of our $\hat{a}$. Also instead of the expression (**), they consider the reverse of this word.

- $\nu_j = 00$ if $\sigma(x_j) = a_i$ and $m_j = 0$.

By comparing the matrix formulation given in Corollary 2.11 in [34] used to compute $R_\omega(U_i)$ with the matrix formulation given in Proposition 4.1 in [6], we leave it to the reader to verify the following:

PROPOSITION 4.7. $R_\omega(U_i) = R_2(\nu(U_i))$.

In other words computing the multiplicities of representations in a generalized Ostrowski numeration system may be reduced to a computation of the multiplicities of representations in the Fibonacci base.

EXAMPLE 4.8. We consider the example originally started in Berstel's paper [6] and later revisited by Justin and Pirillo as Example 2.3 in [34] of the Ostrowski numeration system associated to the infinite word $\omega = a, a, b, b, a, a, a, b, b, a, a, b, b, a, a, a, b, \ldots$. It is readily verified that

$$\mathcal{N}(\omega) = \{1, 3, 7, 24, 55, 134, 323, \ldots\},$$

$\sigma(1) = \sigma(7) = \sigma(55) = \sigma(323) = a$, $\sigma(3) = \sigma(24) = \sigma(134) = b$, and $\rho(1) = 2$, $\rho(3) = 2$, $\rho(7) = 3$, $\rho(24) = 2$, $\rho(55) = 2$, $\rho(134) = 2$, $\rho(323) = 3$. Applying the greedy algorithm we obtain the following representation of the number 660

$$660 = 2(323) + 0(134) + 0(55) + 0(24) + 2(7) + 0(3) + 0(1).$$

So $Z_\omega(660) = \hat{a}\hat{a}abbaabb\hat{a}\hat{a}abbaa$. which is a semigood multiblock based at a. We deduce that

$$\nu(Z_\omega(660)) = 10 \cdot \varepsilon \cdot 00 \cdot \varepsilon \cdot 010 \cdot \varepsilon \cdot 00$$

or simply $\nu(Z_\omega(660)) = 100001000$.

Following the algorithm of Corollary 2.11 of [34] due to Justin and Pirillo, we obtain $q_1 = 2$, $q_2 = 4$, $p_1 = 2$, $p_2 = 2$, $c_1 = c_2 = 1$, so that

$$R_\omega(660) = (1,0)\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 6$$

In contrast, applying the algorithm in Proposition 4.1 of [6] due to Berstel to the Zeckendorff word $\nu(Z_\omega(660)) = 100001000$, we obtain $d_1 = 4$, $d_2 = 3$ so that

$$R_2(\nu(Z_\omega(660))) = (1,1)\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6$$

as required[2].

[2]In [6], Berstel computes $R_\omega(660)$ in a different way by using the matrix formulation of Proposition 5.1 in [6] which applies to an Ostrowski numeration system.

CHAPTER 5

TILINGS OF CERTAIN FACTORS OF THE FIBONACCI WORD

5.1. Introduction

Sturmian words are infinite words on a two-letter alphabet that have exactly $n+1$ factors of length n. The *Fibonacci* word $\omega = 0100101001001010010100100101001 0\ldots$ is probably the most common example of a Sturmian word [7] and is the limit of the sequence of words defined as follows.

$$u_0 = 0, u_1 = 01, \text{ and } u_k = u_{k-1}u_{k-2} \text{ for all } k \geq 2.$$

Alternately, ω is the fixed point of the substitution

$$\begin{aligned} \tau : 0 &\mapsto 01 \\ 1 &\mapsto 0. \end{aligned}$$

It is a straightforward inductive proof to show that $\tau^k(0) = u_k$, for each $k \geq 1$.

Jean Berstel, while attending a conference in 2006, suggested that others at the conference attempt to give combinatorial descriptions of all the various identities related to the Fibonacci numbers. In this chapter, I will give a description of the following identities in terms of tilings of certain factors of the Fibonacci word.

Let the Fibonacci numbers be defined as $f_0 = 0, f_1 = 0$, and set $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Then

$$f_{2n} = \sum_{i=0}^{n} \binom{n}{i} f_{n-i} \quad \text{and} \quad f_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} f_{n-i+1}.$$

5.2. Palindromic Prefixes of the Fibonacci Word

A subword w of a Sturmian word s is *left special* if $0w$ and $1w$ both occur in s, and we say that an infinite binary word s is a characteristic Sturmian word if all the prefixes of s are left special. In addition, the *palindromic closure* of an finite word u is the shortest palindrome $u^{(+)}$ having u as a prefix. A. de Luca, in [15], gives a characterization of Sturmian words in terms of palindromic closures.

Proposition 5.1. *(de Luca) An infinite binary word s is a characteristic Sturmian word if there exists an infinite binary word $\Delta(s) = a_0a_1\cdots$ with infinitely many occurrences of both letters such that $s = \lim_{n\to\infty} w_n$ where $w_0 = \epsilon$ and $w_{n+1} = (w_na_n)^{(+)}$, for $n \geq 0$.*

$\Delta(s) = a_0a_1\cdots$ is called the *directive sequence* of s. In the case of Fibonacci, $\Delta(\omega) = 01010101\ldots$. Figure 5.1 shows the first few palindromic prefixes of ω.

$$
\begin{aligned}
w_0 &= \epsilon \\
w_1 &= 0 \\
w_2 &= 010 \\
w_3 &= 010010 \\
w_4 &= 01001010010 \\
w_5 &= 0100101001001010010 \\
w_6 &= 01001010010010100101001001010010 \\
\vdots \quad &\vdots \quad \vdots
\end{aligned}
$$

FIGURE 5.1. The First Few Palindromic Prefixes of the Infinite Fibonacci Word

In [47], R. Risley and L.Q. Zamboni give another construction of these palindromic prefixes based on suffix replication. Let $\Delta = a_0a_1a_2a_3\ldots = 0101\ldots$ be the directive sequence of ω. As the letters of Δ are added, they are marked by $\hat{}$. So, $w_1 = \hat{0}$ and $w_{n+1} = w_n\hat{a_n}v_n$, where v_n is the longest suffix of w_n containing no $\hat{a_n}$. Figure 5.2 shows the palindromic prefixes of ω obtained using the method in [47].

$$
\begin{aligned}
w_0 &= \epsilon \\
w_1 &= \hat{0} \\
w_2 &= \hat{0}\hat{1}0 \\
w_3 &= \hat{0}\hat{1}\hat{0}010 \\
w_4 &= \hat{0}\hat{1}\hat{0}010\hat{1}0010 \\
w_5 &= \hat{0}\hat{1}\hat{0}010\hat{1}0010\hat{0}1010010 \\
w_6 &= \hat{0}\hat{1}\hat{0}010\hat{1}0010\hat{0}1010010\hat{1}001001010010 \\
\vdots \quad &\vdots \quad \vdots
\end{aligned}
$$

FIGURE 5.2. Palindromic Prefixes of the Infinite Fibonacci Word Using the "Hat" Algorithm

LEMMA 5.2. *For n even, $w_n = w_{n-1}10w_{n-2}$ and for n odd, $w_n = w_{n-1}01w_{n-2}$.*

PROOF. Clearly, the proposition is true for $n = 2, 3$. We proceed by induction on n.

Suppose $n+1$ is odd. We show that $w_{n+1} = w_n01w_{n-1}$.

We know, since $w_{n+1} = (w_n0)^+$, that $w_{n+1} = w_nu$, and by the hat algorithm, u is the suffix of w_n that begins in the last $\hat{0}$ of w_n. So, $w_n = zu = z\hat{0}v_1\hat{1}v_2$, where $\hat{0}v_1$ is the last $\hat{0}$ of w_{n-1} and $\hat{1}v_2$ is the last $\hat{1}$ of w_n. Again, by the hat algorithm, $w_n = w_{n-1}\hat{1}v_2$, and by the induction hypothesis, $w_n = w_{n-1}10w_{n-2}$. So, $\hat{1}v_2 = 10w_{n-2}$. Similarly, $\hat{0}v_1 = 01w_{n-3}$. Together, this gives us that $u = \hat{0}1w_{n-3}\hat{1}0w_{n-2}$. However, by the induction hypothesis, $w_{n-1} = w_{n-2}01w_{n-3}$, but $w_{n-1} = w_{n-3}10w_{n-2}$ since each w_i is a palindrome. So, $u = 01w_{n-1}$ and thus, $w_{n+1} = w_n01w_{n-1}$ as needed.

If $n+1$ is even, a completely symmetric argument shows $w_{n+1} = w_n10w_{n-1}$.

□

LEMMA 5.3. *For each $n \geq 1, |w_n| = f_{n+2} - 2 = f_1 + f_2 + \cdots + f_n$.*

PROOF. We proceed by induction on n. By Lemma 5.2, $w_n = w_{n-1}abw_{n-2}$, where $a, b \in \{0,1\}, a \neq b$. Then,

$$\begin{aligned} |w_n| &= |w_{n-1}| + |w_{n-2}| + 2 \\ &= f_{n+1} - 2 + f_n - 2 + 2 \\ &= f_{n+2} - 2 \\ &= f_1 + \cdots + f_n \end{aligned}$$

□

In [15], de Luca gives the following proposition.

PROPOSITION 5.4. *A palindrome word w has the period $p < |w|$ if and only if it has a palindrome prefix (suffix) of length $|w| - p$.*

We use Proposition 5.4 to establish that w_n has periods f_n and f_{n+1}.

PROPOSITION 5.5. *For each $n \geq 1, w_n$ is the unique word (up to isomorphism) of length $f_n - 2$ beginning in 0 and having periods f_n and f_{n+1}.*

PROOF. By Lemma 5.2, $w_n = w_{n-1}abw_{n-2}$, where $a, b \in \{0,1\}, a \neq b$, and by definition, w_{n-1} and w_{n-2} are palindromes. By Proposition 5.4, to show that w_n has period f_n, it remains to show that $|w_{n-1}| = |w_n| - f_n$.

$$\begin{aligned}|w_{n-1}| &= f_{n+1} - 2\\ &= f_{n+1} + f_n - 2 - f_n\\ &= f_{n+2} - 2 - f_n\\ &= |w_n| - f_n\end{aligned}$$

Again, by Proposition 5.4, to show that w_n has period f_{n+1}, it remains to show that $|w_{n-2}| = |w_n| - f_{n+1}$.

$$\begin{aligned}|w_{n-2}| &= f_n - 2\\ &= f_{n+1} + f_n - 2 - f_{n+1}\\ &= f_{n+2} - 2 - f_{n+1}\\ &= |w_n| - f_{n+1}\end{aligned}$$

□

5.3. Singular Factors of the Fibonacci Word

Let $u_n = z_1 z_2 \cdots z_{f_{n+1}}$ be the prefix of ω of length f_{n+1}. The kth *conjugate* of u_n is defined to be $z_{k+1} \cdots z_{f_{n+1}} z_1 \cdots z_k$, and set $C_n = \{w | w \text{ is a conjugate of } u_n\}$. In [52], it is shown that all the conjugates of u_n are factors of ω and that $|C_n| = f_n$. In other words, each element in C_n is distinct. In addition, it is shown that if n is odd, 01 is a suffix of u_n and if n is even, 10 is a suffix of u_n.

Since there are f_{n+1}-many factors of length f_n in ω, there is one factor of length f_n that is not an element of C_n. Call this factor the nth *singular* factor and denote it s_n. In [52], s_n is defined in the following way. If $u_n = zab$, where $a, b \in \{0, 1\}$ and $a \neq b$, then the singular word s_n of length f_n is azb^{-1}. In addition, it is shown that s_n defined in this way is a factor of ω but is not an element of C_n. Thus, the two definitions are equivalent.

Set $S_0 = 0$, and for all $n > 0, S_n = \begin{cases} 0w_{n-1}0 & (n \text{ odd}) \\ 1w_{n-1}1 & (n \text{ even}) \end{cases}$ and $F_n = 0w_{n-1}1.$

PROPOSITION 5.6. *S_n is the singular factor of the ω of length f_{n+1}.*

PROOF. Since w_{n-1} has length $f_{n+1} - 2, w_{n-1}$ is a prefix of u_n. If n is odd, 01 is a suffix of u_n, and if n is even, 10 is a suffix of u_n. Therefore,

$$u_n = \begin{cases} w_{n-1}01 & (n \text{ even}) \\ w_{n-1}10 & (n \text{ odd}) \end{cases}$$

Thus if n is even, $s_n = 1w_{n-1}1 = S_n$ and if n is odd, $s_n = 0w_{n-1}0 = S_n$. $\square$

REMARK 5.7.

- Since S_n is the singular factor of length f_{n+1}, F_n must be a congugate of u_n.
- S_n is a palindrome since w_{n-1} is.

5.4. Tilings

We now describe the identities

$$f_{2n} = \sum_{i=0}^{n} \binom{n}{i} f_{n-i} \quad \text{and} \quad f_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} f_{n-i+1}$$

with tilings.

For each n, we say S_n is a tile of *type n*. We show that F_{2n} can be tiled by $\binom{n}{0}$-many tiles of type n, $\binom{n}{1}$-many tiles of type $n-1, \ldots,$ and $\binom{n}{n}$-many tiles of type 0, and that F_{2n+1} can be tiled by $\binom{n}{0}$-many tiles of type $n+1$, $\binom{n}{1}$-many tiles of type $n, \ldots,$ and $\binom{n}{n}$-many tiles of type 1.

First, we show that $\{S_n\}$ and $\{F_n\}$ satisfy the following recursive relations.

PROPOSITION 5.8.

- $S_n = S_{n-2}S_{n-3}S_{n-2}$.
- $F_{2n} = F_{2n-2}F_{2n-1}$.
- $F_{2n+1} = F_{2n}F_{2n-1}$.

PROOF. The proof depends on Lemma 5.2.

For n odd,

$$\begin{aligned} S_n &= 0w_{n-1}0 \\ &= 0w_{n-2}10w_{n-3}0 \\ &= 0w_{n-3}01w_{n-4}10w_{n-3}0 \\ &= S_{n-2}S_{n-3}S_{n-2} \end{aligned}$$

For n even,

$$
\begin{aligned}
S_n &= 1w_{n-1}1 \\
&= 1w_{n-2}01w_{n-3}1 \\
&= 1w_{n-3}10w_{n-4}01w_{n-3}1 \\
&= S_{n-2}S_{n-3}S_{n-2}
\end{aligned}
$$

$$
\begin{aligned}
F_{2n} &= 0w_{2n-1}1 \\
&= 0w_{2n-2}01w_{2n-3}1 \\
&= 0w_{2n-3}10w_{2n-2}1 \text{ since } w_{2n-1} \text{ is a palindrome.} \\
&= F_{2n-2}F_{2n-1}
\end{aligned}
$$

$$
\begin{aligned}
F_{2n} &= 0w_{2n}1 \\
&= 0w_{2n-1}10w_{2n-2}1 \\
&= F_{2n}F_{2n-1}
\end{aligned}
$$

$\square$

Next, we define some new words recursively in terms of $\{S_n\}$. We will show then that $\{F_n\}$ can be defined in terms of these new words, and making this definition will allow us to complete the proof of the main theorem.

DEFINITION 5.9. First, we define $y_0 = S_1S_2, \widetilde{y_0} = (\overline{y_0})^{+1} = S_3S_2$, and for $n \geq 1, y_n = \widetilde{y_{n-1}}y_{n-1}$, and $\widetilde{y_n} = (\overline{y_n})^{+1}$ where $(y_n)^{+1}$ is an increase in the subscripts of y_n.

Next, we define $z_0 = S_1S_0, \widetilde{z_0} = (\overline{z_0})^{+1}$, and for $n \geq 1, z_n = z_{n-1}\widetilde{z_{n-1}}$, and $\widetilde{z_n} = (\overline{z_n})^{+1}$ where $(z_n)^{+1}$ is an increase in the subscripts of z_n.

LEMMA 5.10. $z_n\widetilde{z_n} = \widetilde{y_n}$

PROOF. We proceed by induction on n.
For the base cases, $z_0\widetilde{z_0} = S_1S_0S_1S_2 = S_3S_2 = \widetilde{y_0}$, and
$z_1\widetilde{z_1} = S_1S_0S_1S_2S_3S_2S_1S_2 = S_3S_2S_3S_4 = \widetilde{y_1}$.

$$
\begin{aligned}
z_{n+1}\widetilde{z_{n+1}} &= z_n\widetilde{z_n}(\overline{z_{n+1}})^{+1} \\
&= z_n\widetilde{z_n}(\overline{z_n\widetilde{z_{n+1}}})^{+1} \\
&= \widetilde{y_n}(\overline{\widetilde{y_n}})^{+1} \\
&= (\overline{y_n})^{+1}(\overline{\widetilde{y_n}})^{+1} \\
&= (\overline{\widetilde{y_n}y_n})^{+1} \\
&= (\overline{y_{n+1}})^{+1} \\
&= \widetilde{y_{n+1}}.
\end{aligned}
$$

□

LEMMA 5.11. $\widetilde{y_n}y_n = \widetilde{z_{n+1}}$

PROOF. Again, we proceed by induction.

For the base case, $\widetilde{y_0}y_0 = S_3S_2S_1S_2 = \widetilde{v_1}$.

$$
\begin{aligned}
\widetilde{y_{n+1}}y_{n+1} &= (\overline{y_{n+1}})^{+1}y_{n+1} \\
&= (\overline{\widetilde{y_n}y_n})^{+1}\widetilde{y_n}y_n \\
&= (\overline{\widetilde{z_{n+1}}})^{+1}\widetilde{z_{n+1}} \\
&= (\overline{\widetilde{z_{n+1}}})^{+1}(\overline{z_{n+1}})^{+1} \\
&= (\overline{z_{n+1}\widetilde{z_{n+1}}})^{+1} \\
&= (\overline{z_{n+2}})^{+1} \\
&= \widetilde{z_{n+2}}.
\end{aligned}
$$

□

DEFINITION 5.12. Let $F_0' = 0, F_1' = 01, F_2' = z_0, F_3' = y_0$, and for all $k > 3$,

$$
F_k' = \begin{cases} \widetilde{y_j}y_j & j = \frac{k-5}{2}, k \text{ odd}, \\ z_j\widetilde{z_j} & j = \frac{k-4}{2}, k \text{ even}. \end{cases}
$$

PROPOSITION 5.13. *$F_n = F_n'$ for all $n \geq 0$.*

PROOF. To complete the proof, we show that the $\{F_n'\}$ satisfy the same recursive relations as the $\{F_n\}$ in Proposition 5.8. First, we show for $n \geq 2, F_{2n+1}' = F_{2n}'F_{2n-1}'$; we use Lemma 5.10.

$$
\begin{aligned}
F'_{2n+1} &= \widetilde{y_{n-2}}y_{n-2} \\
&= z_{n-2}\widetilde{z_{n-2}}\widetilde{y_{n-3}}y_{n-3} \\
&= F'_{2n}F'_{2n-1}.
\end{aligned}
$$

Next, we show for $n \geq 2, F'_{2n} = F'_{2n-2}F'_{2n-1}$; we use Lemma 5.11.

$$
\begin{aligned}
F'_{2n+1} &= z_{n-2}\widetilde{z_{n-2}} \\
&= z_{n-3}\widetilde{z_{n-3}}\widetilde{y_{n-3}}y_{n-3} \\
&= F'_{2n-2}F'_{2n-1}.
\end{aligned}
$$

$\square$

LEMMA 5.14. $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$.

PROOF.

$$
\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \tfrac{(n-1)!}{(k-1)!(n-k)!} + \tfrac{(n-1)!}{k!(n-1-k)!} \\
&= (n-1)!(\tfrac{k}{k!(n-k)!} + \tfrac{(n-k)}{k!(n-k)!}) \\
&= (n-1)!(\tfrac{n}{k!(n-k)!}) \\
&= \tfrac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}
$$

$\square$

THEOREM 5.15. *For $n \geq 2$,*
(i) F_{2n} can be tiled by $\binom{n}{0}$-many tiles of type n, $\binom{n}{1}$-many tiles of type $n-1, \ldots,$ and $\binom{n}{n}$-many tiles of type 0.
(ii) F_{2n+1} can be tiled by $\binom{n}{0}$-many tiles of type $n+1, \binom{n}{1}$-many tiles of type $n, \ldots,$ and $\binom{n}{n}$-many tiles of type 1.

PROOF. We first show by induction that (i) holds.

By induction, we have that F_{2n-2} can be tiled by $\binom{n-1}{0}$-many tiles of type $n-1$, $\binom{n-1}{1}$-many tiles of type $n-2, \ldots,$ and $\binom{n-1}{n-1}$-many tiles of type 0. We also have that F_{2n-1} can be tiled by $\binom{n-1}{0}$-many tiles of type n, $\binom{n-1}{1}$-many tiles of type $n-1, \ldots,$ and $\binom{n-1}{n-1}$-many tiles of type 1. Since $F_{2n} = F_{2n-2}F_{2n-1}$, we combine the number of tiles for F_{2n-2} and F_{2n-1}, making use of Lemma 5.14.

We have $\binom{n}{0}$-many tiles of type n, $\binom{n-1}{0}+\binom{n-1}{1}=\binom{n}{1}$-many tiles of type $n-1$, $\binom{n-1}{1}+\binom{n-1}{2}=\binom{n}{2}$-many tiles of type $n-2,\ldots,$ $\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}$-many tiles of type $n-k,\ldots,\binom{n-1}{n-2}+\binom{n-1}{n-1}=\binom{n}{n-1}$-many tiles of type 1, and $\binom{n}{n}$-many tiles of type 0, as needed.

We next show by induction that (ii) holds.

Since $F_{2n+1}=F_{2n}F_{2n-1}$, we begin by checking the tilings of F_{2n} and F_{2n-1}. By induction, F_{2n-1} can be tiled as $\binom{n-1}{0}$-many tiles of type S_n, $\binom{n-1}{1}$-many tiles of type $n-1,\ldots,$ and $\binom{n-1}{n-1}$-many tiles of type 1. By Proposition 5.13,

$$F_{2n}=z_{n-2}\widetilde{z_{n-2}}=\widetilde{y_{n-2}}=(\overline{y_{n-2}})^{+1}=(\overline{\widetilde{y_{n-3}}y_{n-3}})^{+1}.$$

But, $F_{2n-1}=\widetilde{y_{n-3}}y_{n-3}$. This implies that F_{2n} can be tiled with the same number of tiles as F_{2n-1} with the tile types increased one unit. Thus we have $\binom{n}{0}$-many tiles of type $n+1$, $\binom{n-1}{0}+\binom{n-1}{1}=\binom{n}{1}$-many tiles of type n, $\binom{n-1}{1}+\binom{n-1}{2}=\binom{n}{2}$-many tiles of type $n-1,\ldots,\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}$-many tiles of type $n-k-1$, $\binom{n-1}{k}+\binom{n-1}{k+1}=\binom{n}{k+1}$-many tiles of type $n-k,\ldots,\binom{n-1}{n-2}+\binom{n-1}{n-1}=\binom{n}{n-1}$-many tiles of type 2, and $\binom{n}{n}$-many tiles of type 1, as needed. $\square$

BIBLIOGRAPHY

[1] J.C. Alexander and D.B. Zagier, *The entropy of certain infinitely convolved Bernoulli measures*, J. London Math. Soc. 44 (1991), 121–134.

[2] F. Ardila, *The coefficients of a Fibonacci power series*, Fib. Quart. 42 (2003), 202–204.

[3] P. Arnoux and G. Rauzy, *Représentations géométrique de suites de complexité* $2n+1$, Bull. Soc. Math. France 119 (1991), 199–215.

[4] M. Baake, *A guide to mathematical quasicrystals, in Quasicrystals, an Introduction to Structure, Physical Properties, adn Applications*, Springer Series in Materials Science, Volume 55, pp. 17–46, Springer, Berlin, 2002.

[5] J. Bernoulli III, *Sur une nouvelle espece de calcul*, Recueil pour les Astronome 1 (1772), 255–284.

[6] J. Berstel, *An exercise on Fibonacci representations, a tribute to Aldo de Luca*, RAIRO, Theor. Inform. Appl. 35 (2001), 491–498.

[7] J. Berstel and P. Séébold, *Sturmian words, in algebraic combinatorics on words*, Lothaire Series, pp. 40–97, Cambridge University Press, Cambridge, UK, 2002.

[8] J. Bertel and J. Karhumaki, *Combinatorics on words-a tutorial*, Bull. Eur. Assoc. Theor. Comput. Sci. EATCS 79 (2003), 178–228.

[9] V. Berthé, *Discrete geometry and symbolic dynamics*, Preprint.

[10] ——, *Autour du systéme de numération d'Ostrowski, journées montoises d'informatique thérorique (marne-la-valée, 2000)*, Bull. Belg. Math. Soc. Simon Stevin 8 (2001), 209–239.

[11] J.E. Bressenham, *Algorithm for computer control of a digital plotter*, IBM Systems J. 4 (1965), 25–30.

[12] L. Carlitz, *Fibonacci representations*, Fib. Quart. 6 (1968), no. 4, 193–220.

[13] E.B. Christoffel, *Observatio arithmetica*, Ann. Mat. 6 (1875), 145–152.

[14] A. de Luca, *A combinatorial property of the Fibonacci words*, Inf. Proc. Lett. 12 (1981), no. 4, 193–195.

[15] ——, *Sturmian words: structure, combinatorics, and their arithmetics*, Theoret. Comp. Sci. 183 (1997), 45–82.

[16] X. Droubay and G. Pirillo, *Palindromes and Sturmian words*, Theoret. Comput. Sci. 223 (1999), 73–85.

[17] M. Edson and L.Q. Zamboni, *On the representation of numbers in the Fibonacci base*, Theoret. Comput. Sci. 326 (2004), 241–260.

[18] ______, *On the number of partitions of an integer in the m-bonacci base*, Ann. Inst. Fourier 56 (2006), 2271–2283.

[19] P. Erdős, M. Joo, and I. Joo, *On a problem of Tamas Varga*, Bull. Sox. Math. France 120 (1992), 507–521.

[20] H.H. Ferns, *On the representation of integers as sums of distinct Fibonacci numbers*, Fib. Quart. 1 (1963), 1–14.

[21] N.J. Fine and H.S. Wilf, *Uniqueness theorem for periodic functions*, Proc. Amer. Math. Soc. 16 (1965), 109–114.

[22] A.S. Fraenkel, *Systems of numeration*, Amer. Math. Monthly 92 (1985), 105–114.

[23] C. Frougny, *Representations of numbers and finite automata*, Math. Sys. Theory 25 (1992), 37–60.

[24] ______, *Numeration systems, in algebraic combinatorics on words*, Lothaire Series, pp. 203–236, Cambridge University Press, Cambridge, UK, 2002.

[25] P.J. Grabner, P. Kirschenhofer, and R.F. Tichy, *Combinatorial arithmetical properties of linear numeration systems*, Combinatorica 22 (2002), 245–267.

[26] P.J. Grabner, P. Liardet, and R.F. Tichy, *Odometers and systems of numeration*, Acta Arith. 70 (1995), 103–123.

[27] P.J. Grabner and R.F. Tichy, *Contibutions to digit expansions with respect to linear recurrences*, J. Number Th. 36 (1990), 160–169.

[28] P.J. Grabner and R.F. Tichy, *α-expansions, linear recurrences and the sum-of-digit function*, Manu. Math. 70 (1991), 311–324.

[29] D. Haylock, *The golden section in Beethoven's Fifth*, MT: Mathematics Teaching 84 (1978), 56–57.

[30] G.A. Hedlund and M. Morse, *Symbolic dynamics ii: Sturmian sequences*, Amer. J. Math. 61 (1940), 1–42.

[31] O. Jenkinson and L.Q. Zamboni, *Characterizations of balanced words via orderings*, Theoret. Comp. Sci. 130 (2004), 247–271.

[32] J. Justin, *Episturmian words and morphisms (results and conjectures)*, pp. 533–539, Springer Italia, Milan, 2001.

[33] J. Justin and G. Pirillo, *Episturmian words and episturmian morphisms*, Theoret. Comput. Sci. 302 (2003), 1–34.

[34] ———, *Episturmian words: shifts, morphisms and numeration systems*, Internat. J. Found. Comput. Sci 15 (2004), 329–348.

[35] J. Justin and L. Vuillon, *Return words in Sturmian and episturmian words*, RAIRO, Theor. Inform. Appl. 34 (2000), 343–356.

[36] D. Klarner, *Represenations of N as a sum or distinct elements from special sequences*, Fib. Quart. 4 (1966), 289–306.

[37] R. Knotts, *Fibonacci numbers and the golden section*, 2007, Online; accessed 22-March-2007.

[38] P. Kocábová, Z. Masácová, and E. Pelantová, *Ambiguity in the m-bonacci numeration system*, preprint.

[39] P. L'Ecuyer, *Uniform random number generation*, Ann. Oper. Res. 53 (1994), 77–120.

[40] C.G. Lekkerkerker, *Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci*, Simon Stevin 29 (1952), 190–195.

[41] A. Lindenmayer and P. Prusinkiewicz, *The algorithmic beauty of plants*, Springer-Verlag, New York, 1996.

[42] M. Lothaire, *Algebraic combinatorics on words*, Cambridge University Press, 2002.

[43] W.F. Lunnon and P.A.B. Pleasants, *Quasicrystallographic tilings*, J. Math. Pures Appl. 66 (1987), 217–263.

[44] A. Markoff, *Sur une question de Jean Bernoulli*, Math. Ann. 19 (1881), 27–36.

[45] A. Ostrowski, *Bemerkungen zur theorie der diophantischen approximationen*, Abh. Math. Sem. Hamburg 1 (1922), 77–98.

[46] M. Queffélec, *Substitution dynamical systems-spectral analysis*, Springer-Verlag, 1987, Volume 1294 of Lecture Notes Math.

[47] R. Risley and L.Q. Zamboni, *A generization of Sturmian sequences: combinatorial structure and transcendence*, Acta Arith. 95 (2000), no. 2, 167–184.

[48] J. Sakarovitch, *Eléments de théorie des automates*, Vuibert, Paris, 2003.

[49] J.I. Tamura, *Some properties of certain infinite products*, preprint.

[50] R. Tijdeman and L.Q. Zamboni, *Fine and Wilf words for any periods*, Indag. Math. 14 (2003), 135–147.

[51] L. Vuillon, *A characterisation of Sturmian words by return words*, European J. Combin. 22 (2001), 263–275.

[52] Z.X. Wen and Z.Y. Wen, *Some properties of the singular words of the Fibonacci word*, European J. Combin. 15 (1994), no. 6, 587–598.

[53] Wikipedia, *Fibonacci number, in Wikipedia, The Free Encyclopedia*, 2007, Online; accessed 22-March-2007.

[54] E. Zeckendorff, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Royale Sci. Liège 42 (1972), 179–182.