

BOREL DETERMINACY AND METAMATHEMATICS

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Thesis Prepared for the Degree of
MASTER OF ARTS

UNIVERSITY OF NORTH TEXAS

December 2001

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Bryant, Ross, Borel Determinacy and Metamathematics. Master of Arts (Mathematics), December 2001, 99 pp., bibliography, 40 titles.

Borel determinacy states that if $G(T, X)$ is a game and X is Borel, then $G(T, X)$ is determined. Proved by Martin in 1975, Borel determinacy is a theorem of ZFC set theory, and is, in fact, the best determinacy result in ZFC. However, the proof uses sets of high set theoretic type (\aleph_1 many power sets of ω). Friedman proved in 1971 that these sets are necessary by showing that the Axiom of Replacement is necessary for any proof of Borel Determinacy. To prove this, Friedman produces a model of ZC and a Borel set of Turing degrees that neither contains nor omits a cone; so by another theorem of Martin, Borel Determinacy is not a theorem of ZC. This paper contains three main sections: Martin's proof of Borel Determinacy; a simpler example of Friedman's result, namely, (in ZFC) a coanalytic set of Turing degrees that neither contains nor omits a cone; and finally, the Friedman result.

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ACKNOWLEDGEMENTS

The author gratefully thanks Dr. Steve Jackson for the many long hours of meetings and without whom this would have been impossible. Also, the author thanks Dr. Dan Mauldin for his help with the historical portions of the document.

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CHAPTER 1

Introduction

When he first listed his axioms for set theory [Zer08], Zermelo omitted the axiom which states that the image of a set under any function is a set, the Axiom of Replacement. Zermelo's axioms of Infinity and Power set suffice to prove the existence of $\omega, \mathcal{P}(\omega), \mathcal{P}(\mathcal{P}(\omega)), \dots$, but the existence $\{\omega, \mathcal{P}(\omega), \mathcal{P}(\mathcal{P}(\omega)), \dots\}$ necessitates Replacement. Fraenkel [Fra21, Fra22] and Skolem [Sko23] later added Replacement thus eliminating this pathology. This revised list of axioms has been dubbed Zermelo-Fraenkel set theory (ZF) and is widely accepted as the standard axioms for set theory.

For most mathematicians, the inability of Zermelo's set theory (Z) to form the above set causes little concern because the overwhelming majority of mathematical objects can be found within a finite number of iterations of the power set operation. Yet for set theorists, Replacement is vital. Von Neumann [vN23] used Replacement to define the ordinal numbers which became the backbone for the higher type sets (i.e., sets formed by the unrestricted use of the Power Set operator). However, the full usage of Replacement remained latent; no proof of a theorem requiring higher type sets existed. Coincidentally, Zermelo's interest in games was the genesis of such a theorem.

Besides his innovations in set theory, Zermelo also introduced the modern mathematical investigation of finite games [Zer12] and showed that chess is determined. That is, either both players have drawing strategies or one player has a winning strategy. Borel [Bor21, Bor24, Bor27], von Neumann [vN28] and Steinhaus [Ste25] continued the analysis of finite games which culminated in von Neumann's and Morgenstern's monograph [vNM44]. Meanwhile, adumbrating the notion of infinite games, Sierpiński used game concepts such as strategy and play to prove that every uncountable Borel subset of \mathbb{R} contains a copy of the Cantor space [Sie24]. But, according to Ulam [Ula60] and Oxtoby [Oxt71], Mazur pioneered the application of infinite games to real analysis.

In 1928, while considering a problem relating to the Baire Category Theorem, Mazur imagined a game in which both players made ω many moves. He later described this to Banach during a conversation at the Scottish Cafe. The infinite game, along with Banach's reference to a solution dated 4 August 1935, became known as the Banach-Mazur game (number 43 in the Scottish Book [Mau81]). Subsequent comments to problem 43 contain a variation suggested by Ulam: A subset X of the unit interval is chosen and two players alternate playing 0 or 1. Player I wins if, and only if $\sum_{n=0}^{\infty} \frac{x_n}{2^n} \in X$ and Player II wins otherwise. Ulam then asked, for which X does Player I (or II) have a winning strategy? This question would remain unanswered for nearly twenty years.

Gale and Stewart [GS53], and independently Mycielski and Zieba [MZ55], partially answered the question by proving that on the one hand, open and closed games are determined but on the other, a nondetermined game can be constructed using the Axiom of Choice (AC). The search for a more robust answer to Ulam's question prompted the following two refining questions:

1. Is determinacy invariant under unions and intersections?
2. For what classes of sets are games determined? G_δ ? F_σ ? Borel? Analytic?

G_δ and F_σ determinacy was proven by Wolfe [Wol55], and independently by Mycielski, Świerczkowski, and Zieba [MŚZ56]. Davis proved $F_{\sigma\delta}$ and $G_{\delta\sigma}$ determinacy [Dav64]. Moreover, he proved that determinacy is not invariant under union and intersection, and thus eliminated the feasibility of a direct proof of Borel Determinacy. Even though Davis' proof was far from full Borel Determinacy, it remained the best result for almost ten more years. During this time, determinacy hypotheses gained popularity.

Mycielski and Steinhaus first formulated [MS62] the hypothesis that all subsets of reals are determined, the Axiom of Determinacy (AD). In light of the aforementioned non-determined game and its heavy dependence on AC, AD blatantly contradicted AC. Coupled with the paucity of determinacy proofs then available, AD seemed to be a dubious proposition. What prompted Mycielski to propose AD was its consequences on subsets of \mathbb{R} , collected in [Myc64]. These are: (1) every subset of \mathbb{R} is Lebesgue

measurable [MŚ64]; (2) every subset of the Cantor space has the Baire property (attributed to Banach and proven in [Oxt71], p. 27, using the above Banach-Mazur game); (3) every uncountable subset of the Cantor space has a perfect subset [Dav64]; and, (4) the Axiom of Countable Choice (CC) [Myc64]. These regularity properties for subsets of reals precipitated a new program to discover, via consistency results and large cardinal hypotheses, the full potential of AD. In 1970 Martin [Mar70] proved that if a measurable cardinal exists, then all analytic games are determined. Therefore, because every Borel set is analytic, Borel determinacy follows. But the existence of measurable cardinals is not provable in ZF with AC (ZFC). Consequently, a ZFC proof of Borel Determinacy remained elusive.

During this period of investigation into the relationship between large cardinals and AD, an unexpected development occurred. Without a proof of Borel Determinacy, Friedman [Fri71] proved that ZC ($Z + AC$) was insufficient to prove Borel Determinacy. Thus, a proof of Borel Determinacy, a simple statement about sets of reals numbers, would require uncountably many iterations of power sets of integers and would become the first known theorem realizing the full potency of the Axiom of Replacement. Following this development, Paris [Par72] proved that Σ_4^0 games are determined and concluded that no further determinacy progress was possible using the methods of analysis.

Fifty years after Replacement's introduction as an axiom of set theory, Martin proved Borel Determinacy [Mar75]. Refinements of the proof in [Mar85] revealed a purely inductive proof and the full use of Replacement in the notion of a covering. A covering reduces the Borel game to a closed (and hence determined) game; the size of the covering needed to reduce the game on a Borel set of rank $\alpha < \omega_1$ was roughly the size of $\mathcal{P}^\alpha(\omega)$.

This paper contains Martin's proof of Borel Determinacy and Friedman's related metamathematical result, and is accordingly divided into two parts. The first part sets forth preliminary definitions and notations and concludes with the proof for Borel Determinacy. The second part constructs in ZFC a non-determined coanalytic game, mimicking the Friedman result (a nondetermined Borel game in ZC) which follows.

Part I

Borel Determinacy

CHAPTER 2

Preliminaries

In this chapter we establish the definitions, notations, and basic results necessary for our reproduction of Martin's inductive proof of Borel Determinacy in Chapter 3. As much of this material is standard, we suppress proofs, referring the reader to [Mos80] and [Kec95] for details.

2.1 Polish spaces

Though many of the results of classical descriptive set theory pertain to \mathbb{R} , they also work in a more general context. Let (X, \mathcal{T}) be a topological space. (X, \mathcal{T}) is **separable** if X contains a countable dense subset. If ρ is a metric on (X, \mathcal{T}) , then (X, \mathcal{T}) is **complete** if every ρ -Cauchy sequence converges in X . (X, \mathcal{T}) is **Polish** if it is separable and completely metrizable. Thus, \mathbb{R} , \mathbb{R}^n , $\mathbb{R}^{\mathbb{N}}$, \mathbb{C} , \mathbb{C}^n , $\mathbb{C}^{\mathbb{N}}$, \mathbb{I} (the unit interval $[0, 1]$), \mathbb{I}^n (the n -dimensional cube), $\mathbb{I}^{\mathbb{N}}$ (the Hilbert cube), \mathbb{T} (the unit circle), \mathbb{T}^n (the n -dimensional torus), $\mathbb{T}^{\mathbb{N}}$ (the infinite dimensional torus) are all Polish.

Two Polish spaces of particular interest are the Baire space and the Cantor space. Given sets A, B , B^A denotes the set of all functions $f : A \rightarrow B$. Let $\omega = \{0, 1, 2, \dots\}$. With ω having the discrete topology, give ω^ω the product topology to form the **Baire space**, \mathcal{N} . The **Cantor space** \mathcal{C} or 2^ω is constructed similarly. It is a theorem that \mathcal{N} is homeomorphic to the irrationals as a subspace of \mathbb{R} . The following proposition shows that either \mathcal{N} or \mathcal{C} can be found in any Polish space and thus justifies the use of \mathcal{N} and \mathcal{C} as the canonical Polish spaces,

Proposition 2.1.1. *For any Polish space X ,*

1. *there is a continuous surjection $f : \mathcal{N} \rightarrow X$*
2. *if X has no isolated points, then there is a continuous injection $f : \mathcal{C} \rightarrow X$*

Products of Polish spaces can be avoided when convenient by the next proposition and consequently dimension plays virtually no role in descriptive set theory.

Proposition 2.1.2. \mathcal{N} is homeomorphic to \mathcal{N}^n , for all $n \in \omega$ and \mathcal{N}^ω .

We shall work almost exclusively with the Baire space throughout this paper. Subsets of \mathcal{N} are called **pointsets**; sets of subsets of \mathcal{N} are called **pointclasses**. Given a pointclass Γ , the **dual** pointclass is given by $\check{\Gamma} = \{X^c : X \in \Gamma\}$ where X^c denotes the **complement** of X .

2.2 Borel sets

Suppose $X \neq \emptyset$ and let $\mathcal{P}(X)$ denote the power set of X . $\mathcal{A} \subset \mathcal{P}(X)$ is a σ -**algebra** of sets if $X \in \mathcal{A}$, and \mathcal{A} is closed under countable unions, countable intersections, and complements. For (X, \mathcal{T}) Polish, $Y \subset X$ is **Borel** if Y is contained in the smallest σ -algebra of X containing \mathcal{T} . The Borel sets of X , denoted $\mathbf{B}(X)$, can be put in a hierarchy as follows. Let ω_1 be the first uncountable ordinal and define by transfinite recursion for $1 \leq \alpha < \omega_1$ the following pointclasses:

$$\begin{aligned} Y \in \Sigma_1^0(X) &\Leftrightarrow Y \in \mathcal{T}, \\ Y \in \Sigma_\alpha^0(X) &\Leftrightarrow Y = \bigcup_n A_n \text{ such that } A_n \in \Pi_{\alpha_n}^0(X), \text{ each } \alpha_n < \alpha \\ Y \in \Pi_\alpha^0(X) &\Leftrightarrow Y^c \in \Sigma_\alpha^0(X) \\ Y \in \Delta_\alpha^0(X) &\Leftrightarrow Y \in \Sigma_\alpha^0(X) \cap \Pi_\alpha^0(X) \end{aligned}$$

It is a theorem that $\mathbf{B}(X) = \bigcup_{\xi < \omega_1} \Sigma_\xi^0(X) = \bigcup_{\xi < \omega_1} \Pi_\xi^0(X) = \bigcup_{\xi < \omega_1} \Delta_\xi^0(X)$. Moreover, the following diagram illustrates this hierarchy (we suppress the X for readability)

$$\begin{array}{ccccccc} & \Sigma_1^0 & & \Sigma_2^0 & & \Sigma_\beta^0 & & \Sigma_\alpha^0 \\ \Delta_1^0 & & \Delta_2^0 & & \dots & \Delta_\beta^0 & & \dots & \Delta_\alpha^0 & & \dots \\ & \Pi_1^0 & & \Pi_2^0 & & \Pi_\beta^0 & & \Pi_\alpha^0 \end{array}$$

⏟
B

where $\beta \leq \alpha$ and any class is contained in every class to the right of it. These containments are proper for uncountable X and thus, all of these classes are distinct.

In the classical notation, Σ_2^0 is the class of F_σ sets, Π_2^0 the G_δ sets, Σ_3^0 the $G_{\delta\sigma}$ sets, Π_3^0 the $F_{\sigma\delta}$ sets, etc. Closure properties for these pointclasses are summarized in the next proposition.

Proposition 2.2.1. *For each $\alpha \geq 1$, the pointclasses Σ_α^0 , Π_α^0 , and Δ_α^0 are closed under finite intersections, finite unions, and continuous preimages. Moreover,*

1. Σ_α^0 is closed under countable unions,
2. Π_α^0 is closed under countable intersections, and
3. Δ_α^0 is closed under complements.

Thus, the Borel sets are closed under the Boolean operations, countable unions, countable intersections, and continuous preimages. The continuous image of a Borel set, however, need not be Borel. The hierarchy of projective sets extends the Borel hierarchy to accommodate this.

2.3 Projective sets

For any Polish space X , a set $A \subseteq X$ is **analytic** if there is a continuous function $f : \mathcal{N} \rightarrow X$ such that $A = f(\mathcal{N})$. The complement of an analytic is **coanalytic**. Define recursively for each $n \in \omega$, the following pointclasses:

$$\begin{aligned} Y \in \Sigma_1^1(X) &\Leftrightarrow Y \text{ is analytic} \\ Y \in \Sigma_{n+1}^1(X) &\Leftrightarrow Y \text{ is the continuous image of a } \Pi_n^1 \text{ set} \\ Y \in \Pi_n^1(X) &\Leftrightarrow Y^c \in \Sigma_n^1(X) \\ Y \in \Delta_n^1(X) &\Leftrightarrow Y \in \Sigma_n^1(X) \cap \Pi_n^1(X) \end{aligned}$$

Define the **projective sets** of X $\mathbf{P}(X) = \bigcup_{n=1}^{\infty} \Delta_n^1$. A famous result of Souslin relates the projective hierarchy to the Borel hierarchy.

Theorem 2.3.1 (Souslin). *For any Polish space X , $\mathbf{B}(X) = \Delta_1^1$.*

Thus, we have the following diagram

$$\begin{array}{ccccccc}
 & & \Sigma_1^1 & & \Sigma_2^1 & & \Sigma_n^1 \\
 \mathbf{B} = \Delta_1^1 & & & \Delta_2^1 & & \dots & \Delta_n^1 & & \dots \\
 & & \Pi_1^1 & & \Pi_2^1 & & & & \Pi_n^1
 \end{array}$$

where each class is contained in every class to the right of it; it is a theorem that each of these inclusions is proper.

2.4 Sequences and Trees

Let $A \neq \emptyset$. Given $s \in A^n$, we consider s as a finite sequence from A of length n and write $s = \langle s_0, \dots, s_{n-1} \rangle$. In the case $n = 0$, $A^0 = \{\emptyset\}$, where \emptyset here denotes the empty sequence. We indicate the **length** of a finite sequence s by $\text{length}(s)$. Given $s \in A^n$ and $m \leq n$, $s \upharpoonright m = \langle s_0, \dots, s_{m-1} \rangle$, the **restriction** of s to m . Given finite sequences s, t from A , s is an **initial segment** of t (equivalently, t is an **extension** of s) if $s = t \upharpoonright m$, for some $m \leq \text{length}(t)$. We write $s \subseteq t$ to denote that s is an initial segment of t . Two finite sequences are **compatible** if one is an initial segment of the other. If s and t are incompatible, we write $s \perp t$. Given two finite sequences s, t we write $s \hat{\ } t$ to denote the **concatenation** of s and t .

Given $x \in A^\omega$ and $n \in \omega$, $x \upharpoonright n = \langle x_0, \dots, x_{n-1} \rangle$, the **restriction** of x to n . Given $x \in A^\omega$, we say that $s \in A^n$ is an **initial segment** of $x \in A^\omega$ if $s = x \upharpoonright n$. We write $s \subseteq x$ to denote that s is an initial segment of x . Note the difference in notation for finite versus infinite sequences: (a_n) denotes the infinite sequence $\{a_n\}_{n \in \omega}$ whereas $\langle a_n \rangle$ is the length 1 sequence whose only member is a_n .

A^ω can be viewed as the product topology of infinitely many copies of A , each having the discrete topology. The standard basis for this topology on A^ω consists of the sets $N_s = \{x \in A^\omega : s \subseteq x\}$, where $s \in A^{<\omega}$. Note that $s \subseteq t \Leftrightarrow N_s \supseteq N_t$ and $s \perp t \Leftrightarrow N_s \cap N_t = \emptyset$.

For a nonempty set A , let $A^{<\omega} = \bigcup_n A^n$, the set of all finite sequences from A . A **tree** on a set A is a subset $T \subseteq A^{<\omega}$ closed under initial segments. Each $s \in T$ is called

a **node** of T . A tree T **pruned** if every $s \in T$ has a proper extension $t \supsetneq s$, $t \in T$. A **branch** of T is a sequence $x \in A^\omega$ such that $x \upharpoonright n \in T$, for all n . The set of all branches of T is written as $[T] = \{x \in A^\omega : \forall n(x \upharpoonright n \in T)\}$. The primacy of pruned trees in descriptive set theory follows from the following foundational proposition.

Proposition 2.4.1. *Let $A \neq \emptyset$ and $T \subseteq A^{<\omega}$ a nonempty pruned tree. Then $[T] \subseteq A^\omega$ is closed.*

Proof. Let A, T be as above and let $x \notin [T]$. Then there is $n \in \omega$ such that $x \upharpoonright n \notin T$. So $x \in N_{x \upharpoonright n} \subset [T]^c$. Hence, $[T]^c$ is open, and $[T]$ is therefore closed. \square

It is a theorem that there is a bijection between pruned trees and closed sets. Given a closed set F , T_F denotes the tree of F .

Finally, for trees S, T (on sets A, B , resp.), a map $\varphi : S \rightarrow T$ is called **monotone** if $s \subseteq t$ implies $\varphi(s) \subseteq \varphi(t)$. For a tree T on A and any $s \in A^{<\omega}$, define $T_s = \{t \in A^{<\omega} : s \hat{\ } t \in T\}$, the **subtree of T at s** ; for $X \subseteq A^\omega$ define $X_s = \{x \in X : s \subseteq x\}$.

2.5 Infinite games of perfect information

An infinite two-player game of perfect information with rules, or simply **game**, is a contest between two players, I and II, played using two sets, A and X , and a prescribed set of legal moves, or **rules**. The nonempty set A specifies those objects used to play the game, while $X \subseteq A^\omega$ determines the winner of the game. A **run** of the game begins with Player I making a **play** by choosing $a_0 \in A$ in compliance with the rules as his first move. Player II makes her play by choosing $a_1 \in A$ in compliance with the rules. (We will maintain this convention of referring to Player I as masculine and Player II as feminine to improve readability.) Player I then chooses $a_2 \in A$ as his second play. Player II responds, and so on. At any time during a run of the game, each player is able to see all, including the opponent's, previous moves. (Thus it is a game of **perfect information**.) Play alternates along these lines for infinitely many (ω many) moves. The winner of a particular run $(a_n) \in A^\omega$ of the game is determined by the **payoff set** $X \subseteq A^\omega$. Player I wins if, and only if $a \in X$, otherwise Player II wins.

More formally, let A be a nonempty set, $X \subseteq A^\omega$ be the payoff set, and $T \subseteq A^{<\omega}$ a nonempty pruned tree. Insisting that each player's moves must occur in T , T serves as the rules of the game. We denote a game on a set A with rules T and payoff set X by $G(T, X)$, or if T is understood, we will write $G(X)$. A run $(a_n) \in A^\omega$ of the game $G(T, X)$ (illustrated below) begins with Player I playing $a_0 \in A$ such that $\langle a_0 \rangle \in T$, followed by Player II playing $a_1 \in A$ such that $\langle a_0, a_1 \rangle \in T$, followed by I playing $a_2 \in A$ such that $\langle a_0, a_1, a_2 \rangle \in T$, etc.

I	a_0	a_2	\dots
II	a_1	a_3	

I wins if, and only if $(a_n) \in X$. II wins if, and only if $(a_n) \in X^c$.

A **strategy** for a player in a game is a way of determining the player's next move from the previous moves. We view a strategy for I in the game $G(T, X)$ as a nonempty, pruned subtree $\sigma \subseteq T$ such that

1. if $\langle a_0, a_1, \dots, a_{2j} \rangle \in \sigma$, then for all $\langle a_0, a_1, \dots, a_{2j+1} \rangle \in T$, $\langle a_0, \dots, a_{2j}, a_{2j+1} \rangle \in \sigma$;
2. if $\langle a_0, a_1, \dots, a_{2j-1} \rangle \in \sigma$, then there is a *unique* $\langle a_0, a_1, \dots, a_{2j} \rangle \in T$ such that $\langle a_0, \dots, a_{2j-1}, a_{2j} \rangle \in \sigma$.

We denote the set of all strategies from a tree T by $S(T)$. We say that I **follows** a strategy σ if I begins by playing the unique $a_0 \in A$ such that $\langle a_0 \rangle \in \sigma$, then, regardless of II's legal response $a_1 \in A$, I plays the unique $a_2 \in A$ according to $\langle a_0, a_1, a_2 \rangle \in \sigma$, and so on. A strategy for player II is defined *mutatis mutandis*.

A strategy σ for I is **winning** if $[\sigma] \subseteq X$; that is, if every run (a_n) of the game $G(T, X)$ in which I follows σ results in $(a_n) \in X$. A winning strategy for II is defined similarly. Clearly, I and II cannot simultaneously have winning strategies in the same game $G(T, X)$. We say that a game $G(T, X)$ is **determined** if one of the players has a winning strategy. For a pointclass Γ , $Det(\Gamma)$ denotes that for every $A \in \Gamma$, $G(A)$ is determined.

[Tel87] and [Myc92] survey games and their history.

CHAPTER 3

$$ZFC \vdash \text{Det}(\mathbf{\Delta}_1^1)$$

This chapter contains Martin's proof by induction of Borel Determinacy. We present Gale and Stewart's result, the determinacy of open and closed games in the first section, followed by Martin's proof in the section.

3.1 The Gale-Stewart Theorem

Given a game $G(T, X)$ and $p \in T$, the **subgame of X at p** is $G(T_p, X_p)$ where T_p, X_p are as in Section 2.4. If during a run of the game $G(T, X)$ a position $p \in T$ is reached with I to play next and such that II has no winning strategy in the game $G(T_p, X_p)$, then we say that p is **not losing** for I. We define not losing for II mutatis mutandis. The following lemma contains the germ of the proof of the Gale-Stewart theorem.

Lemma 3.1.1. *Let $G(T, X)$ be a game on a set A . If p is not losing for I, then there is $a \in A$ such that for all $b \in A$, $p \hat{a} \hat{b}$ is not losing for I.*

Proof. Suppose not. Let $p = \langle a_0, a_1, \dots, a_{2j-1} \rangle \in T$ be not losing for I, and suppose that for each $a \in A$ such that $p \hat{a} \in T$, there is $b \in A$ such that $p' = p \hat{a} \hat{b} \in T$ and II has a winning strategy in the game $G(T_{p'}, X_{p'})$. So, for each $a \in A$, choose $b_a \in A$ and a winning strategy σ_a for II in the game $G(T_{p'}, X_{p'})$ where $p' = p \hat{a} \hat{b}_a$. Hence, II now has a winning strategy in $G(T_p, X_p)$ defined as follows: if I plays $a_{2j} \in A$, II responds by playing $b_{a_{2j}} \in A$ and then follows $\sigma_{a_{2j}}$ to win. Thus, p is losing for I, a contradiction. \square

Theorem 3.1.2 (Gale-Stewart). *Let T be a non-empty pruned tree on A . Let $X \subseteq [T]$ be open or closed in $[T]$. Then $G(T, X)$ is determined.*

Proof. Assume X is closed and that II has no winning strategy in $G(T, X)$. We construct a strategy σ for I as follows: \emptyset is not losing for I since II has no winning strategy in $G(T, X)$. By Lemma 3.1.1 there is $a_0 \in A$ such that for all $a_1 \in A$ such that

$\langle a_0, a_1 \rangle \in T$, $\langle a_0, a_1 \rangle$ is not losing for I. Since $\langle a_0, a_1 \rangle$ is not losing for I, there is $a_2 \in A$ such that for all $a_3 \in A$ such that $\langle a_0, a_1, a_2, a_3 \rangle \in T$, $\langle a_0, a_1, a_2, a_3 \rangle$ is not losing for I, again by Lemma 3.1.1. In general, given $p = \langle a_0, a_1, \dots, a_{2n-1} \rangle \in T$ which is not losing for I, choose $a_{2n} \in A$ so that for any $a_{2n+1} \in A$ such that $p \hat{\ } \langle a_{2n}, a_{2n+1} \rangle \in T$, $p \hat{\ } \langle a_{2n}, a_{2n+1} \rangle$ is not losing for I. It is clear that a subtree $\sigma \subset T$ formed in this fashion is a strategy for I. We claim this strategy is winning for I.

Consider a run of the game (a_n) in which I followed σ so that every position of even length is not losing for I and suppose that $(a_n) \notin X$. Then as X^c is open, there is k such that $N_{\langle a_0, a_1, \dots, a_{2k-1} \rangle} \cap [T] \subseteq X^c$ and hence, $\langle a_0, a_1, \dots, a_{2k-1} \rangle$ is losing for I as II can win by playing arbitrarily for the rest of the game. Therefore, σ is a winning strategy for I.

In the case that X is open, assume that I has no winning strategy in $G(T, X)$; use the above argument to construct a winning strategy for II. \square

3.2 An Inductive Proof of Borel Determinacy

Martin's proof of Borel determinacy associates to each Borel game $G(T, X)$ an auxiliary closed or open game $G(\tilde{T}, \tilde{X})$ in such a way that a winning strategy in $G(\tilde{T}, \tilde{X})$ gives rise to a winning strategy in $G(T, X)$. As the auxiliary game is determined by the Gale-Stewart Theorem, so is the Borel game. The notion of a covering makes this association rigorous.

Let T be a nonempty pruned tree on a set A . Recall that $S(T)$ denotes the set of all strategies from T . A **covering** of T is a triple $(\tilde{T}, \pi, \varphi)$ such that

1. \tilde{T} is a nonempty pruned tree (on some \tilde{A});
2. $\pi : [\tilde{T}] \rightarrow [T]$ is continuous;
3. $\varphi : S(\tilde{T}) \rightarrow S(T)$ maps strategies for player I (resp. II) in \tilde{T} to strategies for player I (resp. II) in T , in such a way that $\varphi(\tilde{\sigma})$ restricted to positions of length $\leq n$ depends only on $\tilde{\sigma}$ restricted to positions of length $\leq n$, for all n .
4. If $\tilde{\sigma}$ is a strategy for I (resp. II) in \tilde{T} and $x \in [T]$ such that $x \in \varphi(\tilde{\sigma})$ then there is $\tilde{x} \in [\tilde{T}]$ such that $\tilde{x} \in [\tilde{\sigma}]$ and $\pi(\tilde{x}) = x$.

Note that the map π naturally arises from a monotone map $\pi' : \tilde{T} \rightarrow T$ such that $\text{length}(\pi'(s)) = \text{length}(s)$. Condition three is stated informally for simplicity. Formally, φ is a monotone map on partial strategies $\tilde{\sigma} \upharpoonright n$ and $\varphi(\tilde{\sigma})$ is defined by $\varphi(\tilde{\sigma}) \upharpoonright n = \varphi(\tilde{\sigma} \upharpoonright n)$.

A covering (\tilde{T}, π, σ) of T **unravels** a set $X \subseteq [T]$ if $\pi^{-1}(X)$ is clopen. The next proposition follows immediately from the Gale-Stewart Theorem.

Proposition 3.2.1. *Let T be a nonempty pruned tree on a set A and $X \subseteq [T]$. If (\tilde{T}, π, σ) is a covering of $G(T, X)$ such that (\tilde{T}, π, σ) unravels X , then $G(T, X)$ is determined.*

If $(\tilde{T}, \pi, \varphi)$ is a covering such that for $k \in \omega$, $T \upharpoonright 2k = \tilde{T} \upharpoonright 2k$ and $\pi \upharpoonright (\tilde{T} \upharpoonright 2k)$ is the identity, then $(\tilde{T}, \pi, \varphi)$ is a **k -covering**.

Lemma 3.2.2. *Let T be a nonempty pruned tree and $X \subseteq [T]$. If $(\tilde{T}, \pi, \varphi)$ is a k -covering that unravels X , then $(\tilde{T}, \pi, \varphi)$ also unravels X^c .*

Proof. If T, X , and $(\tilde{T}, \pi, \varphi)$ are as above, then $\pi^{-1}(X^c) = (\pi^{-1}(X))^c$ is also clopen as π is continuous. \square

Relaxing the uniqueness condition in the definition of a strategy yields the notion of a quasistrategy. A **quasistrategy** for I in $G(T, X)$ is a nonempty, pruned subtree $\Sigma \subseteq T$ such that $\langle a_0, a_1, \dots, a_{2j} \rangle \in \Sigma$ and $\langle a_0, a_1, \dots, a_{2j+1} \rangle \in T$ implies $\langle a_0, \dots, a_{2j}, a_{2j+1} \rangle \in \Sigma$. Since Σ is pruned, if $\langle a_0, a_1, \dots, a_{2j-1} \rangle \in \Sigma$, then there is a $a_{2j} \in A$ such that $\langle a_0, a_1, \dots, a_{2j-1}, a_{2j} \rangle \in \Sigma$.

By modifying our definition of not losing, we can isolate the quasistrategy constructed in the Gale-Stewart Theorem. Let $p \in T$ be of arbitrary length. We say that p is not losing for I if II has no winning strategy in the game $G(T_p, X_p)$. So if $p = \langle a_0, a_1, \dots, a_{2n} \rangle$, then $G(T_p, X_p)$ is the subgame at p in which II plays first. Let Σ be the quasistrategy for I given by $\Sigma = \{p \in T : p \text{ is not losing for I}\}$. This special quasistrategy we call the **canonical quasistrategy** for I in $G(T, X)$. Define the canonical quasistrategy for II mutatis mutandis.

The next lemma is the heart of the inductive proof of Borel Determinacy and constitutes the bulk of this section.

Lemma 3.2.3. *Let T be a nonempty pruned tree. For every $X \in \Pi_1^0([T])$ and for each $k \in \omega$ there is a k -covering of T that unravels X .*

Proof. Let T be a nonempty pruned tree, $X \subseteq [T]$ closed, and $k \in \omega$. Let T_X be the tree of the closed set X . So $G(T, X)$ is of the form

$$\begin{array}{ccccccc} \text{I} & & x_0 & & & x_2 & & & \dots \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \text{II} & & & & x_1 & & & x_3 & \end{array}$$

where for all I, $\langle x_0, \dots, x_i \rangle \in T$. I wins if, and only if $(x_n) \in X$. The k -covering $(\tilde{T}, \pi, \varphi)$ that we will define is an auxiliary game in which players I and II play according to a run of $G(T, X)$ except at the k^{th} turn (moves $2k$ and $2k + 1$) where they play their usual moves along with some additional objects which simplify the game $G(T, X)$. The moves described below define \tilde{T} .

In \tilde{T} both players play as in T for the first $k - 1$ turns:

$$\begin{array}{ccccccc} \text{I} & & x_0 & & & & x_{2k-2} \\ & & & & & & \\ & & & & & \dots & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ \text{II} & & & & x_1 & & & & x_{2k-1} \end{array}$$

In his next move in \tilde{T} , I plays $\langle x_{2k}, \Sigma_I \rangle$ where $\langle x_0, \dots, x_{2k} \rangle \in T$ and Σ_I is a quasistrategy for I in $T_{\langle x_0, \dots, x_{2k} \rangle}$. By offering this quasistrategy, I obliges that he will play according to Σ_I for the duration of $G(T, X)$. So we have

$$\begin{array}{ccccccc} \text{I} & & x_0 & & & x_{2k-2} & & & \langle x_{2k}, \Sigma_I \rangle \\ & & & & & & & & \\ & & & & & & & \dots & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \text{II} & & & & x_1 & & & & x_{2k-1} \end{array}$$

II responds with x_{2k+1} and either accepting or rejecting I's offer so that the game

thus far is

$$\begin{array}{cccc}
\text{I} & x_0 & & x_{2k-2} & & \langle x_{2k}, \Sigma_I \rangle \\
& & & \dots & & \\
\text{II} & & x_1 & & x_{2k-1} & \langle x_{2k+1}, \bullet \rangle
\end{array}$$

II accepts: In this case, II responds by playing $\langle x_{2k+1}, u \rangle$ where $\langle x_0, \dots, x_{2k+1} \rangle \in T$ and u is an even length sequence such that $u \in T_{\langle x_0, \dots, x_{2k+1} \rangle}$ and $u \in (\Sigma_I)_{\langle 2k+1 \rangle} \setminus (T_X)_{\langle x_0, \dots, x_{2k+1} \rangle}$. Both players continue playing $x_{2k+2}, x_{2k+3}, \dots$ so that all moves are in T and compatible with u .

II rejects: In this case, II responds by playing $\langle x_{2k+1}, \Sigma_{II} \rangle$ where $\langle x_0, \dots, x_{2k+1} \rangle \in T$ and Σ_{II} is a quasistrategy for II in $(\Sigma_I)_{\langle x_{2k+1} \rangle}$ with $\Sigma_{II} \subseteq (T_X)_{\langle x_0, \dots, x_{2k+1} \rangle}$. In this case, both players continue playing $x_{2k+2}, x_{2k+3}, \dots$ so that $\langle x_{2k+2}, x_{2k+3}, \dots, x_l \rangle \in \Sigma_{II}$, for all $l \geq 2k+2$.

Formally, \tilde{T} is the set of all finite sequences of the form:

$$\langle x_0, \dots, x_{2k-1}, \langle x_{2k}, \Sigma_I \rangle, \langle x_{2k+1}, \bullet \rangle, x_{2k+2}, \dots, x_l \rangle$$

such that

1. $\langle x_0, \dots, x_i \rangle \in T$ for all $i \leq l$,
2. Σ_I is a quasistrategy for I in $T_{\langle x_0, \dots, x_{2k} \rangle}$,
3. $\bullet = \langle 1, u \rangle$ where u is of even length, $u \in T_{\langle x_0, \dots, x_{2k+1} \rangle}$, $u \in (\Sigma_I)_{\langle 2k+1 \rangle} \setminus (T_X)_{\langle x_0, \dots, x_{2k+1} \rangle}$ or $\bullet = \langle 2, \Sigma_{II} \rangle$ where Σ_{II} is a quasistrategy for II in $(\Sigma_I)_{\langle x_{2k+1} \rangle}$ with $\Sigma_{II} \subseteq (T_X)_{\langle x_0, \dots, x_{2k+1} \rangle}$.

Since both players have always have legal moves at each turn, it is clear that \tilde{T} is pruned. Moreover, $T \neq \emptyset$ implies $\tilde{T} \neq \emptyset$.

The map $\pi : \tilde{T} \rightarrow T$ is given in the obvious way:

$$\pi(\langle x_0, \dots, x_{2k-1}, \langle x_{2k}, \Sigma_I \rangle, \langle x_{2k+1}, \bullet \rangle, x_{2k+2}, \dots, x_l \rangle) = \langle x_0, \dots, x_l \rangle.$$

Again, π induces a map from $[\tilde{T}]$ to $[T]$ which we also refer to as π ; no confusion will result from this slight abuse of notation.

As π is clearly continuous, $\pi^{-1}(X) \in \mathbf{\Pi}_1^0([\tilde{T}])$. Moreover, observe that

$$\tilde{x} \in \pi^{-1}(X) \Leftrightarrow \tilde{x}(2k+1) \text{ is of the form } \langle x_{2k+1}, \langle 2, \Sigma_{II} \rangle \rangle$$

so that $\pi^{-1}(X)$ is also open in $[\tilde{T}]$ since for any $\tilde{x} \in \pi^{-1}(X)$, the cone $N_{\langle x_0, \dots, x_{2k+1} \rangle} \subset \pi^{-1}(X)$ contains \tilde{x} . Thus, $\pi^{-1}(X) \in \mathbf{\Delta}_1^0([\tilde{T}])$.

To complete the definition of $(\tilde{T}, \pi, \varphi)$, the k -covering of T , we now define, informally, φ which maps for each player a strategy $\tilde{\sigma}$ in \tilde{T} to a strategy $\sigma = \varphi(\tilde{\sigma})$ in T in such a way that if $x \in [\sigma]$ is a run of a game on T , then there is a run $\tilde{x} \in [\tilde{\sigma}]$ such that $\pi(\tilde{x}) = x$. Let $\tilde{\sigma} \subset \tilde{T}$ be a strategy. We argue two cases as $\tilde{\sigma}$ can be a strategy for I or II.

Case 1 – $\tilde{\sigma}$ is a strategy for I: For the first $2k$ moves, $\sigma \upharpoonright 2k - 1 = \tilde{\sigma} \upharpoonright 2k - 1$. Next, $\tilde{\sigma}$ produces a unique $\langle x_{2k}, \Sigma_I \rangle$ where $\Sigma_I \subseteq T_{\langle x_0, \dots, x_{2k} \rangle}$ is a quasistrategy for I; σ corresponds with x_{2k} . II then responds with x_{2k+1} in T .

Consider now the game on $(\Sigma_I)_{\langle x_{2k+1} \rangle}$ having payoff set $[(\Sigma_I)_{\langle x_{2k+1} \rangle}] \setminus X_{\langle x_0, \dots, x_{2k+1} \rangle}$. As $T_{X_{\langle x_0, \dots, x_{2k+1} \rangle}}$ is a nonempty pruned tree, $X_{\langle x_0, \dots, x_{2k+1} \rangle}$ is closed, and thus, $[(\Sigma_I)_{\langle x_{2k+1} \rangle}] \setminus X_{\langle x_0, \dots, x_{2k+1} \rangle}$ is open. Hence, by the Gale-Stewart Theorem, the game $G((\Sigma_I)_{\langle x_{2k+1} \rangle}, [(\Sigma_I)_{\langle x_{2k+1} \rangle}] \setminus X_{\langle x_0, \dots, x_{2k+1} \rangle})$ is determined. There are two subcases according to which player has a winning strategy in this game.

Subcase 1A: I has a winning strategy in $G((\Sigma_I)_{\langle x_{2k+1} \rangle}, [(\Sigma_I)_{\langle x_{2k+1} \rangle}] \setminus X_{\langle x_0, \dots, x_{2k+1} \rangle})$. σ then requires I to follow this strategy. For in this case, after a finite number of moves, a position $u \in (\Sigma_I)_{\langle x_{2k+1} \rangle} \subseteq T_{\langle x_0, \dots, x_{2k+1} \rangle}$ of even length is reached such that u is winning for I. That is, there is $u = \langle x_{2k+2}, \dots, x_{2l-1} \rangle$ such that $u \in (\Sigma_I)_{\langle x_{2k+1} \rangle} \setminus \langle T_X \rangle_{\langle x_0, \dots, x_{2k+1} \rangle}$. Thus,

$$\langle x_0, \dots, x_{2k-1}, \langle x_{2k}, \Sigma_I \rangle, \langle x_{2k+1}, \langle 1, u \rangle \rangle, x_{2k+2}, \dots, x_{2l-1} \rangle \in \tilde{T}$$

and henceforth σ requires I to follow $\tilde{\sigma}$. So if $x \in [\sigma]$ is a run of $G(T, X)$, it is clear that there is $\tilde{x} \in [\tilde{\sigma}]$ such that $\pi(\tilde{x}) = x$, namely \tilde{x} in which II plays $\langle x_{2k+1}, \langle 1, u \rangle \rangle$ in her $2k+1$ move.

Subcase 1B: II has a winning strategy in $G((\Sigma_I)_{\langle x_{2k+1} \rangle}, [(\Sigma_I)_{\langle x_{2k+1} \rangle}] \setminus X_{\langle x_0, \dots, x_{2k+1} \rangle})$. Let $\Sigma_{II} \subseteq (\Sigma_I)_{\langle x_{2k+1} \rangle}$ be her canonical quasistrategy in this game. Provided that in the game on \tilde{T} , II plays $\langle x_{2k+1}, \Sigma_{II} \rangle$, I follows σ by playing $\tilde{\sigma}$. (For if II plays otherwise, I now has a winning strategy in this game and can proceed via Subcase 1A.) As long as II plays $\langle x_{2k+2}, \dots, x_{2l-1} \rangle \in (\Sigma_{II})_{\langle x_0, \dots, x_{2k-1} \rangle}$, I continues to play σ by following $\tilde{\sigma}$. However, if at any point II plays $\langle x_{2k+2}, \dots, x_{2l-1} \rangle \notin (\Sigma_{II})_{\langle x_0, \dots, x_{2k+1} \rangle}$, then it follows that $\langle x_{2k+2}, \dots, x_{2l-1} \rangle$ is losing for II, consequently winning for I, in the game $G((\Sigma_I)_{\langle x_{2k+1} \rangle}, [(\Sigma_I)_{\langle x_{2k+1} \rangle}] \setminus X_{\langle x_0, \dots, x_{2k+1} \rangle})$ and I can again continue as in Subcase IA.

Case 2 – $\tilde{\sigma}$ is a strategy for II: Again, for the first $2k$ moves, $\sigma \upharpoonright 2k - 1 = \tilde{\sigma} \upharpoonright 2k - 1$. Next I plays x_{2k} in $G(T, X)$. Define

$$\mathcal{S} = \{ \Sigma_I \subseteq T_{\langle x_0, \dots, x_{2k} \rangle} : \Sigma_I \text{ is a quasistrategy for I} \}$$

and

$$U = \{ \langle x_{2k+1} \rangle \hat{u} \in T_{\langle x_0, \dots, x_{2k} \rangle} : u \text{ has even length and} \\ \exists \Sigma_I \in \mathcal{S} (\tilde{\sigma} \text{ requires II to play} \\ \langle x_{2k+1}, \langle 1, u \rangle \rangle \text{ when I plays } \langle x_{2k}, \Sigma_I \rangle) \}$$

and $\mathcal{U} = \{ x \in [T_{\langle x_0, \dots, x_{2k} \rangle}] : \exists \langle x_{2k+1} \rangle \hat{u} \in U (\langle x_{2k+1} \rangle \hat{u} \subseteq x) \}$. So then, $\mathcal{U} \subseteq [T_{\langle x_0, \dots, x_{2k} \rangle}]$ is open in $[T_{\langle x_0, \dots, x_{2k} \rangle}]$.

Consider now the game $G(T_{\langle x_0, \dots, x_{2k} \rangle}, \mathcal{U})$

$$\begin{array}{ccccccc} \text{I} & & x_{2k+2} & & & & \dots \\ & & & & & & \\ \text{II} & x_{2k+1} & & & x_{2k+3} & & \end{array}$$

where II plays first and wins if, and only if $\langle x_{2k+1}, x_{2k+2}, \dots \rangle \in \mathcal{U}$. As \mathcal{U} is open, this game is determined by Gale-Stewart, and hence there are two subcases.

Subcase 2A: II has a winning strategy in this game. Define a strategy σ for II as follows: II should follow this winning strategy in $G(T_{\langle x_0, \dots, x_{2k} \rangle}, \mathcal{U})$ until a position

$\langle x_{2k+1} \rangle \hat{u} \in U$ is reached, for some even length $u = \langle x_{2k+2}, \dots, x_{2l-1} \rangle$. By the definition of U , let Σ_I witness that $\langle x_{2k+1} \rangle \hat{u} \in U$. So from x_{2l} on, II returns to playing σ in $G(T, X)$ according to $\tilde{\sigma}$ on

$$\langle x_0, \dots, x_{2k-1}, \langle x_{2k}, \Sigma_I \rangle, \langle x_{2k+1}, \langle 1, u \rangle \rangle, x_{2k+2}, \dots, x_l \rangle \in \tilde{T}.$$

It is then clear that if $x \in [\sigma]$, there is $\tilde{x} \in [\tilde{\sigma}]$ such that $\pi(\tilde{x}) = x$. Let $\Sigma_{II} \subseteq (\Sigma_I)_{\langle x_{2k+1} \rangle} \cap (T_X)_{\langle x_0, \dots, x_{2k+1} \rangle}$ be the quasistrategy for II that $\tilde{\sigma}$ produces as II response to $\langle x_{2k}, \Sigma_I \rangle$ in the game on \tilde{T} .

Subcase 2B: I has a winning strategy in this game. Let Σ_I be his canonical quasistrategy in this game. Since Σ_I is winning for I, $[\Sigma_I] \subseteq [T_{\langle x_0, \dots, x_{2k} \rangle}] \setminus \mathcal{U}$, that is, $\Sigma_I \subseteq T_{\langle x_0, \dots, x_{2k} \rangle} \setminus U$ so that no sequence of Σ_I is in U . Suppose then that I plays $\langle x_{2k}, \Sigma_I \rangle$ in the game on \tilde{T} ; $\tilde{\sigma}$ must tell II to respond with something of the form $\langle x_{2k+1}, \langle 2, \Sigma_{II} \rangle \rangle$. (Otherwise, if $\tilde{\sigma}$ produced something of the form $\langle x_{2k+1}, \langle 1, u \rangle \rangle$ where $u \in (\Sigma_I)_{\langle 2k+1 \rangle} \setminus (T_X)_{\langle x_0, \dots, x_{2k+1} \rangle}$ by the rules of \tilde{T} , then by the definition of U , Σ_I would be a witness that $\langle x_{2k+1} \rangle \hat{u} \in U$, a contradiction.) Let $\langle x_{2k+1}, \langle 2, \Sigma_{II} \rangle \rangle$ be II's response to $\langle x_{2k}, \Sigma_I \rangle$ according to $\tilde{\sigma}$. Then II plays x_{2k+1} in $G(T, X)$ and plays σ according to $\tilde{\sigma}$ on

$$\langle x_0, \dots, x_{2k-1}, \langle x_{2k}, \Sigma_I \rangle, \langle x_{2k+1}, \langle 2, \Sigma_{II} \rangle \rangle, x_{2k+2}, \dots, x_{2l} \rangle$$

provided that $\langle x_{2k+2}, \dots, x_{2k} \rangle \in \Sigma_{II}$. If for some $l \geq k+1$, I plays such that $\langle x_{2k+2}, \dots, x_{2k} \rangle \notin \Sigma_{II}$, then $\langle x_{2k+2}, \dots, x_{2k} \rangle \notin (\Sigma_I)_{\langle 2k+1 \rangle}$ since Σ_{II} is a quasistrategy for II in $(\Sigma_I)_{\langle 2k+1 \rangle}$. Hence, $\langle x_{2k+2}, \dots, x_{2k} \rangle$ is losing for I, and we are back in Subcase 2A. \square

The following technical lemma is the final fact needed to carry out the induction.

Lemma 3.2.4. *Let $k \in \omega$ and suppose $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ is a $(k+i)$ -covering of T_i , for each $i \in \omega$. Then there is a pruned tree T_∞ and $\pi_{\infty, i} : [T_\infty] \rightarrow [T_i]$, $\varphi_{\infty, i} : S(T_\infty) \rightarrow S(T_i)$ such that $(T_\infty, \pi_{\infty, i}, \varphi_{\infty, i})$ is a $(k+i)$ -covering of T_i , $\pi_{i+1} \circ \pi_{\infty, i+1} = \pi_{\infty, i}$, and $\varphi_{i+1} \circ \varphi_{\infty, i+1} = \varphi_i$.*

Proof. Let $k \in \omega$, and for each $i \in \omega$, let $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ be a $(k+i)$ -covering of T_i . Define T_∞ as follows:

$$s \in T_\infty \Leftrightarrow \exists i \in \omega [s \in T_i \wedge \text{length}(s) \leq 2(k+i)].$$

Since for each I , T_i is a nonempty pruned tree, it easily follows that so is T_∞ . Moreover, it is clear that $T_\infty \upharpoonright 2(k+i) = T_i \upharpoonright 2(k+i)$.

Define $\pi_{\infty,i} : T_\infty \rightarrow T_i$ as follows:

$$\pi_{\infty,i}(s) = \begin{cases} s & \text{if } \text{length}(s) \leq 2(k+i) \\ (\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_j)(s) & \text{if } 2(k+i) < \text{length}(s) \leq 2(k+j) \text{ for some } j \end{cases}$$

It should be clear that $\pi_{\infty,i}$ is well-defined because in the second case, $\pi_{\infty,i}(s)$ is independent of the choice of j . As each π_i is monotone with $\text{length}(\pi(s)) = \text{length}(s)$, it follows from the definition that $\pi_{\infty,i}$ is also. Moreover, it is clear that for each i , $\pi_{\infty,i} = \pi_{i+1} \circ \pi_{\infty,i+1}$.

Define $\varphi_{\infty,i}$ from the set of strategies in T_∞ to the set of strategies in T_i as follows:

$$\varphi_{\infty,i}(\sigma_\infty) \upharpoonright 2(k+i) = \sigma_\infty \upharpoonright 2(k+i)$$

and for all $j > i$,

$$\varphi_{\infty,i}(\sigma_\infty) \upharpoonright 2(k+j) = (\varphi_{i+1} \circ \varphi_{i+2} \circ \cdots \circ \varphi_j)(\sigma_\infty \upharpoonright 2(k+j))$$

Similarly, it is clear that $\varphi_{\infty,i}$ maps strategies for player I (resp. II) in T_∞ to strategies for player I (resp. II) in T_i , in such a way that $\varphi_{\infty,i}(\sigma)$ restricted to positions of length $\leq n$ depends only on σ restricted to positions of length $\leq n$, for all n . Moreover, it is clear that for each I , $\varphi_{\infty,i} = \varphi_{i+1} \circ \varphi_{\infty,i+1}$. Thus, for $(T_\infty, \pi_{\infty,i}, \varphi_{\infty,i})$ to be a $(k+i)$ -covering of T_i , it remains to show that if $\sigma_\infty \subseteq T_\infty$ is a strategy and $x_i \in [\varphi_{\infty,i}(\sigma_\infty)]$, then there is $x_\infty \in [\sigma_\infty]$ such that $\pi_{\infty,i}(x_\infty) = x_i$. We argue the case $i = 0$; it should be clear that the argument easily generalizes to any $i \in \omega$.

Let $\sigma_\infty \subseteq T_\infty$ be a strategy, and let $x_0 \in [\varphi_{\infty,0}(\sigma_\infty)] \subseteq T_0$. As $\varphi_{\infty,0} = \varphi_1 \circ \varphi_{\infty,1}$, $x_0 \in [\varphi_1(\varphi_{\infty,1}(\sigma_\infty))]$. Since (T_1, π_1, φ_1) is a k -covering of T_0 , let $x_1 \in [\varphi_{\infty,1}(\sigma_\infty)]$ be such that $\pi_1(x_1) = x_0$. Moreover, as $T_1 \upharpoonright 2k = T_0 \upharpoonright 2k$, for any sequence s having $\text{length}(s) \leq 2k$, π_1 is the identity. Next, as $\varphi_{\infty,1} = \varphi_2 \circ \varphi_{\infty,2}$, $x_1 \in [\varphi_2(\varphi_{\infty,2}(\sigma_\infty))]$. Since (T_2, π_2, φ_2) is a $(k+1)$ -covering of T_1 , let $x_2 \in [\varphi_{\infty,2}(\sigma_\infty)]$ be such that $\pi_2(x_2) = x_1$. Moreover, as $T_2 \upharpoonright 2(k+1) = T_1 \upharpoonright 2(k+1)$, for any sequence s having $\text{length}(s) \leq 2(k+1)$, π_2 is the identity. In this way, we define for each $i \in \omega$, $x_{i+1} \in [\varphi_{\infty,i+1}(\sigma_\infty)] \subseteq [T_{i+1}]$ such that $\pi_{i+1}(x_{i+1}) = x_i$. Recall from our definition of T_∞ that $T_\infty \upharpoonright 2(k+i) = T_i \upharpoonright 2(k+i)$. As a result, it is clear that x_0, x_1, x_2, \dots converges to $x_\infty \in [T_\infty]$ given by $x_\infty \upharpoonright 2(k+j) = x_0 \upharpoonright 2(k+j)$ for each $j \geq 0$. Furthermore, $x_\infty \in [\sigma_\infty]$ since $\sigma_\infty \upharpoonright 2(k+j) = \varphi_{\infty,j}(\sigma_\infty) \upharpoonright 2(k+j)$ for all $j \geq 0$. Finally, it remains to show that $\pi_{\infty,0}(x_\infty) = x_0$. From the definition of $\pi_{\infty,0}$ it follows that $\pi_{\infty,0}(x_\infty) \upharpoonright 2k = x_0 \upharpoonright 2k$. For $j > 0$ we have the following

$$\pi_{\infty,0}(x_\infty \upharpoonright 2(k+j)) = \pi_1(x_\infty \upharpoonright 2(k+j)) = \pi_1(x_1 \upharpoonright 2(k+j)) = x_0 \upharpoonright 2(k+j).$$

Thus, it follows that $\pi_{\infty,0}(x_\infty) = x_0$. It is clear that the above argument is completely general for any $i \in \omega$. \square

Theorem 3.2.5 (Martin). *If T is a nonempty pruned tree on A and $X \subseteq [T]$ is Borel, then for each $k \in \omega$ there is a k -covering of T that unravels X .*

Proof. Let T be a nonempty pruned tree on some set A and let $X \subseteq [T]$ be Borel. Suppose $X \in \mathbf{\Pi}_1^0([T])$ and let $k \in \omega$. By Lemma 3.2.3, there is a k -covering of T which unravels X . Moreover, Lemma 3.2.2 insures that such a k -covering also unravels X^c so that the result holds for $X \in \mathbf{\Sigma}_1^0([T])$.

Now suppose that $1 < \alpha < \omega_1$ and that for all $\beta < \alpha$, if $Y \in \mathbf{\Sigma}_\beta^0([T])$ and $k \in \omega$, then there is a k -covering of T that unravels Y ; by Lemma 3.2.2, the result holds for all $\beta < \alpha$, $Y \in \mathbf{\Pi}_\beta^0([T])$.

Let $X \in \mathbf{\Sigma}_\alpha^0([T])$. Thus, $X = \bigcup_{i \in \omega} X_i$ such that for each i , $X_i \in \mathbf{\Pi}_{\beta_i}^0([T])$ some $\beta_i < \alpha$. By the induction hypothesis, let (T_1, π_1, φ_1) be a k -covering of $T_0 = T$ that unravels X_0 ; that is, $\pi_1^{-1}(X_0) \in \mathbf{\Delta}_1^0$. Moreover, since π_1 is continuous, and since

any pointclass is closed under continuous pre-images, it follows that $\pi_1^{-1}(X_i) \in \mathbf{\Pi}_{\beta_i}^0$ for each $i > 0$. Again by the induction hypothesis, there is a $(k+1)$ -covering of T_1 , (T_2, π_2, φ_2) that unravels $\pi_1^{-1}(X_1)$. Thus, $(\pi_2^{-1} \circ \pi_1^{-1})(X_i) \in \mathbf{\Pi}_{\beta_i}^0([T_2])$ for all $i > 1$, and $(\pi_2^{-1} \circ \pi_1^{-1})(X_i) \in \mathbf{\Delta}_1^0([T_2])$ for $i = 0, 1$. In this fashion, define recursively for each I, $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ to be a $(k+i)$ -covering of T_i , that unravels $(\pi_i^{-1} \circ \pi_{i-1}^{-1} \circ \dots \circ \pi_1^{-1})(X_i)$. Now, for each $i \in \omega$, let $(T_\infty, \pi_{\infty, i}, \varphi_{\infty, i})$ be a $(k+i)$ -covering of T_i , as in Lemma 3.2.4. Then $(T_\infty, \pi_{\infty, 0}, \varphi_{\infty, 0})$ unravels X_i for each I. That is, for each I, $\pi_{\infty, 0}^{-1}(X_i) \in \mathbf{\Delta}_1^0([T_\infty])$. Thus, $\pi_{\infty, 0}^{-1}(X) = \pi_{\infty, 0}^{-1}(\bigcup_{i \in \omega} X_i) = \bigcup_{i \in \omega} \pi_{\infty, 0}^{-1}(X_i) \in \mathbf{\Sigma}_1^0([T_\infty])$. Finally, using Lemma 3.2.3 and Lemma 3.2.4 again, let $(\tilde{T}, \pi, \varphi)$ be a k -covering of T_∞ that unravels $\pi_{\infty, 0}^{-1}(X)$. Then, $(\tilde{T}, \pi_{\infty, 0} \circ \pi, \varphi_{\infty, 0} \circ \varphi)$ is a k -covering of T that unravels X . \square

Corollary 3.2.6 (Martin). $ZFC \vdash Det(\mathbf{\Delta}_1^1)$

Part II

The Metamathematics of Borel Determinacy

Prior to Martin's proof of Borel Determinacy, Friedman showed $ZC \not\equiv Det(\Pi_{\omega+2}^0)$ [Fri71]. Using an earlier result of Martin [Mar68], Friedman established the necessity of the Axiom of Replacement to any proof of Borel Determinacy.

If there exists an algorithm that computes x from y , then x is **recursive in** y , written $x \leq_T y$. Two reals x and y are **Turing equivalent**, denoted $x \equiv_T y$, when $x \leq_T y$ and $y \leq_T x$. The equivalence classes of reals in \equiv_T are called **Turing degrees** and we denote the set of Turing degrees by \mathbf{D} . For every real x , the degree of x is denoted by \mathbf{x} . Given a pointclass Γ and $\mathbf{A} \subseteq \mathbf{D}$, we say that \mathbf{A} is a Γ -subset if $\{x \in \omega^\omega : \mathbf{x} \in \mathbf{A}\} \in \Gamma$. Let (\mathbf{D}, \leq) be the partial order induced by $\mathbf{x} \leq \mathbf{y} \Leftrightarrow x \leq_T y$. For $\mathbf{x} \in \mathbf{D}$, the **cone** of \mathbf{x} is $C_{\mathbf{x}} = \{\mathbf{y} \in \mathbf{D} : \mathbf{x} \leq \mathbf{y}\}$. If $\mathbf{A} \subseteq \mathbf{D}$ and $\mathbf{x} \in \mathbf{D}$, \mathbf{A} **contains** the cone of \mathbf{x} if, and only if $C_{\mathbf{x}} \subseteq \mathbf{A}$ and **omits** the cone of \mathbf{x} if $C_{\mathbf{x}} \subseteq \mathbf{D} \setminus \mathbf{A}$.

Theorem (Martin). *Assuming $Det(\Gamma)$, every Γ -subset of \mathbf{D} either contains or omits a cone.*

Proof. Let Γ be a pointclass and assume $Det(\Gamma)$. Let $\mathbf{A} \subseteq \mathbf{D}$ be such that $A = \{x \in \omega^\omega : \mathbf{x} \in \mathbf{A}\} \in \Gamma$. Consider the game $G(A)$

$$\begin{array}{ccccccc} \text{I} & & a_0 & & & a_2 & & & \\ & & & & & & & \dots & \\ & & & & & & & & \\ \text{II} & & & & a_1 & & & a_3 & \end{array}$$

where $a_i \in \omega$. I wins $G(A)$ if, and only if $a \in A$; II wins otherwise. By assumption, $G(A)$ is determined. Suppose $\varphi : \omega^{<\omega} \rightarrow \omega$ is a winning strategy for I in $G(A)$. Fix a recursive bijection $\psi : \omega \rightarrow \omega^{<\omega}$ and define $x = \varphi \circ \psi$. We claim that $C_{\mathbf{x}} \subseteq \mathbf{A}$. Suppose $\mathbf{y} \in C_{\mathbf{x}}$; thus $x \leq_T y$. Consider a run $a \in \omega^\omega$ of the game $G(A)$ in which II plays $y = (a_1, a_3, \dots)$ and I responds by playing (a_0, a_2, \dots) according to φ so that $a \in A$. Now, $y \leq_T a$, hence $\mathbf{y} \leq \mathbf{a}$. But also, $a \leq_T y$ as $x \leq_T y$. So, $\mathbf{a} \leq \mathbf{y}$, and thus $\mathbf{y} = \mathbf{a} \in \mathbf{A}$. A symmetric argument shows that if II has a winning strategy in $G(A)$, then $C_{\mathbf{x}} \subseteq \mathbf{D} \setminus \mathbf{A}$. \square

Thus, producing a set $\mathbf{A} \in \Gamma$ of degrees that neither contains nor omits a cone implies $Det(\Gamma)$ is false. Both Friedman's proof of $ZC \not\equiv Det(\Pi_{\omega+2}^0)$ and our proof

of $ZFC \not\equiv Det(\Pi_1^1)$ use Martin's theorem in this way. Simpler counterexamples to $Det(\Pi_1^1)$ exist, but it is not known at this time whether the literature contains any incidence of the following proof of $ZFC \not\equiv Det(\Pi_1^1)$.

CHAPTER 4

$ZFC \not\vdash Det(\Pi_1^1)$

Arguing in ZFC , we produce a Π_1^1 set of degrees of reals that neither contains nor omits a cone. Each real in this set is the real coding the theory of a limit stage of \mathbf{L} in which a new real occurs. In the first three sections of this chapter, we establish the existence of an $L(\lambda)$ -definable function from ω onto $L(\lambda)$; the reader familiar with these standard \mathbf{L} arguments should skip to the final section containing the construction of the Π_1^1 set.

4.1 Properties of \mathbf{L}

Using the notation of Kunen ([Kun80]), we define the following by transfinite recursion on the ordinals

$$\begin{aligned} L(0) &= \emptyset \\ L(\alpha + 1) &= \mathcal{D}(L(\alpha)), \text{ for } \alpha \text{ successor} \\ L(\lambda) &= \bigcup_{\xi < \lambda} L(\xi), \text{ for } \lambda \text{ limit} \end{aligned}$$

where \mathcal{D} is the **definable power set** operator. Informally, $\mathcal{D}(A)$ is the set of subsets of A definable from a finite number of elements from A by a formula relativized to A . A formal definition of \mathcal{D} follows shortly. So $\mathbf{L} = \bigcup_{\alpha \in \mathbf{ON}} L(\alpha)$. A set x is said to be **constructible** if $x \in \mathbf{L}$. “ $\mathbf{V} = \mathbf{L}$ ” abbreviates the sentence $\forall x(x \in \mathbf{L})$. This section’s goal is the proof of the statement: for every $\lambda > \omega$ limit, $L(\lambda) \models \mathbf{V} = \mathbf{L}$.

A few \mathbf{L} facts are needed. A set is **transitive** if every element is a subset. For every ordinal α , $L(\alpha)$ is transitive; it follows then that \mathbf{L} is transitive. Given $x \in \mathbf{L}$, the **rank** of x , denoted $\rho(x)$ is the least α such that $x \in L(\alpha + 1)$. For every ordinal α , $\rho(\alpha) = \alpha$. We write ZF to denote Zermelo-Fraenkel set theory without the Axiom of Choice. The following is an important theorem in its own right; the proof is standard and we omit it.

Theorem 4.1.1. $\mathbf{L} \models ZF$

We prove in the second section that \mathbf{L} is a model of ZF plus Choice (ZFC). The following lemma catalogs the ranks of some commonly formed sets. We write (x, y) to denote the ordered pair of x, y and x^y denotes the set of all functions from y to x .

Lemma 4.1.2. *Let $x, y \in \mathbf{L}$ be such that $\rho(x), \rho(y) = \alpha$, for some $\alpha > \omega$. Then*

1. $\rho(x \cap y), \rho(x \setminus y) = \alpha$,
2. $\rho(\{x, y\}) = \alpha + 1$,
3. $\rho((x, y)) = \alpha + 2$,
4. $\forall k \in \omega$, if $f : k \rightarrow x$, $\rho(f) = \alpha + 2$,
5. $\forall k \in \omega$, $\rho(x^k) = \alpha + 3$,
6. $\rho(x \times y) = \alpha + 3$.

Proof. Let $x, y \in L(\alpha + 1)$ for some α . As $\alpha = 0$ is trivial and as the limit case easily follows from the successor case, suppose α is a successor. Let φ, ψ be such that $x = \{z \in L(\alpha) : \varphi^{L(\alpha)}(z)\}$ and $y = \{z \in L(\alpha) : \psi^{L(\alpha)}(z)\}$. Then

$$x \cap y = \{z \in L(\alpha) : L(\alpha) \models (\varphi \wedge \psi)(z)\} \in L(\alpha + 1).$$

Similar reasoning shows $x \setminus y \in L(\alpha + 1)$.

Since $\{x, y\} = \{z \in L(\alpha + 1) : L(\alpha + 1) \models (z = x \vee z = y)\}$, it follows that $\{x, y\} \in L(\alpha + 2)$. Hence, $(x, y) = \{\{x\}, \{x, y\}\} \in L(\alpha + 3)$ and thus, $x \times y \in L(\alpha + 4)$.

Now suppose $k \in \omega$ and $f : k \rightarrow x$. Then $f = \{(i, f_i) : i < k \wedge f_i \in x\}$. By the transitivity of $L(\alpha + 1)$, for each $i < k$ and $f_i \in x$, we have $i, f_i \in L(\alpha)$. (Note that this is the point where we use the hypothesis $\alpha > \omega$ for convenience.) Thus, $(i, f_i) \in L(\alpha + 2)$. Hence, $f \in L(\alpha + 3)$, and $x^k \in L(\alpha + 4)$ for every k . \square

We now formalize the definition of the \mathcal{D} operator in order to aid our discussion of certain absoluteness results, beginning with the satisfaction relation.

Every formula in the language of set theory (**LST**) can be rewritten using only the membership (\in) and equality ($=$) predicates, the logical relations of negation (\neg), conjunction (\wedge), and existential quantification (\exists). Exploiting this fact, our first definition inductively captures every k -ary relation on a set A definable from a formula relativized to A in a manner that codes every formula by an integer. Let $A \neq \emptyset$ and $k \in \omega$. For $i, j \in \omega$ and $i, j < k$, define

$$\begin{aligned} \text{Diag}_{\in}(A, i, j, k) &= \{s \in A^k : s(i) \in s(j)\}, \\ \text{Diag}_{=} (A, i, j, k) &= \{s \in A^k : s(i) = s(j)\}, \\ \text{Proj}(A, R, k) &= \{s \in A^k : \exists t \in R(t \upharpoonright k = s)\}. \end{aligned}$$

where $t \upharpoonright k$ denotes the **restriction** of t to k . Define $Df(A, n)$ by recursion on n as follows:

1. $Df(A, n) = \emptyset$, if $n = 0$
2. $Df(A, n) = \text{Diag}_{\in}(A, i, j, k)$, if $n = 2 \cdot 3^i \cdot 5^j \cdot 7^k$, where $i, j < k$
3. $Df(A, n) = \text{Diag}_{=} (A, i, j, k)$, if $n = 2^2 \cdot 3^i \cdot 5^j \cdot 7^k$, where $i, j < k$
4. $Df(A, n) = A^k \setminus Df(A, i)$, if $n = 2^3 \cdot 3^i \cdot 5^j \cdot 7^k$, where $i = 2^p \cdot 3^q \cdot 5^r \cdot 7^k$ for $p = 1, 2, 3, 4, 5$ and some q, r , and $j = 0$
5. $Df(A, n) = Df(A, i) \cap Df(A, j)$, if $n = 2^4 \cdot 3^i \cdot 5^j \cdot 7^k$, where $i = 2^p \cdot 3^q \cdot 5^r \cdot 7^k$, $j = 2^{p'} \cdot 3^{q'} \cdot 5^{r'} \cdot 7^k$ for $p, p' = 1, 2, 3, 4, 5$ and some q, q', r, r'
6. $Df(A, n) = \text{Proj}(A, Df(A, i), k)$, if $n = 2^5 \cdot 3^i \cdot 5^j \cdot 7^k$ where $i = 2^p \cdot 3^q \cdot 5^r \cdot 7^{k+1}$ for $p = 1, 2, 3, 4, 5$ and some q, r , and $j = 0$
7. $Df(A, n) = \emptyset$, if n is not of one of these forms.

A simple induction on n shows that $\{Df(A, n) : n \in \omega\}$ enumerates all relations on A definable by a formula relativized to A . This is possible because, given n , we can recursively recover both the **LST** formula φ_n and its arity, denoted $Ar(n)$. As φ_n involves no parameters of A , the set of formulas is countable.

At last, we have the following for $A \neq \emptyset$:

$$\mathcal{D}(A) = \{X \subset A : \exists n, k, s, R [n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1} \\ \wedge R = Df(A, n) \wedge X = \{x \in A : s \hat{\ } x \in R\}]\}$$

where $s \hat{\ } x$ denotes the **concatenation** of s with x . Our immediate goal is the proof that \mathcal{D} is absolute for $L(\lambda)$, $\lambda > \omega$ limit; what we mean by absolute will become clear in the lemmas to follow. To this end, we begin by establishing the absoluteness of the $Diag_{\in}$, $Diag_{=}$, $Proj$, and Df relations for $L(\lambda)$, $\lambda > \omega$ limit.

A formula in which all quantifiers are bound is called Δ_0 . Any formula equivalent to a Δ_0 formula is absolute for any transitive, well-founded set. Note that each of the formulas in Lemma 4.1.2 is equivalent to a Δ_0 formula.

Lemma 4.1.3. *Let $\lambda > \omega$ be limit and $A \in L(\lambda)$ be such that $\rho(A) = \alpha < \lambda$.*

1. *For all $i, j, k \in \omega$, $\rho(Diag_{\in}(A, i, j, k))$, $\rho(Diag_{=}(A, i, j, k)) = \alpha + 3$.*
2. *For all relations R on A , if $\rho(R) = \beta$, then $\rho(Proj(A, R, k)) = \beta$.*
3. *$Diag_{\in}$, $Diag_{=}$, and $Proj$ are absolute for $L(\lambda)$.*

Proof. Let $\lambda > \omega$ be limit and $A \in L(\lambda)$ such that $\rho(A) = \alpha$. We prove the first claim for $Diag_{\in}$; the proof for $Diag_{=}$ is similar. Let $i, j, k \in \omega$ with $i, j < k$. By Lemma 4.1.2, for every $s \in A^k$, $\rho(s) = \alpha + 2$. Thus,

$$Diag_{\in}(A, i, j, k) = \{s \in A^k : s_i \in s_j\} \\ = \{z \in L(\alpha + 3) : L(\alpha + 3) \models z : k \rightarrow A \wedge z_i \in z_j\}$$

Hence, $\rho(Diag_{\in}(A, i, j, k)) = \alpha + 3$.

For the second claim, let R a relation on A be such that $\rho(R) = \beta$ and let $k \in \omega$. Thus,

$$Proj(A, R, k) = \{s \in A^k : \exists t \in R(t \upharpoonright k = s)\} \\ = \{z \in L(\beta) : L(\beta) \models z : k \rightarrow A \wedge \exists y \in R(y \upharpoonright k = z)\}$$

So $Proj(A, R, k) \in L(\beta + 1)$.

Finally, we prove that $Proj$ is absolute for $L(\lambda)$; the proof is similar for $Diag_{\in}$ and $Diag_{=}$. First observe that “ $z \in Proj(A, R, k)$ ” $\Leftrightarrow z \in A^k \wedge \exists x \in R(x \upharpoonright k = s)$, which is Δ_0 . Thus, “ $z \in Proj(A, R, k)$ ” is absolute for $L(\lambda)$, as $L(\lambda)$ is transitive. Now we want to show that for all $y, A, R, k \in L(\lambda)$,

$$y = Proj(A, R, k) \Leftrightarrow L(\lambda) \models y = Proj(A, R, k)$$

where “ $y = Proj(A, R, k)$ ” abbreviates $\forall z(z \in y \Leftrightarrow z \in Proj(A, R, k))$. Let $y, A, R, k \in L(\lambda)$. Suppose $y = Proj(A, R, k)$. As universal quantification is downward absolute and as “ $z \in Proj(A, R, k)$ ” is absolute for $L(\lambda)$, it follows that $L(\lambda) \models y = Proj(A, R, k)$. Now suppose $L(\lambda) \models y = Proj(A, R, k)$. We must show $\forall z(z \in y \Leftrightarrow z \in Proj(A, R, k))$. Suppose $z \in y$. By the transitivity of $L(\lambda)$, $z \in y \in L(\lambda)$. Since $L(\lambda) \models y = Proj(A, R, k)$, we have $L(\lambda) \models z \in Proj(A, R, k)$, which is absolute for $L(\lambda)$. Thus, $z \in Proj(A, R, k)$. Now suppose $z \in Proj(A, R, k)$. As $R \in L(\lambda)$, $Proj(A, R, k) \in L(\lambda)$; again by the transitivity of $L(\lambda)$, $z \in L(\lambda)$. Since $L(\lambda) \models y = Proj(A, R, k)$, we have $L(\lambda) \models z \in y$, which is Δ_0 and hence absolute for $L(\lambda)$. Thus, $z \in y$. \square

The next lemma proves that the Df function is absolute for $L(\lambda)$ for $\lambda > \omega$ limit. Due to the inductive definition of Df , “ $R = Df(A, n)$ ” abbreviates the formula $\exists g \nu(g, R, A, n)$ where g is a finite function building up the Df sets such that $g(n) = Df(A, n)$.

Lemma 4.1.4. *For all $\lambda > \omega$ limit, $A \in L(\lambda)$, and $n \in \omega$,*

1. *if $\rho(A) = \alpha$, then $\rho(Df(A, n)) = \alpha + 3$, and*
2. *Df is absolute for $L(\lambda)$*

Proof. Let $A \in L(\lambda)$ for $\lambda > \omega$ limit with $\rho(A) = \alpha$ for some $\omega < \alpha < \lambda$. We first show by induction on n that $Df(A, n) \in L(\alpha + 4)$. The case $n = 0$ is trivial as $Df(A, n) = \emptyset$. Suppose now that $n > 0$ and that for all $i < n$, $Df(A, i) \in L(\alpha + 4)$.

Case 1: $n = 2 \cdot 3^i \cdot 5^j \cdot 7^k$ where $i, j < n$.

So $Df(A, n) = \text{Diag}_\in(A, i, j, k) \in L(\alpha + 3)$, by Lemma 4.1.3.

Case 2: $n = 2^2 \cdot 3^i \cdot 5^j \cdot 7^k$ where $i, j < n$.

So $Df(A, n) = \text{Diag}_=(A, i, j, k) \in L(\alpha + 3)$, by Lemma 4.1.3.

Case 3: $n = 2^3 \cdot 3^i \cdot 5^j \cdot 7^k$ for $j = 0$ and some k and where $i = 2^p \cdot 3^q \cdot 5^r \cdot 7^k$ for $p = 1, 2, 3, 4, 5$ and some q, r .

As $i < n$, $Df(A, i) \in L(\alpha + 3)$ by the IH. By part 5 of Lemma 4.1.2, $A^k \in L(\alpha + 3)$; thus, by the first part of the same lemma $A^k \setminus Df(A, i) = Df(A, n) \in L(\alpha + 3)$.

Case 4: $n = 2^4 \cdot 3^i \cdot 5^j \cdot 7^k$ for some k and where $i = 2^p \cdot 3^q \cdot 5^r \cdot 7^k$ and $j = 2^{p'} \cdot 3^{q'} \cdot 5^{r'} \cdot 7^k$ for $p, p' = 1, 2, 3, 4, 5$ and some q, q', r, r' .

As $i, j < n$, $Df(A, i), Df(A, j) \in L(\alpha + 3)$ by the IH. By the first part of Lemma 4.1.2, $Df(A, i) \cap Df(A, j) = Df(A, n) \in L(\alpha + 3)$.

Case 5: $n = 2^5 \cdot 3^i \cdot 5^j \cdot 7^k$ for $j = 0$ and some k and where $i = 2^p \cdot 3^q \cdot 5^r \cdot 7^{k+1}$ for $p = 1, 2, 3, 4, 5$ and some q, r .

As $i < n$, $Df(A, i) \in L(\alpha + 3)$ by the IH. Thus, $Df(A, n) = \text{Proj}(A, Df(A, i), k) \in L(\alpha + 3)$ by Lemma 4.1.3.

Case 6: Suppose n is not of the above forms. Then $Df(A, n) = \emptyset \in L(\alpha + 3)$.

Therefore, $Df(A, n) \in L(\alpha + 3)$ for every n .

Next, to show the absoluteness of the relation “ $R = Df(A, n)$ ” for $L(\lambda)$, let $R, A \in L(\lambda)$ and let $n \in \omega$. “ $R = Df(A, n)$ ” abbreviates $\exists g \nu(g, R, A, n)$ where $\nu(g, R, A, n)$ is the formula

$$\begin{aligned}
&g \text{ is a function } \wedge \text{dom } g = n + 1 \wedge \forall l \in \text{dom } g [\forall i, j, k, p, p', q, q', r, r' \in \omega \\
&[[(l = 2 \cdot 3^i \cdot 5^j \cdot 7^k \wedge i, j < k) \Rightarrow g(l) = \text{Diag}_\in(A, i, j, k)] \\
&\wedge [(l = 2^2 \cdot 3^i \cdot 5^j \cdot 7^k \wedge i, j < k) \Rightarrow g(l) = \text{Diag}_=(A, i, j, k)] \\
&\wedge [(l = 2^3 \cdot 3^i \cdot 5^j \cdot 7^k \wedge i = 2^p \cdot 3^q \cdot 5^r \cdot 7^k \wedge j = 0 \wedge 1 \leq p \leq 5) \Rightarrow g(l) = A^k \setminus g(i)] \\
&\wedge [(l = 2^4 \cdot 3^i \cdot 5^j \cdot 7^k \wedge i = 2^p \cdot 3^q \cdot 5^r \cdot 7^k \wedge j = 2^{p'} \cdot 3^{q'} \cdot 5^{r'} \cdot 7^k \wedge 1 \leq p, p' \leq 5) \\
&\quad \Rightarrow g(l) = g(i) \cap g(j)] \\
&\wedge [(l = 2^5 \cdot 3^i \cdot 5^j \cdot 7^k \wedge i = 2^p \cdot 3^q \cdot 5^r \cdot 7^{k+1} \wedge j = 0 \wedge 1 \leq p \leq 5) \\
&\quad \Rightarrow g(l) = \text{Proj}(A, g(i), k)] \\
&\wedge [(l \neq 2^p \cdot 3^i \cdot 5^j \cdot 7^k \wedge 1 \leq p \leq 5) \Rightarrow g(l) = \emptyset]] \wedge R = g(n)
\end{aligned}$$

By inspection, ν faithfully represents the inductive definition of Df . Moreover, ν is absolute for $L(\lambda)$ since it is comprised of subformulas which either are Δ_0 or are absolute for $L(\lambda)$ by Lemma 4.1.3. First, $L(\lambda) \models \exists g \nu(g, R, A, n) \Rightarrow \exists g \nu(g, R, A, n)$ is immediate as existential quantification reflects upward. Next, suppose g is such that $\nu(g, R, A, n)$. As $A \in L(\alpha + 1)$, $Df(A, n) \in L(\alpha + 4)$ for every n ; so by Lemma 4.1.2 $(n, Df(A, n)) \in L(\alpha + 6)$ for every n . Thus, $g \in L(\alpha + 7) \subset L(\lambda)$. By the absoluteness of ν we have $L(\lambda) \models \exists g \nu(g, R, A, n)$. \square

As a consequence of Lemma 4.1.4, we have the following closure property.

Corollary 4.1.5. *For $\lambda > \omega$ limit, let $A, R \in L(\lambda)$ be such that $\rho(A), \rho(R) = \alpha$. Then $\rho(g) = \alpha + 6$ where g is from the relation “ $R = Df(A, n)$ ”.*

We write “ $X \in \mathcal{D}(A)$ ” to abbreviate

$$\begin{aligned} X \subset A \wedge \exists n, k, s, R [n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1} \wedge \\ R = Df(A, n) \wedge \forall x \in A (x \in X \Leftrightarrow s \hat{x} \in R)] \end{aligned}$$

Lemma 4.1.6. “ $X \in \mathcal{D}(A)$ ” is absolute for $L(\lambda)$ for $\lambda > \omega$ limit.

Proof. Let $\lambda > \omega$ be limit and let $X, A \in L(\alpha)$ for some $\alpha < \lambda$. We want to show “ $X \in \mathcal{D}(A)$ ” $\Leftrightarrow L(\lambda) \models$ “ $X \in \mathcal{D}(A)$ ”.

Suppose that “ $X \in \mathcal{D}(A)$ ”. Thus, $X \subset A$ and there exist n, k, s, R such that $n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1} \wedge R = Df(A, n) \wedge \forall x \in A (x \in X \Leftrightarrow s \hat{x} \in R)$. Replacing “ $R = Df(A, n)$ ” with $\exists g \nu(g, R, A, n)$, let n, k, s, R, g be such that “ $X \in \mathcal{D}(A)$ ”. As $\lambda > \omega$, $n, k \in L(\lambda)$. Since $A \in L(\alpha)$, $A^{k-1} \in L(\alpha + 3)$ and $s \in L(\alpha + 2)$ by Lemma 4.1.2. Lemma 4.1.3 insures that $R \in L(\alpha + 3)$. Since $\nu(g, R, A, n)$, we have $g = \{(i, Df(A, i)) : i \leq n\}$. By Lemma 4.1.4, for all i , $Df(A, i) \in L(\alpha + 3)$. Thus, $g \in L(\alpha + 6) \subset L(\lambda)$. Since each subformula of “ $X \in \mathcal{D}(A)$ ” is Δ_0 , except for ν which is absolute for $L(\lambda)$ by Lemma 4.1.4, we have $L(\lambda) \models$ “ $X \in \mathcal{D}(A)$ ”.

Conversely, suppose $L(\lambda) \models$ “ $X \in \mathcal{D}(A)$ ”. As existential quantification is upward absolute, $X \in \mathcal{D}(A)$. \square

We can now prove that \mathcal{D} is absolute for $L(\lambda)$, $\lambda > \omega$ limit. In order to insure the existence of $\mathcal{D}(A)$ for each A , we abbreviate “ $Y = \mathcal{D}(A)$ ” by

$$\begin{aligned} \forall X [X \in Y \Rightarrow X \in \mathcal{D}(A)] \wedge \forall n, k, s [(n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1}) \\ \Rightarrow \exists R, X [R = Df(A, n) \wedge \forall x \in A (x \in X \Leftrightarrow s^{\hat{}}x \in R) \wedge X \in Y]] \end{aligned}$$

Lemma 4.1.7. *For all $\lambda > \omega$ limit and $A \in L(\lambda)$*

1. *if $\rho(A) = \alpha$, then $\mathcal{D}(A) \in L(\alpha + 8)$, and*
2. *\mathcal{D} is absolute for $L(\lambda)$.*

Proof. Let $A \in L(\lambda)$ for $\lambda > \omega$ limit and suppose $\rho(A) = \alpha$. Using the definition of \mathcal{D} and Corollary 4.1.5 we have

$$\begin{aligned} \mathcal{D}(A) = \{X \in L(\alpha + 7) : \\ L(\alpha + 7) \models X \subset A \wedge \exists n, k, s, R [n \in \omega \wedge k = Ar(n) \wedge \\ s \in A^{k-1} \wedge R = Df(A, n) \wedge \forall x \in A (x \in X \Leftrightarrow s^{\hat{}}x \in R)]\}. \end{aligned}$$

Thus, $\mathcal{D}(A) \in L(\alpha + 8) \subset L(\lambda)$.

In order to show “ $Y = \mathcal{D}(A)$ ” $\Leftrightarrow L(\lambda) \models$ “ $Y = \mathcal{D}(A)$ ”, let $Y, A \in L(\alpha)$ for some $\omega < \alpha < \lambda$.

Suppose “ $Y = \mathcal{D}(A)$ ”. As universal quantification is downward absolute and as “ $X \in \mathcal{D}(A)$ ” is absolute for $L(\lambda)$ by Lemma 4.1.6, $L(\lambda) \models \forall X [X \in Y \Rightarrow X \in \mathcal{D}(A)]$. For the second conjunction of “ $Y = \mathcal{D}(A)$ ”, suppose $n, k, s \in L(\lambda)$ are such that $L(\lambda) \models n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1}$. As these formulas are Δ_0 , $n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1}$. By assumption, $Y = \mathcal{D}(A)$. So let g, R, X be such that $\nu(g, R, A, n) \wedge \forall x \in A (x \in X \Leftrightarrow s^{\hat{}}x \in R) \wedge X \in Y$ where ν is as in Lemma 4.1.4. As $A \in L(\lambda)$, Corollary 4.1.5 implies that $g \in L(\lambda)$. Now $R \in \bigcup \bigcup g$; by the transitivity of $L(\lambda)$, $R \in L(\lambda)$. Similarly, $X \in Y \in L(\lambda)$ implies that $X \in L(\lambda)$.

Since ν is absolute for $L(\lambda)$ by Lemma 4.1.4 and the rest is Δ_0 , it follows that

$$\begin{aligned} L(\lambda) \models \forall n, k, s [(n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1}) \\ \Rightarrow \exists g, X [\nu(g) \wedge \forall x \in A (x \in X \Leftrightarrow s \hat{x} \in g(n)) \wedge X \in Y]] \end{aligned}$$

Thus, $L(\lambda) \models "Y = \mathcal{D}(A)"$.

Conversely, suppose $L(\lambda) \models "Y = \mathcal{D}(A)"$. First, let $X \in Y$. Since $L(\lambda)$ is transitive, $X \in L(\lambda)$. So by assumption, $L(\lambda) \models X \in \mathcal{D}(A)$. But " $X \in \mathcal{D}(A)$ " is absolute for $L(\lambda)$. So $X \in \mathcal{D}(A)$, and thus, $\forall X [X \in Y \Rightarrow X \in \mathcal{D}(A)]$. For the second conjunction of $Y = \mathcal{D}(A)$, let n, k, s be such that $n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1}$. As $\lambda > \omega$, $n, k \in L(\lambda)$. As $A \in L(\lambda)$, Lemma 4.1.2 gives $s \in L(\lambda)$. Since these formulas are Δ_0 , $L(\lambda) \models n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1}$. As existential quantification is upward absolute and since the following formula is absolute for $L(\lambda)$,

$$\exists g, R, X (\nu(g, R, A, n) \wedge \forall x \in A (x \in X \Leftrightarrow s \hat{x} \in R) \wedge X \in Y)$$

hence, " $Y = \mathcal{D}(A)$ ". Therefore, " $Y = \mathcal{D}(A)$ " is absolute for $L(\lambda)$. \square

Consider the map $\alpha \mapsto L(\alpha)$; we write $\mathcal{L} \upharpoonright \alpha + 1 = \{(\beta, L(\beta)) : \beta \leq \alpha\}$. Thus,

$$\begin{aligned} "f = \mathcal{L} \upharpoonright \alpha + 1" \equiv \alpha \in \mathbf{ON} \wedge f \text{ is a function} \wedge \text{dom } f = \alpha + 1 \wedge \\ \forall \beta \in \text{dom } f [(\beta = 0 \Rightarrow f(\beta) = \emptyset) \wedge (\beta \text{ limit} \Rightarrow f(\beta) = \bigcup_{\gamma < \beta} f(\gamma)) \wedge \\ (\beta \text{ successor} \Rightarrow f(\beta) = \mathcal{D}(f(\beta - 1)))] \end{aligned}$$

The following proposition is the germ of the proof that $\alpha \mapsto L(\alpha)$ is absolute for $L(\lambda)$, λ limit.

Proposition 4.1.8. *For each $\alpha \in \mathbf{ON}$, $\mathcal{L} \upharpoonright \alpha + 1 \in L(\alpha + \omega)$.*

Proof. By transfinite induction on α . The case $\alpha = 0$ follows from $\mathcal{L} \upharpoonright 1 = \{(0, \emptyset)\} \in L(3) \subset L(\alpha + \omega)$. Now suppose $\alpha > 0$ and that for all $\beta < \alpha$, $\mathcal{L} \upharpoonright \beta + 1 \in L(\beta + \omega)$.

If $\alpha = \beta + 1$ then $\mathcal{L} \upharpoonright \beta + 1 \in L(\beta + k)$ for some $k \in \omega$. Since $\beta + 1, L(\beta + 1) \in L(\beta + 2)$, it follows that $\{(\beta + 1, L(\beta + 1))\} \in L(\beta + 5)$. Thus,

$$\mathcal{L} \upharpoonright \alpha + 1 = \mathcal{L} \upharpoonright \beta + 1 \cup \{(\beta + 1, L(\beta + 1))\} \in L(\beta + k + 5) \subset L(\alpha + \omega).$$

If α is limit, we claim that $\mathcal{L} \upharpoonright \alpha + 1 \in L(\alpha + 1)$. It suffices to show that

$$\bigcup_{\beta < \alpha} \mathcal{L} \upharpoonright \beta + 1 = \{z \in L(\alpha) : L(\alpha) \models \exists \beta, g (g = \mathcal{L} \upharpoonright \beta + 1 \wedge z = (\beta, g(\beta)))\}$$

Suppose $z \in \bigcup_{\beta < \alpha} \mathcal{L} \upharpoonright \beta + 1$. Then for some $\beta < \alpha$ we have $z \in \mathcal{L} \upharpoonright \beta + 1$. Thus, $z = (\beta, L(\beta)) \in L(\beta + 4) \subset L(\alpha)$. By the induction hypothesis, there is a $g \in L(\beta + k) \subset L(\alpha)$ such that $g = \mathcal{L} \upharpoonright \beta + 1$ for some k . Now the formula “ $g = \mathcal{L} \upharpoonright \beta + 1$ ” is Δ_0 , except for the occurrence of the \mathcal{D} function. As α is limit, \mathcal{D} is absolute for $L(\alpha)$. Thus, $L(\alpha) \models g = \mathcal{L} \upharpoonright \beta + 1$, and hence

$$L(\alpha) \models \exists \beta, g (g = \mathcal{L} \upharpoonright \beta + 1 \wedge z = (\beta, g(\beta)))$$

The reverse inclusion follows from the fact that existential quantification is upward absolute. \square

Lemma 4.1.9. *For all $\lambda > \omega$ limit and $\alpha < \lambda$, $\mathcal{L} \upharpoonright \alpha + 1$ is absolute for $L(\lambda)$.*

Proof. Let $\lambda > \omega$ limit and $\alpha < \lambda$. Letting $f = \mathcal{L} \upharpoonright \alpha + 1$, Proposition 4.1.8 implies that $f \in L(\alpha + \omega) \subseteq L(\lambda)$. Moreover, the formula “ $f = \mathcal{L} \upharpoonright \alpha + 1$ ” is Δ_0 , except for the occurrence of the \mathcal{D} function, which is absolute for $L(\lambda)$ by Lemma 4.1.7. \square

Abbreviating $\exists f [f = \mathcal{L} \upharpoonright \alpha + 1 \wedge x \in f(\alpha)]$ by “ $x \in L(\alpha)$ ” produces the following.

Proposition 4.1.10. *“ $x \in L(\alpha)$ ” is absolute for $L(\lambda)$, $\lambda > \omega$ limit.*

Proof. Combine Proposition 4.1.8 and Lemma 4.1.9. \square

Finally, we reach the goal of this section. Recall that

$$\mathbf{“V = L”} \Leftrightarrow \forall x (x \in \mathbf{L}) \Leftrightarrow \forall x \exists \alpha (x \in L(\alpha))$$

Theorem 4.1.11. *For every limit ordinal $\lambda > \omega$, $L(\lambda) \models \mathbf{V=L}$.*

Proof. Let $x \in L(\lambda)$ for some $\lambda > \omega$ limit; so $x \in L(\alpha)$ for some $\omega < \alpha < \lambda$. Letting $f = \mathcal{L} \upharpoonright \alpha + 1$, it follows that $f \in L(\alpha + \omega) \subseteq L(\lambda)$ by Proposition 4.1.8. The absoluteness of “ $f = \mathcal{L} \upharpoonright \alpha + 1$ ” for $L(\lambda)$ from Lemma 4.1.9 insures that $L(\lambda) \models “f = \mathcal{L} \upharpoonright \alpha + 1”$. As $x \in f(\alpha)$ is Δ_0 we have $L(\lambda) \models \forall x \exists \alpha, f[f = \mathcal{L} \upharpoonright \alpha + 1 \wedge x \in f(\alpha)]$. \square

4.2 Consequences of $\mathbf{V=L}$

In this section, we present a few important consequences of $\mathbf{V=L}$. First, we show under what circumstances will a set $M = L(\lambda)$ for some $\lambda > \omega$ limit. Next, we prove that there is a parameter-free uniformly $L(\lambda)$ -definable well ordering of $L(\lambda)$; hence, $L(\lambda) \models AC$. Finally, we define Skolem functions and Skolem hulls and prove some fundamental absoluteness and definability results.

4.2.1 Transitive models of $\mathbf{V=L}$

In this section we show that if M is a transitive set modeling $\mathbf{V=L}$ and a finite fragment of ZFC, then $M = L(\lambda)$ for some limit λ . We define Ψ_1 to be the conjunction of Axioms of Extensionality, Infinity, Pairing, Foundation, and the following sentences:

$$\begin{aligned} \forall x \exists y [x \in \mathbf{ON} \Rightarrow y = \text{Suc}(x)] \\ \forall x, n \exists y [n \in \omega \Rightarrow y = x^n] \\ \forall x \exists y [y = \mathcal{D}(x)] \end{aligned}$$

It should be clear that for all $\lambda > \omega$ limit, $L(\lambda) \models \Psi_1$.

Let M be any set. We define the **ordinal** of M , denoted $o(M)$, to be the least ordinal not in M . Given $\alpha \in \mathbf{ON}$ we denote the **successor** of α by $\text{Suc}(\alpha)$.

Proposition 4.2.1. *For all transitive M such that $M \models \mathbf{V=L} \wedge \Psi_1$,*

1. $o(M) > \omega$ is limit,
2. \mathcal{D} is absolute for M , and

3. $M = L(o(M))$.

Proof. Let M be a transitive set modeling $\mathbf{V}=\mathbf{L}$ and Ψ_1 . That M is well-founded follows from $M \models \text{Foundation}$. $M \models \text{Infinity} \wedge \forall x \exists y [x \in \mathbf{ON} \Rightarrow y = \text{Suc}(x)]$ implies that $o(M) > \omega$ is limit. Also, the transitivity of M implies that any Δ_0 formula is absolute for M ; hence, all of the defined functions in Lemma 4.1.2 are absolute for M .

To show \mathcal{D} is absolute for M , let $A \in M$. Since $M \models \forall x \exists y [y = \mathcal{D}(x)]$, let $Y \in M$ be such that $M \models Y = \mathcal{D}(A)$; we show that $Y = \mathcal{D}(A)$. For the definition of “ $Y = \mathcal{D}(A)$ ”, see the comments immediately preceding Lemma 4.1.7. For the first conjunction, suppose $X \in Y$. As M is transitive, $Y \in M$ implies that $X \in M$. This is Δ_0 , thus $M \models X \in Y$. By assumption, $M \models X \in \mathcal{D}(A)$. We claim that “ $X \in \mathcal{D}(A)$ ” is absolute for M ; the argument is similar to the proof of Lemma 4.1.6. To see this, we first observe that since $M \models \forall x, n \exists y [n \in \omega \Rightarrow y = x^n]$, then for all n , $Df(A, n) \in M$. Since $M \models \text{Pairing}$, the function g from the inductive definition of “ $R = Df(A, n)$ ” is also in M . The rest follows from the fact that all of the subformulas in “ $X \in \mathcal{D}(A)$ ” are Δ_0 . Thus, $X \in \mathcal{D}(A)$. For the second conjunction, let n, k, s be such that $n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1}$. As $o(M) > \omega$, it follows that $n, k \in M$. Since $M \models \forall x, n \exists y [n \in \omega \Rightarrow y = x^n]$, it follows that $s \in M$. Moreover, $M \models n \in \omega \wedge k = Ar(n) \wedge s \in A^{k-1}$, since this formula is absolute for M . So by assumption, $\exists g, R, X \in M [\nu(g, R, A, n) \wedge \forall x \in A [x \in X \Leftrightarrow s \hat{x} \in R] \wedge X \in Y]$. As ν is absolute for M and the rest is Δ_0 , it follows that $Y = \mathcal{D}(A)$. Similar arguments show that

$$Y = \mathcal{D}(A) \Rightarrow M \models Y = \mathcal{D}(A)$$

Finally, we show $M = L(o(M))$. Let $o(M) = \lambda$ for some $\lambda > \omega$ limit.

Suppose $x \in M$. By assumption, $M \models \mathbf{V}=\mathbf{L}$. So let $\alpha, f \in M$ be such that $M \models f = \mathcal{L} \upharpoonright \alpha + 1 \wedge x \in f(\alpha)$. The formula “ $f = \mathcal{L} \upharpoonright \alpha + 1$ ” is Δ_0 , except for the \mathcal{D} function, which we proved is absolute for M . Thus, as M is transitive, “ $f = \mathcal{L} \upharpoonright \alpha + 1$ ” is absolute for M . So $f = \mathcal{L} \upharpoonright \alpha + 1$ and $x \in f(\alpha)$. As $\alpha \in M$ it must be that $\alpha < \lambda$. Thus, $\alpha \in L(\alpha + 1) \subset L(\lambda)$. By Proposition 4.1.8 $f \in L(\alpha + \omega) \subseteq L(\lambda)$. Hence $x \in f(\alpha)$ implies $x \in L(\lambda)$. Therefore, $M \subseteq L(\lambda)$.

Suppose $x \in L(\lambda)$. As $\lambda > \omega$ is limit, Theorem 4.1.11 gives $L(\lambda) \models \mathbf{V}=\mathbf{L}$. So let $\alpha, f \in L(\lambda)$ be such that $L(\lambda) \models f = \mathcal{L} \upharpoonright \alpha + 1 \wedge x \in f(\alpha)$. Since $\alpha < \lambda = o(M)$, it follows that $\alpha \in M$. By assumption, $M \models \mathbf{V}=\mathbf{L}$. So let $\beta, g \in M$ be such that $M \models g = \mathcal{L} \upharpoonright \beta + 1 \wedge \alpha \in g(\beta)$. Now $\alpha \in g(\beta)$ implies that $\alpha \leq \beta$. We claim that for all $\gamma \leq \alpha$, $f(\gamma) = g(\gamma)$ (by induction on γ). For $\gamma = 0$ this is immediate. Suppose that for all $\delta < \gamma$, $f(\delta) = g(\delta)$. If γ is a limit ordinal, then

$$f(\gamma) = \bigcup_{\delta < \gamma} f(\delta) = \bigcup_{\delta < \gamma} g(\delta) = g(\gamma).$$

If γ is a successor, it follows that

$$f(\gamma) = \mathcal{D}(f(\gamma - 1)) = \mathcal{D}(g(\gamma - 1)) = \mathcal{D}^M(g(\gamma - 1)) = g(\gamma)$$

as the \mathcal{D} function is absolute for M . Thus, $f(\gamma) = g(\gamma)$ for all $\gamma \leq \alpha$. Consequently, $x \in f(\alpha) = g(\alpha)$. Moreover, $\alpha \leq \beta$ implies $g(\alpha) \subseteq g(\beta)$. Thus, $x \in g(\beta) \in M$. Therefore, $L(\lambda) \subseteq M$ and we are done. \square

4.2.2 AC in \mathbf{L}

Another consequence of $L(\lambda) \models \mathbf{V} = \mathbf{L}$ is a uniformly definable well-ordering of $L(\lambda)$ for $\lambda > \omega$ limit. To improve the readability of the formula defining this well-order, we make the following abbreviations and notations.

Concerning the numbering of formulas, it should be clear from our definition of the Df function in Section 4.1 that for every formula $\varphi(x_0, x_1, \dots, x_{k-1})$, there is a canonical n such that $\{s \in A^k : \varphi^A(s_0, s_1, \dots, s_{k-1})\} = Df(A, n)$. As a result, we write φ_n to denote the n^{th} formula in this enumeration. Moreover, this enumeration is primitive in the sense that if φ_m is a subformula of φ_n , then $m < n$. Finally, recall that the Ar function recovers the arity of the free variables of φ_n .

Given $x \in \mathbf{L}$, recall that $\rho(x)$ denotes the rank of x ; that is, “ $\alpha = \rho(x)$ ” $\Leftrightarrow \alpha \in \mathbf{ON} \wedge x \notin L(\alpha) \wedge x \in L(\alpha + 1)$. For $x \in L(\alpha + 1)$, let $n(x)$ be the least n such that x is definable from the n^{th} formula for some parameter $s \in L(\alpha)^{k-1}$ where $k = Ar(n)$.

Formally, “ $m = n(x)$ ” if, and only if

$$\begin{aligned} m \in \omega \wedge \exists \alpha [\alpha = \rho(x) \\ \wedge \exists s \in L(\alpha)^{Ar(m)-1} (x = \{z \in L(\alpha) : \varphi_m^{L(\alpha)}(z, s_0, \dots, s_{Ar(m)-1})\}) \\ \wedge \forall l < m \forall s \in L(\alpha)^{Ar(l)-1} (x \neq \{z \in L(\alpha) : \varphi_l^{L(\alpha)}(z, s_0, \dots, s_{Ar(l)-1})\})]. \end{aligned}$$

Given $x \in L(\alpha)$, let $Ar(n(x)) - 1 = a_x$; let $\xi(x, y, j, f, \beta)$ be the formula $x \upharpoonright j = y \upharpoonright j \wedge (x(j), y(j)) \in f(\beta - 1)$. These are Δ_0 and hence absolute for $L(\lambda)$. Since “ $x \in L(\alpha)$ ” and the satisfaction relation are absolute for $L(\lambda)$, the formulas “ $\alpha = \rho(x)$ ” and “ $m = n(x)$ ” are absolute for $L(\lambda)$.

We fix a formula θ of two free variables. $\theta(x, y) \Leftrightarrow \exists f, \alpha \bar{\theta}(x, y, f, \alpha)$ where $\bar{\theta}(x, y, f, \alpha)$ is the formula :

$$\begin{aligned} f \text{ is a function} \wedge \alpha \in \mathbf{ON} \wedge \text{dom } f = \alpha + 1 \wedge f(0) = \emptyset \wedge \\ \forall \beta \in \text{dom } f \\ [(\beta \text{ limit} \wedge \beta > 0) \Rightarrow f(\beta) = \bigcup_{\delta < \beta} f(\delta) \wedge \\ \beta \text{ successor} \Rightarrow f(\beta) = \{(x, y) \in L(\beta) \times L(\beta) : \\ (x, y) \in f(\beta - 1) \vee (x \in L(\beta - 1) \wedge y \notin L(\beta - 1)) \vee \\ (x, y \notin L(\beta - 1) \wedge n(x) < n(y)) \vee \\ (x, y \notin L(\beta - 1) \wedge n(x) = n(y) \wedge \\ \exists s, t \in L(\beta - 1)^{a_x} \\ [x = \{z \in L(\beta - 1) : \varphi_{n(x)}^{L(\beta-1)}(z, s)\}] \wedge \\ \forall r \in L(\beta - 1)^{a_x} \forall j < a_x \\ [\xi(r, s, j, f, \beta) \rightarrow x \neq \{z \in L(\beta - 1) : \varphi_{n(x)}^{L(\beta-1)}(z, r)\}] \wedge \\ y = \{z \in L(\beta - 1) : \varphi_{n(y)}^{L(\beta-1)}(z, t)\}] \wedge \\ \forall r \in L(\beta - 1)^{a_y} \forall j < a_y \\ [\xi(r, t, j, f, \beta) \rightarrow y \neq \{z \in L(\beta - 1) : \varphi_{n(y)}^{L(\beta-1)}(z, r)\}] \wedge \\ \exists j < a_x (\xi(s, t, j, f, \beta)))] \wedge (x, y) \in f(\alpha) \end{aligned}$$

Proposition 4.2.2. *The relation $\theta(x, y)$ uniformly defines a well order $\triangleleft_{L(\lambda)}$ of $L(\lambda)$ for $\lambda > \omega$ limit.*

Proof. Let $\lambda > \omega$ be limit. θ induces a well-order $\triangleleft_{L(\lambda)}$ of $L(\lambda)$ in the following way. Given two elements of $L(\lambda)$, first compare by rank using the usual well-order of the ordinals. If they have the same rank, then compare by formula using the canonical enumeration of formulas and the usual well-order on the integers. If two sets of the same rank are definable by the same formula, then compare the set of parameters defining each set using the lexicographic order on the set of k -tuples. This ordering is necessarily a linear order such that every nonempty subset of $L(\lambda)$ has a $\triangleleft_{L(\lambda)}$ -least element. It remains verify the absoluteness of θ for $L(\lambda)$.

Let $x, y \in L(\lambda)$. From Corollary 4.1.5 and Proposition 4.1.10, we have the absoluteness for $L(\lambda)$ of the formulas “ $R \in Df(A, n)$ ” and “ $x \in L(\beta)$ ”. It follows that $\bar{\theta}(x, y, f, \alpha)$ is absolute for $L(\lambda)$. Thus, $\theta^{L(\lambda)}(x, y) \Rightarrow \theta(x, y)$ is immediate, as existential quantification is upward absolute.

Now suppose $\theta(x, y)$. Let f, α be such that $\bar{\theta}(x, y, f, \alpha)$. Without loss of generality we assume that $\alpha < \lambda$. (If $\alpha > \lambda$, then let $\alpha' = \max\{\rho(x), \rho(y)\} + 1$ so that $\alpha' < \lambda$. Thus, $\bar{\theta}(x, y, f', \alpha')$ follows by taking $f' = f \upharpoonright \alpha'$.) Since $\alpha \in L(\alpha + 1) \subset L(\lambda)$ it remains to show that $f \in L(\lambda)$. Reasoning similarly as in Proposition 4.1.8, it should be clear that $f \in L(\alpha + \omega)$. As $\bar{\theta}$ is absolute for $L(\lambda)$, it follows that $L(\lambda) \models \theta(x, y)$. \square

Hereafter, when we write $\triangleleft_{L(\lambda)}$ we mean the $L(\lambda)$ -definable well order of $L(\lambda)$ induced by θ . As all of \mathbf{L} can be well ordered by θ , we can now complete Theorem 4.1.1.

Theorem 4.2.3. $\mathbf{L} \models ZFC$

4.2.3 Skolem functions and Skolem hulls

Given formula $\varphi_n(v_1, \dots, v_k)$ with free variables among v_1, \dots, v_k , a **Skolem function** for φ_n over A is a function $f : A^k \rightarrow A$ such that

1. if φ_n is $\exists u \varphi_i(u, v_1, \dots, v_k)$ and there is $y \in A$ such that $\varphi_i(y, x_1, \dots, x_k)$, then $f(x_1, \dots, x_k)$ is the \triangleleft_A -least such y , or
2. if φ_n is $\exists u \varphi_i(u, v_1, \dots, v_k)$ and there is not $y \in A$ such that $\varphi_i(y, x_1, \dots, x_k)$, then $f(x_1, \dots, x_k) = \emptyset$, or
3. if φ_n is not of the form $\exists u \varphi_i(u, v_1, \dots, v_k)$ or if $k = 0$, then $f(x_1, \dots, x_k) = \emptyset$.

Recall from Section 4.1 that $Proj$ codes the existential formulas. That is, $Df(A, n) = Proj(A, Df(A, i), k)$, if $n = 2^5 \cdot 3^i \cdot 5^j \cdot 7^k$ where $i = 2^p \cdot 3^q \cdot 5^r \cdot 7^{k+1}$ for $p = 1, 2, 3, 4, 5$ and some q, r , and $j = 0$. So we have “ $f_n(x_1, \dots, x_k) = y$ ” $\Leftrightarrow \sigma(n, x_1, \dots, x_k, y)$ where σ abbreviates the formula

$$\begin{aligned}
& \exists i, j, k, p, q, r \in \omega \\
& [(n = 2^5 \cdot 3^i \cdot 5^j \cdot 7^k \wedge i = 2^p \cdot 3^q \cdot 5^r \cdot 7^{k+1} \wedge j = 0 \wedge k > 0 \wedge p < 6) \wedge \\
& ((\varphi_i(x_1, \dots, x_k, y) \wedge \forall z [\theta(z, y) \Rightarrow \neg \varphi_i(x_1, \dots, x_k, z)]) \vee \\
& \forall z (\neg \varphi_i(x_1, \dots, x_k, z) \wedge y = \emptyset))] \vee \\
& \neg \exists i, j, k, p, q, r \in \omega \\
& [(n = 2^5 \cdot 3^i \cdot 5^j \cdot 7^k \wedge i = 2^p \cdot 3^q \cdot 5^r \cdot 7^{k+1} \wedge j = 0 \wedge k \geq 0 \wedge p < 6) \\
& \wedge y = \emptyset]
\end{aligned}$$

As θ is $L(\lambda)$ -definable for $\lambda > \omega$ limit and since the satisfaction relation is absolute for $L(\lambda)$, the relation “ $f_n(x_1, \dots, x_k) = y$ ” is $L(\lambda)$ -definable. Furthermore, any finite set of Skolem functions f_1, \dots, f_N is $L(\lambda)$ -definable.

Suppose $N \in \omega$ and f_1, \dots, f_N are the Skolem functions for A corresponding to the formulas $\varphi_1, \dots, \varphi_N$. Consider the following subset $H \subset A$:

$$\begin{aligned} H_0 &= \omega \\ H_{l+1} &= H_l \cup \left(\bigcup_{i=1}^N \{f_i(x_1, \dots, x_{k_i}) : x_1, \dots, x_{k_i} \in H_l\} \right) \\ H &= \bigcup_{l \in \omega} H_l \end{aligned}$$

H is the **Skolem hull** of ω inside A under f_1, \dots, f_N . It should be clear that the formulas $\varphi_1, \dots, \varphi_N$ corresponding to the Skolem functions f_1, \dots, f_N are absolute for H, A . This is critical since a hull H need not be transitive so that even Δ_0 formulas are not absolute for H, A . As we can close H under f_1, \dots, f_N for any N , by choosing N large enough, we can make any finite number of formulas absolute for H, A . Note that given an LST-formula φ , we write $\ulcorner \varphi \urcorner$ to denote the Gödel number of φ according to our enumeration given by the Df function on page 28.

Lemma 4.2.4. *Let $\lambda > \omega$ be a limit ordinal. Suppose H the Skolem hull of ω inside $L(\lambda)$ under f_1, \dots, f_N where $N > \ulcorner \mathbf{V} = \mathbf{L} \wedge \Psi_1 \urcorner$. Then, for $1 \leq n \leq N$, “ $f_n(x_1, \dots, x_k) = y$ ” is absolute for $H, L(\lambda)$.*

Proof. Let λ, N, H , and n be as above and let $x_1, \dots, x_k, y \in H$ where $k = Ar(n)$. In order to show $H \models \sigma(n, x_1, \dots, x_k, y) \Leftrightarrow L(\lambda) \models \sigma(n, x_1, \dots, x_k, y)$ we concentrate on that part of σ , denoted here by $\psi(i, x_1, \dots, x_k, y)$,

$$\begin{aligned} (\varphi_i(x_1, \dots, x_k, y) \wedge \forall z[\theta(z, y) \Rightarrow \neg\varphi_i(x_1, \dots, x_k, z)]) \vee \\ \forall z[\neg\varphi_i(x_1, \dots, x_k, z) \wedge y = \emptyset] \end{aligned}$$

as the absoluteness of the rest of σ easily follows.

First suppose $H \models \psi(i, x_1, \dots, x_k, y)$ where $i < n$. In order to show that $L(\lambda)$ models the first half of the disjunction, observe first that φ_i is absolute for $H, L(\lambda)$ since H is closed under the i^{th} Skolem function. Thus, $\varphi_i^{L(\lambda)}(x_1, \dots, x_k, y)$. Now suppose $z \in L(\lambda)$ is such that $\theta^{L(\lambda)}(z, y)$. If $z \in H$, then $\theta^H(z, y)$ as θ is absolute

for $H, L(\lambda)$. Thus, by assumption $\neg\varphi_i^H(x_1, \dots, x_k, z)$, hence $\neg\varphi_i^{L(\lambda)}(x_1, \dots, x_k, z)$. So suppose $z \notin H$ and suppose for contradiction that $\varphi_i^{L(\lambda)}(x_1, \dots, x_k, z)$. Now, $\theta^{L(\lambda)}(z, y)$ implies that y must not be the $\triangleleft_{L(\lambda)}$ -least element of $L(\lambda)$ such that $\varphi_i^{L(\lambda)}(x_1, \dots, x_k, z_0)$. Let $z_0 \in L(\lambda)$ be $\triangleleft_{L(\lambda)}$ -least such that $\varphi_i^{L(\lambda)}(x_1, \dots, x_k, z_0)$. Since H is closed under the i^{th} Skolem function, $z_0 \in H$, and hence,

$$H \models \varphi_i(x_1, \dots, x_k, z_0) \wedge \forall z[\theta(z, z_0) \Rightarrow \neg\varphi_i(x_1, \dots, x_k, z)]$$

contradicting the hypothesis $H \models \psi(i, x_1, \dots, x_k, y)$. Thus, $\neg\varphi_i^{L(\lambda)}(x_1, \dots, x_k, z)$.

For the second part of the disjunction, suppose that $z \in L(\lambda)$ is such that $\varphi_i^{L(\lambda)}(x_1, \dots, x_k, y)$. So $\varphi_i^H(x_1, \dots, x_k, y)$ follows by the absoluteness of φ_i for $H, L(\lambda)$. As $\emptyset \in H$, “ $y = \emptyset$ ” is absolute for $H, L(\lambda)$; hence, $L(\lambda) \models \forall z[\neg\varphi_i(x_1, \dots, x_k, z) \wedge y = \emptyset]$. So $H \models \psi(i, x_1, \dots, x_k, y) \Rightarrow L(\lambda) \models \psi(i, x_1, \dots, x_k, y)$.

$L(\lambda) \models \psi(i, x_1, \dots, x_k, y) \Rightarrow H \models \psi(i, x_1, \dots, x_k, y)$ follows from $H \subseteq L(\lambda)$ and since universal quantification is upward absolute. \square

Next, we show there is an $L(\lambda)$ -definable surjection $F : \omega \rightarrow H$. Let (n_1, \dots, n_k) be a finite sequence of integers and let $\{p_k\}_{k \in \omega}$ be the usual enumeration of the primes. We say that the **code** of (n_1, \dots, n_k) is

$$\langle n_1, \dots, n_k \rangle = 2^{n_1+1} \cdot 3^{n_2+1} \cdot \dots \cdot p_k^{n_k+1}$$

It is clear that the coding function $\langle \cdot \rangle : \omega^{<k} \rightarrow \omega$ is injective. Furthermore, $\langle \cdot \rangle$ is a recursive function as is the decoding function that maps a k -tuple to its code. Thus, we will use $\langle \cdot \rangle$ freely in future formulas with the understanding that it can be replaced by a formula that is absolute for $L(\lambda)$, $\lambda > \omega$ limit.

Define $F : \omega \rightarrow H$ recursively as follows:

$$F(n) = \begin{cases} c & \text{if } n = \langle 0, c \rangle; \\ f_i(x_1, \dots, x_{k_i}) & \text{if } n = \langle l + 1, i, c_1, \dots, c_{k_i} \rangle \text{ for } 1 \leq i \leq N \text{ and} \\ & \text{for each } 1 \leq j \leq k_i, x_j = F(c_j) \text{ for } x_j \in H_l; \\ \emptyset & \text{if } n \text{ is not one of the above forms.} \end{cases}$$

Clearly, F is surjective. Moreover, “ $F(n) = x$ ” is $L(\lambda)$ -definable as a relation by the formula $\Phi(n, x)$:

$$\begin{aligned} & \exists g[g \text{ is a function} \wedge \text{dom } g = n + 1 \\ & \forall m \leq n[\forall l, c \leq n[(l = 0 \wedge m = \langle l, c \rangle) \Rightarrow g(m) = c] \wedge \\ & \quad \forall l, i, c_1, \dots, c_{k_i} \leq n, \forall x_1, \dots, x_{k_i} \\ & \quad [(l > 0 \wedge \forall j [1 \leq j \leq k_i \Rightarrow g(c_j) = x_j] \\ & \quad \wedge m = \langle l, i, c_1, \dots, c_{k_i} \rangle) \Rightarrow \sigma(i, x_1, \dots, x_{k_i}, g(m))] \wedge \\ & \quad \forall l, i, c_1, \dots, c_{k_i} \leq n, \forall x_1, \dots, x_{k_i} \\ & \quad [((l > 0 \wedge \exists j [1 \leq j \leq k_i \wedge g(c_j) \neq x_j] \wedge m = \langle l, i, c_1, \dots, c_{k_i} \rangle) \\ & \quad \vee (l > 0 \wedge m = \langle l, i \rangle) \vee m = \langle l \rangle \vee m = 0, 1) \Rightarrow g(m) = \emptyset] \\ & \wedge g(n) = x] \end{aligned}$$

where σ is as in Lemma 4.2.4. We claim that “ $F(n) = x$ ” is absolute for $H, L(\lambda)$. Upon examination $\Phi(n, x)$, we see that we need to close the hull under the Skolem functions corresponding to the formulas insuring that the following are absolute for $H, L(\lambda)$: ordered pairing; given an ordered pair (z_0, z_1) , each coordinate z_0, z_1 exists; and given a finite function g , every image $g(n)$ exists. Define Ψ_2 to be the conjunction of Ψ_1 and the sentences that insure the absoluteness of the above formulas.

Lemma 4.2.5. *Let $\lambda > \omega$ be a limit ordinal. Suppose H is the Skolem hull of ω inside $L(\lambda)$ under f_1, \dots, f_N where $N > \lceil \mathbf{V} = \mathbf{L} \wedge \Psi_2 \rceil$. Then, for $1 \leq n \leq N$, $\Phi(n, x)$ is absolute for $H, L(\lambda)$.*

Proof. Similar to Lemma 4.1.4. Let λ be limit. The absoluteness of Φ for $H, L(\lambda)$ follows from the absoluteness of σ for $H, L(\lambda)$ from Lemma 4.2.4. \square

4.3 The theory of $L(\lambda)$

A **structure** (in the language of set theory) is an ordered pair (A, E) such that A is a nonempty set and E is a binary relation on A . For a structure (A, E) , the **theory** of (A, E) is the set of all sentences that are true in (A, E) ; that is, $Th(A, E) = \{n \in \omega : (A, E) \models \varphi_n\}$. When E is understood, we simply write $Th(A)$. We show in this section that if $x \in \omega^\omega \cap \mathbf{L}$ is such that $\rho(x) = \alpha$, then $Th(L(\alpha + \omega)) \in L(\alpha + \omega + 2)$.

Suppose $x \in \omega^\omega \cap \mathbf{L}$, $\rho(x) = \alpha$. A priori, x may have parameters, that is infinite ordinals, figuring in its definition. Our first two lemmas of this section shows that we can eliminate the parameters. Note that we write $Lim(\alpha)$ to abbreviate the formula “ α is a limit ordinal.”

Lemma 4.3.1. *Let $\lambda = \alpha + \omega$ and let λ' be the largest limit ordinal less than λ . Then,*

1. *for all $l \in \omega$, $\lambda' + l$ is $L(\lambda)$ -definable without parameters, and*
2. *for all $y \in L(\lambda)$ and $l \in \omega$, if y is $L(\lambda' + l)$ -definable without parameters, then y is $L(\lambda)$ -definable without parameters.*

Proof. We prove the first claim by induction on l . For $l = 0$, consider $\varphi(\alpha) \equiv Lim(\alpha) \wedge \forall \beta [Lim(\beta) \Rightarrow (\beta \leq \alpha)]$. Clearly, $\varphi(\alpha)$ is parameter-free and $L(\lambda) \models \varphi(\lambda')$. Suppose now that the result holds for $\lambda' + l$; let $\psi(\alpha)$ be the parameter-free formula such that $L(\lambda) \models \psi(\lambda' + l)$. Consider $\varphi(\alpha) \equiv \exists \beta [\psi(\beta) \wedge \alpha = \beta + 1]$. Clearly, φ is parameter-free and $L(\lambda) \models \varphi(\lambda' + l + 1)$.

For the second claim, let $\psi(\alpha)$ be the parameter-free formula defining $\lambda' + l$ over $L(\lambda)$. Define a new formula $\bar{\varphi}(z, \alpha)$ to be the formula $\varphi(z)$ having all unbound quantifiers bound by $L(\alpha)$. Then, it is clear that

$$y = \{z \in L(\lambda) : L(\lambda) \models \exists \alpha [\psi(\alpha) \wedge “z \in L(\alpha)” \wedge \bar{\varphi}(z, \alpha)]\}$$

as “ $z \in L(\alpha)$ ” is absolute for $L(\lambda)$ by Proposition 4.1.10. \square

Lemma 4.3.2. *Suppose $x \in \omega^\omega \cap \mathbf{L}$ is such that $x \in L(\alpha+1) \setminus L(\alpha)$ and let $\lambda = \alpha + \omega$. Then there is a parameter-free $L(\lambda)$ -definable $y \in \omega^\omega \cap \mathbf{L}$ such that $y \notin L(\alpha)$.*

Proof. Let x, α, λ be as above. Define $S \subset L(\lambda)$ as follows:

$$S = \{(n, b_1, \dots, b_k) : n \in \omega \wedge b_1, \dots, b_k \in L(\alpha) \wedge \{z \in \omega : \varphi_n^{L(\alpha)}(z, b_1, \dots, b_k)\} \notin L(\alpha)\}$$

As x witnesses that $S \neq \emptyset$, let $\vec{a} = (n, a_1, \dots, a_k)$ be the $\triangleleft_{L(\lambda)}$ -least element of S . We claim that for some $l \in \omega$, $y = \{z \in \omega : \varphi_n^{L(\alpha)}(z, a_1, \dots, a_k)\} \notin L(\alpha)$ is $L(\alpha + l)$ -definable; thus, by Lemma 4.3.1, y is $L(\lambda)$ -definable without parameters, and we are done.

To prove the claim, let $\xi(\beta)$ be the parameter-free formula from Lemma 4.3.1 defining α over $L(\lambda)$. Let $\psi(\beta, \vec{b}, v)$ be the formula $\forall m \in \omega [m \in v \Leftrightarrow \varphi_n^{L(\beta)}(m, b_1, \dots, b_k)]$. Let $\zeta(z)$ be the formula

$$\begin{aligned} \exists \beta, \vec{b}, v [\xi(\beta) \wedge \varphi_n^{L(\beta)}(z, b_1, \dots, b_k) \wedge \psi(\beta, \vec{b}, v) \wedge v \notin L(\beta) \wedge \\ \forall \vec{c}, w [(\theta(\vec{c}, \vec{b}) \wedge \psi(\beta, \vec{c}, w)) \Rightarrow w \in L(\beta)]] \end{aligned}$$

where θ is as in Proposition 4.2.2. Recall that “ $v \in L(\beta)$ ” abbreviates the formula $\exists f [f = \mathcal{L} \upharpoonright \beta + 1 \wedge v \in f(\beta)]$. The formula “ $\theta(\vec{c}, \vec{b})$ ” abbreviates a similar existential formula (see Section 4.2). Next, recall that $\varphi_{b_0}^{L(\beta)}(z, b_1, \dots, b_n) \equiv \exists R [R = Df(L(\beta), b_0) \wedge b_1 \hat{\ } \dots \hat{\ } b_n \hat{\ } z \in R]$. Recall that the absoluteness of these formulas for $L(\lambda)$ required going up a finite number of levels beyond $L(\alpha)$. Thus, for some $k \in \omega$, $y = \{z \in \omega : L(\alpha + k) \models \zeta(z)\}$. \square

Hereafter, ζ is used to designate the parameter-free real y from Lemma 4.3.2 so that $y = \{n \in \omega : L(\lambda) \models \zeta(n)\}$. Clearly, y is different for each λ .

In order to achieve our goal $Th(L(\lambda)) \in L(\lambda + 2)$, we need an $L(\lambda)$ -definable map from ω onto $L(\lambda)$, $\lambda < \omega_1$. From the previous section, we have Φ defining a surjection of ω onto a Skolem hull H so it might seem natural to use Proposition 4.2.1 to get $H = L(\lambda)$, except that H need not be transitive. Yet this is easily remedied by

the Mostowski Collapsing Lemma. Briefly put, the Lemma states that every well-founded extensional structure is isomorphic to a unique \in -structure. Moreover, the isomorphism is unique. Once we close H under a sufficient number of functions, we collapse H to a transitive M . To insure $M = L(\lambda)$, we define Ψ_3 to be the conjunction of Ψ_2 and $\exists z \forall n \in \omega [n \in z \Leftrightarrow \zeta(n)]$ where ζ is as in Lemma 4.3.2.

Proposition 4.3.3. *Suppose $x \in \omega^\omega \cap \mathbf{L}$ is such that $x \in L(\alpha + 1) \setminus L(\alpha)$ and let $\lambda = \alpha + \omega$. Let H be the Skolem hull of ω inside $L(\lambda)$ under f_1, \dots, f_N where $N > \ulcorner \mathbf{V} = \mathbf{L} \wedge \Psi_3 \urcorner$ and let M be the transitive collapse of H . Then, $M = L(\lambda)$.*

Proof. Let $x, \alpha, \lambda, H, N, M$ be as above. Let $y = \{n \in \omega : \zeta^{L(\lambda)}(n)\}$ be such that $y \in L(\lambda) \setminus L(\alpha)$ as in Lemma 4.3.2. We observe that as $H \cong M$, every formula that is absolute for $H, L(\lambda)$ is also absolute for $M, L(\lambda)$. As $M \models \mathbf{V} = \mathbf{L} \wedge \Psi_1$, it follows from Proposition 4.2.1 that $M = L(\beta)$ for some limit β . Clearly, $\beta \leq \lambda$ as $H \subseteq L(\lambda)$. By the absoluteness of ζ for $M, L(\lambda)$, it follows that $y \in M$. As $y \notin L(\alpha)$, we have $\alpha \leq \beta$. If α is a successor, then β limit implies $\alpha + \omega = \lambda \leq \beta$. If α is limit, then $\alpha + 1 \leq \beta$ and thus $\alpha + \omega \leq \beta$. \square

Corollary 4.3.4. *Suppose $x \in \omega^\omega \cap \mathbf{L}$ is such that $x \in L(\alpha + 1) \setminus L(\alpha)$ and let $\lambda = \alpha + \omega$. Then, there is an $L(\lambda)$ -definable surjection $G : \omega \rightarrow L(\lambda)$.*

Proof. Use the formula $\Phi(n, x)$. \square

Theorem 4.3.5. *Suppose $x \in \omega^\omega \cap \mathbf{L}$ is such that $x \in L(\alpha + 1) \setminus L(\alpha)$ and let $\lambda = \alpha + \omega$. Then $Th(L(\lambda)) \in L(\lambda + 2)$.*

Proof. Let x, α, λ be as above. By definition, $Th(L(\lambda)) = \{n \in \omega : L(\lambda) \models \varphi_n\}$. Rather than explicitly writing the formula that defines $Th(L(\lambda))$ over $L(\lambda + 1)$, we observe that by using the $L(\lambda)$ -definable map $G : \omega \rightarrow L(\lambda)$, we can refer to any set in $L(\lambda)$ by its code. So even though the satisfaction relation over $L(\lambda)$ amounts to a formula involving the Df function and other sets in $L(\lambda)$, we can replace each such reference by an integer. Thus, each $n \in \omega$ such that $L(\lambda) \models \varphi_n$ is definable over $L(\lambda)$, and hence $Th(L(\lambda))$ is definable over $L(\lambda + 1)$. Hence, $Th(L(\lambda)) \in L(\lambda + 2)$. \square

4.4 A non-determined Π_1^1 set

We conclude this chapter with our construction of a Π_1^1 set of Turing degrees which neither contains nor omits a cone. Readers familiar with effective descriptive set theory will recognize this set as being Π_1^1 . In the first part of this final section, we give a brief sketch of the rudiments of effective descriptive set theory (see [Mar77] or [Mos80] for details) in order to motivate this change from boldface to lightface notation. We then define our set of reals which codes the theories of structures that fulfill the conditions of Theorem 4.3.5 and proceed to prove that this set is Π_1^1 . This set of reals gives rise to a set of degrees which we prove neither contains or omits a cone, and thus, by Martin's theorem, this set must be non-determined.

Recall from Sections 2.2 and 2.3 the Borel and projective hierarchies. Because they are topological in nature, these hierarchies constitute the domain of **classical descriptive set theory**; for uncountable Polish spaces, the classical results are rich and deep. Yet when defined on the countable Polish space ω , the Borel and projective hierarchies collapse as every subset of ω is open. By replacing the topological notion of open set with that of a semirecursive set, a nontrivial Borel and projective hierarchy that retains much of the classical character emerges. Descriptive set theory on ω from this approach is generally referred to as **effective descriptive set theory**. Defining the relationship between classical and effective descriptive set theory requires a brief sketch of recursion theory.

Though the definitions of recursive and semirecursive functions can be rigorously developed (see [End77] and [Sho67]), we omit these rudiments for the sake of proceeding directly to the main result and approach these definitions intuitively. A function $F : \omega^k \rightarrow \omega$ is **semirecursive** if there is an algorithm such that given $(n_1, \dots, n_k) \in \omega^k$, the algorithm eventually halts and produces $F(n_1, \dots, n_k) \Leftrightarrow (n_1, \dots, n_k) \in \text{dom } F$. A relation R is semirecursive if there is an algorithm which when applied to the inputs (n_1, \dots, n_k) gives an output iff $R(n_1, \dots, n_k)$; that is, the algorithm will eventually produce a 'yes' or 1 iff $(n_1, \dots, n_k) \in R$. A function $F : \omega^k \rightarrow \omega$ is **recursive** if there is an algorithm which accepts $(n_1, \dots, n_k) \in \text{dom } F$

as input and eventually produces $F(n_1, \dots, n_k)$. In other words, a function is recursive if its graph is semirecursive. A relation $R \subseteq \omega^k$ is recursive iff its characteristic function is recursive; that is, given $(n_1, \dots, n_k) \in \omega^k$, the algorithm produces a 'yes' or 1 iff $R(n_1, \dots, n_k)$ and a 'no' or 0 iff $\neg R(n_1, \dots, n_k)$.

A relation P is **arithmetical** if it has an explicit definition

$$P(x) \Leftrightarrow Q_1 x_1 \dots Q_n x_n R(x, x_1, \dots, x_n)$$

where R is a recursive relation and each $Q_i x_i$ is an existential or universal integer quantifier. Blocks of like quantifiers can be contracted so that the $Q_i x_i$'s can be considered alternating from \exists to \forall . For $n \geq 1$, an arithmetical relation is Σ_n^0 (resp. Π_n^0) if, and only if its explicit definition has n integer quantifiers, the first being existential (resp. universal). If a relation P is both Σ_n^0 and Π_n^0 , P is said to be Δ_n^0 . So the recursive relations are precisely the Δ_1^0 relations. Closure properties of the arithmetical pointclasses are listed in the following theorem, stated without proof.

Theorem 4.4.1. 1. If P is Σ_m^0 or Π_m^0 , then P is Σ_n^0 and Π_n^0 for all $n > m$.

2. If P is Σ_n^0 (or resp. Π_n^0), then $\neg P$ is Π_n^0 (or resp. Σ_n^0).

3. For each $n \geq 1$, both Σ_n^0 and Π_n^0 are closed under recursive substitution, union, intersection, and bounded existential or universal integer quantification.

4. For each $n \geq 1$, Σ_n^0 is closed under existential integer quantification.

5. For each $n \geq 1$, Π_n^0 is closed under universal integer quantification.

6. For each $n \geq 1$, there is $P \in \Sigma_n^0$ such that $P \notin \Pi_n^0$ and there is $P \in \Pi_n^0$ such that $P \notin \Sigma_n^0$.

The upshot of Theorem 4.4.1 is that the finite levels of Borel hierarchy illustration from Section 2.2 correspond to the arithmetical hierarchy; the boldface notation is simply changed to lightface. (We will have more to say later about the direct relationship between boldface and lightface notation.) Also, just as the Borel hierarchy extends well beyond the finite levels by taking unions at limit stages, the arithmetical

hierarchy is extended by the hyperarithmetical hierarchy. Though we will encounter these sets in the results to follow, we will develop only the theory germane to our result.

Replacing the integer quantifiers with real number quantifiers yields the lightface version of the projective hierarchy, the analytical hierarchy. A relation P is **analytical** if it has an explicit definition

$$P(x) \Leftrightarrow Q_1 x_1 \dots Q_n x_n R(x, x_1, \dots, x_n)$$

where R is a recursive relation and each $Q_i x_i$ is an existential or universal real number quantifier. Blocks of like quantifiers can be contracted so that the $Q_i x_i$'s can be considered alternating from \exists to \forall . For $n \geq 1$, a analytical relation is Σ_n^1 (resp. Π_n^1) iff its explicit definition has n real number quantifiers, the first being existential (resp. universal). If a relation P is both Σ_n^1 and Π_n^1 , P is said to be Δ_n^1 . Closure properties of the analytical pointclasses are listed in the following theorem, stated without proof.

Theorem 4.4.2. 1. If P is Σ_m^1 or Π_m^1 , then P is Σ_n^1 and Π_n^1 for all $n > m$.

2. If P is Σ_n^1 (or resp. Π_n^1), then $\neg P$ is Π_n^1 (or resp. Σ_n^1).

3. For each $n \geq 1$, both Σ_n^1 and Π_n^1 are closed under recursive substitution, union, intersection, and existential or universal integer quantification.

4. For each $n \geq 1$, Σ_n^1 is closed under existential real number quantification.

5. For each $n \geq 1$, Π_n^1 is closed under universal real number quantification.

6. For each $n \geq 1$, there is $P \in \Sigma_n^1$ such that $P \notin \Pi_n^1$ and there is $P \in \Pi_n^1$ such that $P \notin \Sigma_n^1$.

The upshot of Theorem 4.4.2 is that the Projective hierarchy illustration from Section 2.3 corresponds to the analytical hierarchy; the boldface notation is simply changed to lightface.

To complete our comments about effective descriptive set theory, we have the following definition. Given a lightface pointclass Γ and a real $z \in \omega^\omega$, the **relativization**

$\Gamma(z)$ of Γ to z is the pointclass containing all Γ sets computable using z as an oracle. That is, the algorithm has z sitting on a tape and can reference it finitely many times for any calculation. With this in hand, we can finally state the relationship between the lightface and boldface hierarchies: for any pointclass Γ ,

$$\Gamma = \bigcup_{z \in \mathcal{N}} \Gamma(z)$$

In the spirit of the pointclass hierarchies we have defined, a similar hierarchy can be defined for the formulas in the language of set theory. Let φ be a formula in LST. Then φ is Δ_0 (or equivalently, Σ_0 or Π_0) if it does not contain any unbounded quantifiers. That is, every quantifier in φ is of the form $\exists x \in y$ or $\forall x \in y$. For each $n \geq 0$, φ is Σ_{n+1} if φ is of the form $\exists x \psi(x)$ for some Π_n formula ψ . For each $n \geq 0$, φ is Π_{n+1} if φ is of the form $\forall x \psi(x)$ for some Σ_n formula ψ . For each $n \geq 0$, φ is Δ_{n+1} if φ is both Σ_{n+1} and Π_{n+1} . Closure properties for each of these collections of formulas are similar to those of the arithmetical hierarchy; we refer the reader to Theorem 4.4.1 for details.

Let (ω, E) be a structure. We code (ω, E) by a real $x \in 2^\omega$ in the following way:

$$x(k) = \begin{cases} 1, & \text{if } k = \langle n, m \rangle \text{ and } nEm; \\ 0, & \text{otherwise.} \end{cases}$$

We will often refer to a structure and its code interchangeably. For this reason, we write $(\omega, E) = (\omega, E_x)$ where x is the real coding (ω, E) . For a structure (ω, E_x) , the following lemma details the relationship between the arithmetical hierarchy and the formula hierarchy.

Lemma 4.4.3. *If $\varphi(y_1, \dots, y_k)$ is a Σ_n (resp. Π_n) formula and $n_1, \dots, n_k \in \omega$, then $\{x \in 2^\omega : (\omega, E_x) \models \varphi(n_1, \dots, n_k)\} \in \Sigma_n^0$ (resp. Π_n^0)*

Proof. By induction on the complexity of φ .

If $\varphi(y_1, y_2)$ is an atomic formula, then for $n, m \in \omega$,

$$\begin{aligned} \{x \in 2^\omega : (\omega, E_x) \models \varphi(n, m)\} &= \{x \in 2^\omega : nE_x m\} \\ &= \{x \in 2^\omega : x(\langle n, m \rangle) = 1\} \in \Delta_1^0. \end{aligned}$$

Now suppose $\varphi(y_1, \dots, y_k)$ is a Σ_{n+1} formula and that $n_1, \dots, n_k \in \omega$. Also, suppose that for every $\Sigma_n(\Pi_n)$ formula $\psi(y_1, \dots, y_k)$ and all $n_1, \dots, n_k \in \omega$,

$$\{x \in 2^\omega : (\omega, E_x) \models \psi(n_1, \dots, n_k)\} \in \Sigma_n^0(\Pi_n^0).$$

Case 1: $\varphi(y_1, \dots, y_k)$ is of the form $\psi(y_1, \dots, y_k) \wedge \chi(y_1, \dots, y_k)$

Then $\{x \in 2^\omega : (\omega, E_x) \models \varphi(n_1, \dots, n_k)\} =$

$$\{x \in 2^\omega : (\omega, E_x) \models \psi(n_1, \dots, n_k)\} \cap \{x \in 2^\omega : (\omega, E_x) \models \chi(n_1, \dots, n_k)\}$$

which is $\Sigma_{n+1}^0(\Pi_{n+1}^0)$ since it is the intersection of two $\Sigma_{n+1}^0(\Pi_{n+1}^0)$ sets.

Case 2: $\varphi(y_1, \dots, y_k)$ is of the form $\neg\psi(y_1, \dots, y_k)$

Then $\{x \in 2^\omega : (\omega, E_x) \models \varphi(n_1, \dots, n_k)\} =$

$$2^\omega \setminus \{x \in 2^\omega : (\omega, E_x) \models \psi(n_1, \dots, n_k)\}$$

which is $\Pi_n^0(\Sigma_n^0) \subset \Sigma_{n+1}^0(\Pi_{n+1}^0)$ since it is the complement of a $\Sigma_n^0(\Pi_n^0)$ set.

Case 3: $\varphi(y_1, \dots, y_k)$ is of the form $\exists y_0 \in \omega \psi(y_0, y_1, \dots, y_k)$

Then $\{x \in 2^\omega : (\omega, E_x) \models \varphi(n_1, \dots, n_k)\} =$

$$\begin{aligned} \{x \in 2^\omega : (\omega, E_x) \models \exists n_0 \in \omega \psi(n_0, n_1, \dots, n_k)\} &= \\ \{x \in 2^\omega : \exists n_0 \in \omega [(\omega, E_x) \models \psi(n_0, n_1, \dots, n_k)]\}. \end{aligned}$$

which is $\Sigma_{n+1}^0(\Pi_{n+1}^0)$ by definition. □

A structure (A, E) is **well-founded** if there are no infinite descending E -chains.

For a structure (A, E) we define the **first order definable over** (A, E) sets by

$$FODO(A, E) = \{x \subset A : \exists n \in \omega \forall y \in A (y \in x \Leftrightarrow (A, E) \models \varphi_n(y))\}.$$

Let (A^0, E^0) be the following fixed structure: $A^0 = \{n \in \omega : n \text{ is odd}\}$, E^0 is a recursive relation and $(A^0, E^0) \cong (V(\omega), \in)$ by some fixed isomorphism so that each odd integer codes a hereditarily finite set. Moreover, there is a recursive procedure so that given any $n \in A^0$, we can recover the formula defining that element of $V(\omega)$. To distinguish between an actual integer (ie: an odd integer coding an element of ω) and an integer as an element of the structure, we use the symbol \bar{n} to denote the code of that element of A^0 which is satisfied in (A^0, E^0) to be the integer n .

Given a structure (ω, E_x) , we code its theory $Th(\omega, E_x)$ by a real $y \in 2^\omega$ as follows:

$$y(k) = \begin{cases} 1, & \text{if } (\omega, E_x) \models \varphi_k; \\ 0, & \text{otherwise.} \end{cases}$$

We will not distinguish between $Th(\omega, E_x)$ and its code y , unless absolutely necessary.

Fix an integer $N > \ulcorner \mathbf{V} = \mathbf{L} \wedge \Psi_3 \urcorner$ (see Prop. 4.3.3) and define $T \subset 2^\omega$ as follows:

$$\begin{aligned} y \in T \Leftrightarrow & \exists x \in 2^\omega [x \text{ codes a structure } (\omega, E_x) \wedge y = Th(\omega, E_x) \\ & \wedge (\omega, E_x) \models \varphi_1, \dots, \varphi_N \wedge E_x \upharpoonright A^0 = E^0 \\ & \wedge \forall n \in \omega \setminus A^0 \exists i \in \omega [n = 2i \wedge n \text{ is the unique element of } (\omega, E_x) \\ & \quad \text{such that } (\omega, E_x) \models \varphi_i(n) \wedge i \text{ is the least such that } (\omega, E_x) \models \varphi_i(n)] \\ & \wedge y \in FODO(FODO(\omega, E_x), \in) \wedge (\omega, E_x) \text{ is well-founded}] \end{aligned}$$

We will show that all of the formulas inside the quantifier, except for the well-foundedness condition which is Π_1^1 , define relatively simple (Π_ω^0) sets. But the real number quantifier $\exists x \in 2^\omega$ seems to make $T \in \Sigma_2^1$. Our first lemma shows that the real quantifier $\exists x \in 2^\omega$ can be replaced by an existential integer quantifier.

Lemma 4.4.4. *For all $y \in T$, if x is such that $y = Th(\omega, E_x)$, then $x \leq_T y$.*

Proof. Let $y \in T$ and let $x \in 2^\omega$ be such that $y = Th(\omega, E_x)$. We seek a recursive f such that for all $k \in \omega$, $f(k, y) = x(k)$.

Let $k \in \omega$. If there do not exist $n, m \in \omega$ such that $k = \langle n, m \rangle$, then $f(k, y) = 0$. Now suppose $n, m \in \omega$ are such that $k = \langle n, m \rangle$. Note that given φ in LST, we let $\ulcorner \varphi \urcorner$ denote the Gödel number of φ .

Case 1: $n = 2i + 1$ and $m = 2j + 1$ for some $i, j \in \omega$.

$f(k, y) = 1$ if nE^0m and 0 otherwise.

Case 2: $n = 2i + 1$ and $m = 2j$ for some $i, j \in \omega$.

Since there is a recursive procedure that accepts n as input and produces the formula defining the corresponding element of $V(\omega)$ coded by n , let P_n be this formula defining n in (A^0, E^0) . Then $f(k, y) = y(\ulcorner \exists u P_n(u) \wedge \exists v \psi_j(v) \wedge u \in v \urcorner)$.

Case 3: $n = 2i$ and $m = 2j + 1$ for some $i, j \in \omega$.

Then $f(k, y) = 0$.

Case 4: $n = 2i$ and $m = 2j$ for some $i, j \in \omega$.

Let $f(k, y) = y(\ulcorner \exists u \varphi_i(u) \wedge \exists v \psi_j(v) \wedge u \in v \urcorner)$.

It is clear that for all k , $f(k, y) = x(k)$ and that f is recursive. \square

Since there is one algorithm that produces the code of a structure from its theory, we henceforth fix an index e of this algorithm. Given $y \in 2^\omega$, we will write $(\omega, E_{f_e^y})$ to represent the structure having $y = Th(\omega, E_{f_e^y})$. Thus, $T(y)$ can be reformulated as

$$T(y) \Leftrightarrow e \text{ codes a total function} \wedge f_e^y \text{ codes a structure } (\omega, E_{f_e^y}) \wedge \dots$$

replacing every occurrence of x with f_e^y . Before we embark on the proof that $T \in \Pi_1^1$, we isolate here for the reader's benefit a delicate part of that proof.

Lemma 4.4.5. *If $\varphi(y_0, \dots, y_k)$ is a Σ_n (resp. Π_n) formula and $n_0, \dots, n_k \in \omega$, then*

$$\{y \in 2^\omega : (\omega, E_{f_e^y}) \models \text{Extensionality} \wedge (FODO(\omega, E_{f_e^y}), \epsilon) \models \varphi(n_0, \dots, n_k)\} \in \Sigma_n^0 \text{ (resp. } \Pi_n^0 \text{)}.$$

Proof. Let φ be Σ_n , $n_0, \dots, n_k \in \omega$, and suppose $y \in 2^\omega$. (The proof for Π_n formulas

is similar.) We consider each element of $p \in FODO(\omega, E_{f_e^y})$ as finite sequence of integers p_0, p_1, \dots, p_k where

$$p = \{z \in A : (\omega, E_{f_e^y}) \models \varphi_{p_0}(z, p_1, \dots, p_k)\}.$$

coding this finite sequence of integers in the usual way: $p = \langle p_0, p_1, \dots, p_k \rangle$. Thus, with the integers coding the elements of $FODO(\omega, E_{f_e^y})$, we think of $\varphi(n_0, \dots, n_k)$ as a tree T on ω . All non-terminal nodes of a given rank in T correspond to a block of existential or universal quantifiers. Consequently, as φ_n is Σ_n , T has rank n . Each terminal node of T corresponds to a quantifier-free statement built up from the logical connectives and atomic statements involving the finite number of integers along the node. So the lemma easily follows if we can show that for all $p, q \in FODO(\omega, E_{f_e^y})$, we can verify recursively in $Th(\omega, E_{f_e^y})$ that $(FODO(\omega, E_{f_e^y}), \in) \models "p = q"$ and $(FODO(\omega, E_{f_e^y}), \in) \models "p \in q"$.

Let $p, q \in \omega$ code elements of $FODO(\omega, E_{f_e^y})$, say $p = \langle p_0, p_1, \dots, p_k \rangle$ and $q = \langle q_0, q_1, \dots, q_l \rangle$ for some $k, l \in \omega$. That is,

$$p = \{z : \varphi_{p_0}^{(\omega, E_{f_e^y})}(z, p_1, \dots, p_k)\} \text{ and } q = \{z : \varphi_{q_0}^{(\omega, E_{f_e^y})}(z, q_1, \dots, q_l)\}.$$

So, $(FODO(\omega, E_{f_e^y}), \in) \models "p = q" \Leftrightarrow$

$$(\omega, E_{f_e^y}) \models \forall z [\varphi_{p_0}(z, p_1, \dots, p_k) \Leftrightarrow \varphi_{q_0}(z, q_1, \dots, q_l)]$$

Next, $(FODO(\omega, E_{f_e^y}), \in) \models "p \in q" \Leftrightarrow$

$$(\omega, E_{f_e^y}) \models \exists y \forall z [(z \in y \Leftrightarrow \varphi_{p_0}(z, p_1, \dots, p_k)) \wedge \varphi_{q_0}(y, q_1, \dots, q_l)]$$

In either case, as $(\omega, E_{f_e^y})$ models Extensionality, it is clear that

$$(FODO(\omega, E_{f_e^y}), \in) \models "p = q" \text{ and } (FODO(\omega, E_{f_e^y}), \in) \models "p \in q"$$

are recursive in $Th(\omega, E_{f_e^y})$. □

The next proof contains the only instance of a hyperarithmetical sets in this chapter. Despite the absence of any formal definition of a hyperarithmetical set, it will nevertheless be clear that these sets is not arithmetical.

Proposition 4.4.6. $T \in \Pi_1^1$.

Proof. Let $y \in 2^\omega$. The formula “ e codes a total function” can be replaced by

$$\forall k \in \omega \exists m \in \omega (f_e^y(k) = m).$$

Thus, $\{y \in 2^\omega : \forall k \in \omega \exists m \in \omega (f_e^y(k) = m)\} \in \Pi_2^0$. It is clear that the formula “ f_e^y codes a structure $(\omega, E_{f_e^y})$ ” merely asserts that $f_e^y \in 2^\omega$ and hence is Δ_1^0 . Similarly, the formula “ $(\omega, E_{f_e^y}) \models \varphi_1, \dots, \varphi_N$ ” can be replaced with $\forall i \leq N [y(i) = 1]$. Thus, $\{y \in 2^\omega : (\omega, E_{f_e^y}) \models \varphi_1, \dots, \varphi_N\} \in \Delta_1^0$. Next, the formula “ $E_{f_e^y} \upharpoonright A^0 = E^0$ ” can be replaced with

$$\forall n, m, k \in \omega [(n, m \text{ odd} \wedge k = \langle n, m \rangle) \Rightarrow (f_e^y(k) = 1 \Leftrightarrow nE^0 m)].$$

Since E^0 is recursive by definition, $\{y \in 2^\omega : E_{f_e^y} \upharpoonright A^0 = E^0\} \in \Pi_1^0$.

Now the formula

$$\forall n \in \omega \setminus A^0 \exists i \in \omega [n = 2i$$

$$\wedge \text{“}n \text{ is the unique element of } (\omega, E_{f_e^y}) \text{ such that } (\omega, E_{f_e^y}) \models \varphi_i(n)\text{”}$$

$$\wedge \text{“}i \text{ is the least such } i\text{”}]$$

abbreviates

$$\forall n \exists i [n \text{ is even} \Rightarrow (n = 2i$$

$$\wedge (\omega, E_{f_e^y}) \models \varphi_i(n) \wedge \forall m \in \omega [(\omega, E_{f_e^y}) \models \varphi_i(m) \Rightarrow n = m]$$

$$\wedge \forall j < i [(\omega, E_{f_e^y}) \not\models \varphi_j(n)]]]$$

By Lemma 4.4.3, each instance of $(\omega, E_{f_e^y}) \models \varphi_i(n)$ is $\Sigma_{i'}^0$ where φ_i is a $\Sigma_{i'}$ formula.

Thus, as n ranges over ω , i' increases without bound. Hence,

$$\begin{aligned} \{y \in 2^\omega : \forall n \exists i [n \text{ is even} \Rightarrow (n = 2i) \\ \wedge (\omega, E_{f_\varepsilon^y}) \models \varphi_i(n) \wedge \forall m \in \omega [(\omega, E_{f_\varepsilon^y}) \models \varphi_i(m) \Rightarrow n = m] \\ \wedge \forall j < i [(\omega, E_{f_\varepsilon^y}) \not\models \varphi_j(n)]]\} \in \Pi_\omega^0 \end{aligned}$$

By similar reasoning, the formula

$$"y = Th(\omega, E_{f_\varepsilon^y})" \Leftrightarrow \forall n \in \omega [n \in y \Leftrightarrow (\omega, E_{f_\varepsilon^y}) \models \varphi_n]$$

also defines a Π_ω^0 set of reals.

We claim next that $\{y \in 2^\omega : y \in FODO(FODO(\omega, E_{f_\varepsilon^y}), \varepsilon)\} \in \Sigma_\omega^0$. First, we observe that

$$\begin{aligned} "y \in FODO(FODO(\omega, E_{f_\varepsilon^y}), \varepsilon)" \Leftrightarrow \\ \exists n, n_1, \dots, n_k \forall m [y(m) = 1 \Leftrightarrow (FODO(\omega, E_{f_\varepsilon^y}), \varepsilon) \models \varphi_n(m, n_1, \dots, n_k)] \end{aligned}$$

It suffices to show that if $\varphi_n(y_0, y_1, \dots, y_k)$ is a $\Sigma_{n'}$ formula and $n_0, \dots, n_k \in \omega$, then

$$\{y \in 2^\omega : (FODO(\omega, E_{f_\varepsilon^y}), \varepsilon) \models \varphi_n(n_0, \dots, n_k)\} \in \Sigma_{n'}^0.$$

Thus, as n ranges over ω , n' increases without bound and the claim easily follows.

Finally, the formula " $(\omega, E_{f_\varepsilon^y})$ is well-founded" can be replaced by the formula

$$\forall \alpha \in \omega^\omega [\forall n \in \omega [\alpha(n+1) E_{f_\varepsilon^y} \alpha(n)] \Rightarrow \exists n \in \omega [\alpha(n+1) = \alpha(n)]]$$

which is clearly Π_1^1 . As Π_1^1 is closed under intersections, $T \in \Pi_1^1$. □

Our next result shows that every structure whose theory is in T is isomorphic to a limit stage of the \mathbf{L} hierarchy. Recall that the ordinal of a set M is $o(M) = M \cap \mathbf{ON}$.

Lemma 4.4.7. *For all $y \in T$, there is a unique limit ordinal λ such that if $y = Th(\omega, E_x)$ for some x then $(\omega, E_x) \cong (L(\lambda), \varepsilon)$.*

Proof. Let $y \in T$ and let x be such that $y = Th(\omega, E_x)$. Since (ω, E_x) is a well-founded structure, we can collapse (ω, E_x) to a transitive structure $(M, \in) \cong (\omega, E_x)$. So (M, \in) is a transitive, well-founded modeling $\mathbf{V}=\mathbf{L}$ and Ψ_1 . Thus, $\lambda = o(M)$ is limit by Proposition 4.2.1. Hence $(M, \in) \cong (L(\lambda), \in)$. \square

Henceforth, for $y \in T$, we write λ_y to denote the unique limit ordinal such that $(\omega, E_x) \cong (L(\lambda), \in)$ where $y = Th(\omega, E_x)$.

For $x \in 2^\omega$, $J(x)$ denotes the **(Turing) jump of x** . We view $J(x)$ as the complete $\Sigma_1^0(x)$ set of integers. It is immediate that for any real x , $x <_T J(x)$. For each n , define inductively $J^n(x)$, the **n^{th} jump of x** as $J^0(x) = x$ and $J^{n+1}(x) = J(J^n(x))$. Note that we always consider the jump of any real as another real by associating $J(x)$ with its characteristic function.

Proposition 4.4.8. *For all $y_1, y_2 \in T$, if $\lambda_{y_1} < \lambda_{y_2}$, then for all n , $J^n(y_1) \leq_T y_2$.*

Proof. Let $y_1, y_2 \in T$ and let $x_1, x_2 \in 2^\omega$ be such that $y_1 = Th(\omega, E_{x_1})$ and $y_2 = Th(\omega, E_{x_2})$. Let $\lambda_{y_1} < \lambda_{y_2}$ be limit ordinals such that $(\omega, E_{x_1}) \cong (L(\lambda_{y_1}), \in)$ and $(\omega, E_{x_2}) \cong (L(\lambda_{y_2}), \in)$. As $\lambda_{y_1} < \lambda_{y_2}$, it follows that $L(\lambda_{y_1}) \in L(\lambda_{y_2})$ so that for some least $i \in \omega$, $(\omega, E_{x_2}) \models \varphi_i(2i) \Leftrightarrow (L(\lambda_{y_2}), \in) \models \varphi_i(L(\lambda_{y_1}))$. We prove the result by induction on n .

Suppose first that $n = 0$. To see that $J^0(y_1) = y_1 \leq_T y_2$, we observe that for any $k \in \omega$, $y_1(k) = y_2(\ulcorner \exists z(\varphi_i(z) \wedge (z, E_{x_2} \upharpoonright z) \models \varphi_k) \urcorner)$.

Now suppose that $n > 0$ and let $k \in \omega$ be fixed. Since $y_1 \in T$, we have $y_1 \in FODO(FODO(\omega, E_{x_1}))$. That is, $Th(L(\lambda_{y_1}), \in) \in L(\lambda_{y_1} + 2) \subset L(\lambda_{y_2})$ since λ_{y_2} is limit. So let $j \in \omega$ be the least such that

$$(\omega, E_{x_2}) \models \varphi_j(2j) \Leftrightarrow (L(\lambda_{y_2}), \in) \models \varphi_j(Th(L(\lambda_{y_1}), \in)).$$

So then, we observe that $(J^n(y_1))(k) = y_2(\ulcorner \psi(k) \urcorner)$ where $\psi(k)$ is the formula

$$\exists z, y[\varphi_i(z) \wedge k \text{ codes a } \Sigma_n \text{ formula} \wedge \varphi_j(y) \wedge \varphi'_k(y)]$$

with φ'_k is the formula $\varphi_k(y)$ with all instances of the \in relation replaced with E_{x_1}

and all existential quantifiers replaced with existential number quantifiers. So then $J^n(y_1) \leq_T y_2$. \square

Now we define $\mathbf{A} \subset \mathbf{D}$ as follows: $\mathbf{x} \in \mathbf{A} \Leftrightarrow \exists y \in T(x \equiv_T y)$.

Lemma 4.4.9. \mathbf{A} is Π_1^1 .

Proof. Replace “ $\mathbf{x} \in \mathbf{A}$ ” with $\exists y \in T \exists e_1, e_2 \in \omega(x = f_{e_1}(y) \wedge y = f_{e_2}(x))$. \square

Proposition 4.4.10. For all $\mathbf{x} \in \mathbf{A}$, there is a unique λ such that if $x \equiv_T y$ for some $y \in T$, then $\lambda_y = \lambda$.

Proof. Let $\mathbf{x} \in \mathbf{A}$ and let $y \in T$ be such that $x \equiv_T y$. So there is $x' \in 2^\omega$ such that $y = Th(\omega, E_{x'})$. Let λ_y be the unique limit ordinal such that $(\omega, E_{x'}) \cong (L(\lambda_y), \in)$ as in Lemma 4.4.7. Now suppose $y' \in T$ is such that $x \equiv_T y'$. Furthermore, suppose for a contradiction that $\lambda_y < \lambda_{y'}$. By Proposition 4.4.8, $J(y) \leq_T y'$. But then,

$$y \leq_T J(y) \leq_T y' \equiv_T x \equiv_T y$$

and hence, $J(y) \equiv_T y$, a contradiction. A symmetric argument shows that $\lambda_{y'} < \lambda_y \Rightarrow y' \equiv_T J(y')$, again, a contradiction. Thus, $\lambda_y = \lambda_{y'}$. \square

Henceforth, for $\mathbf{x} \in \mathbf{A}$, we denote the unique limit ordinal from Proposition 4.4.10 by $\lambda_{\mathbf{x}}$. As a corollary, we have the following:

Corollary 4.4.11. For all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{A}$, if $x_1 <_T x_2$, then $\lambda_{\mathbf{x}_1} < \lambda_{\mathbf{x}_2}$.

Given $\mathbf{x} \in \mathbf{D}$, we define $\mathbf{J}(\mathbf{x})$, the **jump of \mathbf{x}** , as $\mathbf{J}(\mathbf{x}) = \{y : y \equiv_T J(x)\}$. Our next proposition is analogous to Proposition 4.4.8.

Proposition 4.4.12. For all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{A}$, if $x_2 \not\leq_T x_1$, then for all n , $J^n(x_1) \leq_T x_2$.

Proof. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{A}$ be such that $x_2 \not\leq_T x_1$. Let $\lambda_{\mathbf{x}_1}$ be the unique limit ordinal (as in Proposition 4.4.10) such that for all $y_1 \in T$ such that $y_1 \equiv x_1$,

$$(L(\lambda_{y_1}), \in) \cong (L(\lambda_{\mathbf{x}_1}), \in).$$

Define $\lambda_{\mathbf{x}_2}$ mutatis mutandis. Since $x_2 \not\leq_T x_1$, it follows from Corollary 4.4.11 that $\lambda_{\mathbf{x}_1} < \lambda_{\mathbf{x}_2}$. Thus, for all $n \in \omega$, $J^n(x_1) \leq_T x_2$, by Proposition 4.4.8. \square

A set of degrees \mathbf{A} contains a cone if there is $\mathbf{x} \in \mathbf{D}$ such that $C_{\mathbf{x}} \subseteq \mathbf{A}$. That is, for all \mathbf{y} such that $\mathbf{x} \leq \mathbf{y}$, $\mathbf{y} \in \mathbf{A}$. The next theorem show that \mathbf{A} does not contain any cone.

Theorem 4.4.13. *For all $\mathbf{x} \in \mathbf{D}$ there is \mathbf{y} such that $\mathbf{x} \leq_T \mathbf{y}$ and $\mathbf{y} \notin \mathbf{A}$.*

Proof. Let $\mathbf{x} \in \mathbf{A}$ and let $\mathbf{y} = \mathbf{J}(\mathbf{x})$. Suppose for a contradiction that $\mathbf{y} \in \mathbf{A}$. Let $\lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}$ be the unique limit ordinals from Proposition 4.4.10. There are three cases.

Case 1: $\lambda_{\mathbf{x}} = \lambda_{\mathbf{y}}$.

Because $x \equiv_T y$, it follows that $y \equiv_T J(x) \equiv_T J(y)$, a contradiction.

Case 2: $\lambda_{\mathbf{y}} < \lambda_{\mathbf{x}}$.

By Proposition 4.4.8, $J(y) \leq_T x$. Hence, $y \leq_T J(y) \leq_T x \leq_T y$. Thus $y \equiv_T J(y)$, again, a contradiction.

Case 3: $\lambda_{\mathbf{x}} < \lambda_{\mathbf{y}}$

By Proposition 4.4.8, $J(J(x)) \leq_T y$. Hence, $y \equiv_T J(x) \leq_T J(J(x)) \leq_T y$. Thus $y \equiv_T J(y)$, again, a contradiction.

Hence, $\mathbf{y} \notin \mathbf{A}$. \square

A set of degrees \mathbf{A} omits a cone if there is $\mathbf{x} \in \mathbf{D}$ such that $C_{\mathbf{x}} \subseteq \mathbf{D} \setminus \mathbf{A}$. That is, for all $\mathbf{y} \in \mathbf{A}$, $\mathbf{x} \not\leq \mathbf{y}$. The next theorem show that \mathbf{A} does not omit any cone.

Theorem 4.4.14. *For all $\mathbf{x} \in \mathbf{D}$ there is $\mathbf{y} \in \mathbf{A}$ such that $\mathbf{x} \leq_T \mathbf{y}$.*

Proof. Let $\mathbf{x} \in \mathbf{D}$, for some $x \in \omega^\omega \cap \mathbf{L}$. So for some $\alpha < \omega_1^{\mathbf{L}}$, $\rho(x) = \alpha$. Let $\lambda = \rho(x) + \omega$. We define an isomorphism $\pi : (L(\lambda), \in) \rightarrow (\omega, E_{x_1})$ for some $x_1 \in 2^\omega$ so that $y = Th(\omega, E_{x_1}) \in T$, and thus $\mathbf{y} \in \mathbf{A}$. Because $L(\lambda_x) \in L(\lambda_y)$, it follows that $x \leq_T y$ using the argument from Proposition 4.4.8. Hence $\mathbf{x} \leq \mathbf{y}$.

First, define $\pi \upharpoonright V(\omega)$ to be the fixed isomorphism from $(V(\omega), \in)$ to (A^0, E^0) . If $a \in L(\lambda) \setminus V(\omega)$, let $\pi(a) = 2i$ where i is the least such that $(L(\lambda), \in) \models \varphi_i(a)$ and for any $a' \in L(\lambda) \setminus V(\omega)$, $(L(\lambda), \in) \not\models \varphi_i(a')$. Let E_{x_1} be the relation induced by π on ω , and let $x_1 \in 2^\omega$ be the characteristic function of E_{x_1} . It is clear that $(L(\lambda), \in) \cong (\omega, E_{x_1})$ via π and that $y = Th(\omega, E_{x_1}) \in T$. \square

Theorem 4.4.15. $ZFC \not\vdash Det(\Pi_1^1)$

Proof. Use **A** and the contrapositive of Martin's Theorem. □

CHAPTER 5

$ZC \not\equiv Det(\Pi_{\omega+2}^0)$

This final chapter is devoted to the main metamathematical result concerning Borel Determinacy. We argue in ZC set theory (ZFC - Replacement) the existence of a Borel set of Turing degrees that neither contains nor omits a cone. We will structure the argument similar to that of Chapter 4.

5.1 Properties of $L^{\omega+\omega}$

Because Borel Determinacy is a theorem of ZFC, \mathbf{L} is an insufficient model for the results of this chapter. Instead, we will work exclusively in the ZC model $L^{\omega+\omega}$. Recall that for a given structure (A, E) , $FODO(A, E) = \{x \subset A : \exists n \in \omega \forall y \in A (y \in x \Leftrightarrow (A, E) \models \varphi_n(y))\}$. Define a structure $(L^{\omega+\omega}, \in)$ by transfinite recursion on α as follows:

$$\begin{aligned} L^{\omega+\omega}(0) &= \mathbf{V}(\omega) \\ L^{\omega+\omega}(\alpha + 1) &= FODO((L^{\omega+\omega}(\alpha), \in)) \cap \mathbf{V}(\omega + \omega), \text{ for } \alpha \text{ successor} \\ L^{\omega+\omega}(\lambda) &= \bigcup_{\alpha < \lambda} L^{\omega+\omega}(\alpha), \text{ for } \lambda \text{ limit} \\ L^{\omega+\omega} &= \bigcup_{\alpha \in \mathbf{ON}} L^{\omega+\omega}(\alpha) \end{aligned}$$

Note that $L^{\omega+\omega}$ is similar to \mathbf{L} in that for each α , $L^{\omega+\omega}(\alpha)$ is transitive. Thus, $L^{\omega+\omega}$ is transitive. Yet, unlike \mathbf{L} , $L^{\omega+\omega}(0)$ is the entire finite part of the structure. Moreover, there is the liability of $L^{\omega+\omega} \subset \mathbf{V}(\omega + \omega)$. One important consequence of this, which we will overcome, is that $L^{\omega+\omega}$ contains no ordinal $\geq \omega + \omega$, and hence $ZC \not\equiv Replacement$. The other consequence is contained in our first lemma.

Lemma 5.1.1. *There is $\alpha \in \mathbf{ON}$ such that $L^{\omega+\omega}(\alpha + 1) = L^{\omega+\omega}(\alpha)$.*

Proof. Suppose not. Thus, for each $\alpha \in \mathbf{ON}$ there is $x_\alpha \in L^{\omega+\omega}$ such that $x_\alpha \in L^{\omega+\omega}(\alpha+1) \setminus L^{\omega+\omega}(\alpha)$. But then, as $L^{\omega+\omega} \subset V(\omega+\omega)$, it would follow that $V(\omega+\omega)$ is a proper class, a contradiction. \square

We will occasionally need to calculate the \mathbf{V} -rank of a set x , denoted $\rho^{\mathbf{V}}(x)$. The next lemma is nearly identical to Lemma 4.1.2, and we omit the proof.

Lemma 5.1.2. *Let x, y be such that $\rho^{\mathbf{V}}(x) = \alpha$ and $\rho^{\mathbf{V}}(y) = \beta$ for some $\alpha, \beta > \omega$. Then,*

1. $\rho^{\mathbf{V}}(\bigcup x) = \alpha$,
2. $\rho^{\mathbf{V}}(\mathcal{P}(x)) = \alpha + 1$,
3. $\rho^{\mathbf{V}}(\{x\}) = \alpha + 1$,
4. $\rho^{\mathbf{V}}(\{x, y\}) = \alpha + \beta + 1$,
5. $\rho^{\mathbf{V}}(\langle x, y \rangle) = \alpha + \beta + 2$,
6. $\rho^{\mathbf{V}}(x \times y) = \alpha + \beta + 3$

The next proposition verifies that $L^{\omega+\omega}$ is a transitive model of Z (ZF - Replacement); we will show in section 5.2.2 that $L^{\omega+\omega} \models ZC$.

Proposition 5.1.3. $L^{\omega+\omega} \models Z$.

Proof. Let $\alpha_0 \in \mathbf{ON}$ be as in Lemma 5.1.1. Hence, $L^{\omega+\omega} = L^{\omega+\omega}(\alpha_0)$.

Extensionality: Follows since $L^{\omega+\omega}$ is transitive.

Foundation: Follows from $L^{\omega+\omega} \subset V(\omega+\omega)$.

Infinity: $\omega \in L^{\omega+\omega}(\omega+1)$.

Now note that since $\omega+\omega$ is limit, $V(\omega+\omega)$ is closed under the operations of Pairing, Union, and Power Set.

Pairing: For any $x, y \in L^{\omega+\omega}(\alpha_0)$,

$$\{z \in L^{\omega+\omega}(\alpha_0) : z = x \vee z = y\} \in L^{\omega+\omega}(\alpha_0+1) = L^{\omega+\omega}.$$

Union: For any $x \in L^{\omega+\omega}(\alpha_0)$,

$$\{z \in L^{\omega+\omega}(\alpha_0) : \exists w \in x(z \in w)\} \in L^{\omega+\omega}(\alpha_0 + 1) = L^{\omega+\omega}.$$

Power Set: For any $x \in L^{\omega+\omega}(\alpha_0)$,

$$\{z \in L^{\omega+\omega}(\alpha_0) : z \subset x\} \in L^{\omega+\omega}(\alpha_0 + 1) = L^{\omega+\omega}.$$

Comprehension: Fix a formula $\varphi(v_0, \dots, v_n)$ of $n + 1$ free variables and let $z, w_1, \dots, w_n \in L^{\omega+\omega} = L^{\omega+\omega}(\alpha_0)$. Let $y = \{x \in z : \varphi^{L^{\omega+\omega}}(x, w_1, \dots, w_n)\}$. Now as $L^{\omega+\omega}(\alpha_0)$ is transitive, if $x \in z$, $x \in L^{\omega+\omega}(\alpha_0)$. Evidently then,

$$y = \{x \in L^{\omega+\omega}(\alpha_0) : L^{\omega+\omega} \models x \in z \wedge \varphi(x, w_1, \dots, w_n)\} \in L^{\omega+\omega}(\alpha_0 + 1) = L^{\omega+\omega}$$

□

In order to show $L^{\omega+\omega}$ also models Choice, we need to show that “ $\mathbf{V} = L^{\omega+\omega}$ ” holds in $L^{\omega+\omega}$. The real crux of the problem is that $L^{\omega+\omega}$ only contains ordinals $< \omega + \omega$, not enough to carry out all of the construction of $L^{\omega+\omega}$. Using the concepts of coded hierarchies and pure ordinals, we will show that $L^{\omega+\omega}$ contains sets that look like each $L^{\omega+\omega}(\alpha)$ built up from sets which look like the ordinals. The bulk of this section is devoted to definitions and absoluteness results necessary to develop these two concepts. Arguments that are similar to those in Chapter 4 will be suppressed.

As in Section 4.4, we fix the following: a structure (A^0, E^0) such that $A^0 = \{i \in \omega : i \text{ is odd}\}$, E^0 is a recursive binary relation on A^0 , and an isomorphism π such that $(A^0, E^0) \cong_{\pi} (\mathbf{V}(\omega), \in)$. In order to distinguish between an integer n and the coded object of (A^0, E^0) that stands for n , we write \bar{n} for that element of (A^0, E^0) that is satisfied in (A^0, E^0) to be the integer n .

Rather than repeat the arguments from Section 4.1 that enumerated the formulas of LST, we assume a fixed primitive recursive total one-one onto Gödel enumeration of the formulas of LST. φ_n denotes the n^{th} formula according to this enumeration; $\ulcorner \varphi \urcorner$ denotes the Gödel number of the formula φ .

We say that $x = (A, <)$ is a **linear order** if $A \neq \emptyset$, $A \cap \mathbf{V}(\omega) = \emptyset$, and $<$ is transitive and connected binary relation on A . We write $LO(x)$ for “ x is a linear order”; if $x = (A, <)$, we write $A = Field(x)$.

Suppose $LO(x)$. If $x = (A, <)$ and $y \in A$ is such that for all $z \in A$, $z \not< y$ then y is the **zero** of $(A, <)$ and we write “ $y = 0_x$ ”. If $y, z \in A$ and $z < y$ and there does not exist $a \in A$ such that $z < a$ and $a < y$, then y is the **successor** of z and we write “ $y = Suc_x(z)$ ”. If $y \in A$ is such that for all $z < y$, there is $a \in A$ such that $z < a < y$, then y is a **limit** element of x and we write “ $Lim_x(y)$ ”.

Given $A \neq \emptyset$, define the set of all finite sequences of A , $Seq(A) =$

$$\{y : y \text{ is a function } \wedge \exists k \in \omega (k \neq 0 \wedge \text{dom}(y) = k \wedge \text{range}(y) \subseteq A)\}.$$

For $a \in Seq(A)$, we write $len(a) = k$ to denote the length of the sequence and we write $a \upharpoonright i$ for some $i \leq len(a)$ to denote the **restriction** of a to I . If $a, b \in Seq(A)$, then $a \hat{\ } b$ is the **concatenation** of a and b . \emptyset denotes the **empty sequence**, and it is the only sequence of length 0.

Suppose $LO(x)$ and $a, b \in Seq(Field(x))$. We write $a <_{lex}^x b$ if a precedes b in the lexicographic order on $Seq(Field(x))$ induced by $<_x$. The next lemma establishes the absoluteness of $<_{lex}^x$ for limit stages of $L^{\omega+\omega}$ and for other transitive structures that model enough of \mathbf{Z} . Note that we use the sentence Q_1 to keep track of how much \mathbf{Z} we are using for absoluteness. We will maintain this convention throughout the rest of this section, as it will become vital in later arguments (Cf. Prop. 4.2.1, Lemma 4.2.4, Lemma 4.2.5, and Prop. 4.3.3). For the remainder of this chapter, our absoluteness proofs will be brief, relying on the work we did in Chapter 4 for the details.

Lemma 5.1.4. *There is a formula $P_1(x, a, b)$ and a sentence Q_1 such that*

1. *for all λ limit, $(L^{\omega+\omega}(\lambda), \in) \models Q_1$, and*
2. *for all transitive A such that $(A, \in) \models Q_1$ and for all $x, a, b \in A$,*
 $(A, \in) \models P_1(x, a, b) \Leftrightarrow a <_{lex}^x b$ and
 $(A, \in) \models \forall x \exists w \forall z [z \in w \Leftrightarrow z = (a, b) \wedge P_1(x, a, b)]$.

Proof. Let λ be limit. We first define $P_1(x, a, b)$ and argue the $L^{\omega+\omega}(\lambda)$ case, then we will define the sentence Q_1 . It will be clear the $L^{\omega+\omega}(\lambda) \models Q_1$, and that if (A, \in) is any transitive structure modeling Q_1 , then the second result follows.

Define $P_1(x, a, b) =$

$$LO(x) \wedge a, b \in Seq(Field(x)) \wedge \\ [a = \emptyset \vee \exists i \leq len(a)[a \upharpoonright i = b \upharpoonright i \wedge a_i <_x b_i] \vee (len(a) < len(b) \wedge a = b \upharpoonright len(a))]$$

First note that $L^{\omega+\omega}(\lambda)$ is transitive and models Extensionality, Pairing, Union, and Infinity. Consequently, given $x \in L^{\omega+\omega}(\lambda)$ and any $k \in \omega$, $x^k \in L^{\omega+\omega}(\lambda)$. Upon inspection of the definition of P_1 it should be clear that it is Δ_0 , and hence absolute for $L^{\omega+\omega}(\lambda)$ by transitivity. Moreover, as λ is limit, for every $x \in L^{\omega+\omega}(\lambda)$, $\{(a, b) : a <_{lex}^x b\} \in L^{\omega+\omega}(\lambda)$.

Define Q_1 to be the conjunction of the Axioms of Pairing, Extensionality, Union, Infinity, and the instance of Comprehension insuring the existence of $\{(a, b) : a <_{lex}^x b\}$. It should be clear from the above argument that if (A, \in) is a transitive structure modeling Q_1 , the second result follows. \square

We say that $x = (A, E)$ is a **coded structure** if A is a non-empty set and E is a binary relation on A . We write $CS(x)$ for “ x is a coded structure”; if $x = (A, E)$, we write $A = Field(x)$.

Suppose $CS(x)$ for some $x = (A, E)$. If $n \in \omega$ and $y = (a_0, a_1, \dots, a_k) \in Seq(A)$, then we write $Sat_x(n, y)$ if y satisfies φ_n in the coded structure x ; that is,

$$Sat_x(n, y) \Leftrightarrow x \models \varphi_n(a_0, a_1, \dots, a_k)$$

Note that the Sat relation is the analog of the Df relation from Section 4.1. The following lemma is similar to Lemma 4.1.4 and Corollary 4.1.5.

Lemma 5.1.5. *There is a formula $P_2(x, n, y)$ and a sentence Q_2 such that*

1. *for all λ limit, $(L^{\omega+\omega}(\lambda), \in) \models Q_2$, and*

2. for all transitive A such that $(A, \in) \models Q_2$ and for all $x, n, y \in A$,
 $(A, \in) \models P_2(x, n, y) \Leftrightarrow Sat_x(n, y)$ and
 $(A, \in) \models \forall x \exists w \forall z [z \in w \Leftrightarrow z = (n, y) \wedge P_2(x, n, y)]$.

Proof. Let λ be limit. We first define $P_2(x, n, y)$ and argue the $L^{\omega+\omega}(\lambda)$ case, then we will define the sentence Q_2 . It will be clear the $L^{\omega+\omega}(\lambda) \models Q_2$, and that if (A, \in) is any transitive structure modeling Q_2 , then the second result follows.

Define $P_2(x, n, y) =$

$$CS(x) \wedge n \in \omega \wedge y \in Seq(Field(x)) \wedge y = (a_0, \dots, a_k) \wedge 0 \leq k \wedge \\ x = (A, E) \wedge (A, E) \models \varphi_n(a_0, \dots, a_k)$$

where “ $(A, E) \models \varphi_n(a_0, \dots, a_k)$ ” is replaced with a formula similar to the formula ν in Lemma 4.1.4. By reasoning similar to that in Lemma 4.1.4, it should be clear that the finite function building up the satisfaction relation can be found in $L^{\omega+\omega}(\lambda)$ as λ is limit. The rest of P_2 is clearly Δ_0 , and hence is absolute for $L^{\omega+\omega}(\lambda)$. Moreover, as λ is limit, for every $x \in L^{\omega+\omega}(\lambda)$, $\{(n, y) : Sat_x(n, y)\} \in L^{\omega+\omega}(\lambda)$.

Define Q_2 to be the conjunction of Q_1 and the instances of Comprehension insuring the existence of the above finite function and the existence of $\{(n, y) : Sat_x(n, y)\}$. It should be clear from the above argument that if (A, \in) is a transitive structure modeling Q_2 , the second result follows. \square

If $A \neq \emptyset$ and $E, <$ are binary relations on A such that $LO((A, <))$ and $CS((A, E))$ and if $F : A \rightarrow \omega$, then we say that $(A, E, <, F)$ is a **structured linear order**. Note that F serves the purpose of keeping track of the $\mathbf{V}(\omega + k)$ rank of the sets in $L^{\omega+\omega}$. If $x = (A, E, <, F)$ is a structured linear order, we write $SLO(x)$ and $A = Field(x)$.

Suppose $SLO(x)$ for some $x = (A, E, <, F)$; suppose also that $n \in \omega$ and that $(b_0, \dots, b_m) \in Seq(A)$ and $a = (n) \hat{\ } (b_0, \dots, b_m)$. Then for $k \neq 0$ we write $Defn_x(a, k)$ if, and only if

$$Y = \{b : Sat_{(A, E)}(n, (b, b_0, \dots, b_m))\}$$

meets the following conditions:

1. $k - 1 \in \text{Range}(F \upharpoonright Y) \subseteq \{0, 1, 2, \dots, k - 1\}$,
2. for all $c \in A$, $Y \neq \{b : bEc\}$,
3. for all $a' = (j, c_0, \dots, c_r)$ where $j \in \omega$ and $(c_0, \dots, c_r) \in \text{Seq}(A)$,
if $Y = \{b : \text{Sat}_{(A,E)}(j, (b, c_0, \dots, c_r))\}$, then $a <_{lex}^{(A, <)} a'$.

So, $\text{Defn}_x(a, k)$ is the subset of $\mathbf{V}(\omega + k)$ defined by $a = (n, a_0, \dots, a_m)$ where a is the “least” (in the sense of being defined first by the least formula n , then by the least parameter (a_0, \dots, a_m)) in the structured linear order x .

Lemma 5.1.6. *There is a formula $P_3(x, a, k)$ and a sentence Q_3 such that*

1. for all λ limit, $(L^{\omega+\omega}(\lambda), \in) \models Q_3$, and
2. for all transitive A such that $(A, \in) \models Q_3$ and for all $x, a, k \in A$,
 $(A, \in) \models P_3(x, a, k) \Leftrightarrow \text{Defn}_x(a, k)$ and
 $(A, \in) \models \forall x \exists w \forall z [z \in w \Leftrightarrow z = (a, k) \wedge P_3(x, a, k)]$.

Proof. Let λ be limit. We argue similarly as in Lemmas 5.1.4 and 5.1.5.

Define $P_3(x, a, k) =$

$$\begin{aligned}
& SLO(x) \wedge k > 0 \wedge \exists n, m \in \omega \exists a_0, \dots, a_m \in \text{Field}(x) [a = (n, a_0, \dots, a_m) \wedge \\
& \quad \forall b [\text{Sat}_{(A_x, E_x)}(n, (b, a_0, \dots, a_m)) \Rightarrow \exists l < k (F_x(b) = l)] \wedge \\
& \quad \exists b [\text{Sat}_{(A_x, E_x)}(n, (b, a_0, \dots, a_m)) \wedge F_x(b) = k - 1] \wedge \\
& \quad \forall c \in \text{Field}(x) \exists b [(bE_x c \wedge \neg \text{Sat}_{(A_x, E_x)}(n, (b, a_0, \dots, a_m))) \vee \\
& \quad \quad (\neg bE_x c \wedge \text{Sat}_{(A_x, E_x)}(n, (b, a_0, \dots, a_m)))] \wedge \\
& \quad \forall j \in \omega \forall a' \in \text{Seq}(\text{Field}(x)) [(j)^\wedge(a') <_{lex}^{(A_x, E_x)} a \\
& \quad \Rightarrow \exists b [(\text{Sat}_{(A_x, E_x)}(j, (b)^\wedge(a')) \wedge \neg \text{Sat}_{(A_x, E_x)}(n, (b, a_0, \dots, a_m))) \vee \\
& \quad \quad (\neg \text{Sat}_{(A_x, E_x)}(j, (b)^\wedge(a')) \wedge \text{Sat}_{(A_x, E_x)}(n, (b, a_0, \dots, a_m)))]
\end{aligned}$$

The absoluteness of P_3 for $L^{\omega+\omega}(\lambda)$ follows from the same reasoning as in Lemma 4.1.6.

Define Q_3 to be the conjunction of the sentence Q_2 and the instance of Comprehension that insures the existence of the set $\{(a, k) : Defn_x(a, k)\}$ for every $x \in L^{\omega+\omega}(\lambda)$. It should be clear that for any transitive structure (A, \in) that models Q_3 , the second result follows. \square

Up to this point, our development of $L^{\omega+\omega}$ has differed little from that of \mathbf{L} , except for keeping track of the $\mathbf{V}(\omega + k)$ rank of the sets involved. Now, our discussion diverges, due to the lack of enough ordinals in $L^{\omega+\omega}$. Without the ordinals $\geq \omega + \omega$, we cannot construct the actual $L^{\omega+\omega}(\alpha)$ sets inside $L^{\omega+\omega}$. Instead, we need construct sets isomorphic, or nearly isomorphic, to the $L^{\omega+\omega}(\alpha)$.

Suppose $LO(x)$. We say that f is a **coded hierarchy** on x (equivalently, f codes a hierarchy on x) if f is a function, $dom(f) = Field(x)$, and

$$\forall y \in Field(x)[SLO(y) \wedge Field(f(y)) \subseteq A^0 \cup Seq(V(\omega) \cup x)]$$

and f is recursively defined as follows:

1. if $y = 0_x$, then $f(y) = (A^0, E^0, \in \upharpoonright A^0, \mathbf{0})$; that is, for all $a \in A$, $F(a) = 0$,
2. if $z = Suc_x(y)$ then $f(z) = (A, E, <, F)$ where

$$A = Field(f(y)) \cup \{(\{1\}, y, n) \hat{\ } b_0 \hat{\ } \cdots \hat{\ } b_m \hat{\ } (\{2\})\} :$$

$$Defn_{f(y)}((n, b_0, \dots, b_m), k) \text{ for some } k\}$$

$$E = E_{f(y)} \cup \{(a, s) : a \in Field(f(y)) \wedge s \in A \setminus Field(f(y)) \wedge$$

$$s = (\{1\}, y, n) \hat{\ } b_0 \hat{\ } \cdots \hat{\ } b_m \hat{\ } (\{2\}) \wedge Sat_{f(y)}(n, (a, b_0, \dots, b_m))\}$$

$$< = <_{f(y)} \cup \{(a, s) : a \in Field(f(y)) \wedge s \in A \setminus Field(f(y))\}$$

$$\cup \{(a, s) : a, s \in A \setminus Field(f(y)) \wedge a = (\{1\}, y, n) \hat{\ } b_0 \hat{\ } \cdots \hat{\ } b_l \hat{\ } (\{2\})$$

$$\wedge s = (\{1\}, y, m) \hat{\ } c_0 \hat{\ } \cdots \hat{\ } b_r \hat{\ } (\{2\}) \wedge (n, b_0, \dots, b_l) <_{lex}^{f(y)} (m, c_0, \dots, c_l)\}$$

And, for all $a \in A$, if $a \in Field(f(y))$, then $F(a) = F_{f(y)}(a)$; otherwise, if $a \in A \setminus Field(f(y))$, say $a = (\{1\}, y, n) \hat{\ } b_0 \hat{\ } \cdots \hat{\ } b_l \hat{\ } (\{2\})$, then $F(a) = k$ where $Defn_{f(y)}((n, b_0, \dots, b_m), k)$.

3. if $Lim_x(y)$, then $f(y) = (A, E, <, F)$ where

$$\begin{aligned} A &= \bigcup_{z <_x y} Field(f(z)) & E &= \bigcup_{z <_x y} E_{f(z)} \\ < &= \bigcup_{z <_x y} <_{f(z)} & F &= \bigcup_{z <_x y} F_{f(z)} \end{aligned}$$

If f codes a hierarchy on x , we write $Chy_x(f)$. Note that we did not insist that x be a well order. Because of the lack of ordinals in $L^{\omega+\omega}$, we will rely sets in $L^{\omega+\omega}$ that on one hand resemble ordinals enough to build these coded hierarchies, but on the other may potentially contain an ill founded part. Note also we have unique readability of the finite sequences that constitute the field of a given structured linear order.

Lemma 5.1.7. *There is a formula $P_4(x, f)$ and a sentence Q_4 such that*

1. for all λ limit, $(L^{\omega+\omega}(\lambda), \in) \models Q_4$, and
2. for all transitive A such that $(A, \in) \models Q_4$ and for all $x, f \in A$,
 $(A, \in) \models P_4(x, f) \Leftrightarrow Chy_x(f)$

Proof. Let λ be limit. Rather than write out the cumbersome formula $P_4(x, f)$ which says f codes a hierarchy on x , we simply observe that P_4 is similar to the “ $f = \mathcal{L} \upharpoonright \alpha + 1$ ” formula on page 34 from Chapter 4. Define Q_4 to be the conjunction of Q_3 along with the instances of Comprehension insuring the existence of the sets of ordered pairs in the definition of a coded hierarchy. Again, λ limit implies that $L^{\omega+\omega}(\lambda) \models Q_4$. Moreover, if (A, \in) is any transitive structure modeling Q_4 , then P_4 is absolute for (A, \in) . \square

Our next lemmas contain the machinery to prove the $L^{\omega+\omega}$ analog of Proposition 4.1.8.

If $LO(x)$ is such that for all $y \subseteq Field(x)$, if $y \neq \emptyset$ implies that there is $a \in y$ such that for all $b \in y$, $b \not<_x a$, then x is a **well order**; we write $WO(x)$ to denote that x is a well order. Given a well order x , if α is the unique ordinal such that $(Field(x), <_x) \cong (\alpha, \in)$, then we write $x \cong \alpha$. If $WO(x)$ and $a \in Field(x)$, we write

x_a to denote the **initial segment** a of x . For $a \in Field(x)$, if $len(a) = \beta$, we write x_β to denote the initial segment of x of length β .

Using the closure properties of \mathbf{V} from Lemma 5.1.2, our next lemma shows that given a well order in $\mathbf{V}(\omega + \omega)$, we can code a unique hierarchy $f \in \mathbf{V}(\omega + \omega)$ that imitates the construction of the levels of $L^{\omega+\omega}$.

Lemma 5.1.8. *For all $x \in \mathbf{V}(\omega + \omega)$ such that $WO(x)$, there is a unique $f \in \mathbf{V}(\omega + \omega)$ such that $Chy_x(f)$. Moreover, for all $a \in Field(x)$,*

1. *there is a unique isomorphism g_a such that if $x_a \cong \beta$,*

$$(A_a, E_a) \stackrel{g_a}{\cong} (L^{\omega+\omega}(\beta), \in)$$

2. $WO(A_a, <_a)$

3. *for all $c \in A_a$, $F_a(c)$ is the least n such that $g_a(c) \in \mathbf{V}(\omega + n)$.*

Proof. Let $x \in \mathbf{V}(\omega + \omega)$ be such that $WO(x)$ and $len(x) = \alpha$. Define f such that $Chy_x(f)$ in the obvious way. A simple transfinite induction shows that if \bar{f} is such that $Chy_x(\bar{f})$, then $\bar{f} = f$.

To prove $f \in \mathbf{V}(\omega + \omega)$, we will show that there is $l \in \omega$ such that for all $a \in Field(x)$, $f(a) \in \mathbf{V}(\omega + l)$; it then follows that $f \in \mathbf{V}(\omega + l + 3) \subset \mathbf{V}(\omega + \omega)$. The key observation is that since for some $k \in \omega$, $x \in \mathbf{V}(\omega + k)$, then for all $a \in Field(x)$, $x_a \in \mathbf{V}(\omega + k)$. For $a \in Field(x)$ such that $len(a) = \beta$, we write $f(a) = (A_\beta, E_\beta, <_\beta, F_\beta)$. Suppose $\beta \leq \alpha$.

If $\beta = 0$, then $f(a) = (A^0, E^0, \in \upharpoonright A^0, \mathbf{0})$. As $A^0, E^0, \in \upharpoonright A^0, \mathbf{0} \in \mathbf{V}(\omega)$, it follows that $f(0) \in \mathbf{V}(\omega + 4)$. Because $x_0 \in \mathbf{V}(\omega + k)$, $(x_0, f(0)) \in \mathbf{V}(\omega + k + 4)$.

If $\beta > 0$, and β is a successor, then we claim that $f(a) \in \mathbf{V}(\omega + k + 12)$. Examining the definition of Chy , we see that $A_\beta \subset A^0 \cup Seq(\mathbf{V}(\omega) \cup x_\beta)$. Since $\mathbf{V}(\omega) \cup x_\beta \in \mathbf{V}(\omega + k)$ it follows that $A^0 \cup Seq(\mathbf{V}(\omega) \cup x_\beta) \in \mathbf{V}(\omega + k + 4)$. As $A_\beta \subset \mathbf{V}(\omega + k + 4)$, we have that $A_\beta \in \mathbf{V}(\omega + k + 5)$. Now, $E_\beta, <_\beta \subset A_\beta \times A_\beta \in \mathbf{V}(\omega + k + 8)$, thus $E_\beta, <_\beta \in \mathbf{V}(\omega + k + 9)$. Finally, $F_\beta \in \omega^{A_\beta} \in \mathbf{V}(\omega + k + 9)$. So since $A_\beta, E_\beta, <_\beta, F_\beta \in \mathbf{V}(\omega + k + 9)$, we have $f(a) \in \mathbf{V}(\omega + k + 12)$.

If $\beta > 0$ and is a limit, we observe that the coordinates of $f(a)$ involve taking unions, which does not increase \mathbf{V} -rank. So the limit case follows from the successor case, and is the end of the first claim.

Now let $a \in Field(x)$.

Claim 1: For each $a \in Field(x)$, we will define the maps $g_a : (A_\beta, E_\beta) \rightarrow (L^{\omega+\omega}(\beta), \in)$ by transfinite recursion and show that these maps are isomorphisms.

If $a = 0_x$, then we let g_0 be the fixed isomorphism π (p. 64) from (A^0, E^0) to $(\mathbf{V}(\omega), \in)$.

Now suppose that $a = Suc_x(b)$ and the isomorphism g_b has been defined appropriately. Let $s \in A_a$. Recall that

$$A_a = A_b \cup \{(\{1\}, b, n) \hat{b}_0 \hat{\dots} \hat{b}_m \hat{(\{2\})} : Defn_{f(b)}((n, b_0, \dots, b_m), k) \text{ for some nonzero } k \in \omega\}$$

If $s \in A_b$, then $g_a(s) = g_b(s)$. Otherwise, let $s = (\{1\}, b, n) \hat{b}_0 \hat{\dots} \hat{b}_m \hat{(\{2\})}$ be such that $Defn_{f(b)}((n, b_0, \dots, b_m), k)$ for some nonzero $k \in \omega$. Define

$$g_a(s) = \{z \in \mathbf{V}(\omega + (k - 1)) : (L^{\omega+\omega}(\beta), \in) \models \varphi_n(z, b_0, \dots, b_m)\}$$

Observe that this map is well-defined because of the unique readability of sequences. Evidently, g_a is a surjection. Moreover, the definition of $Defn$ insures that g_a is injective. To show that g_a is an isomorphism, we must show that g_a preserves the E_a relation. Let $t, s \in A_a$ be such that $tE_a s$. If $t, s \in A_b$, then we are done since g_b is an isomorphism. Otherwise, $t \in A_b$ and $s \in A_a \setminus A_b$ where $s = (\{1\}, b, n) \hat{b}_0 \hat{\dots} \hat{b}_m \hat{(\{2\})}$ and $Sat_{f(b)}(n, (t, b_0, \dots, b_m))$. Since $(A_b, E_b) \cong (L^{\omega+\omega}(\beta), \in)$, we have that

$$Sat_{f(b)}(n, (t, b_0, \dots, b_m)) \Leftrightarrow (L^{\omega+\omega}(\beta), \in) \models \varphi_n(g_b(t), b_0, \dots, b_m)$$

Thus, $g_a(t) \in g_a(s)$. Similar reasoning shows that $g_a(t) \in g_a(s) \Rightarrow tE_a s$. Hence, for all successor $a \in Field(x)$, g_a is an isomorphism.

Now suppose that $Lim_x(a)$ and that for all $b <_x a$, the isomorphism g_b has been

defined appropriately. Since $A_a = \bigcup_{b <_x a} A_b$, if $s \in A_a$, then there is a $<_x$ -least $b \in Field(x)$ such that $s \in A_b$. Define $g_a : A_a \rightarrow L^{\omega+\omega}(\beta)$ where $\beta = \bigcup_{b <_x a} \beta_b$ as follows: $g_a(s) = g_b(s)$ where b is the $<_x$ -least $b \in Field(x)$ such that $s \in A_b$. As each g_b is an isomorphism, so is g_a .

Claim 2: Let $a \in Field(x)$. We will show that $WO(A_a, <_a)$.

If $a = 0_x$, then $(A_0, <_0) = (A^0, \in \upharpoonright A^0)$, a well order, as this is the usual ordering of the integers.

Now suppose that $a \neq 0_x$ and that for all $b <_x a$, $WO(A_b, <_b)$. Let $S \subseteq A_a$ be nonempty. If $a = Suc_x(b)$ and $S \cap A_b \neq \emptyset$, then we are done as we can choose a $<_a$ -least element of S using the well order $<_b$. If $S \subseteq A_a \setminus A_b$, then we observe that $<_b$ induces a well ordering $<_{lex}^b$ of $Seq(\mathbf{V}(\omega) \cup x_b)$. Using this well order, choose the $<_{lex}^b$ -least element of S ; clearly then, this is the $<_a$ -least element of S . Now suppose $Lim_x(a)$ so that $A_a = \bigcup_{b <_x a} A_b$ and $<_a = \bigcup_{b <_x a} <_b$. Thus, for all $s \in S$, there is $b <_x a$ such that $s \in A_b$; that is, $\{b \in Field(x) : \exists s \in S (s \in A_b)\} \neq \emptyset$. Since $WO(x)$, choose a $<_x$ -least element of this set, say b_0 , and then use $<_{b_0}$ to choose a least element of S .

Claim 3: We will show that for all $a \in Field(x)$ and for all $c \in A_a$, $F_a(c)$ is the least $k \in \omega$ such that $g_a(c) \in \mathbf{V}(\omega + k)$. Let $a \in Field(x)$ and let $c \in A_a$. If $a = 0_x$, then $c \in A^0$. From the definition of g_0 , $g_0(c) \in L^{\omega+\omega}(0) = \mathbf{V}(\omega)$. As $F_a = \mathbf{0}$, then $F_a(c) = 0$ is the least k such that $g_a(c) \in \mathbf{V}(\omega + k)$.

Suppose now that $a \neq 0_x$, and suppose that for all $b <_x a$ that the result holds. If $a = Suc_x(b)$, and $c \in A_b$, then $F_a(c) = F_b(c)$, and we are done by the induction hypothesis. Otherwise, $c \in A_a \setminus A_b$ and is of the form $(\{1\}, b, n) \hat{\ } b_0 \hat{\ } \dots \hat{\ } b_m \hat{\ } (\{2\})$ such that $Defn_{f(b)}((n, b_0, \dots, b_m), k)$ for some nonzero $k \in \omega$. By definition, $F_a(c) = k$, and $g_a(c) = \{z \in \mathbf{V}(\omega + (k - 1)) : (L^{\omega+\omega}(\beta), \in) \models \varphi_n(z, b_0, \dots, b_m)\}$ and thus, $g_a(c) \in \mathbf{V}(\omega + k)$. Recall from the definition of $Defn$ that $k - 1$ is an element of the range of F restricted to the set defined by (n, b_0, \dots, b_m) . This implies that for some $z \in g_a(c)$, $z \in \mathbf{V}(\omega + (k - 1))$ witnessing that k is the least such k . The case $Lim_x(a)$ follows from the successor case. \square

Fix the following notation: given a structured linear order x , we write $R_x^k = \{b \in$

$Field(x) : F_x(b) \leq k\}$.

Proposition 5.1.9. *Suppose $x \in L^{\omega+\omega}(\beta)$ is such that $LO(x)$ and $x \cong \alpha$, for some $\alpha, \beta \in \mathbf{ON}$. Then, there exists $f \in L^{\omega+\omega}(\beta + \alpha + \omega)$ such that $Chy_x(f)$. Moreover, for each $a \in Field(x)$ and each $k \in \omega$, there is an isomorphism $g_a^k \in L^{\omega+\omega}(\beta + \alpha + \omega)$ such that if $x_a \cong \gamma$,*

$$(A_a \cap R_x^k, E_a \upharpoonright (A_a \cap R_x^k)) \stackrel{g_a^k}{\cong} (L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k), \in)$$

and $L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k) \in L^{\omega+\omega}(\beta + \alpha + \omega)$.

Proof. Let $x \in L^{\omega+\omega}(\beta)$ be a linear order such that $x \cong \alpha$. Fix n such that $\rho^{\mathbf{V}}(x) = \omega + n$. By Lemma 5.1.8, there is a unique $f \in \mathbf{V}(\omega + \omega)$ such that $Chy_x(f)$. We show by induction on α that $f \in L^{\omega+\omega}(\beta + \alpha + \omega)$.

Suppose $\alpha = 0$. Because $A^0, E^0, \in \upharpoonright A^0, \mathbf{0} \in L^{\omega+\omega}(1)$, it follows that $(A^0, E^0, \in \upharpoonright A^0, \mathbf{0}) \in L^{\omega+\omega}(4)$. Thus, $f \in L^{\omega+\omega}(\beta + 7) \subset L^{\omega+\omega}(\beta + \alpha + \omega)$.

Now suppose $\alpha > 0$ and for all $\gamma < \alpha$, $\exists g \in L^{\omega+\omega}(\beta + \gamma + \omega)$ such that $Chy_{x_\gamma}(g)$.

Suppose $\alpha = \gamma + 1$. Thus, for some k , there is $g \in L^{\omega+\omega}(\beta + \gamma + k)$ such that g codes a hierarchy on x_γ . To see that the $L^{\omega+\omega}$ ranks of the sets $A_x, E_x, <_x, F_x$ are bounded, note that the proof for Lemma 4.1.4 also works in $L^{\omega+\omega}$. Moreover, the \mathbf{V} -ranks of these sets are also bounded using Lemma 5.1.2. Thus, since $x \in L^{\omega+\omega}(\beta)$ and $(A_x, E_x, <_x, F_x) \in L^{\omega+\omega}(\beta + \gamma + l)$ for some l , it follows that

$$f = g \cup \{(x, (A_x, E_x, <_x, F_x))\} \in L^{\omega+\omega}(\beta + \gamma + k + l + 4) \subset L^{\omega+\omega}(\beta + \alpha + \omega).$$

Now suppose α is limit. It suffices to show that for some k

$$f(x) = \left(\bigcup_{\gamma < \alpha} A_\gamma, \bigcup_{\gamma < \alpha} E_\gamma, \bigcup_{\gamma < \alpha} <_\gamma, \bigcup_{\gamma < \alpha} F_\gamma \right) \in L^{\omega+\omega}(\beta + \alpha + k).$$

Observe that

$$\bigcup_{\gamma < \alpha} A_\gamma = \{z \in L^{\omega+\omega}(\beta + \alpha) : L^{\omega+\omega}(\beta + \alpha) \models \exists y, g[y \text{ is an initial segment of } x \wedge Chy_y(g) \wedge z \in Field(f(y))]\}$$

by the induction hypothesis and Lemma 5.1.7 which guarantees the absoluteness of “ $Chy_y(g)$ ” for limit stages of $L^{\omega+\omega}$. The sets $E_x, <_x, F_x$ follow similarly. Moreover, Lemma 5.1.8 insures that \mathbf{V} -ranks of these four of these sets are bounded. Using reasoning similar to the limit case of Proposition 4.1.8, each of the four sets can be shown to be in $L^{\omega+\omega}(\beta + \alpha + k)$ for some k .

Now let $a \in Field(x)$ be such that $x_a \cong \gamma$. We show by induction on k that there is an isomorphism $g_a^k \in L^{\omega+\omega}(\beta + \alpha + \omega)$ such that

$$(A_a \cap R_x^k, E_a \upharpoonright (A_a \cap R_x^k)) \stackrel{g_a^k}{\cong} (L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k), \in)$$

and that $L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k) \in L^{\omega+\omega}(\beta + \alpha + \omega)$.

Let $k = 0$. Then $(A_a \cap R_x^0, E_a \upharpoonright (A_a \cap R_x^0)) = (A^0, E^0)$ and $(L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + 0), \in) = (L^{\omega+\omega}(0), \in) = (\mathbf{V}(\omega), \in)$. So let $g_a^0 = \pi$ where π is the fixed recursive isomorphism from (A^0, E^0) to $(\mathbf{V}(\omega), \in)$ (p. 64). Clearly, both g_a^0 and $L^{\omega+\omega}(0)$ are elements of $L^{\omega+\omega}(1) \subset L^{\omega+\omega}(\beta + \alpha + \omega)$.

Now let $k > 0$ and suppose for induction that there is an isomorphism $g_a^k \in L^{\omega+\omega}(\beta + \alpha + \omega)$ such that $(A_a \cap R_x^k, E_a \upharpoonright (A_a \cap R_x^k)) \stackrel{g_a^k}{\cong} (L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k), \in)$ and that $L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k) \in L^{\omega+\omega}(\beta + \alpha + \omega)$. Given $s \in A_a \cap R_x^{k+1}$ define the map $g_a^{k+1} : (A_a \cap R_x^{k+1}, E_a \upharpoonright (A_a \cap R_x^{k+1})) \rightarrow (L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k + 1), \in)$ by

$$g_a^{k+1}(s) = \begin{cases} g_a^k(s), & \text{if } s \in R_x^k; \\ \{z \in L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k) : \exists t \in A_a (F_a(t) = k \wedge tE_a s)\}, & \text{otherwise} \end{cases}$$

Evidently, g_a^{k+1} is an isomorphism. Also, since $A_a, E_a, F_a, L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k) \in L^{\omega+\omega}(\beta + \alpha + \omega)$, it follows that for each $s \in A_a \cap R_x^{k+1}$, $g_a^{k+1}(s) \in L^{\omega+\omega}(\beta + \alpha + \omega)$. Consequently, $g_a^{k+1} = \{(s, g_a^{k+1}(s)) : s \in A_a \cap R_x^{k+1}\} \in L^{\omega+\omega}(\beta + \alpha + \omega)$. Finally,

observe that

$$L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k + 1) = \{z \in L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k) : \exists s \in A_a(z = g_a^{k+1}(s))\}$$

so that $L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega + k + 1) \in L^{\omega+\omega}(\beta + \alpha + \omega)$.

□

In Chapter 4, we dealt with certain structures isomorphic to $L(\lambda)$ for some λ limit. In light of Proposition 5.1.9, we will need to give up little ground. If λ is an ordinal such that for all $\beta < \lambda$, $\beta + \beta < \lambda$, then we say that λ is **additively closed**.

We say that $L^{\omega+\omega}(\alpha)$ is **pure** if $\omega < \alpha$ and for all $\beta < \alpha$,

1. $L^{\omega+\omega}(\beta) \neq L^{\omega+\omega}(\beta + 1)$
2. there is $x \in L^{\omega+\omega}(\alpha)$ such that $LO(x)$ and $x \cong \beta$.

So pure $L^{\omega+\omega}(\alpha)$ contain well orders that are copies of every ordinal less than α . Lemma 5.1.11 will show that for pure $L^{\omega+\omega}(\lambda)$, if λ is additively closed and $L^{\omega+\omega}(\lambda) \models WO(x)$ and x is not a well order, then the length of the well-founded part of x must be at least λ . The proof requires the following lemma.

Lemma 5.1.10. *Let M be a transitive nonempty structure and let $x \in M$ be such that $\neg WO(x)$, but $M \models WO(x)$. If \tilde{x} is the maximal well-founded initial segment of x , then $\tilde{x} \notin M$.*

Proof. Let M, x, \tilde{x} be as above and suppose $\tilde{x} \in M$. By assumption, $x \setminus \tilde{x} \neq \emptyset$. Let $a \in x$ be such that $M \models a$ is the least element of $x \setminus \tilde{x}$. But then, $\tilde{x} \cup \{a\}$ is a well-order, contradicting the maximality of \tilde{x} . □

Lemma 5.1.11. *Suppose λ is additively closed and $L^{\omega+\omega}(\lambda)$ is pure. If x is such that $(L^{\omega+\omega}(\lambda), \in) \models WO(x)$, then either $WO(x)$ or for all $\beta < \lambda$, there is $a \in \text{Field}(x)$ such that $(\beta, \in) \cong (\{b : b <_x a\}, <_x \upharpoonright \{b : b <_x a\})$.*

Proof. Let $L^{\omega+\omega}(\lambda)$ be pure for some additively closed λ , and let $x \in L^{\omega+\omega}(\lambda)$ be such that $L^{\omega+\omega}(\lambda) \models WO(x)$. Suppose x is not a well order. Let $\beta < \lambda$, and suppose

for a contradiction that β is the order type of \tilde{x} , the maximal well ordered initial segment of x . Thus, $\tilde{x} \neq x$, by assumption. Since $L^{\omega+\omega}(\lambda)$ is pure, let $y \in L^{\omega+\omega}(\lambda)$ be such that $LO(y)$ and $y \cong \beta$. Choose $\gamma < \lambda$ such that $\tilde{x}, y \in L^{\omega+\omega}(\gamma)$, and build an isomorphism $h : \tilde{x} \rightarrow y$ in the usual way. Clearly, $h \in L^{\omega+\omega}(\gamma + \beta + \omega)$, thus $\tilde{x} \in L^{\omega+\omega}(\gamma + \beta + \omega)$, contradicting Lemma 5.1.10. \square

Proposition 5.1.12. *Suppose λ additively closed, $L^{\omega+\omega}(\lambda)$ is pure, $(L^{\omega+\omega}(\lambda), \in) \models WO(x)$, and $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda + 1)$. Then there exists $f \in L^{\omega+\omega}(\lambda)$ such that $Chy_x(f)$ if there is $\alpha < \lambda$ such that $x \cong \alpha$.*

Proof. (\Leftarrow): Follows from Lemma 5.1.9.

(\Rightarrow): For some $\beta < \lambda$, let $f \in L^{\omega+\omega}(\beta)$ be such that $Chy_x(f)$. Suppose for a contradiction that there does not exist $\alpha < \lambda$ such that $x \cong \alpha$; that is, $\neg WO(x)$. By Lemma 5.1.11, the maximal well-founded initial segment of x , denoted \tilde{x} , must have at least length λ . Now consider $S \subset x$:

$$S = \{a \in Field(x) : \exists k \exists b \in Range(g_a^k) \forall c <_x a \forall p (b \notin Range(g_c^p))\}$$

where the g_a^k are as in Lemma 5.1.9. Observe from the proof of Lemma 5.1.9 that as $f \in L^{\omega+\omega}(\beta)$, for each k and each $a \in Field(x)$, $g_a^k \in L^{\omega+\omega}(\beta + \omega)$. Thus, $S \in L^{\omega+\omega}(\beta + \omega + 1) \subset L^{\omega+\omega}(\lambda)$. Moreover, it is clear that $\tilde{x} \subset S$. We now have two cases:

Case 1: $\tilde{x} \cong \lambda$. First note that $S \cap \tilde{x} \neq \emptyset$. (Otherwise, $\tilde{x} = S \in L^{\omega+\omega}(\lambda)$, contradicting Lemma 5.1.10.) Since $L^{\omega+\omega}(\lambda) \models WO(x)$, let $a \in S$ be the $<_x$ -least such that $a \notin \tilde{x}$. By definition of S , let $k \in \omega$ and $b \in Range(g_a^k)$ be such that for all $c <_x a$ and $p \in \omega$, $b \notin Range(g_c^p)$. On the one hand, $b \in Range(g_a^k) \subset L^{\omega+\omega}(\beta + \omega) \subset L^{\omega+\omega}(\lambda)$. But, on the other, for all $c \in \tilde{x}$ and all $p \in \omega$, $b \notin Range(g_c^p) \cong (L^{\omega+\omega}(\gamma_c) \cap \mathbf{V}(\omega + p))$. So $\tilde{x} \cong \lambda$ implies that $b \notin L^{\omega+\omega}(\lambda)$, a contradiction.

Case 2: $length(\tilde{x}) > \lambda$. So there is a well-founded initial segment of x of length $\lambda + 1$, also denoted \tilde{x} . Since $\tilde{x} \subset S$, let $k \in \omega$ and $b \in Range(g_{a_{\lambda+1}}^k)$ be as in the definition of S . On the one hand, all of the $g_a^k \in L^{\omega+\omega}(\lambda)$ by Lemma 5.1.9; in particular, $b \in Range(g_{a_{\lambda+1}}^k) \subset L^{\omega+\omega}(\beta + \omega) \subset L^{\omega+\omega}(\lambda)$. But on the other, $b \in$

$Range(g_{a_{\lambda+1}}^k) \cong L^{\omega+\omega}(\lambda+1) \cap \mathbf{V}(\omega+k)$ and by definition of S , $b \notin L^{\omega+\omega}(\gamma) \cap \mathbf{V}(\omega+j)$ for all $\gamma < \lambda+1$ and all $j \in \omega$. Hence, $b \notin L^{\omega+\omega}(\lambda)$, a contradiction. \square

We can now formulate our statement “ $\mathbf{V} = L^{\omega+\omega}$ ”. This statement must insure the existence of the g_a^k isomorphisms from Proposition 5.1.9. Note in the following formula that the predicate $P(s, w)$ refers to the fixed recursive isomorphism π from page 64. So we have “ $\mathbf{V} = L^{\omega+\omega}$ ” \Leftrightarrow

$$\begin{aligned} \forall x \exists y, f[Chy_y(f) \wedge WO(y) \wedge \forall a \in Field(y) \forall k \in \omega \exists g_a^k \\ [g_a^k \text{ is a function} \wedge dom(g_a^k) = A_a \wedge \\ \forall s \in dom(g_a^k)[s \in A_a \wedge F_a(s) \leq k \wedge \\ \forall l \leq k[(l = 0 \wedge F_a(s) = l) \Rightarrow \exists w(P(s, w) \wedge g_a^k(s) = w)] \wedge \\ ((l > 0 \wedge F_a(s) = l) \Rightarrow \exists w(z \in w \Leftrightarrow \\ \exists t \in A_a(F_a(t) = l - 1 \wedge g_a^{l-1}(t) = z \wedge tE_a s)))]]] \\ \wedge \exists a \in Field(y) \exists k \in \omega(x \in Range(g_a^k))] \end{aligned}$$

Theorem 5.1.13. *For every additively closed λ , if $L^{\omega+\omega}(\lambda)$ is pure and $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda+1)$, then $L^{\omega+\omega}(\lambda) \models \mathbf{V} = L^{\omega+\omega}$.*

Proof. Immediate from Proposition 5.1.12 \square

5.2 Consequences of $\mathbf{V} = L^{\omega+\omega}$

Our goal for this section is the same as in Section 4.2.

5.2.1 Transitive models of $\mathbf{V} = L^{\omega+\omega}$

Recall from Section 4.2.1 the following definition: given a structure M , define the ordinal of M , denoted $o(M)$, to be the least ordinal not in M . Clearly, this definition of $o(M)$ no longer suffices as $o(L^{\omega+\omega}) = \omega + \omega$. So we redefine $o(M)$ as follows: let $o(M)$ be the least ordinal not witnessed by some well-order in M . To insure that $o(M)$ is additively closed, let Q_5 be the conjunction of the sentence Q_4 and the sentence $\forall x, n \exists y[(WO(x) \wedge n \in \omega) \Rightarrow WO(y) \wedge y = x \times n]$. Note that when we

write “ $y = x \times n$ ” we mean $y = \{(i, a) : 0 \leq i < n \wedge a \in \text{Field}(x)\}$ with the reverse lexicographic ordering.

Proposition 5.2.1. *For all transitive A such that $A \models \mathbf{V} = L^{\omega+\omega} \wedge Q_5$ and for all $x \in A$, $[A \models \exists f(\text{Chy}_x(f)) \wedge \text{WO}(x)] \Rightarrow \text{WO}(x)$, then $\exists \lambda(\lambda \text{ is additively closed} \wedge A = L^{\omega+\omega}(\lambda))$.*

Proof. Let A be as above and let $\lambda = o(A)$. First, to show that λ is additively closed, let $\alpha < \lambda$. By definition, there is $x \in A$ of order type α . As $A \models Q_5$, let $y \in A$ be such that $y = x \times 2$. So y is a well order of order type $\alpha \times 2$. Thus, $\alpha \times 2 < \lambda = o(A)$.

We claim that $A = L^{\omega+\omega}(o(A))$. Suppose $x \in A$. As $A \models \mathbf{V} = L^{\omega+\omega}$, there exist $y, f \in A$ such that $A \models \text{Chy}_y(f) \wedge x \in f(y)$. As $[A \models \exists f(\text{Chy}_x(f))] \Rightarrow \text{WO}(x)$, we have $\text{WO}(y)$. Moreover, as $A \models Q_4$, it follows from Lemma 5.1.7 that $\text{Chy}_y(f)$. Proposition 5.1.8 guarantees, for each $a \in \text{Field}(y)$, the existence of unique isomorphisms $g_{y_a} : (A_{y_a}, E_{y_a}) \rightarrow (L^{\omega+\omega}(\beta_{y_a}), \in)$. So it follows that $x \in L^{\omega+\omega}(\beta_{y_a})$, for some $\beta_{y_a} < o(A)$. Thus, $A \subset L^{\omega+\omega}(o(A))$. The reverse inclusion follows by similar reasoning. \square

5.2.2 AC in $L^{\omega+\omega}$

Theorem 5.2.2. *There is a formula $P_5(x, y)$ such that for pure $L^{\omega+\omega}(\lambda)$, λ additively closed, and $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda + 1)$ we have $\text{WO}((L^{\omega+\omega}(\lambda), R))$ where $R = \{(a, b) : L^{\omega+\omega}(\lambda) \models P_5(a, b)\}$.*

Proof. Rather than write out P_5 (it is similar to $\theta(x, y)$ from page 39), we simply define the R .

$$R = \{(g_y^k(a), g_y^p(b)) : \exists x, y, f[\text{WO}(x) \wedge f \in L^{\omega+\omega}(\lambda) \wedge \text{Chy}_x(f) \wedge y \in \text{Field}(x) \wedge a, b \in A_y \wedge a <_y b \wedge F_y(a) = k \wedge F_y(b) = p]\}$$

Note that the g_y^k, g_y^p depend on x, f as in Lemma 5.1.9. \square

Corollary 5.2.3. $L^{\omega+\omega} \models \text{ZC}$

5.2.3 Skolem functions and Skolem hulls

Let $L^{\omega+\omega}(\lambda)$ be pure, λ additively closed, $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda+1)$, and recall from Section 4.2.3 the definition of a Skolem function for φ_n over a structure A . In light of the uniformly $L^{\omega+\omega}(\lambda)$ -definable well-ordering of $L^{\omega+\omega}(\lambda)$ given in Theorem 5.2.2, it should be clear that a formula analogous to $\sigma(n, x_1, \dots, x_k, y) \equiv "f_n(x_1, \dots, x_k) = y"$ (p. 41) can be defined for $L^{\omega+\omega}(\lambda)$. (This uses the recursive enumeration of all LST formulas (p. 64) and the satisfaction relation for $L^{\omega+\omega}(\lambda)$ from Lemma 5.1.5.) Consequently, any finite set of Skolem functions over $L^{\omega+\omega}(\lambda)$ is $L^{\omega+\omega}(\lambda)$ -definable. Given $x \subset L^{\omega+\omega}(\lambda)$ and a finite set of Skolem functions f_1, \dots, f_N over $L^{\omega+\omega}(\lambda)$, we form the Skolem hull H of x inside $L^{\omega+\omega}(\lambda)$ under f_1, \dots, f_N similarly to the definition of H on page 42 by replacing ω with x .

Given a set $x \subset L^{\omega+\omega}(\lambda)$, define the **transitive closure** of x , denoted $TC(x)$, by recursion on n

$$\begin{aligned} \bigcup^0 x &= x \\ \bigcup^{n+1} x &= \bigcup \left(\bigcup^n x \right) \\ TC(x) &= \bigcup \left\{ \bigcup^n x : n \in \omega \right\} \end{aligned}$$

In the following lemma, we form the Skolem hull of $TC(x)$ inside $L^{\omega+\omega}(\lambda)$ under a finite set of Skolem functions f_1, \dots, f_N . Note that by choosing N large enough ($N > \lceil \mathbf{V} = L^{\omega+\omega} \wedge Q_5 \rceil$), we have the $L^{\omega+\omega}(\lambda)$ analog of Lemma 4.2.4, the absoluteness for $H, L^{\omega+\omega}(\lambda)$ of the formula σ defining the Skolem functions. Having defined H , we can define the function G , the analog of our surjection $F : \omega \rightarrow H$. However, in this case, we define $G : \omega \times TC(x) \rightarrow H$, and insist that it need only be a partial function. In the definition of G , we use the same coding of finite sequences of integers (p. 43.)

Lemma 5.2.4. *Let $L^{\omega+\omega}(\lambda)$ be pure, λ additively closed, $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda+1)$, and $x \in L^{\omega+\omega}(\lambda+1)$ where $x = \{a : L^{\omega+\omega}(\lambda) \models \varphi(a)\}$. Then, there is a transitive $A \subset L^{\omega+\omega}(\lambda)$ such that*

1. $(A, \in) \models Q_5 \wedge \mathbf{V} = L^{\omega+\omega}$

2. $TC(x) \subset A \wedge \forall a \in x[(A, \in) \models \varphi(a) \Leftrightarrow (L^{\omega+\omega}(\lambda), \in) \models \varphi(a)]$
3. $(A, \in) \models \forall x[\exists f(Chy_x(f)) \Rightarrow WO(x)]$
4. for all $y \in A$, $(A, \in) \models WO(x) \Leftrightarrow (L^{\omega+\omega}(\lambda), \in) \models WO(x)$ and $(A, \in) \models \exists f(Chy_y(f)) \Leftrightarrow (L^{\omega+\omega}(\lambda), \in) \models \exists f(Chy_y(f))$
5. there is a partial onto function $G : \omega \times TC(x)^{<\omega} \rightarrow A$ and a formula $P_6(a, b, c, x)$ such that $G(a, b) = c \Leftrightarrow (L^{\omega+\omega}(\lambda), \in) \models P_6(a, b, c, x)$

Proof. Let λ and $x \in L^{\omega+\omega}(\lambda)$ be as above, say $x = \{a : L^{\omega+\omega}(\lambda) \models \varphi(a)\}$. Form the Skolem hull $H \subset L^{\omega+\omega}(\lambda)$ of $TC(x)$ under the Skolem functions f_1, \dots, f_N for the finite number of formulas needed. Now define the partial function $G : \omega \times TC(x)^{<\omega} \rightarrow H$ recursively as follows:

$$G(a, \vec{b}) = \begin{cases} \vec{b}_i & \text{if } a = \langle a_0, a_1 \rangle, a_0 = 0, \text{ and } a_1 = 1; \\ f_i(x_1, \dots, x_l) & \text{if } a = \langle a_0, \dots, a_l \rangle, a_0 = k, l = Ar(k), \text{ and } x_i = G(a, \vec{b}) \\ & \text{for } 1 \leq i \leq N \text{ and} \\ \emptyset & \text{if } a \text{ is not one of the above forms.} \end{cases}$$

By inspection, G is onto. The formula $P_6(a, b, c, x)$ is defined similar to the formula $\Phi(n, x)$ from Lemma 4.2.5 and is absolute for $H, L^{\omega+\omega}(\lambda)$.

Finally, collapse H to $A \subset L^{\omega+\omega}(\lambda)$ via the isomorphism $\eta : H \rightarrow A$ given by $\eta(z) = \{\eta(y) : y \in z\}$. Note first that η is $L^{\omega+\omega}(\lambda)$ -definable. Next, A is evidently transitive and preserves $TC(x)$. Thus, our choice of N large insures that 1, 2, and 3 are satisfied. Moreover, this isomorphism clearly preserves well-orderings so that 4 is satisfied. As $H \stackrel{\eta}{\cong} A$, G is onto A and the absoluteness of P_6 is preserved, satisfying 5. \square

Lemma 5.2.5. *Let $L^{\omega+\omega}(\lambda)$ be pure, λ additively closed, $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda + 1)$. Then there is a partial function G and P_6 such that 5. holds in Lemma 5.2.4 and $A = L^{\omega+\omega}(\lambda)$.*

Proof. Let λ be as above. Because $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda+1)$, there is $w \in L^{\omega+\omega}(\lambda+1) \setminus L^{\omega+\omega}(\lambda)$, not necessarily parameter-free. Using the well-ordering given by the formula P_5 in Lemma 5.2.2, order the set of tuples of parameters that define a new set in $L^{\omega+\omega}(\lambda+1) \setminus L^{\omega+\omega}(\lambda)$, then define a parameter-free $x \in L^{\omega+\omega}(\lambda)$, $x \notin L^{\omega+\omega}(\lambda)$ via $x = \{a : L^{\omega+\omega}(\lambda) \models \varphi(a)\}$ by choosing the least tuple of parameters. Note that x is $L^{\omega+\omega}(\lambda)$ -definable arguing similarly as in Lemma 4.3.2.

Now let $A \subset L^{\omega+\omega}(\lambda)$ be as in Lemma 5.2.4. Note that since $A \models \mathbf{V} = L^{\omega+\omega}$, $A = L^{\omega+\omega}(\beta)$, for some additively closed β , by Proposition 5.2.1. By the absoluteness of φ for $A, L^{\omega+\omega}(\lambda)$, $x \in L^{\omega+\omega}(\beta+1)$. But also, $x \notin L^{\omega+\omega}(\lambda)$, so $\beta \not\leq \lambda$. Thus, $L^{\omega+\omega}(\lambda) \subseteq L^{\omega+\omega}(\beta)$, and hence, $\lambda \leq \beta$. Now suppose $z \in A = L^{\omega+\omega}(\beta)$. Since $A \models \mathbf{V} = L^{\omega+\omega}$, $A \models \exists f, y (Chy_y(f) \wedge z \in f(y))$. As $H \cong A$, $H \models \exists f, y (Chy_y(f) \wedge z \in f(y))$. Let $f, y \in H \subset L^{\omega+\omega}(\lambda)$ be as such. By 3. of Lemma 5.2.4, $H \models WO(y)$. So by 4. of Lemma 5.2.4, $L^{\omega+\omega}(\lambda) \models WO(y)$. Similarly, $L^{\omega+\omega}(\lambda) \models \exists f (Chy_y(f))$. So $L^{\omega+\omega}(\lambda) \models \exists f, y (Chy_y(f) \wedge z \in f(y))$. Thus, $z \in L^{\omega+\omega}(\lambda)$. \square

Up to this point, all of our results have been predicated on the purity of the $L^{\omega+\omega}(\lambda)$. The next proposition shows that if a new set gets constructed at $L^{\omega+\omega}(\lambda+1)$, then $L^{\omega+\omega}(\lambda+1)$ is pure.

Proposition 5.2.6. *Let $L^{\omega+\omega}(\lambda)$ be pure, λ additively closed, $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda+1)$. Then $L^{\omega+\omega}(\lambda+1)$ is pure.*

Proof. Let λ be as above. As $L^{\omega+\omega}(\lambda)$ is pure, it is enough to show that there is a well order $(A, R) \in L^{\omega+\omega}(\lambda)$ of order type λ .

Since $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda+1)$, there is $y \in L^{\omega+\omega}(\lambda+1) \setminus L^{\omega+\omega}(\lambda)$. We first claim that there is a parameter-free $x \in L^{\omega+\omega}(\lambda+1) \setminus L^{\omega+\omega}(\lambda)$. Fix $k \in \omega$, the least arity of a tuple of parameters from $L^{\omega+\omega}(\lambda)$ that define some $x \in L^{\omega+\omega}(\lambda+1) \setminus L^{\omega+\omega}(\lambda)$ (y witnesses the existence of at least one such tuple), and fix $n \in \omega$ the least formula φ_n that defines some $x \in L^{\omega+\omega}(\lambda+1) \setminus L^{\omega+\omega}(\lambda)$ from some k -tuple of parameters. Now define $y = \{z \in L^{\omega+\omega}(\lambda) : \psi(z)\}$, where $\psi(z)$ is the formula

$$\begin{aligned} & \exists \vec{b} [\vec{b} \text{ is a } k\text{-tuple} \wedge \varphi_n(z, \vec{b}) \wedge \\ & \neg \exists w \forall p (p \in w \Leftrightarrow \varphi_n(p, \vec{b})) \wedge \forall \vec{c} (P_5(\vec{c}, \vec{b}) \Rightarrow \exists w \forall p (p \in w \Leftrightarrow \varphi_n(p, \vec{c})))] \end{aligned}$$

and P_5 is as in Theorem 5.2.2. Clearly, y is a parameter-free $L^{\omega+\omega}(\lambda)$ -definable subset of $L^{\omega+\omega}(\lambda)$ and $y \notin L^{\omega+\omega}(\lambda)$. So by Lemma 5.2.4 form the hull $H \subset L^{\omega+\omega}(\lambda)$ of $TC(y)$ under a finite number of Skolem functions, and let $A \subset L^{\omega+\omega}(\lambda)$ be the transitive collapse of H . By Lemma 5.2.5, $A = L^{\omega+\omega}(\lambda)$. Let $G : \omega \times TC(y) \rightarrow A$ be the partial surjection guaranteed by Lemma 5.2.5, and let P_6 be the formula also from Lemma 5.2.5. Let $B = Dom(G)$ and define a binary relation R on B by

$$(n_1, x_1)R(n_2, x_2) \Leftrightarrow L^{\omega+\omega}(\lambda) \models P_5(G(n_1, x_1), G(n_2, x_2))$$

Clearly, $(B, R) \in L^{\omega+\omega}(\lambda)$. Moreover, (B, R) has order type at least λ . If (B, R) has order type $< \lambda$, then the initial segment of (B, R) of length λ proves the proposition. \square

The next proposition shows that everything is pure.

Proposition 5.2.7. *If $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$ for $\omega < \alpha$, then $L^{\omega+\omega}(\alpha)$ and $L^{\omega+\omega}(\alpha+1)$ are pure.*

Proof. Let $\alpha > \omega$ be such that $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$. If α is additively closed, then the purity of $L^{\omega+\omega}(\alpha+1)$ follows from Proposition 5.2.6. If α not additively closed, then let $\beta < \alpha$ be the largest additively closed limit ordinal below α . So $\alpha = (\beta \cdot n) + \gamma$ for some $\gamma < \beta$. Since $L^{\omega+\omega}(\beta+1)$ is pure, $L^{\omega+\omega}(\beta+1)$ contains well orders of length β and γ . Build a well order of length α in the usual way. Clearly, this well-order is in $L^{\omega+\omega}(\alpha)$. \square

The following proposition insures that the $L^{\omega+\omega}$ construction ends at a sufficiently large ordinal.

Proposition 5.2.8. *If $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$, then $L^{\omega+\omega}(\alpha \cdot \omega) \neq L^{\omega+\omega}(\alpha \cdot \omega + 1)$.*

Proof. Suppose $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$, but $L^{\omega+\omega}(\alpha \cdot \omega) = L^{\omega+\omega}(\alpha \cdot \omega + 1)$. By Proposition 5.2.7, there is a well order of length α in $L^{\omega+\omega}(\alpha+1)$. So we have $y \in L^{\omega+\omega}(\alpha \cdot \omega)$ a well order of length $(\alpha \cdot \omega) + 1$. As $(L^{\omega+\omega}(\alpha \cdot \omega), \in) \models \mathbf{Z}$, Proposition 5.1.9 holds in $L^{\omega+\omega}(\alpha \cdot \omega)$. So there must be $f \in L^{\omega+\omega}(\alpha \cdot \omega)$ such that $Chy_y(f)$. That is, f

codes the entire model since $L^{\omega+\omega}(\alpha \cdot \omega) = L^{\omega+\omega}(\alpha \cdot \omega + 1)$. So, $TC(f) \in L^{\omega+\omega}(\alpha \cdot \omega)$ and thus, $L^{\omega+\omega}(\alpha \cdot \omega)$ must satisfy that every set has smaller cardinality than $TC(f)$. But as $L^{\omega+\omega}(\alpha \cdot \omega)$ models the power set axiom and Cantor's theorem holds in $L^{\omega+\omega}(\alpha \cdot \omega)$, this is a contradiction. \square

5.3 The theory of $L^{\omega+\omega}(\lambda)$

Our first lemma shows that there is an $L^{\omega+\omega}(\lambda)$ -definable map from ω onto $L^{\omega+\omega}(\lambda)$

Lemma 5.3.1. *Let $x \in \omega^\omega \cap L^{\omega+\omega}$. Then there is a limit λ such that $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda + 1)$ and a formula $P_7(n, x, a)$ such that $L^{\omega+\omega}(\lambda) \models \forall x \exists! n \in \omega (P_7(x, n, a))$ for some $a \in L^{\omega+\omega}(\lambda)$.*

Proof. \square

The next lemma shows that we can eliminate the parameter from the formula P_7 .

Lemma 5.3.2. *Let $x \in \omega^\omega \cap L^{\omega+\omega}$. Then there is a limit λ such that $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda + 1)$ and a formula $P_8(n, x)$ such that $L^{\omega+\omega}(\lambda) \models \forall x \exists! n \in \omega (P_8(x, n))$.*

Proof. \square

Recall that given a structure A , $Th(A)$ denotes the theory of A , the set of all sentences true in A .

Lemma 5.3.3. *Suppose $P_9(n, x)$ is a formula such that $(L^{\omega+\omega}(\lambda), \in) \models \forall x \exists! n \in \omega (P_9(x, n))$. Then $Th(L^{\omega+\omega}(\lambda)) \in L^{\omega+\omega}(\lambda + 2)$.*

Proof. \square

Given a structure (A, E) , $n \in \omega$ and $x \in A$, we write $Def((A, E), n, x)$ if φ_n is a formula of exactly one free variable, and x is the unique element of A such that $(A, E) \models \varphi_n(x)$, and furthermore n is the least integer with this property that x is the unique element of A such that $(A, E) \models \varphi_n(x)$.

Theorem 5.3.4. *For every $x \in \omega^\omega \cap L^{\omega+\omega}$, there is a limit ordinal λ and formulas $\psi_1(v_0, v_1), \psi_2(v_0, v_1), \psi_3(v_0, v_1)$ whose free variables are shown such that*

1. $x \in L^{\omega+\omega}(\lambda)$,
2. for all $y \in L^{\omega+\omega}(\lambda)$, there is $n \in \omega$ such that $Def((L^{\omega+\omega}(\alpha), \in), n, y)$, for $\alpha < \lambda$,
3. $Th(L^{\omega+\omega}(\lambda), \in) \in L^{\omega+\omega}(\lambda + 2)$
4. $(L^{\omega+\omega}, \in) \models \psi_1(v_0, v_1) \Leftrightarrow \rho(v_0) < \rho(v_1)$,
5. $(L^{\omega+\omega}, \in) \models \psi_2(v_0, v_1) \Leftrightarrow \rho(v_0) = \rho(v_1)$, and
6. $(L^{\omega+\omega}, \in) \models \psi_3(v_0, v_1) \Leftrightarrow v_1 = \rho^{\mathbf{V}}(v_0)$.

5.4 The non-determined $\Pi_{\omega+2}^0$ set

We conclude our final chapter with the construction of a $\Pi_{\omega+2}^0$ set of degrees which neither contains nor omits a cone. The reals in these degrees are the theories of structures called towered structures which are similar to those in Section 4.4 except that a towered structure retreats from full well-foundedness to Σ_{ω}^0 well-foundedness (clause 11 of the definition to follow). Note that the formulas ψ_1, ψ_2, ψ_3 occurring in the following definition are as in Theorem 5.3.4.

Definition 1. Given $A \subseteq \omega$, $A \neq \emptyset$ and a binary relation E on A , let

$$\begin{aligned} < &= \{(x, y) \in A \times A : \psi_1^{(A, E)}(x, y)\}, \\ \sim &= \{(x, y) \in A \times A : \psi_2^{(A, E)}(x, y)\}, \\ F &= \{(x, n) \in A \times \omega : \psi_3^{(A, E)}(x, n)\}. \end{aligned}$$

Then (A, E) is a **towered structure** provided that:

1. \sim is an equivalence relation on A ,
2. $<$ is a strict linear order preserving \sim and having no maximal element
3. $A^0 = \{i \in A : \forall j \in A(j \not< i)\}$ and $E^0 = E \upharpoonright A^0$,
4. $\forall x \in A \exists ! n \in \omega [(x \in A^0 \Rightarrow F(x) = \bar{0}) \wedge F(x) = \bar{n}]$,

5. $\forall x \in A \setminus A^0 \exists n \in \omega [F(x) = \bar{n} \wedge (\forall y \in A \exists m \in \omega (yEx \Rightarrow F(y) = \bar{m} \wedge m <_I n))]$
where $<_I$ is the usual order on ω ,
6. $\forall x, y \in A [xEy \Rightarrow x < y]$,
7. (A, E) models *Extensionality* where the \in relation is interpreted as E ,
8. $\forall i \in A \setminus A^0 [Def((A, E), \frac{i}{2}, i)]$,
9. $\forall x, z \in A [z \in FODO(I_x, E \upharpoonright I_x) \Leftrightarrow (z \subset I_x \wedge \exists j [(j < x \vee j \sim x) \wedge z = \{k : kEj\}])]$, where $I_x = \{i : i < x\}$,
10. $Th(A, E) \in FODO(FODO((A, E)), \in)$, and
11. for every Π_ω^0 relation $Q(n, f)$, if $\exists n \in A (\neg Q(n, Th(A, E)))$ then
 $\exists n \in A [\neg Q(n, Th(A, E)) \wedge \forall m < n (Q(m, Th(A, E)))]$

Now define T as follows:

$$T(y) \Leftrightarrow \exists x [x \text{ is a towered structure} \wedge y = Th(x)].$$

At first, the initial existential real quantifier seems to make $T \in \Sigma_1^1$. The next lemma shows that we can eliminate this quantifier.

Lemma 5.4.1. *For all $y \in T$, if $x \in 2^\omega$ is such that $y = Th(\omega, E_x)$ for some towered structure (ω, E_x) , then $x \leq_T y$.*

Proof. The proof is similar to the proof of Lemma 4.4.4. Clauses 3 and 9 in the definition of a towered structure provide the necessary definability conditions. \square

Again, as in Lemma 4.4.4, it is clear that there is one algorithm for all towered structures (ω, E_x) that computes x from $Th(\omega, E_x)$. We fix an index of this algorithm, say e . So, we reformulate T : $T(y) \Leftrightarrow f_e^y$ codes a towered structure $\wedge y = Th(f_e^y)$.

Lemma 5.4.2. $T \in \Pi_{\omega+2}^0$

Proof. Let $y \in 2^\omega$. We show that

$$\begin{aligned} T(y) &\Leftrightarrow T_1(y) \wedge T_2(y) \\ &\Leftrightarrow y = Th(f_e^y) \wedge f_e^y \text{ codes a towered structure } \in \Pi_{\omega+2}^0 \end{aligned}$$

by showing that $T_1 \in \Pi_\omega^0(y)$ and $T_2 \in \Pi_{\omega+2}^0(y)$.

To see that $T_1 \in \Pi_\omega^0(y)$, observe that

$$“y \in Th(f_e^y)” \Leftrightarrow \forall k [k \in y \Leftrightarrow (\omega, E_{f_e^y}) \models \varphi_k]$$

Let $k \in \omega$. “ $k \in y$ ” is evidently $\Delta_1^0(y)$. Recall Lemma 4.4.3 which insures that if φ_k is $\Sigma_{n(k)}(\Pi_{n(k)})$, then $\{x \in 2^\omega : (\omega, E_x) \models \varphi_k\} \in \Sigma_{n(k)}^0(\Pi_{n(k)}^0)$. Thus it follows that $(\omega, E_{f_e^y}) \models \varphi_k \in \Sigma_{n(k)}^0(y)(\Pi_{n(k)}^0(y))$. Thus, as k ranges over ω , $n(k)$ increases without bound. Hence, $T_1 \in \Pi_\omega^0(y)$.

Next, we claim that $T_2(y) \in \Pi_{\omega+2}^0(y)$. Specifically, clauses 1-9 are arithmetic, clause 10 is $\Sigma_\omega^0(y)$ and clause 11 is $\Pi_{\omega+2}^0(y)$.

It is easy to verify that clauses 1-7 are $\Sigma_n^0(y)$ for some n . Given $i \in \omega$, it should be clear from Lemma 5.4.1 that $Def((A, E), \frac{i}{2}, i)$ is $\Delta_1^0(y)$, because we can consult y to verify whether or not $\ulcorner Def((A, E), \frac{i}{2}, i) \urcorner \in y$. Consequently, clause 8 is $\Pi_1^0(y)$. As for clause 9, let $x, z \in \omega$. Recall that $I_x = \{m : \psi_1(m, x)\}$. The formula “ $z \in FODO(I_x, E_{f_e^y} \upharpoonright I_x)$ ” abbreviates the formula $\exists n, z_1, \dots, z_k \forall m [m \in z \Leftrightarrow ((I_x, E_{f_e^y} \upharpoonright I_x) \models \varphi_n(m, z_1, \dots, z_k))]$. Now there is a recursive procedure that, given a formula φ_n , produces the relativization of φ_n to $(I_x, E_{f_e^y} \upharpoonright I_x)$. Namely, replace each instance of the \in relation in φ_n with the $E_{f_e^y}$ relation, and replace every unbound quantifier $\exists l$ with $\exists l[\psi_1(l, x) \wedge \dots]$. Thus, $z \in FODO(I_x, E_{f_e^y} \upharpoonright I_x)$ is $\Sigma_2^0(y)$. It is easy to verify that “ $z \subset I_x$ ” is $\Pi_1^0(y)$ and that $\exists j[(j < x \vee j \sim x) \wedge z = \{k : k E_{f_e^y} j\}]$ is $\Sigma_2^0(y)$. Consequently, clause 9 is $\Pi_3^0(y)$.

Next, we claim that clause 10 is Σ_ω^0 . Observe that the formula

$$\exists n, z_1, \dots, z_m \forall k [k \in y \Leftrightarrow (FODO((A, E_{f_e^y}), \in) \models \varphi_n(k, z_1, \dots, z_m))]$$

faithfully represents clause 10. Let $n, k \in \omega$ and $z_1, \dots, z_m \in FODO((A, E_{f_e^y}), \epsilon)$. From our above argument for clause 9, it should be clear that each incidence of $z_i \in FODO((A, E_{f_e^y}), \epsilon)$ is arithmetic in y . Using the reasoning found in Lemma 4.4.5, we claim that if φ_n is $\Sigma_{p(n)}(\Pi_{p(n)})$, then $(FODO((A, E_{f_e^y}), \epsilon) \models \varphi_n(k, z_1, \dots, z_m))$ is $\Sigma_{p(n)}^0(y)(\Pi_{p(n)}^0(y))$. Thus, $\forall k[k \in y \Leftrightarrow (FODO((A, E_{f_e^y}), \epsilon) \models \varphi_n(k, z_1, \dots, z_m))]$ is $\Pi_{p(n)+1}^0(y)$. As n ranges over ω , $p(n)$ increases without bound. Thus, clause 10 is $\Sigma_\omega^0(y)$.

Finally, we claim that clause 11 is $\Pi_{\omega+2}^0$. Observe that the formula

$$\begin{aligned} & \forall k[k \text{ codes a } \Pi_\omega^0 \text{ relation} \wedge [\exists n(\neg\varphi_k(n, Th(A, E)))] \Rightarrow \\ & \exists n(\neg\varphi_k(n, Th(A, E)) \wedge \forall m < n(\varphi_k(m, Th(A, E))))] \end{aligned}$$

faithfully represents clause 11. Counting quantifiers, it is easy to see that this is $\Pi_{\omega+2}^0$. \square

Recall that at this point in Section 4.4, we have Lemma 4.4.7 which assigns a unique limit ordinal to each structure. This is possible because of each structure's well-foundedness, a luxury we lack in towered structures. We must therefore use some other way to compare structures. Our next proposition is the main idea in that direction and shows that if two towered structures are "close" enough, then either they are isomorphic or one is isomorphic to an initial segment of the other. Note that for $x, y \in \omega^\omega$, the **join** of x and y , denoted $x \oplus y$, is given by:

$$(x \oplus y)(k) = \begin{cases} x(n) & \text{if } k = 2n; \\ y(n) & \text{if } k = 2n + 1. \end{cases}$$

Proposition 5.4.3. *If $(A_1, E_1), (A_2, E_2)$ are towered structures such that*

$$Th(A_1, E_1) \leq_T J(Th(A_2, E_2)) \text{ and } Th(A_2, E_2) \leq_T J(Th(A_1, E_1)),$$

then either

1. $(A_1, E_1) \cong (A_2, E_2)$, or

2. $(A_1, E_1) \cong (I_x, E_2 \upharpoonright x)$ for some $x \in A_2$ where $I_x = \{y \in A_2 : y <_2 x\}$, or
3. $(A_2, E_2) \cong (I_x, E_1 \upharpoonright x)$ for some $x \in A_1$ where $I_x = \{y \in A_1 : y <_1 x\}$.

Proof. Let $(A_1, E_1), (A_2, E_2)$ be towered structures as above where $<_i, \sim_i, F_i$ refer to (A_i, E_i) and $T_i = Th(A_i, E_i)$ for $i = 1, 2$. We build a partial map from (A_1, E_1) to (A_2, E_2) inductively on the $\mathbf{V}(\omega + k)$ rank of the elements of A_1, A_2 . Then we prove by induction on the rank of the elements of A_1, A_2 that either the map is an isomorphism or the map is an isomorphism from one structure to an initial segment of the other.

Define $P(n, i, j)$ by recursion on n as follows:

$$\begin{aligned}
P(0, i, j) &\Leftrightarrow i \in A^0 \wedge i = j \\
P(n+1, i, j) &\Leftrightarrow F_1(i) = F_2(j) = \overline{n+1} \wedge \\
&\quad \forall a[aE_1i \Rightarrow \exists b, k(bE_2j \wedge P(k, a, b) \wedge F_1(a) = F_2(b) = \bar{k})] \wedge \\
&\quad \forall b[bE_2j \Rightarrow \exists a, k(aE_1i \wedge P(k, a, b) \wedge F_1(a) = F_2(b) = \bar{k})]
\end{aligned}$$

It is clear that for each k , $P(k, a, b)$ uniformly defines a $\Delta_{2k}^0(T_1 \oplus T_2)$ relation. Consequently, uniformly for each k , $P(k, a, b) \in \Delta_{2k+1}^0(T_1)$ and $P(k, a, b) \in \Delta_{2k+1}^0(T_2)$.

Next, we show by induction on n that for each $i \in A_1$, there is at most one $j \in A_2$ such that $P(n, i, j)$. The case $n = 0$ is trivial because $(A_1 \cap A^0, E_1 \upharpoonright E^0) = (A_2 \cap A^0, E_2 \upharpoonright E^0)$. Now suppose that for all $k \leq n$ and for all $i \in A_1$ there is at most one $j \in A_2$ such that $P(k, i, j)$. Let $i \in A_1, j, j' \in A_2$ be such that $P(n+1, i, j)$ and $P(n+1, i, j')$. We claim that $j = j'$. Suppose $b \in A_2$ is such that bE_2j . By Clause 5 of the definition of a towered structure, $F_2(b) = \bar{k}$ for some $k \leq n$. By the definition of P , there is $a \in A_1$ such that aE_1i and $P(k, a, b)$. Since $P(n+1, i, j')$, for some $b' \in A_2, b'E_2j'$ and $P(k, a, b')$. But by the induction hypothesis, $b = b'$. Thus, bE_2j' . A symmetric argument shows that $\forall b[bE_2j' \Rightarrow bE_2j]$. Since (A_2, E_2) models Extensionality, we must have $j = j'$. A similar argument shows that for each $j \in A_2$, there is at most one $i \in A_1$ such that $P(n, i, j)$. Consequently, P defines a partial isomorphism from A_1 to A_2 . Let π be the partial isomorphism from A_1 to A_2 induced by the predicate $P(n, i, j)$.

Now consider the following $K \subseteq A_1$:

$$\begin{aligned}
i \in K &\Leftrightarrow i \in \text{dom } \pi \wedge \\
&\forall j, a, b [(\pi(i) = a \wedge i \sim_1 j) \Rightarrow (j \in \text{dom } \pi \wedge (\pi(j) = b \Rightarrow a \sim_2 b))] \wedge \\
&\forall j, a, b [(\pi(i) = a \wedge a \sim_2 b) \Rightarrow (b \in \text{range } \pi \wedge (\pi(j) = b \Rightarrow i \sim_1 j))] \wedge \\
&\forall j, a, b [(\pi(i) = a \wedge b <_2 a) \Rightarrow (b \in \text{range } \pi \wedge (\pi(j) = b \Rightarrow j <_1 i))]
\end{aligned}$$

We claim that $K \in \Pi_\omega^0(T_1)$. Observe that

$$\begin{aligned}
“i \in \text{dom } \pi” &\Leftrightarrow \exists n, j P(n, i, j) \Leftrightarrow \forall n [\psi_3(i, n) \Rightarrow \exists j P(n, i, j)] \\
“j \in \text{range } \pi” &\Leftrightarrow \exists n, i P(n, i, j) \Leftrightarrow \forall n [\psi_3(j, n) \Rightarrow \exists i P(n, i, j)]
\end{aligned}$$

It follows that $\text{dom}(\pi), \text{range}(\pi) \in \Delta_\omega^0(T_1)$. The presence of the universal quantifiers makes $K \in \Pi_\omega^0(T_1)$.

We use K to argue by cases that either π is an isomorphism between (A_1, E_1) and (A_2, E_2) , or π is a partial isomorphism between one towered structure and an initial segment of the other. Recall that for $x \in A_i$, $I_x = \{z \in A_i : z <_i x\}$, $i = 1, 2$.

Case 1: $A_1 \setminus K = \emptyset$ and $\forall a \in A_2 \exists n, i P(n, i, a)$.

Then $\text{dom}(\pi) = A_1$ and $\text{range}(\pi) = A_2$. Hence, $(A_1, E_1) \cong (A_2, E_2)$ via π .

Case 2: $A_1 \setminus K = \emptyset$ and $\exists a \in A_2 \forall n, i (\neg P(n, i, a))$.

Again $\text{dom}(\pi) = A_1$. Let $L = \{a \in A_2 : \forall n, i (\neg P(n, i, a))\}$. First, we claim that $L \leq_T J^\omega(T_2)$. Let $a \in A_1$ be given. We can compute $F_1(a)$ recursively in T_1 , say $F_1(a) = n$. Then for any $a \in A_2$, we can verify $\forall i (i <_1 a \Rightarrow \neg P(n, i, a))$ recursively in $J^{2n+1}(T_2)$. Thus, $L \leq_T J^\omega(T_2)$.

Now $L \neq \emptyset$ by assumption. Thus, by clause 11 of the definition of a towered structure, there is $y \in A_2$ such that $\forall n, i (\neg P(n, i, y)) \wedge \forall a <_2 y (\exists n, i P(n, i, a))$. We claim that $(A_1, E_1) \stackrel{\pi}{\cong} (I_y, E_2 \upharpoonright I_y)$. Suppose $a \in A_2$ is such that $\exists n, i P(n, i, a)$. Note that $i \in K$ by the assumption that $A_1 = K$. Thus, if $a \sim_2 y$, then there would be $r \in \omega$ and $d \in A_1$ such that $d \sim_1 i$ and $P(r, d, y)$, violating the definition of y . Similar reasoning shows that if $y <_2 a$ then there exist $r \in \omega$ and $d \in A_1$ such that $d <_1 i$ such that $P(r, d, y)$, again, a contradiction. Consequently, $\forall a \in A_2 (\exists n, i P(n, i, a) \Rightarrow$

$a <_2 y$), and thus $\text{range}(\pi) = I_y$.

Case 3: $A_1 \setminus K \neq \emptyset$.

So there is $x \in A_1 \setminus K$. Because $K \in \Pi_\omega^0(T_1)$, there is a $<_1$ -least such x by Clause 11 of the definition of a towered structure. Let x be as such. Note that $x \notin A^0$ as $A^0 \subset K$. Now consider $L = \{a \in A_2 : \forall n, i (i <_1 x \Rightarrow (\neg P(n, i, a)))\}$. There are two cases: $L = \emptyset$ and $L \neq \emptyset$.

If $L = \emptyset$, then $(I_x, E_1 \upharpoonright I_x) \cong (A_2, E_2)$ via π^{-1} .

Suppose now that $L \neq \emptyset$; we will show that $x \in K$, a contradiction. Let $y \in L$ be the $<_2$ -least element of L ; again, clause 11 of the definition of a towered structure insures that such a y exists. Hence, $\forall n, i (i <_1 x \Rightarrow \neg P(n, i, y))$ and $\forall a <_2 y \exists n, i (i <_1 x \wedge P(n, i, a))$. Thus, $(I_x, E_1 \upharpoonright I_x) \cong (I_y, E_2 \upharpoonright I_y)$ via π . To show $x \in K$, it is enough to show that $\forall i \sim_1 x (\exists n, a (P(n, i, a) \wedge a \sim_2 y))$ and $\forall a \sim_2 y (\exists n, i (P(n, i, a) \wedge i \sim_1 x))$. Let $i \in A_1$ be such that $i \sim_1 x$; let $n \in \omega$ be such that $F_1(i) = \overline{n+1}$. By clause 6 of the definition of a towered structure, $\{j \in A_1 : j E_1 i\} \in FODO(I_x, E_1 \upharpoonright I_x)$. So let $\varphi(z_0, z_1, \dots, z_l)$ be a formula and let $j_1, \dots, j_l \in A_1$ be such that

$$\{j : j E_1 i\} = \{j : (I_x, E_1 \upharpoonright I_x) \models \varphi(j, j_1, \dots, j_l)\}.$$

Since π is an isomorphism between $(I_x, E_1 \upharpoonright I_x)$ and $(I_y, E_2 \upharpoonright I_y)$, let $a \in A_2$ be the unique element such that

$$\{b : b E_2 a\} \Leftrightarrow \{b : (I_y, E_2 \upharpoonright I_y) \models \varphi(b, \pi(j_1), \dots, \pi(j_l))\}$$

again, by Clause 6 of the definition of a towered structure. Moreover, as $i \notin I_x$, it follows that $a \notin I_y$ and thus, $a \not<_2 y$. Since $a \in FODO(I_y, E_2 \upharpoonright I_y)$ by clause 6 of the definition of a towered structure, $a \sim_2 y$. Hence, $P(n+1, i, a)$. A symmetric argument shows that $\forall a \sim_2 y (\exists n, i (P(n, i, a) \wedge i \sim_1 x))$. Thus, $x \in K$, a contradiction. Thus, $L = \emptyset$. \square

The next proposition is in the same spirit as Proposition 4.4.8.

Proposition 5.4.4. *If $(A_1, E_1), (A_2, E_2)$ are towered structures such that for some $y \in A_2$, $(A_1, E_1) \cong (I_y, E_2 \upharpoonright I_y)$, then $J(\text{Th}(A_1, E_1)) <_T \text{Th}(A_2, E_2)$.*

Proof. Let $(A_1, E_1), (A_2, E_2)$ be towered structures as above where $<_i, \sim_i$ refer to (A_i, E_i) and $T_i = \text{Th}(A_i, E_i)$ for $i = 1, 2$. Let $x \in A_2$ be such that $(A_1, E_1) \cong (I_x, E_2 \upharpoonright I_x)$. We observe first that since $<_2$ has no largest element, for any $a \in A_2$, we have $\{b \in A_2 : a <_2 b\} \neq \emptyset$. So let $y_1 \in A_2$ be any $<_2$ -least element of $\{y \in A_2 : x <_2 y\}$; we are assured that such a y_1 exists by clause 11 of the definition of a towered structure. By the same reasoning, let $y_2 \in A_2$ be any $<_2$ -least element of $\{y \in A_2 : y_1 <_2 y\}$. Now define $y \subset A_2$ as follows:

$$z E_2 y \Leftrightarrow \exists k [z = \bar{k} \wedge (I_x, E_2 \upharpoonright I_x) \models \varphi_k].$$

First, since $(A_1, E_1) \cong (I_x, E_2 \upharpoonright I_x)$ and since (A_1, E_1) and (A_2, E_2) are identical on A^0 , it is clear that y is recursively isomorphic to T_1 . Moreover, it follows from clause 10 of the definition of a towered structure that $y \sim_2 y_2$.

Next, note that as $x, y \in A_2 \setminus A^0$, there exist $i, j \in \omega$ such that

$$\text{Def}((A_2, E_2), i, x) \text{ and } \text{Def}((A_2, E_2), j, y),$$

by clause 9 of the definition of a towered structure. Let i, j be as such. Finally, we observe that $(J^2(T_1))(k) = T_2(\ulcorner \psi_k \urcorner)$ where ψ_k is the sentence

$$\exists z_1, z_2 [\varphi_i(z_1) \wedge k \text{ codes a } \Sigma_2^0 \text{ formula of one free variable } \wedge \varphi_j(z_2) \wedge \varphi'_k(z_2)]$$

where $\varphi'_k(v_0)$ is the formula $\varphi_k(v_0)$ modified as follows: every $\exists a$ is replaced with $\exists a$ (“ a is an integer ...”); likewise, every $\forall b$ is replaced with $\forall a$ (“ a is an integer ...”); every instance of “ $n \in y$ ” is replaced with $n E_1 y$; and every instance of “ $n \notin y$ ” is replaced with $\neg n E_1 y$. Thus, $J^2(T_1) \leq_T T_2$. Since $J(T_1) <_T J^2(T_1)$, it follows that $J(T_1) <_T T_2$. \square

We can now improve Proposition 5.4.3 with Proposition 5.4.4; if two towered structures are “close” enough, they are identical.

Proposition 5.4.5. *If $(A_1, E_1), (A_2, E_2)$ are towered structures such that*

$$Th(A_1, E_1) \leq_T J(Th(A_2, E_2)) \text{ and } Th(A_2, E_2) \leq_T J(Th(A_1, E_1)),$$

then $(A_1, E_1) = (A_2, E_2)$.

Proof. Let (A_1, E_1) and (A_2, E_2) be as above where $T_i = Th(A_i, E_i)$ for $i = 1, 2$. By Proposition 5.4.3, it follows either that (A_1, E_1) and (A_2, E_2) are isomorphic or that one is isomorphic to an initial segment of the other. If $(A_1, E_1) \cong (I_x, E_2 \upharpoonright I_x)$ for some $x \in A_2$, then by assumptions and by Proposition 5.4.4 we would have $T_2 \leq_T J(T_1) <_T T_2$, clearly a contradiction. A symmetric argument shows that if (A_2, E_2) is isomorphic to an initial segment of (A_1, E_1) , a similar contradiction is reached. Thus, we must have $(A_1, E_1) \cong (A_2, E_2)$ for some isomorphism π , and consequently, $T_1 = T_2$. We claim that π must be the identity map. Both structures are clearly isomorphic on the odd integers, because they both are (A^0, E^0) on the odd integers. Now suppose $n = 2j$ and $m = 2k$ for some $j, k \in \omega$ and that $\pi(n) = m$. Suppose $j < k$. By Clause 9 of the definition of a towered structure, $Def((A_1, E_1), j, n)$ and $Def((A_2, E_2), k, m)$. But, π an isomorphism implies that m is also definable from j in (A_2, E_2) , contradicting the minimality of k from the definition of Def . A similar contradiction occurs if $j > k$. It follows that $j = k$ and hence, for all k , $\pi(2k) = 2k$. Thus, π is the identity map. \square

Lemma 5.4.6. *If $Y \subset 2^\omega$ is such that $Y \in \Pi_{\omega+2}^0$, then $Y \cap L^{\omega+\omega} \in \Pi_{\omega+2}^0$.*

Proof. Standard absoluteness argument. \square

Corollary 5.4.7. *$T \cap L^{\omega+\omega} \in L^{\omega+\omega}$ and $T \cap L^{\omega+\omega} \in \Pi_{\omega+2}^0$.*

Proof. Follows immediately from Lemmas 5.4.2 and 5.4.6. \square

The next theorem is analogous to Theorem 4.4.14 : T does not omit any cone.

Theorem 5.4.8. *For all $x \in 2^\omega \cap L^{\omega+\omega}$, there is $x' \in T \cap L^{\omega+\omega}$ such that $x \leq_T x'$.*

Proof. Let $x \in 2^\omega \cap L^{\omega+\omega}$. Choose λ to be a limit ordinal in accordance with Theorem 5.3.4. We will show that there is a towered structure (A, E) such that $(A, E) \cong (L^{\omega+\omega}(\lambda), \in)$. First, we define a map g from $L^{\omega+\omega}(\lambda)$ to ω . There are two cases.

Case 1: $L^{\omega+\omega}(\lambda) \cap V(\omega)$. Take $g \upharpoonright V(\omega)$ to be the fixed isomorphism from $(V(\omega), \in)$ to (A^0, E^0) .

Case 2: $L^{\omega+\omega}(\lambda) \setminus V(\omega)$. Suppose $y \in L^{\omega+\omega}(\lambda) \setminus V(\omega)$. Then by Theorem 5.3.4, there is $n \in \omega$ such that $Def((L^{\omega+\omega}(\lambda)), n, y)$. So define $g(y) = 2n$.

Now let $A = Range(g)$ and let E be the relation on induced by g on A . Then it is clear that $(A, E) \stackrel{g}{\cong} (L^{\omega+\omega}(\lambda), \in)$. As $(L^{\omega+\omega}(\lambda), \in)$ easily fulfills clauses 1-10 of the definition of a towered structure, so does (A, E) . Moreover, $<$ is a well-founded relation on $(L^{\omega+\omega}(\lambda), \in)$. Thus, (A, E) fulfills clause 11 of the definition of a towered structure, and hence is a towered structure. Letting $x' = Th(A, E)$, we have $x' \in T \cap L^{\omega+\omega}$ such that $x \leq_T x'$. \square

The next theorem is analogous to Theorem 4.4.13 : T does not contain any cone.

Theorem 5.4.9. *For all $x \in 2^\omega \cap L^{\omega+\omega}$, there is x' such that $x \leq_T x'$ and for all $y \in 2^\omega$, if $y =_T x'$, then $x' \in (2^\omega \setminus T) \cap L^{\omega+\omega}$.*

Proof. Let $x \in 2^\omega \cap L^{\omega+\omega}$. Then by Theorem 5.4.8, there is $z \in T \cap L^{\omega+\omega}$ such that $x \leq_T z$. We claim that $J(z)$, the jump of z , proves the theorem. Since $z \in T$, let (A, E) be a towered structure such that $z = Th(A, E)$. It is clear that $J(z) \in L^{\omega+\omega}$, as $z \in L^{\omega+\omega}$. Moreover, for all $y =_T J(z)$, $y \in L^{\omega+\omega}$. Now suppose that $y \in T$ is such that $y =_T J(z)$. That is, let (A', E') be a towered structure such that $y = Th(A', E')$. Now $y \leq_T J(z)$, so that $y = Th(A', E') \leq_T J(Th(A, E))$. But also, $z \leq_T J(J(z))$, thus, $Th(A, E) \leq_T J(Th(A', E'))$. So by Proposition 5.4.5, $(A, E) = (A', E')$ and hence $z = Th(A, E) = Th(A', E') = y =_T J(z)$, clearly a contradiction. \square

We now define $\mathbf{A} \subset \mathbf{D}$ as follows: $\mathbf{x} \in \mathbf{A} \Leftrightarrow \exists y \in T(x \equiv_T y)$.

Lemma 5.4.10. $\mathbf{A} \cap L^{\omega+\omega} \in L^{\omega+\omega}$ and $\mathbf{A} \cap L^{\omega+\omega} \in \Pi_{\omega+2}^0$.

Proof. Follows from Corollary 5.4.7. \square

Theorem 5.4.11. \mathbf{A} does not contain any cone.

Proof. Follows from Theorem 5.4.9 □

Theorem 5.4.12. ***A** does not omit any cone.*

Proof. Follows directly from Theorem 5.4.8 □

Theorem 5.4.13. $ZC \not\leq Det(\Pi_{\omega+2}^0)$

Proof. Take **A** and the contrapositive of Martin's Theorem. □

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