INFINITE PLANAR GRAPHS

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How many equivalence classes of geodesic rays does a graph contain? How many bounded automorphisms does a planar graph have? Neimayer and Watkins studied these two questions and answered them for a certain class of graphs.

Using the concept of excess of a vertex, the class of graphs that Neimayer and Watkins studied are extended to include graphs with positive excess at each vertex. The results of this paper show that there are an uncountable number of geodesic fibers for graphs in this extended class and that for any graph in this extended class the only bounded automorphism is the identity automorphism.

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CHAPTER 1

Introduction and Preliminary Definitions

1.1 Introduction

How many equivalence classes of geodesic rays does a planar graph contain? How many bounded automorphisms does a planar graph have? The main purpose of this paper is to answer these two questions for the class of planar graphs that are 3-connected, 1-ended, and have positive excess at each vertex. This extends a result by Neimayer and Watkins [NW] where they answered these questions for a certain class of graphs. Chapter 4 contains the answer to the first question and chapter 5 answers the second question. The rest of Chapter 1 introduces the reader to concepts in graph theory that are used throughout the paper. Chapter 2 contains results concerning the concept of excess in planar graphs that are useful in answering the two main questions. Chapter 3 discusses the key concept of a Bilinski Map and extends this concept to constructing Bilinski Lines. Chapter 3 also contains many technical lemmas which are vital in answering the two main questions.

Motivation for answering these two questions comes from Bonnington, Imrich and Watkins [BIW] where they conjecture that if Γ is a simple, locally finite, 1-ended, vertex-transitive graph, then Γ is planar if and only if every geodesic double ray bisects Γ. Bonnington, Imrich, and Seifter [BIS] suggest that this conjecture could be answered by studying the structure of the underlying geodesic rays of the graph. Also, by examining the automorphism group of a graph it gives information on what

sort of automorphisms are possible. This information may be of use in anwering the conjecture and other questions related to infinite planar graphs.

1.2 Preliminary Definitions

Basic concepts of graph theory can be found in [We] [Go]. A *proper embedding* of a graph in the plane is a planar embedding so that the set of vertices has no accumulation point. When a graph Γ is described in this paper as *planar*, it is meant that Γ can be embedded properly in the plane. Γ has *bounded degree* if there is a positive integer d such that every vertex has degree at most d. A planar graph has *bounded codegree* if there is a positive integer ℓ such that each face is bounded by a polygon with at most ℓ sides. A graph is 3-*connected* if there are 3 internally disjoint paths between every pair of vertices in the graph. Graphs considered in this paper are infinite, locally finite, and properly embedded in the plane unless otherwise indicated. The symbols $V(\Gamma)$ and $E(\Gamma)$ will denote respectively the vertex set and edge set of a graph Γ. A *disk* is a finite connected planar graph embedded in the plane so that the union of its closed finite regions is a topological disk.

The *distance* between two vertices x and y in a graph Γ is the length of a shortest path connecting the vertices and will be denoted by $d(x, y)$. A *ray* is a one way infinite path and a *double ray* is a two way infinite path. A path, ray, or double ray P is said to be *geodesic* if the distance between any two vertices along P is the same as the distance between them in the graph. Halin [Ha] defines two rays in a graph to be *end-equivalent* if there exists another ray whose intersection with each of them is infinite. An *end* is an equivalence class of rays with respect to this equivalence

relation. In particular a graph is *1-ended* if the removal of a finite set of vertices yields at most one infinite component.

1.3 Excess of a Vertex

Recall that for a regular k-gon in the Euclidean plane an interior angle has size $(1-\frac{2}{k})\pi$. If there are $deg(v)$ regular polygons incident to a vertex v and these polygons have n_i edges respectively, where $1 \leq i \leq deg(v)$, then the sum of the angles at v is $\sum_{i=1}^{deg(v)}(1-\frac{2}{n_i})\pi$. For convenience the factor of π is ommited and *excess* of a vertex in general is defined as follows. (Loosely speaking the excess measures how far the sum of the angles incident to a vertex deviates from the normal Euclidean sum of 2π.)

The *excess* [BMV] of a vertex v in a disk is given by

$$\mathrm{Ex}(v) = \left[\sum_i (1-\frac{2}{n_i})\right] - 2 + b_v,$$

where n_i is the number of edges bounding the ith face incident with v and b_v is one if v is incident with the unbounded face and zero otherwise. The number n_i is counted with multiplicity. Furthermore, the same face is counted with multiplicity in the sum. Thus, for the examples in this paper

$$\mathrm{Ex}(v) = \left[\sum_i (1-\frac{2}{n_i})\right] - 2,$$

if the vertex v is in the interior of a disk. If the vertex v is on the boundary of a disk D, denote the excess at v as $\mathrm{Ex}^-(v)$ and then by the definition of excess,

$$\mathrm{Ex}^-(v) = \left[\sum_i (1-\frac{2}{n_i})\right] - 1.$$

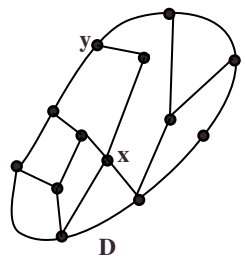

Figure 1.1: A disk D with interior vertex x and boundary vertex y

When there is ambiguity as to which disk that is considered, the boundary excess for the disk D is denoted as $\mathrm{Ex}_D^-(v)$.

Note that if Γ has bounded degree d and codegree ℓ, then for any vertex v in Γ, $\mathrm{Ex}(v) \leq d(1-\frac{2}{\ell})-2$ and $\mathrm{Ex}^-(v) \leq (d-1)(1-\frac{2}{\ell})-1$. Thus for a given planar graph Γ with bounded degree and codegree, both $\mathrm{Ex}(v)$ and $\mathrm{Ex}^-(v)$ are bounded.

Consider the following examples as illustrated in Figure 1.1. Note that D is a disk with x in the interior and y on the boundary. Thus since x is adjacent to faces of codegree 3, 4, 5, and 6,

$$\mathrm{Ex}(x) = (1-\frac{2}{3})+(1-\frac{2}{4})+(1-\frac{2}{5})+(1-\frac{2}{6})-2 = \frac{1}{10}.$$

Since y is on the boundary and adjacent to faces of codegree 5 and 6,

$$\mathrm{Ex}^-(y) = (1-\frac{2}{5})+(1-\frac{2}{6})-1 = \frac{4}{15}.$$

This example illustrates how excess will be calculated throughout the paper.

1.4 Bounded Automorphisms and Whitney's Theorem

An *isomorphism* from a graph Γ to a graph Λ is bijection $\phi : V(\Gamma) \to V(\Lambda)$ such that $(u,v) \in E(\Gamma)$ iff $(\phi(u), \phi(v)) \in E(\Lambda)$. An *automorphism* of a graph is an isomorphism

from Γ to itself. An automorphism is *bounded* if there exists some real number $\rho > 0$ such that the distance between a vertex v and $\phi(v)$ is less than or equal to ρ for all $v \in V(\Gamma)$

By Whitney's Theorem [Wh] and its extension to infinite planar graphs (by [Im] or [Th]), the cyclic orderings around the vertices of a 3-connected planar graph are unique (up to the simultaneous reversal of all orderings) in any planar embedding. An automorphism of the graph is a map from one planar embedding of the graph to another and maps vertices to vertices and edges to edges. Thus, an isomorphism extends to a homeomorphism of the plane. This assures that in any automorphism of a graph Γ that the automorphism preserves the property of excess of a vertex.

CHAPTER 2

Results Concerning Excess

This chapter presents results concerned with excess of vertices and how they relate to planar graphs. These results are key ingredients in many of the proofs that are presented throughout the paper.

2.1 The Euler Characteristic Equation and Excess

Since this paper deals with planar graphs, an interesting idea to investigate is the Euler Characteristic equation as it applies to excess. This result can be found in [BMV] and is included for completeness because of its usefullness in the proofs of many of the results in this paper.

Lemma 2.1 *If D is a disk, then* $\sum_{v \in V(D)} \mathrm{Ex}(v) = -2$

Proof. Let $V_{int}(D)$ denote the set of vertices on the interior of D and $V_{bndy}(D)$ denote the set of vertices on the boundary of D. From the definition of excess,

$$\begin{aligned}
\sum_{v \in D} \mathrm{Ex}(v) &= \sum_{v \in V_{int}(D)} \mathrm{Ex}(v) + \sum_{v \in V_{bndy}(D)} \mathrm{Ex}^{-}(v) \\
&= \sum_{v \in V_{int}(D)} \left(\left[\sum_{i=1}^{deg(v)} \left(1 - \frac{2}{n_i}\right) \right] - 2 \right) + \sum_{v \in V_{bndy}(D)} \left(\left[\sum_{i=1}^{deg(v)-1} \left(1 - \frac{2}{n_i}\right) \right] - 1 \right) \\
&= \sum_{v \in V_{int}(D)} \sum_{i=1}^{deg(v)} \left(1 - \frac{2}{n_i}\right) - 2|V_{int}G| + \sum_{v \in V_{bndy}(D)} \sum_{i=1}^{deg(v)-1} \left(1 - \frac{2}{n_i}\right) - |V_{bndy}G| \\
&= \sum_{\text{all regions } R_j} \left(1 - \frac{2}{n_{R_j}}\right) n_{R_j} - 2|V_{int}(D)| - |V_{bndy}(D)|
\end{aligned}$$

$$= \sum_{\text{all regions } R_j} n_{R_j} - 2|V_{int}(D)| - |V_{bndy}(D)| - 2f$$

where n_{R_j} is the number of edges incident to region R_j and f is the number of regions in D.

Let v be the number of vertices in D, v_x be the number of boundary vertices, v_i be the number of internal vertices, and e be the number of edges. Then $v = v_x + v_i$ and by Euler's formula, $v_x + v_i - e + f = 1$. Since on the boundary of the region $e_x = v_x$, $\sum_{\text{all regions } R_j} n_{R_j} = 2e - v_x$. Thus

$$\sum_{\text{all regions } R_j} n_{R_j} - 2|V_{int}(D)| - |V_{bndy}(D)| - 2f$$

$$\begin{aligned} &= \sum_{\text{all regions } R_j} n_{R_j} - 2v_i - v_x - 2f \\ &= 2e - v_x - 2v_i - v_x - 2f \\ &= 2e - 2v - 2f \\ &= -2. \end{aligned}$$

∎

2.2 Bounds on Positive Excess

The next results are bounds on excess that will be useful throughout the paper.

Theorem 2.2 *If a vertex v in a planar graph has positive excess, then* $\mathrm{Ex}(v) \geq \frac{1}{903}$.

Proof. Let v be a vertex with positive excess in a planar graph. Note that v cannot have degree 2 since this would require $\mathrm{Ex}(v) < 0$. If v has degree 3, start

by supposing one incident face is 3-sided. This forces the other two incident faces to each have codegree greater than or equal to 7, or else v would not have positive excess. Suppose the second face has codegree 7. For the third incident face, the lowest codegree that gives v positive excess is when the third face is codegree 47 (This was done by a computer check). In this case $\mathrm{Ex}(v) = \frac{1}{903}$. If the third face has codegree greater than 47 then $\mathrm{Ex}(v) > \frac{1}{903}$.

Next, consider when the second face size is 8 and repeat he process. Then for this case a computer check reveals that $\mathrm{Ex}(v) > \frac{1}{903}$. Keep increasing the codegree of the second face. When the second face gets to codegree 47, all the cases have been checked for the first face having codegree 3. Increase the first face codegree to 4 and repeat the process. All cases for v having degree 3 will be checked when the first face codegree is increased to codegree 7. In all cases for v having degree 3, a computer check reveals that $\mathrm{Ex}(v) \geq \frac{1}{903}$.

If v has degree 4, adjust the process by starting out with the first two faces having codegree 3, and complete the process as was done for the cases when the degree was 3. Continue the process, if v has degree 5 by letting the first 4 faces have codegree 3, and in the case where v has degree 6, letting the first 5 faces have codegree 3.

In any case for v having degree 6 or less, there is only a finite number of cases for the computer to check and $\mathrm{Ex}(v) \geq \frac{1}{903}$. The only case where there is equality is when v is degree 3 and has face sizes of 3, 7, and 43.

If v has degree d, where $d \geq 7$, then

$$\mathrm{Ex}(v) \geq d(1 - \frac{2}{3}) - 2 > \frac{1}{903}.$$

So in all cases,

$$\mathrm{Ex}(v) \geq \frac{1}{903}.$$

∎

The bound for a vertex on the boundary of a disk may be found in a similar manner and the result is given in Corollary 2.3.

Corollary 2.3 *If a vertex v in a planar graph that is on the boundary of a disk, then* $\mathrm{Ex}^-(v) \geq \frac{1}{21}$

Note that equality for this case only occurs when there are two faces incident to v with face sizes of 3 and 7.

2.3 Geodesic Paths and Excess

In [NW], properties of a *shy* ray, which is a ray that is incident with only one edge of any face in the graph, are exploited. The proof in [NW] that $G_{4,5} \cup G_{5,4}$ has only one bounded automorphism relies on the fact that there is a shy ray S through every vertex and that the geodesic fiber cointaining S is finite (geodesic fibers will be defined in Chapter 5). A logical extension would be to exploit geodesic rays using similar arguments that were used in [NW]. However, to extend the argument in [NW] there would need to be a lower bound on the sum of the $Ex^-(v)$ for any consecutive vertices v on the geodesic path (It is easy to come up with an example to show there is no upper bound). However, this cannot be done. Consider the repeated pattern with geodesic path P starting with v_0 and continuing to $v_2, v_3, ...$ in Figure 2.1.

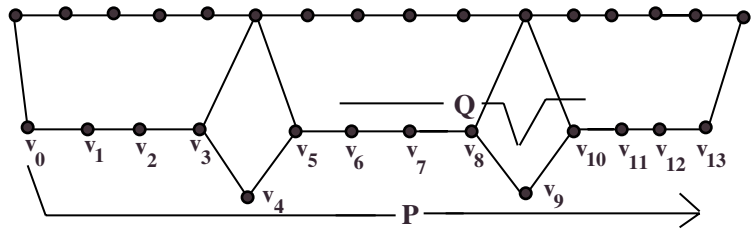

Figure 2.1: A geodesic path P with no lower bound.

Note that if $\mathrm{Ex}^-(v)$ is summed up for vertices v_6 through v_{10}, illustrated as path Q in Figure 2.1, the sum is $\frac{-3}{10}$. This pattern repeats indefinitely for all vertices v_{5k+1} through v_{5k+5} to make the sum of $\mathrm{Ex}^-(v)$ for all v in P have no lower bound. Thus, a geodesic path cannot be characterized in terms of excess without more assumptions.

CHAPTER 3

Technical Lemmas using Bilinski Maps

In this chapter a Bilinski Map is defined and several technical lemmas that are necessary for the main results are proved.

The *regional distance* between two vertices is the least number of adjacent faces in a path between the vertices. A *Bilinski Map* [Bi] [NW] is a way of labeling vertices in a planar graph that measures how far a vertex is from a fixed vertex using the regional distance. For a nicely embedded graph Γ with a specified vertex v_0, let F_1 be the set of faces incident with v_0. For $m \geq 1$ V_m is the set of vertices not in V_{m-1} ($V_0 = \{v_0\}$) that are incident with a face in F_m. For $m \geq 1$ F_{m+1} is the set of faces not in F_m that are incident with a vertex in V_m. An example of a Bilinski Map is given in Figure 3.1 with specified vertex labelled 0.

3.1 Technical Lemmas

Let Γ be a 3-connected, 1-ended, planar graph with positive excess at each vertex. Fix a vertex v_0 of Γ. As shown in [BMV], for each $m \geq 1$ there is a cycle C_m whose vertices are in V_m with the property that v_0 is in the finite component of $\Gamma - C_m$. Furthermore in [BMV], it is shown that $|V(C_m)|$ grows exponentially with m.

For each vertex v of C_m, $\mathrm{Ex}^-(v)$ has been defined to be the excess for the vertex v on the disk bounded by C_m. Let the quantity $\mathrm{Ex}^+(v) = \left[\sum_i (1 - \frac{2}{n_i})\right] - 1$, where in this sum the index i labels faces incident to v which are exterior to C_m. Note that

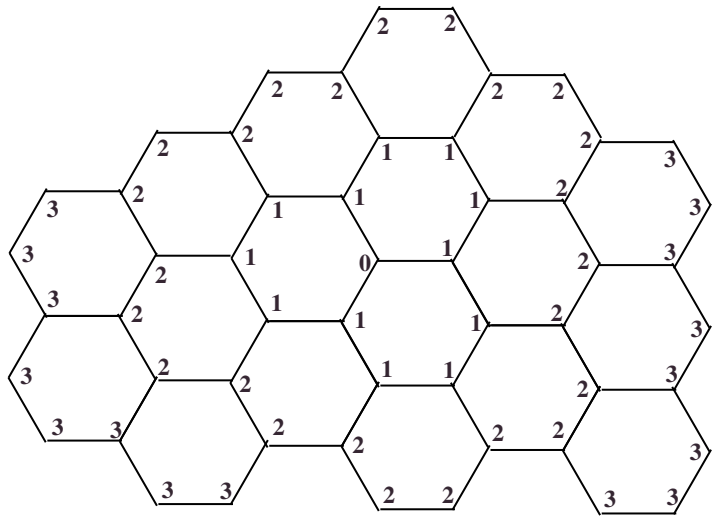

Figure 3.1: An example of a Bilinski Map with specified vertex labelled 0.

$\mathrm{Ex}^-(v) + \mathrm{Ex}^+(v) = \mathrm{Ex}(v)$. Refer to $\mathrm{Ex}^-(v)$ as the *inner excess* and $\mathrm{Ex}^+(v)$ as the *outer excess*. If there is ambiguity about which disk is being considered, the inner excess is denoted as $\mathrm{Ex}^-_D(v)$ and the outer excess is denoted as $\mathrm{Ex}^+_D(v)$ for a disk D.

The following technical lemmas lay the groundwork for the main results.

Lemma 3.1 *For each vertex v in C_m with $m \geq 1$, there is a face incident with v that is also incident with a vertex in C_{m-1}.*

Proof. Since v is in V_m, it is incident with a face containing a vertex w in V_{m-1}. If w is not in C_{m-1}, then there is a path from v_0 to w that does not intersect C_{m-1}. Consequently, there is a path from v_0 to $v \in V(C_m)$ that does not intersect C_{m-1}. But then, $V(C_m)$ is in the component of $\Gamma - C_{m-1}$ containing v_0. This implies that v_0 is in the infinite component of $\Gamma - C_m$ which is a contradiction. ∎

Note that Lemma 3.1 ensures that a geodesic path from a vertex in C_m back to v_0 has length at most $\frac{1}{2}ml$, where l is an upper bound on codegree.

In [BMV] it is shown that for planar graphs where every region is a triangle, if the excess at each vertex is at least 0, then the graph is concentric. That is, all the vertices in V_m are in C_m. A key part of the inductive argument is that $Ex^-(v) \leq 0$ for each vertex in C_m. In this setting, this is certainly not the case. However, it is important to establish an upper bound on the excess sum around consecutive vertices on the cycle C_m. The next few lemmas give an upper bound on the inner excess sum around consecutive vertices on the cycle C_m. Let ℓ be an upper bound on codegree, d be an upper bound on degree, and ϵ be a lower bound on excess. Note that by Theorem 2.2, ϵ is at least $\frac{1}{903}$ for any graph.

Lemma 3.2 *On the cycle C_m, for any consecutive set of vertices R,*

$$\sum_{v \in R} \mathrm{Ex}^-(v) < \frac{2}{3} m\ell.$$

Proof. In the cases where R contains all vertices in C_m or all except one vertex in C_m, the statement follows from Euler's formula involving excess since the excess of each internal vertex is positive. For the remaining cases, let u and w be the end vertices for R. That is, u and w are vertices not in R, but incident in C_m with vertices in R. By Lemma 3.1 there are paths p and q of length at most $\frac{1}{2}\ell m$ starting at u and w respectively and ending at a common vertex x on some C_i, $0 \leq i < m$. Furthermore it may be assumed that x is the only vertex common to the two paths. Let D be the disk bounded by the cycle consisting of the paths p and q together with the part of C_m induced by the vertices of $R \cup \{u, w\}$. Each vertex of the paths p and q have excess at least $-\frac{2}{3}$ and the sum of the disk excess for D is -2. Let $E = \sum_{v \in R} \mathrm{Ex}^-(v)$ and $E' = \sum_{v \in \mathrm{int}(D)} \mathrm{Ex}(v) \geq 0$. Then $E + E' - \frac{2}{3}(\ell m + 1) \leq -2$. Therefore, $E < \frac{2}{3}\ell m$.

∎

In the case that $|R|$ is large, Lemma 3.2 can be improved to insure that the excess sum is negative.

Lemma 3.3 *If R is a set of consecutive vertices on C_m and $|R| \geq \frac{2d\ell^2 m}{3\epsilon}$, then $\sum_{v \in R} \mathrm{Ex}^-(v) < -1$.*

Proof. As in Lemma 3.2 construct paths from end vertices u and w of R that meet at vertex x and each having length at most $\frac{\ell m}{2}$. Also form the disk D as in Lemma 3.2. Let R' denote the vertices interior to D which are on the cycle C_{m-1}.

Each vertex v of R is connected by a face to a vertex in R'. Therefore, $|R'| > \frac{|R|}{d\ell}$. Define E and E' as in Lemma 3.2 and note that $E' > \frac{|R|}{d\ell}\epsilon$. Since

$$E + \frac{|R|}{d\ell}\epsilon - \frac{2}{3}(\ell m + 1) \leq -2,$$

it follows that

$$\begin{aligned} E &< \frac{2}{3}\ell m - \frac{|R|}{d\ell}\epsilon - 1 \\ &< \frac{2}{3}\ell m - \frac{2d\ell^2 m}{3\epsilon}\frac{\epsilon}{d\ell} - 1 \\ &= -1 \end{aligned}$$

∎

Given two vertices v and w in C_m, let $B(v, w)$ denote the vertices encountered in a counterclockwise walk around C_m from v to w, but including neither v nor w. Refer to the set $B(v, w)$ as the set of vertices between v and w. Note that the set of vertices

between v and w is not the same as the set of vertices between w and v. In general, the notation $B(v,w)$ is used as the vertices encountered in a counterclockwise walk around the boundary of any fixed disk D

Let $v \in V(C_m)$ and say that the vertex v *links outward* to w if there is a face incident with both v and w, and $w \in V(C_{m+1})$. A key part of the proof of Theorem 4.1 is to establish how the total excess grows from cycle C_m to cycle C_{m+1}. Not every vertex of C_m links outward. Lemmas 3.4 and 3.5 give bounds on the excess sum between consecutive vertices that link outward and the excess sum between the vertices in C_{m+1} to which they link.

Lemma 3.4 *Suppose that p_1 and p_2 are paths from $x \in V(C_m)$ to $x' \in V(C_{m+1})$ and $y \in V(C_m)$ to $y' \in V(C_{m+1})$ respectively, and there is a face A whose bounding cycle contains p_1 and a face B whose bounding cycle contains p_2. Furthermore, assume there are no vertices incident with either A or B that are in $B(y,x) \cup B(y',x')$. Then*

$$\sum_{v \in B(y',x')} \mathrm{Ex}^-(v) \leq \sum_{v \in B(y,x)} \mathrm{Ex}^-(v) - \epsilon |B(y,x)|.$$

Proof. Let D be the disk whose boundary consists of the cycle C_m restricted to the vertices $\{x,y\} \cup B(y,x)$, the path p_2, the cycle C_{m+1} restricted to $\{x',y'\} \cup B(y',x')$, and the path p_1. Then

$$\begin{aligned} -2 = \sum_{v \in V(p_1)} \mathrm{Ex}_D(v) \quad &+ \sum_{v \in V(p_2)} \mathrm{Ex}_D(v) + \sum_{v \in B(y,x)} \mathrm{Ex}^+(v) \\ &+ \sum_{v \in B(y',x')} \mathrm{Ex}^-(v) + \sum_{v \in \mathrm{int}(D)} \mathrm{Ex}(v). \end{aligned}$$

It is easy to verify that $\sum_{v \in V(p_1)} \mathrm{Ex}_D(v) + \sum_{v \in V(p_2)} \mathrm{Ex}_D(v) \geq -2$. Also, $\sum_{v \in \mathrm{int}(D)} \mathrm{Ex}(v) \geq 0$ since the excess at each vertex of Γ is positive. Consequently, $\sum_{v \in B(y',x')} \mathrm{Ex}^-(v) \leq$

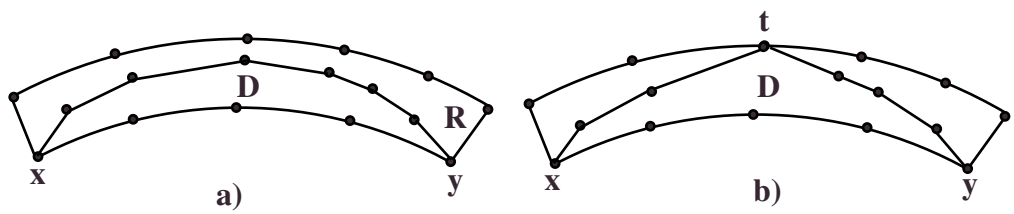

Figure 3.2: Faces having at most a point in common.

$-\sum_{v\in B(y,x)} \mathrm{Ex}^+(v)$. Since $\mathrm{Ex}^+(v)+\mathrm{Ex}^-(v) = \mathrm{Ex}(v) \geq \epsilon$ for every vertex $v \in B(y,x)$, thus

$$\sum_{v\in B(y',x')} \mathrm{Ex}^-(v) \leq \sum_{v\in B(y,x)} \mathrm{Ex}^-(v) - \epsilon|B(y,x)|.$$

∎

Lemma 3.5 *If x and y are vertices in C_m incident with faces containing vertices in C_{m+1} and there is no vertex in $B(y,x)$ incident with a face containing a vertex of C_{m+1}, then*

$$\sum_{v\in B(y,x)} \mathrm{Ex}^+(v) \leq 0.$$

Proof. First note that by a minor modification of the construction of C_m given in [BMV], it is possible to allow only edges in C_m that bound faces incident with both a vertex in V_m and a vertex in V_{m-1}. As a result, there are only two possible cases. Either the face incident with x and incident with a vertex of C_{m+1} is the same face as the face incident with y and incident with a vertex in C_{m+1} or else the two faces intersect in a vertex t on C_{m+1}. See Figures 3.2a and 3.2b respectively.

First consider the case shown in Figure 3.2a where the faces are the same. Let A be the set of vertices on the boundary of the face R in Figure 3.2a. Then $\sum_{v\in A} \mathrm{Ex}(v)+$

$\sum_{v \in B(y,x)} \mathrm{Ex}^+(v) \leq -2$. But $\sum_{v \in A} \mathrm{Ex}(v) \geq -2$, as the sum of the excess over all the vertices bounding the face R is -2. It follows that $\sum_{v \in B(y,x)} \mathrm{Ex}^+(v) \leq 0$.

For the case illustrated in Figure 3.2b, let D be the disk whose boundary contains x, y and t, then $\sum_{v \in D} \mathrm{Ex}_D(v) = -2$. The vertices x, y, and t contribute at least -2 to the sum. Each vertex in the interior of D contributes a positive amount to the sum, and each vertex on the interior of the paths from x to t and t to y on the boundary of D contribute a positive amount to the sum. Therefore, $\sum_{v \in B(y,x)} \mathrm{Ex}^+(v) \leq 0$. ∎

Corollary 3.6 gives a lower bound on how far around the cycle C_m one travels in order to come to a vertex that links outward.

Corollary 3.6 *For any vertex $v \in V(C_m)$, there is a vertex w on C_m such that*

1. *w links outward, and*

2. *the path from v to w on C_m in a clockwise direction has length at most $\lceil \frac{2d\ell^2 m}{3\epsilon} \rceil$.*

Furthermore, in condition 2) clockwise can be replaced with counterclockwise.

Proof. From v travel around C_m in a clockwise direction until you find the first vertex that links outward. Call this vertex x. From v travel counterclockwise to find the first vertex y that links outward. Suppose there are at least $\lceil \frac{2d\ell^2 m}{3\epsilon} \rceil$ vertices in $B(y,x)$. Then $\sum_{v \in B(y,x)} \mathrm{Ex}^-(v) < 0$ by Lemma 3.3. Since for any $v \in V(C_m)$, $\mathrm{Ex}(v) = \mathrm{Ex}^+(v) + \mathrm{Ex}^-(v)$, it follows that $\sum_{v \in B(y,x)} \mathrm{Ex}^+(v) > 0$. This contradicts Lemma 3.5. ∎

Corollary 3.7 gives a convenient summary of Lemma 3.4 and Corollary 3.6.

Corollary 3.7 *There is a $\beta > 0$ such that if R is a set of consecutive vertices around C_m with $|R| > \beta m$, then*

1. $\sum_{v \in R} \mathrm{Ex}^-(v) < -1$, *and*

2. *among the vertices of R at least one vertex links outward.*

Note that β depends on ϵ, d, and ℓ, but does not depend on m.

It is necessary to have some control of how the total excess changes from C_m to C_{m+1}. The excess sum between vertices on C_m gives a measure of the distance between the vertices. Lemma 3.8 gives the desired bound.

Lemma 3.8 *Let x and y be vertices in C_m which link outward to x' and y' respectively. If $\sum_{v \in B(y,x)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\tau m$, where $m > \frac{2\delta}{\epsilon}$, $\delta = 1 + \epsilon$, and τ is some positive number, then $\sum_{v \in B(y',x')} \mathrm{Ex}^-(v) < -\frac{2}{3}\tau(m+1)\delta$.*

Proof. By Lemma 3.4, $\sum_{v \in B(y',x')} \mathrm{Ex}^-(v) \leq \sum_{v \in B(y,x)} \mathrm{Ex}^-(v) - \epsilon|B(y,x)|$. Since $\sum_{v \in B(y,x)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\tau m$, $|B(y,x)| \geq \tau m$. Consequently,

$$\begin{aligned}
\sum_{v \in B(y',x')} \mathrm{Ex}^-(v) &\leq -\frac{2}{3}\tau m - \epsilon\tau m \\
&= -\tau m\left(\frac{2}{3} + \epsilon\right) \\
&= -\frac{2}{3}\tau\,(m+1)\,\delta + \frac{2}{3}\tau\,(m+1)\,\delta - \tau m\left(\frac{2}{3} + \epsilon\right) \\
&= -\frac{2}{3}\tau\,(m+1)\,\delta + \tau m\left(\frac{2}{3}\delta - \frac{2}{3} - \epsilon\right) + \frac{2}{3}\tau\delta \\
&= -\frac{2}{3}\tau\,(m+1)\,\delta + \tau m\left(-\frac{1}{3}\epsilon\right) + \frac{2}{3}\tau\delta \\
&= -\frac{2}{3}\tau\,(m+1)\,\delta + \tau\left(\frac{-\epsilon m}{3} + \frac{2}{3}\delta\right) \\
&< -\frac{2}{3}\tau\,(m+1)\,\delta + \tau\left(-\frac{\epsilon m}{3} + \frac{2}{3}\frac{\epsilon m}{2}\right) \\
&= -\frac{2}{3}\tau\,(m+1)\,\delta
\end{aligned}$$

∎

3.2 Main Technical Lemmas

The proof of Theorem 4.1 relies on an iterative process for selecting special clusters of vertices on cycles of the Bilinski Map. Lemma 3.9 is the first of the main technical lemmas that are needed. Condition 5 will ensure that the inner excess sum of the vertices in each special cluster will depend mainly on the index of the Bilinski cycle it lies on. Conditions 3 and 4 allow the required number of disjoint special clusters to be identified on the next appropriate Bilinski cycle.

Recall that ϵ is the minimum excess of the vertices of the graph, ℓ is the maximum face size and $\delta = 1 + \epsilon$. For Lemma 3.9 it is assumed that $\epsilon < 1$. Since it is only assumed that the excess at each vertex is at least ϵ, ϵ can always be replaced with a smaller positive value.

Lemma 3.9 *Let $\alpha > \beta$ and let $\alpha' = \max\left(\frac{36}{\epsilon}\alpha, \ell\right)$. Suppose that x and y are vertices in C_m with $m > \frac{2\delta}{\epsilon}$, $\sum_{v \in B(y,x)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\alpha' m$, x links outward to x', and y links outward to y'. Then there are vertices $u, w \in B(y', x')$ such that*

1. *both u and w link outward, and*

2. *on C_{m+1} starting at x' and walking clockwise the order in which vertices are traversed is x', u, w, y', and*

3. $-3\alpha\,(m+1) < \sum_{v \in B(u,x')} \mathrm{Ex}^-(v) < -\alpha\,(m+1)$, *and*

4. $-3\alpha\,(m+1) < \sum_{v \in B(y',w)} \mathrm{Ex}^-(v) < -\alpha\,(m+1)$, *and*

5. $\sum_{v \in B(w,u)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\alpha'(m+1)(1+\frac{3}{4}\epsilon)$.

Proof. Let R be the set of vertices between x' and y'. By Lemma 3.4 the inner excess sum for these vertices is at most $-\frac{2}{3}\alpha'(m+1)(1+\epsilon)$. Each vertex has excess at least $-\frac{2}{3}$ so there are at least $\alpha'(m+1)(1+\epsilon)$ vertices in R. Since $\alpha' \geq \frac{36\alpha}{\epsilon}$ it follows that $\alpha'(m+1)(1+\epsilon) > 36\alpha(m+1)$, that is, there are at least $36\alpha(m+1)$ vertices in R. By Corollary 3.8, starting at x' and moving around C_{m+1} in a clockwise direction, there is a first vertex u which links outward and satisfies condition 3. Similarly, by starting at y' and moving around C_{m+1} counterclockwise, there is a first vertex w which links outwards and satisfies condition 4.

Suppose that the order of u and w indicated in condition 2 is reversed. Let

$$\begin{aligned}
r' &= \sum_{v \in B(u,w)} \mathrm{Ex}^-(v) \\
r_1 &= \sum_{v \in B(u,x')} \mathrm{Ex}^-(v) \\
r_2 &= \sum_{v \in B(y',w)} \mathrm{Ex}^-(v) \\
r_3 &= \sum_{v \in B(y',x')} \mathrm{Ex}^-(v).
\end{aligned}$$

Then the sum of the excess for the vertices in $B(u,w)$ is given by

$$\begin{aligned}
r' &= r_1 + r_2 - r_3 \\
&> -6\alpha(m+1) + \frac{2}{3}\alpha'(m+1)(1+\epsilon) \\
&\geq -6\alpha(m+1) + \frac{2}{3}\frac{36\alpha}{\epsilon}(m+1)\epsilon + \frac{2}{3}\alpha'(m+1) \\
&= -6\alpha(m+1) + 24\alpha(m+1) + \frac{2}{3}\alpha'(m+1) \\
&\geq \frac{2}{3}\alpha'(m+1) \\
&\geq \frac{2}{3}\ell(m+1).
\end{aligned}$$

Note that this contradicts Lemma 3.2. Therefore, the order indicated in condition 2 is correct.

It remains to show condition 5. Let r_1, r_2, and r_3 be defined as above, but let $r' = \sum_{v \in B(w,u)} \mathrm{Ex}^-(v)$. A calculation similar to the previous one gives:

$$\begin{aligned}
r' &= r_3 - r_1 - r_2 \\
&< -\frac{2}{3}\alpha'(m+1)(1+\epsilon) + 6\alpha(m+1) \\
&= -\frac{2}{3}\alpha'(m+1) - \frac{2}{3}\alpha'(m+1)\epsilon + 6\alpha(m+1) \\
&\leq -\frac{2}{3}\alpha'(m+1) - \frac{2}{3}\frac{36\alpha}{\epsilon}(m+1)\epsilon + 6\alpha(m+1) \\
&= -\frac{2}{3}\alpha'(m+1) - 18\alpha(m+1) \\
&= -\frac{2}{3}\alpha'(m+1)(1+\frac{3}{4}\epsilon)
\end{aligned}$$

∎

Lemma 3.10 is the second main technical lemma. It shows that any geodesic path which starts at the center z of the Bilinski Map and passes through a special cluster of vertices on C_{m+ik} also passes through a previous such cluster on each of the cycles $C_m, C_{m+k}, ...C_{m+(i-1)k}$. Recall that Lemma 3.1 implies that a geodesic path from z to C_m has at length at most $\frac{\ell m}{2}$ and that if a graph has bounded degree and codegree then excess and inner and outer excess has an upper bound. Call the common bound k.

Lemma 3.10 *Let $\alpha > \beta(\ell m)^4 + k\left(\frac{\ell m+\ell}{4}\right)$ and x, y, u, w be vertices which satisfy the conditions of Lemma 3.9. Any geodesic path from the center of the Bilinski Map to a*

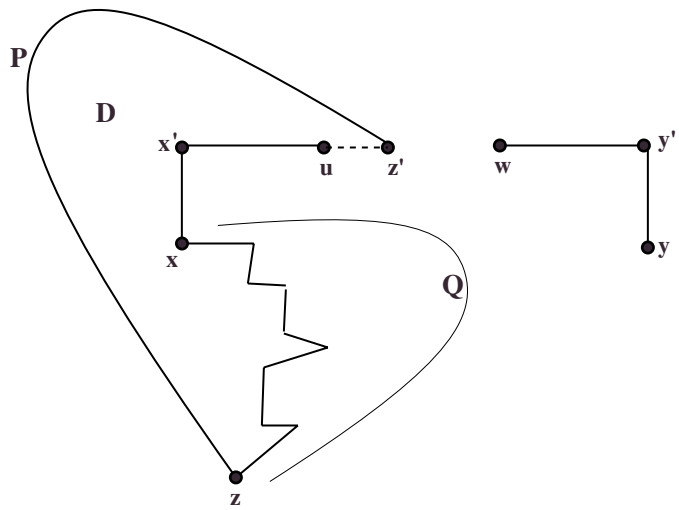

Figure 3.3: The case where P does not intersect $B(u, x')$

vertex in $B(w, u)$ contains a vertex from $B(y, x)$.

Proof. Let z be the center of the Bilinski Map, construct a geodesic path P from a vertex z' in $B(w, u)$ to z. Note that $|P| \leq \frac{\ell(m+1)}{2}$. Suppose that P does not intersect a vertex in $B(y, x)$. There are two main cases to consider.

Case 1: P does not intersect either $B(u, x')$ or $B(y', w)$. This case is illustrated (without loss of generality) in Figure 3.3. Create a disc D by constructing a path Q from x to z, now follow along P from z to z' (if P and Q intersect at a vertex before z, use that vertex in place of z), then along C_{m+1} to x', and finally back to x. By Lemma 3.9 it follows that

$$\sum_{v \in B(u,x')} \mathrm{Ex}^-(v) < -\alpha(m+1)$$

on the boundary of disk C_{m+1}, hence

$$\sum_{v \in B(u,x')} \mathrm{Ex}^-(v) > \alpha(m+1)$$

on the boundary of D.

Case 1-1: Note that if $|B(z', u)| \geq \beta$, by Lemma 3.7 it follows that

$$\sum_{v \in B(z', x')} \mathrm{Ex}^-(v) > \alpha(m+1),$$

thus summing up the excess of the disk D

$$\begin{aligned} -2 &= \sum_{v \in B(z', x')} \mathrm{Ex}^-(v) + \sum_{v \in B(x', z')} \mathrm{Ex}^-(v) + \mathrm{Ex}^-(x') + \mathrm{Ex}^-(z') + \sum_{v \in D_{int}} \mathrm{Ex}(v) \\ &> \alpha(m+1) + \frac{-2}{3}\ell(m+1) \\ &> 0. \end{aligned}$$

Case 1-2: If $|B(z', u)| < \beta$, then summing up the excess of the disk D

$$\begin{aligned} -2 &= \sum_{v \in B(u, x')} \mathrm{Ex}^-(v) + \sum_{v \in B(z', u)} \mathrm{Ex}^-(v) + \sum_{v \in B(z', x')} \mathrm{Ex}^-(v) \\ &\quad + \mathrm{Ex}^-(x') + \mathrm{Ex}^-(z') + \mathrm{Ex}^-(u) + \sum_{v \in D_{int}} \mathrm{Ex}(v) \\ &> \alpha(m+1) + \frac{-2}{3}\beta + \frac{-2}{3}\ell(m+1) \\ &> 0. \end{aligned}$$

Case 2: Suppose that P does intersect either $B(u, x')$ or $B(y', w)$ and consider when it intersects $B(u, x')$ (when P intersects $B(y', w)$ the proof is similar). Create disks by constructing a path Q from x to z, now follow along P from z to z' (if P and Q intersect at a vertex before z, use that vertex in place of z), then along C_{m+1} to x', and finally bach to x. Since $|P| \leq \frac{\ell(m+1)}{2}$, and choosing C_{m+1} as a bounding cycle, there may be $\left(\frac{\ell(m+1)}{2}\right)/2 = \frac{\ell m+\ell}{4}$ points of intersection of P with C_{m+1}, and therefore at most $\frac{\ell m+\ell}{4}$ disks created as illustrated in Figure 3.4. Note that $\alpha > \beta(\ell m)^4 + k\left(\frac{\ell m+\ell}{4}\right)$, so at least one of the regions, call it D,

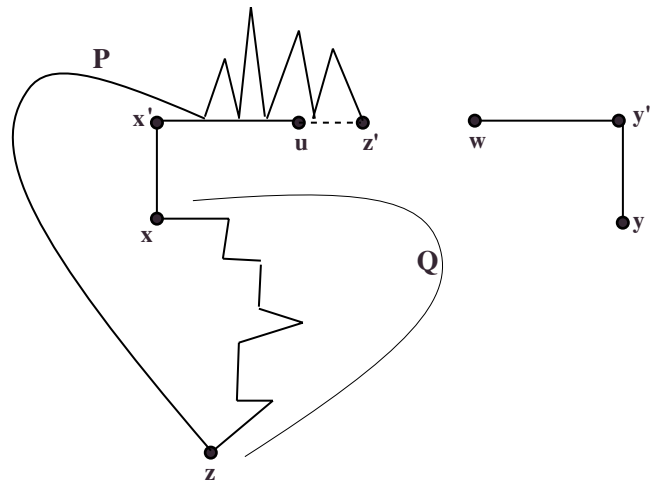

Figure 3.4: The case where P does intersect $B(u, x')$

has the sum of the $\mathrm{Ex}^-(v)$ greater than $(\ell m)^2\beta$ for all vertices v that intersect C_{m+1}. The rest of the region is bounded by P, thus it is less than $\frac{\ell(m+1)}{2}$ edges bounding it.

Case 2-1: If the region D along C_{m+1} is a subset of $B(u, x')$ then along D, as illustrated in Figure 3.5,

$$\begin{aligned} -2 &= \sum_{v \in B(b,a)} \mathrm{Ex}^-(v) + \sum_{v \in B(a,b)} \mathrm{Ex}^-(v) + \mathrm{Ex}^-(a) + \mathrm{Ex}^-(b) + \sum_{v \in D_{int}} \mathrm{Ex}(v) \\ &> (\ell m)^2\beta + \frac{-2}{3}\left(\frac{\ell(m+1)}{2}\right) + 0 + -2\left(\frac{-2}{3}\right) \\ &> 0. \end{aligned}$$

Case 2-2: If this region contains x' then as illusrtated in Figure 3.5,

$$\begin{aligned} -2 &= \sum_{v \in B(b,x')} \mathrm{Ex}^-(v) + \sum_{v \in B(x',b)} \mathrm{Ex}^-(v) + \mathrm{Ex}^-(b) + \mathrm{Ex}^-(x') + \sum_{v \in D_{int}} \mathrm{Ex}(v) \\ &> (\ell m)^2\beta + \frac{-2}{3}\ell(m+1) \\ &> 0 \end{aligned}$$

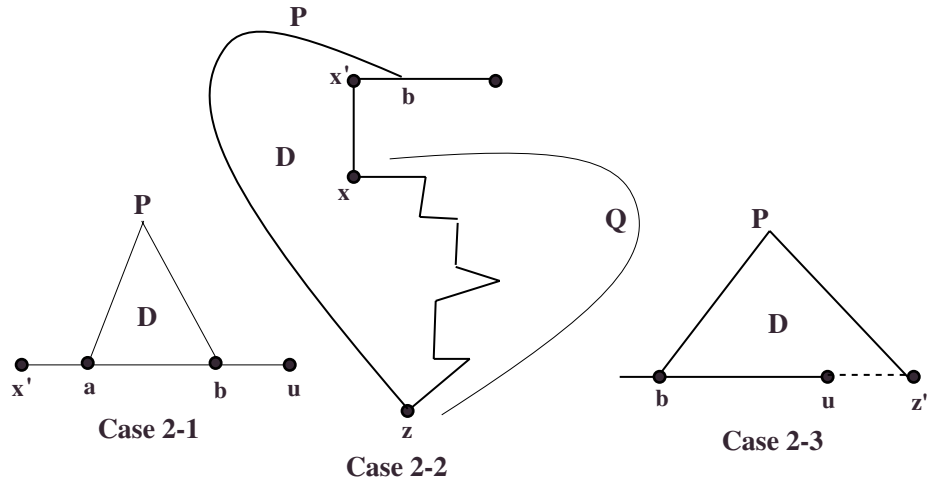

Figure 3.5: Three subcases of case 2

Case 2-3: The third subcase, as illustrated in Figure 3.5 occurs if the region contains u. If $|B(z', u)| \geq \beta$ by Lemma 3.7 $\sum_{v \in B(z',b)} \mathrm{Ex}^-(v) > (\ell m)^2$, thus

$$\begin{aligned} -2 &= \sum_{v \in B(z',b)} \mathrm{Ex}^-(v) + \sum_{v \in B(b,z')} \mathrm{Ex}^-(v) + \mathrm{Ex}^-(z') + \mathrm{Ex}^-(b) + \sum_{v \in D_{int}} \mathrm{Ex}(v) \\ &> (\ell m)^2 \beta + \frac{-2}{3}\left(\frac{\ell(m+1)}{2}\right) - 2\left(\frac{-2}{3}\right) \\ &> 0 \end{aligned}$$

If $|B(z', u)| < \beta$,

$$\begin{aligned} -2 &= \sum_{v \in B(u,b)} \mathrm{Ex}^-(v) + \sum_{v \in B(z',u)} \mathrm{Ex}^-(v) + \sum_{v \in B(b,z')} \mathrm{Ex}^-(v) \\ &\quad + \mathrm{Ex}^-(z') + \mathrm{Ex}^-(b) + \mathrm{Ex}^-(u) + \sum_{v \in D_{int}} \mathrm{Ex}(v) \\ &> (\ell m)^2 \beta + \frac{-2}{3}\left(\frac{\ell(m+1)}{2}\right) + \frac{-2}{3}\beta + 3\left(\frac{-2}{3}\right) \\ &> 0 \end{aligned}$$

For all cases there is a contradiction, so P passes through $B(y, x)$. ∎

The third of the main technical lemmas shows that if two vertices are far apart on

C_m then the distance they are apart in the graph is also large. Recall again that if a graph has bounded degree and codegree then excess, inner excess, and outer excess have an upper bound. Call the common bound k.

Lemma 3.11 *Let ρ be a positive integer. If x and y are on C_m for $m > 2\rho$ with $\sum_{v \in B(y,x)} \mathrm{Ex}^-(v) < -2\rho^2(\ell d)^\rho + k\rho$ then $d(x,y) > \rho$.*

Proof.

Suppose that $d(x,y) \leq \rho$. Since C_m grows exponetially [BMV], for some positive integer m there are vertices x and y on C_m so that $\sum_{v \in B(y,x)} \mathrm{Ex}^-(v) < -2\rho^2(\ell d)^\rho + k\rho$. Construct a geodesic path P between x and y and thus, $|P| \leq \rho$.

Suppose that P intersects $B(y,x)$ on C_m. Then there are at most ρ regions created by P and $B(y,x)$. One of the regions has an x' and a y' so that $\sum_{v \in B(y',x')} \mathrm{Ex}^-(v) < \frac{-2\rho^2(\ell d)^\rho}{\rho} = -2\rho(\ell d)^\rho$. So it is enough to consider the case that a region is bounded by a geodesic ray P where $|P| \leq \rho$ and the vertices $B(y,x)$ where $\sum_{v \in B(y,x)} \mathrm{Ex}^+(v) > 2\rho(\ell d)^\rho$. The region, call it D, stays completely on the interior or exterior of the disk created by C_m.

Suppose first P stays only on the exterior of C_m. Since there is positive excess at each vertex, $\sum_{v \in B(y,x)} \mathrm{Ex}^+(v) > 2\rho(\ell d)^\rho$. For the region D,

$$\begin{aligned} -2 &= \sum_{v \in B(y,x)} \mathrm{Ex}^-(v) + \sum_{v \in B(x,y)} \mathrm{Ex}^-(v) + \mathrm{Ex}^-(x) + \mathrm{Ex}^-(y) + \sum_{v \in D_{int}} \mathrm{Ex}(v) \\ &> 2\rho(\ell d)^\rho + \frac{-2}{3}\rho + 2\left(\frac{-2}{3}\right) \\ &> 0 \end{aligned}$$

which is a contradiction.

Now suppose that P traverses only on the interior of C_m. Since $\sum_{v \in B(y,x)} \mathrm{Ex}^-(v) <$

$-2\rho(\ell d)^\rho$, it follows that $|B(y,x)| > 2\rho(\ell d)^\rho$ on C_m. P is geodesic, so $|P| \leq \rho$. Each vertex on C_m has maximum degree d and is incident with a face with maximum size ℓ on C_{m-1}. Thus, if P does not intersect C_{m-1} there are at least $\frac{2\rho(\ell d)^\rho}{\ell d}$ vertices between x and y along P. However, $|P| \geq \frac{2\rho(\ell d)^\rho}{(\ell d)} > \rho$ which is a contradiction. So P intersects C_{m-1}.

Similarly, each vertex on C_{m-1} has maximum degree d and is incident with a face with maximum size ℓ on C_{m-2}. Thus, if P does not intersect C_{m-2} there are at least $\frac{2\rho(\ell d)^\rho}{(\ell d)^2}$ vertices between x and y along P. However, $|P| \geq \frac{2\rho(\ell d)^\rho}{(\ell d)^2} > \rho$ which is a contradiction. So P intersects C_{m-2}.

Each vertex on C_{m-j} has maximum degree d and is incident with a face with maximum size ℓ on $C_{m-(j+1)}$. Thus, if P does not intersect $C_{m-(j+1)}$ there are at least $\frac{2\rho(\ell d)^\rho}{(\ell d)^j}$ vertices between x and y along P. So when $j = \rho$ it follows that $|P| \geq \frac{2\rho(\ell d)^\rho}{(\ell d)^\rho} > \rho$ which means that P intersects $C_{m-\rho-1}$. However, any path starting and ending in C_m and intersecting $C_{m-\rho-1}$ has length greater than 2ρ. Thus there is a contradiction, so $|P| > \rho$ and $d(x,y) > \rho$. ∎

3.3 Bilinski Lines

In this section, the concept of a Bilinski Map is extended to consider the face distance from a double ray rather than a vertex. Suppose Γ is a planar graph and there is a geodesic double ray R in Γ that separates Γ into two infinite regions (Since the ray is geodesic, it may never intersect itself). Consider one of the regions and call it Γ_1. Let F_1 be the set of faces in Γ_1 incident with R. For $m \geq 1$ V_m is the set of vertices not in V_{m-1} ($V_0 = R$) that are incident with a face in F_m. For $m \geq 1$

F_{m+1} is the set of vertices not in F_m that are incident with a vertex in V_m. Let E_m be the set of edges that have a vertex labelled m and m is the maximum label of its vertices. The following theorem is an extension of the existence of a bounding cycle of the Bilinski Map proved in [BMV]. The theorem uses the idea in [BMV] of a bounding cycle. The corresponding result would be sets of bounding lines L_m for m a positive integer. Each L_m would consist of vertices from V_m and edges on E_m such that the connected component of the complement of L_m not containing R has an infinite number of vertices of Γ_1, but none from $V_0 \cup V_1 \cup ... \cup V_m$.

Theorem 3.12 *Let Γ_1 and R be as above. Then there is a double ray L_m consisting of vertices from V_m and edges on E_m such that connected component of the complement of L_m not containing R has an infinite number of vertices of Γ_1, but none from $V_0 \cup V_1 \cup ... \cup V_m$.*

Proof. Let Γ_1 and R be as above. Construct L_m in the following manner. Let x_0 be a vertex on R. Follow a path along R starting x_0 and label the vertices x_1, x_2, . . . In the opposite direction along the ray R, label the vertices x_{-1}, x_{-2}, . . . Let $R(x_j, x_h)$ denote the vertices x_j to x_h along R. Next, construct the Bilinski Map in Γ for each vertex x_i and a bounding cycle at stage m in the Bilinski Map guaranteed by [BMV]. Label each cycle as $C_{m,i}$ for each vertex x_i and consider its restriction on Γ_1.

The construction will proceed by induction. Consider first the union of the cycles $C_{m,-\ell m}$ through $C_{m,\ell m}$. Note that this union is a finite union of intersecting cycles. Thus, there is a bounding cycle. At the first step of the induction let P_0 be the

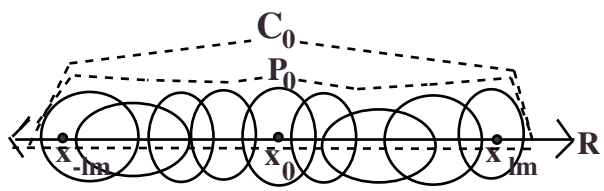

Figure 3.6: The constuction of P_0.

bounding path that comes from restricting the bounding cycle to Γ_1 and C_0 be the cycle that results from P_0 and R. This process is illustrated in Figure 3.6.

Next consider $R(x_{\ell m}, x_{3\ell m})$ and $R(x_{-\ell m}, x_{-3\ell m})$ centered around vertices $x_{2\ell m}$ and $x_{-2\ell m}$ respectively. Consider the union of C_0 and $C_{m,i}$ for every vertex $x_i \in R(x_{\ell m}, x_{3\ell m}) \cup R(x_{-\ell m}, x_{-3\ell m})$ and call this union C_1. Note that this is also a finite union of intersecting cycles, thus there is a bounding cycle. Restrict the cycle to Γ_1 and call its bounding path P_1. Let $\bigcup_{i=i_1}^{i_2} C_{m,i}$ refer to the union of the $C_{m,i}$ for each vertex $x_i \in R(x_{i_1}, x_{i_2})$.

Note that if $\bigcup_{i=\ell m}^{3\ell m} C_{m,i}$ intersects $\bigcup_{i=-3\ell m}^{-\ell m} C_{m,i}$, then for some $x_i \in R(x_{\ell m}, x_{3\ell m})$ and some $x_j \in R(x_{-3\ell m}, x_{-\ell m})$ that the intersection between $C_{m,i}$ and $C_{m,j}$ is not empty. Consider a geodesic path from a point on the intersection to both x_i and x_j. This creates a path from x_i to x_j that is at most length ℓm by Lemma 3.1. This is a contradiction since R is geodesic and the path length along R between x_i and x_j is greater than ℓm. Thus,

$$\bigcup_{i=\ell m}^{3\ell m} C_{m,i} \bigcap \bigcup_{i=-3\ell m}^{-\ell m} C_{m,i} = \emptyset.$$

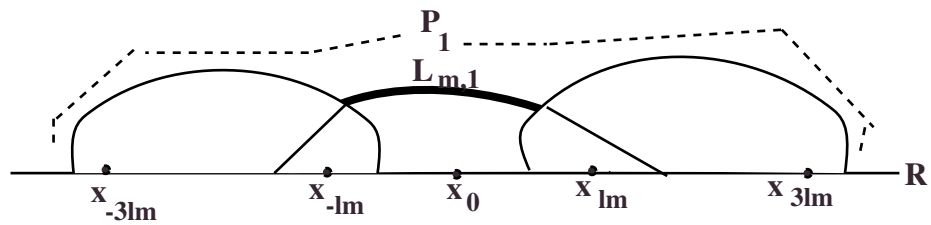

Figure 3.7: The constuction of $L_{m,1}$ and P_1.

Since the intersection is empty, there is subpath of P_0 that is not in the interior of $\bigcup_{i=\ell m}^{3\ell m} C_{m,i}$ or in the interior of $\bigcup_{i=-3\ell m}^{-\ell m} C_{m,i}$. Call this subpath $L_{m,1}$ and note that

$$L_{m,1} = P_0 \bigcap \overline{\left(\bigcup_{i=\ell m}^{3\ell m} C_{m,i} \bigcup \bigcup_{i=-3\ell m}^{-\ell m} C_{m,i} \right)}$$

The construction of $L_{m,1}$ is illustrated in the following Figure 3.7.

So suppose at step k you have a P_k and a $L_{m,k}$. Create a P_{k+1} and $L_{m,k+1}$ in a similar manner. Let $R(x_{(2k+1)\ell m}, x_{(2k+3)\ell m})$ and $R(x_{-(2k+1)\ell m}, x_{-(2k+3)\ell m})$ each centered around vertices $x_{2(k+1)\ell m}$ and $x_{-2(k+1)\ell m}$ respectively. Consider the union of P_k the $C_{m,i}$ for each vertex $x_i \in R(x_{(2k+1)\ell m}, x_{(2k+3)\ell m}) \cup R(x_{-(2k+3)\ell m}, x_{-(2k+1)\ell m})$. Note that this is also a finite union of intersecting cycles, thus there is a bounding cycle. Restrict the cycle to Γ_1 and call its bounding path P_{k+1}. Similarly as in the first step,

$$\bigcup_{i=2k+1)\ell m}^{(2k+3)\ell m} C_{m,i} \bigcap \bigcup_{i=-(2k+3)\ell m}^{-(2k+1)\ell m} C_{m,i} = \emptyset.$$

Since the intersection is empty, there is subpath of P_k that is not in the interior of $\bigcup_{i=(2k+1)\ell m}^{(2k+3)\ell m} C_{m,i}$ or in the interior of $\bigcup_{i=-(2k+3)\ell m}^{-(2k+1)\ell m} C_{m,i}$. Call this subpath $L_{m,k+1}$ and note that

$$L_{m,k+1} = P_k \bigcap \overline{\left(\bigcup_{i=(2k+1)\ell m}^{(2k+3)\ell m} C_{m,i} \bigcup \bigcup_{i=-(2k+3)\ell m}^{-(2k+1)\ell m} C_{m,i} \right)}$$

Note that $L_{m,k}$ is a subpath of $L_{m,k+1}$ since $L_{m,k}$ does not intersect the union of the $C_{m,i}$ for each vertex $x_i \in R(x_{(2k+1)\ell m}, x_{(2k+3)\ell m})$ or the union of the $C_{m,i}$ for each vertex $x_i \in R(x_{-(2k+1)\ell m}, x_{-(2k+3)\ell m})$. This is because $L_{m,k}$ is a subpath of P_{k-2} and which consists of vertices $x_i \in B(x_{(2k-1)\ell m}, x_{-(2k-1)\ell m})$. Without loss of generaltiy, if

$$\bigcup_{i=(2k+1)\ell m}^{(2k+3)\ell m} C_{m,i} \bigcap \bigcup_{i=(2k-1)\ell m}^{-(2k-1)\ell m} C_{m,i} \neq \emptyset,$$

then for some $x_i \in R(x_{(2k+1)\ell m}, x_{(2k+3)\ell m})$ and some $x_j \in R(x_{(2k-1)\ell m}, x_{-(2k-1)\ell m})$ the intersection between $C_{m,i}$ and $C_{m,j}$ is not empty. Consider a geodesic path from a point on the intersection to both x_i and x_j. This creates a path from x_i to x_j that is at most length ℓm by Lemma 3.1. This is a contradiction since R is geodesic and the path length along R between x_i and x_j is greater than ℓm. Thus, $L_{m,k}$ is a subpath of $L_{m,k+1}$.

Continuing by induction, the double ray L_m that results from this process consists of vertices from V_m and edges on E_m such that connected component of the complement of L_m not containing R has an infinite number of vertices of Γ_1, but none from $V_0 \cup V_1 \cup ... \cup V_m$ due to the properties of the Bilinski Maps from [BMV]. ∎

CHAPTER 4

Geodetic Fibers in Graphs with Positive Excess

For a connected graph Γ the concept of distance can be generalized to define the distance between subgraphs X, Y in Γ as

$$d(X,Y) = min\{d(x,y) : x \in X, y \in Y\}.$$

For any non-negative integer N, the *n-neighborhood* of X is the set

$$N_n(X) = \{v \in V(X) : d(v,X) \leq n\}.$$

The *Hausdorff distance* between subgraphs X and Y is defined to be

$$d_{Hsdf}(X,Y) = min\{n : V(X) \subseteq N_n(Y) \text{ and } V(Y) \subseteq N_n(X)\}.$$

Rays P and Q are said to be equivalent, denoted $P \sim Q$, if $d_{Hsdf}(P,Q) < \infty$. For Γ locally finite, $\sim$ is an equivalence relation on the set of rays of Γ, and the equivalence classes are called the *fibers* of Γ [JN]. A *geodesic fiber* is a fiber that contains at least one geodesic ray.

Neimayer and Watkins [NW] ask the question 'How many geodesic fibers does a graph contain?'. They prove that there are uncountably many geodesic fibers for the class $G_{4,6} \bigcup G_{5,4}$ of 1-ended, planar, 3-connected graphs all of whose degrees and codegrees are finite along with the assuptions that degree is at least 4 and codegree is at least 6, or degree is at least 5 and codegree is at least 4. $G_{4,6} \bigcup G_{5,4}$ is a subset of the class of graphs mentioned in the following theorem and thus Theorem 4.1 is an extension of the result by Neimayer and Watkins.

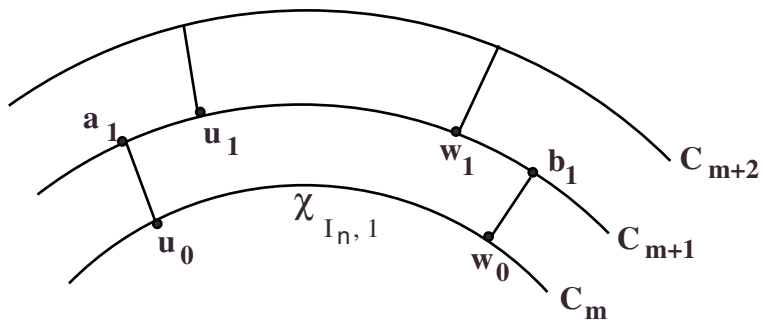

Figure 4.1: The start of the procedure.

Theorem 4.1 *Let Γ be an infinite, 1-ended, locally finite, planar graph with bounded degree and codegree. If every vertex of Γ has positive excess, then Γ has an uncountable number of geodesic fibers.*

The first part of the proof uses Lemma 3.9 to select special vertex clusters and the second part relies on Lemma 3.10 to show that Γ has an uncountable number of geodesic fibers. Recall that $\alpha > \beta(\ell m)^4 + jh$ where $j > \frac{\ell m+\ell}{4}$ and h is an upper bound on the excess of a vertex and that $\alpha' = \max\left(\frac{36}{\epsilon}\alpha, \ell\right)$.

Proof. Choose $z \in V(\Gamma)$ and create the Bilinski Map centered at z. Let α' be as stated above. Since the growth of the cycles of the Bilinski Map is exponential [BMV], by Corollary 3.7 there exists a cycle C_m containing vertices y_1, y_2, and y_3 with the property that $B(y_2, y_1) \cap B(y_3, y_2)$ is empty, $\sum_{v \in B(y_2,y_1)} \mathrm{Ex}^-(v) \leq \frac{-2}{3}\alpha' m$ and $\sum_{v \in B(y_3,y_2)} \mathrm{Ex}^-(v) \leq \frac{-2}{3}\alpha' m$. Let $\chi_0 = B(y_2, y_1)$ and $\chi_1 = B(y_3, y_2)$. Note that $\chi_0 \bigcap \chi_1 = \emptyset$. The sets of vertices χ_0 and χ_1 form two special clusters on C_m which are the beginning of the iterative process. χ_0 will 'spawn' all special clusters $\chi_{0,I}$ and χ_1 will 'spawn' all special clusters $\chi_{1,I}$, where I is any sequence of 0's and 1's.

The general iterative step is now described. Let I_n be the first n tems of a bi-

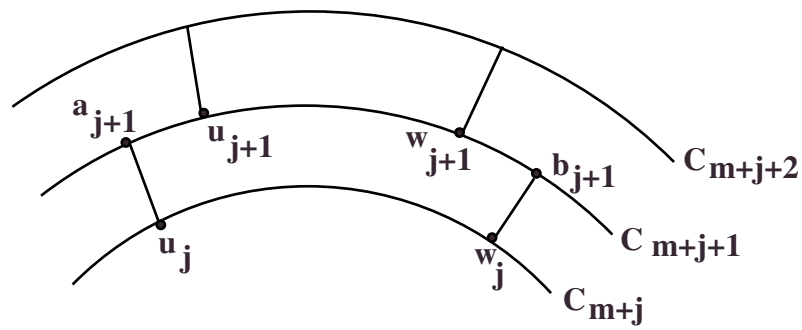

Figure 4.2: After j steps of the procedure.

nary sequence. Without loss of generality begin with $\chi_{I_n,0} = B(u_0, v_0)$ and $\chi_{I_n,1} = B(w_0, u_0)$ in C_m, where v_0, u_0, and w_0 each link outward to C_{m+1}, with $\sum_{v\in B(u_0,v_0)} \mathrm{Ex}^-(v) \leq \frac{-2}{3}\alpha' m$, $\sum_{v\in B(w_0,u_0)} \mathrm{Ex}^-(v) \leq \frac{-2}{3}\alpha' m$, and $\chi_{I_n,0} \bigcap \chi_{I_n,1} = \emptyset$. First consider $\chi_{I_n,1}$, suppose u_0 links outward to a_1 and w_0 links outward to b_1. Determine special clusters $\chi_{I_n,1,0}$ and $\chi_{I_n,1,1}$ by the following procedure.

Take u_1 and w_1 on C_{m+1} (Figure 4.1) as in Lemma 3.9, then

- $\sum_{v\in B(w_1,u_1)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\alpha'(m+1)(1+\frac{3}{4}\epsilon)$,
- $-3\alpha\,(m+1) < \sum_{v\in B(u_1,a_1)} \mathrm{Ex}^-(v) < -\alpha\,(m+1)$, and
- $-3\alpha\,(m+1) < \sum_{v\in B(b_1,w_1)} \mathrm{Ex}^-(v) < -\alpha\,(m+1)$.

Now apply Lemma 3.9 repeatedly by replacing α with $\alpha\left(1+\frac{3}{4}\epsilon\right)$ and use $B(u_k, w_k)$ as the starting set. After j steps this gives (Figure 4.2):

- $\sum_{v\in B(w_j,u_j)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\alpha'(m+j)(1+\frac{3}{4}\epsilon)^j$,
- $-3\alpha\,(m+j)\,(1+\frac{3}{4}\epsilon)^j < \sum_{v\in B(u_j,a_j)} \mathrm{Ex}^-(v) < -\alpha\,(m+j)\,(1+\frac{3}{4}\epsilon)^j$, and
- $-3\alpha\,(m+j)\,(1+\frac{3}{4}\epsilon)^j < \sum_{v\in B(b_j,w_j)} \mathrm{Ex}^-(v) < -\alpha\,(m+j)\,(1+\frac{3}{4}\epsilon)^j$.

Eventually, for some value of j, say $j = k$,

$$\sum_{v\in B(w_k,u_k)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\alpha'(m+k)(1+\frac{3}{4}\epsilon)^k < 3(\frac{-2}{3})\alpha'(m+k).$$

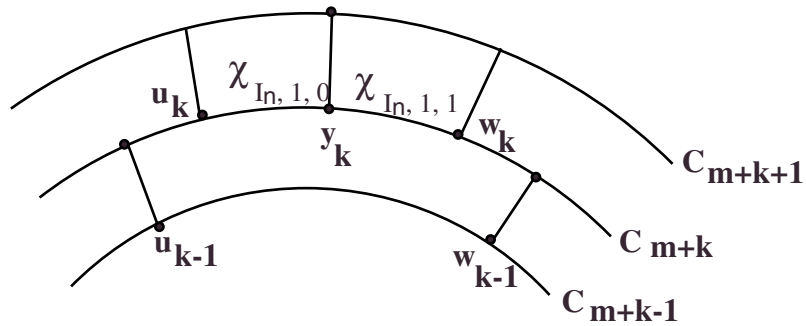

Figure 4.3: The creation of the next step of special clusters.

Repeating these steps starting with $\chi_{I,0}$ gives vertices u'_k, v_k on C_{m+k} such that

$$\sum_{v \in B(v_k, u'_k)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\alpha'(m+k)(1+\frac{3}{4}\epsilon)^k < 3(\frac{-2}{3})\alpha'(m+k)$$

where starting at v_k and walking clockwise the order of these vertices is v_k, u'_k, u_k, w_k. Without loss of generality, choose k large so that C_{m+k} is the same cycle for both $\chi_{I,0}$ and $\chi_{I,1}$.

Corollary 3.7 ensures that there exists a vertex y_k in $B(w_k, u_k)$ (Figure 4.3) so that y_k links outward to C_{m+1}, and

- $\sum_{v \in B(u_k, y_k)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\alpha'(m+k)$ and,
- $\sum_{v \in B(y_k, w_k)} \mathrm{Ex}^-(v) \leq -\frac{2}{3}\alpha'(m+k)$.

Let $B(u_k, y_k) = \chi_{I_n,1,0}$ and $B(y_k, w_k) = \chi_{I_n,1,1}$. Form $\chi_{I_n,0,0}$ and $\chi_{I_n,0,1}$ from $B(u'_k, v_k)$ in a similar fashion. Continue this iterative process indefinitely. At each step Lemma 3.9 ensures that $\chi_{I_n,0,0} \bigcap \chi_{I_n,0,1} = \emptyset$, and the inner excess sum of each special cluster will always be sufficiently negative to ensure that it contains a vertex with an outward link.

It remains to show that there are an uncountable number of geodesic fibers. To do this, first fix an infinite binary sequence I. Let I_n be the first n terms of I, let

P_v denote the set of geodesic paths from a vertex v to z, and let $H_S = \{P_v | v \in S\}$. Define a geodesic ray associated to I in the following manner. For each χ_{I_n} construct $H_{\chi_{I_n}}$.

Note first that there are only a finite number of geodesic paths in $H_{\chi_{I_1}}$. By Lemma 3.10, any geodesic path through $\chi_{I_{n+1}}$ passes through χ_{I_n} on its way back to z. Thus, one of the paths, say $P_1 \in H_{\chi_{I_1}}$, must be a subpath of an infinite number of paths $P \in \bigcup_{n=1}^{\infty} H_{\chi_{I_n}}$. Call this infinite subset of paths K_1 and note that $K_1 \bigcap H_{\chi_{I_n}} \neq \emptyset$ for any positive integer n.

Now consider $H_{\chi_{I_2}}$. Note that $H_{\chi_{I_2}}$ is finite and $K_1 \bigcap H_{\chi_{I_2}} \neq \emptyset$. By Lemma 3.10, there is a $P_2 \in H_{\chi_{I_2}}$ so that P_1 is a subpath of P_2 and P_2 is a subpath of an infinite subset of paths $P \in K_1$. Call this infinite subset of paths K_2 and note that $K_2 \bigcap H_{\chi_{I_n}} \neq \emptyset$ for any positive integer n.

Continue this process for each $H_{\chi_{I_n}}$ to get a path P_n so that P_{n-1} is a subray of P_n. Thus by induction, every binary sequence constructs a geodesic ray.

Now let I_A and I_B be two different binary sequences with associated rays A and B. Suppose that the sequnces first differ on the nth term with I_{n_1} the first n terms of I_A and I_{n_2} the first n terms of of I_B . Let C_m be the cycle associated with $\chi_{I_{n_1}}$ and $\chi_{I_{n_2}}$

Note that Lemma 3.9 implies that the distance on the cycle between any vertices in $A \bigcap C_{m+1}$ and any verices in $B \bigcap C_{m+1}$ is more than $2\alpha m$ (Figure 4.4). Note also that since each special cluster splits at the same cycle for any two sequences I_{n_1} and I_{n_2} of the same length, then $\chi_{I_{n_1}} \bigcap \chi_{I_{n_2}} = \emptyset$. Lemma 3.9 shows that this distance on the cycle grows exponentially in m. Consequently the Hausdorff distance between any

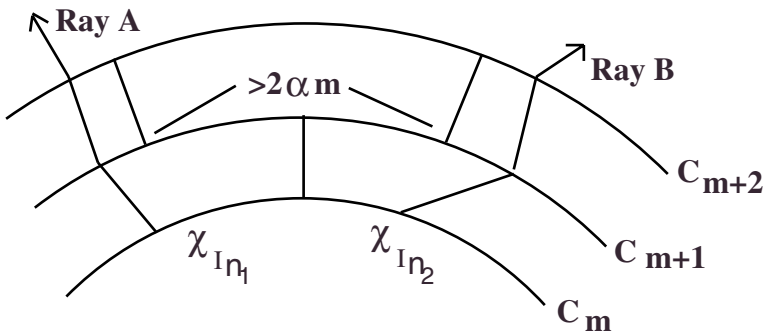

Figure 4.4: The distance between rays A and B.

two rays associated with different binary sequences is infinite and thus these rays are in different Hausdorff classes. Hence Γ contains an uncountable number of geodetic fibers. ∎

Neimayer and Watkins [NW] include examples of planar graphs with quadratic growth where all vertices have negative or zero excess. In Figure 4.5, graph a) is the grid graph where excess at each vertex is zero. In any geodesic ray, all the horizontal edges point in the same direction (right or left) and the same holds for all the vertical edges (up or down). There are two types of geodesic rays. One kind contains finitely many vertical edges or finitely many horizontal edges. The other contains countably many instances of a horizontal edge immediately followed by a vertical edge. An equivalent geodesic ray can be made in an uncountable number of ways. Graph a) contains uncountably many geodesic fibers.

Graph b) has either excess negative or zero at each vertex. There are two kinds of geodesic rays depending on whether the ray contains infinitely many or finitely many vertical edges. If there are infinitely many vertical edges, the geodesic ray contains a vertical subray that points upward or downward. That vertical ray is contained in

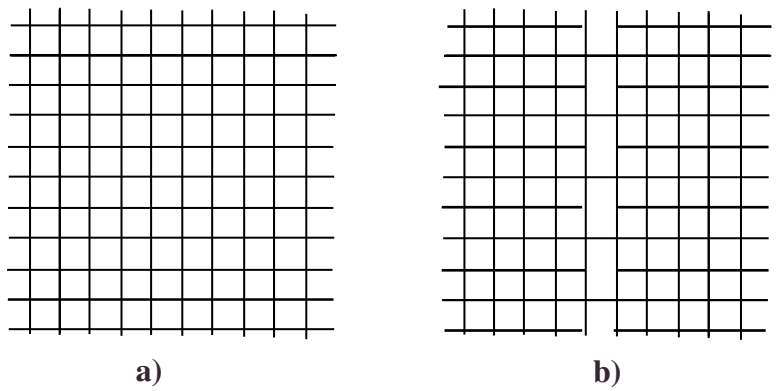

Figure 4.5: Examples of graphs from [NW].

the central vertical line of the figure. Thus, the geodesic rays containing infinitely many vertical edges belong to either of exactly two geodesic fibers. The geodesic rays that contain only finitely many edges contain a horizontal subray which may point to the right or left. Any two such right (or left) pointing rays are in the same fiber. Thus, there are four geodesic fibers in graph c).

Consequently the condition of positive excess is sufficient but not necessary to ensure the existence of an uncountable number of geodesic fibers in infinite, locally finite, planar graphs with bounded degree and codegree.

CHAPTER 5

Bounded Automorphisms of Graphs with Positive Excess

Neimayer and Watkins [NW] discovered that the only bounded automorphism on $G_{4,5} \cup G_{5,4}$, which is the set of planar graphs that are 3-connected, 1-ended, have finite degree and codegree, with codegree at least 5 and degree at least 4 or with codegree at least 4 and degree at least 5, was the identity automorphism. The graphs in $G_{4,5} \cup G_{5,4}$ form a subset of all 3-connected, 1-ended planar graphs that have positive excess at each vertex. A natural extension of the result by Neimayer and Watkins is to prove that if Γ is a 3-connected, 1-ended, planar graph that has positive excess at each vertex, the only bounded automorphism on Γ is the identity automorphism.

The following lemmas construct continuous functions that are useful in certain situations. Let Γ be a 3-connected, 1-ended, planar graph and consider the Bilinski Map with center vertex z.

For the first lemma, suppose there is a path P that intersects C_m and C_{m-1} at intersection vertices x, u, w, and y as illustrated in Figure 5.1 part a). Let $Q_1 = B(y,x) \bigcup \{x\} \bigcup \{y\}$ along C_m, let $Q_2 = B(u,x) \bigcup \{u\} \bigcup \{x\}$ along P, let $Q_3 = B(w,u) \bigcup \{w\} \bigcup \{u\}$ along C_{m-1}, and $Q_4 = B(y,w) \bigcup \{w\} \bigcup \{y\}$ along P.

For the second lemma, suppose there is a path R that intersects C_m and C_{m-1} at intersection vertices a, b, c, and y as illustrated in Figure 5.1 part b). Let $S_1 = B(d,a) \bigcup \{a\} \bigcup \{y\}$ along C_{m-1}, let $S_2 = B(b,a) \bigcup \{a\} \bigcup \{b\}$ along R, let $S_3 = B(c,b) \bigcup \{c\} \bigcup \{b\}$ along C_m, and $S_4 = B(d,c) \bigcup \{d\} \bigcup \{c\}$ along R.

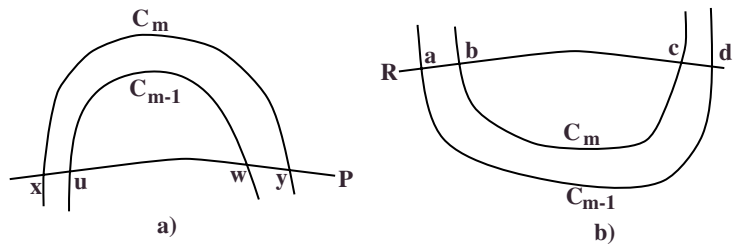

Figure 5.1: The situations for Lemma 5.1 and Lemma 5.2.

Recall that ℓ is an upper bound on codegree.

Lemma 5.1 *Suppose the situation where Q_1, Q_2, Q_3, and Q_4 are constructed as listed above. There exists a continuous orientation preserving map r such that r maps the Q_1 to $Q_2 \bigcup Q_3 \bigcup Q_4$ Furthermore, for any vertex v on Q_1, $d(v, r(v)) \leq \ell$.*

Proof. First note that every vertex in C_m is contained in a face that is adjacent to C_{m-1}. Thus, every vertex on Q_1 is contained on a face adjacent to $Q_2 \bigcup Q_3 \bigcup Q_4$. For each vertex v on Q_1, walk clockwise around the face until the first vertex $\alpha(v)$ on $Q_2 \bigcup Q_3 \bigcup Q_4$ is reached. If there is more than one face, choose the face that is the most counterclockwise direction along Q_1. Note that α defines a mapping from the vertices of Q_1 to the vertices of $Q_2 \bigcup Q_3 \bigcup Q_4$. Extend α to a continuous map r by stretching any edges between consecutive vertices v_1 and v_2 on Q_1 to the path $B(\alpha(v_1), \alpha(v_2))$ along $Q_2 \bigcup Q_3 \bigcup Q_4$. If $\alpha(v_1) = \alpha(v_2) = v$, r maps the edge between v_1 and v_2 to v. Since clockwise ordering of the vertices of Q_1 is preserved under α, r is a continuous orientation preserving map from Q_1 to $Q_2 \bigcup Q_3 \bigcup Q_4$. Note also that for any vertex v on Q_1, $d(r(v), v) = d(\alpha(v), v) \leq \ell$. ∎

Lemma 5.2 *Suppose the situation where S_1, S_2, S_3, and S_4 are constructed as listed earlier. There exists a continuous orientation preserving map s such that s maps the S_1 to $S_2 \bigcup S_3 \bigcup S_4$ Furthermore, for any vertex v on S_1, $d(v, s(v)) \leq \beta + \ell$, where $\beta = \lceil \frac{2d\ell^2 m}{3\epsilon} \rceil$.*

Proof. First note that by Corollary 3.6 every vertex in C_{m-1} is at most distance β along C_{m-1} away from a vertex that is contained by a face that is adjacent to C_m. Thus, every vertex on S_1 is at most β away from a vertex that is contained by a face that is adjacent to $S_2 \bigcup S_3 \bigcup S_4$. For each vertex v on S_1, walk clockwise along S_1 until the a vertex that is on a face adjacent to C_m is reached, and then counterclockwise around the face until the first vertex $\alpha(v)$ on $S_2 \bigcup S_3 \bigcup S_4$ is reached. If there is more than one face, choose the face that is the most clockwise direction along S_1. Note that α defines a mapping from the vertices of S_1 to the vertices of $S_2 \bigcup S_3 \bigcup S_4$. Extend α to a continuous map r by stretching any edges between consecutive vertices v_1 and v_2 on S_1 to the path $B(\alpha(v_1), \alpha(v_2))$ along $S_2 \bigcup S_3 \bigcup S_4$. If $\alpha(v_1) = \alpha(v_2) = v$, s maps the edge between v_1 and v_2 to v. Since clockwise ordering of the vertices of S_1 is preserved under α, s is a continuous orientation preserving map from S_1 to $S_2 \bigcup S_3 \bigcup S_4$. Note also that for any vertex v on S_1, $d(s(v), v) = d(\alpha(v), v) \leq \beta + \ell$. ∎

An automorphism is said to be *orientation preserving* if identifying a face of a graph under the automorphism and ordering the vertices in a clockwise manner, then the vertices under the image are also ordered clockwise. It is said to be *orientation reversing* if the vertices are ordered counterclockwise under the automorphism.

The following theorem proves that a bounded automorphism is orientation preserving. Recall that an automorphism on a planar graph preserves excess and that that if a graph has bounded degree and codegree then excess, inner excess, and outer excess have upper bounds. Call the common bound k.

Lemma 5.3 *Let Γ be a 3-connected, 1-ended, planar graph with positive excess at each vertex. If ϕ is a bounded automorphism on Γ, then ϕ is orientation preserving.*

Proof. Let ϕ be a bounded automorhism with bound ρ on Γ. Suppose by way of contradiction that ϕ is orientation reversing. Choose a vertex z. Construct the Bilinski Map with center z and look at its image under ϕ centered around $\phi(z)$. Choose C_m in the Bilinski Map centered around z with $m > \rho$ and $j = \beta + \ell$ so that

$$\sum_{v \in C_m} \mathrm{Ex}^-(v) < -6(\rho + j\rho)^2(\ell d)^{(\rho + j\rho)} + k(\rho + j\rho).$$

Note also that for vertices in $\phi(C_m)$

$$\sum_{v \in \phi(C_m)} \mathrm{Ex}^-(v) < -6(\rho + j\rho)^2(\ell d)^{(\rho + j\rho)} + k(\rho + j\rho).$$

C_m and $\phi(C_m)$ cannot be concentric since the disk contained by C_m and $\phi(C_m)$ bounds the same number of faces. Thus C_m and $\phi(C_m)$ intersect as illustrated in Figure 5.2 part a). Note that ϕ can be extended to a homeomorphism $\overline{\phi}$ where $\overline{\phi}$ agrees with ϕ on the vertices of C_m and maps the edges homeomorphically to their images under ϕ. Since C_m and $\phi(C_m)$ are homeomorphic and intersect, construct a continuous map r from $\phi(C_m)$ to C_m by projecting $\phi(C_m)$ to C_m in the following manner which is illustrated in Figure 5.2 parts b) c). Note that since ϕ is a bounded automorphism with bound ρ, (C_m) is between $\phi(C_{m+\rho})$ and $\phi(C_{m-\rho})$.

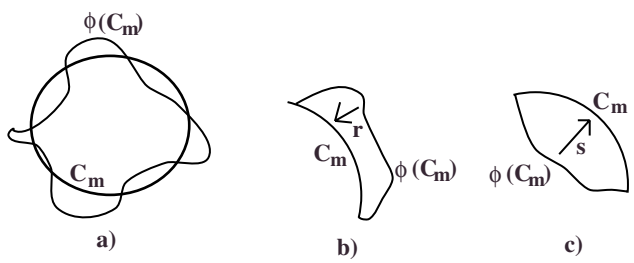

Figure 5.2: The projection of $\phi(C_m)$ to C_m.

Consider first the case where a region R is such that C_m is on the interior of $\phi(C_m)$ between intersection points as illustrated in part b) of Figure 5.2. Note that C_m is between $\phi(C_m)$ and $\phi(C_{m-\rho})$. Construct $\phi(C_{m-1}), \phi(C_{m-2}), ...\phi(C_{m-i})$ where $m-i$ is the smallest index such that $\phi(C_{m-i}) \bigcap C_m = \emptyset$. Project $\phi(C_m)$ to $\phi(C_{m-1})$ using the map r_1 described in Lemma 5.1. Consider the restriction of the map on R. Note that for any $v \in \phi(C_m)$, $d(v, r_1(v)) \leq \ell$. Similarly map C_{m-1} to C_{m-2} by r_2 and restricting it to R. Continue this process. There will be at most ρ maps r_i until $\phi(C_m)$ is mapped to C_m along R. Thus by Lemma 5.1 if r is the composition of the r_i's, then $d(x, r(x)) \leq \rho\ell$ for any vertex x on $\phi(C_m)$.

Consider next the case where a region S is such that $\phi(C_m)$ is on the interior of C_m between intersection points as illustrated in part c) of Figure 5.2. Note that C_m is between $\phi(C_m)$ and $\phi(C_{m+\rho})$. Construct $\phi(C_{m+1}), \phi(C_{m+2}), ...\phi(C_{m+i})$ where $m+i$ is the smallest index such that $\phi(C_{m+i}) \bigcap C_m = \emptyset$. Project $\phi(C_m)$ to $\phi(C_{m+1})$ using the map s_1 described in Lemma 5.2. Consider the resrtiction of the map on S. Note that for any $v \in \phi(C_m)$, $d(v, r_1(v)) \leq \beta + \ell$. Similarly map C_{m+1} to C_{m+2} by s_2 and restricting it to S. Continue this process. There will be at most ρ maps s_i until

$\phi(C_m)$ is mapped to C_m along S. Thus by Lemma 5.2 if s is the composition of the s_i's, then $d(x, r(x)) \leq (\beta + \ell)\rho = j\rho$ for any vertex x on $\phi(C_m)$.

Consider the map $\psi = s \circ r \circ \overline{\phi}$. By the triangle inequality, $d(v, \psi(v)) \leq \rho + j\rho$ for every vertex v on C_m. Also ψ is a continuous map and since $\overline{\phi}$ is orientation reversing, r and s orientation preserving, ψ is orientation reversing. Since $\psi(C_m) = C_m$, Theorem 21.5 in [Mu] gives that some point p on C_m maps to its antipodal point under ψ. Consider the vertices u and w that are closest to the point p (choose $p = u$ and a vertex adjacent to it as w if $\psi(p)$ is a vertex) and look at u, $\psi(u)$, w and $\psi(w)$. Note that $d(u, \psi(u)) \leq \rho + j\rho$ and $d(w, \psi(w)) \leq \rho + j\rho$. Thus, by Lemma 3.11 on the cycle C_m,

$$\sum_{v \in B(u,\psi(u))} \mathrm{Ex}^-(v) < -2(\rho + j\rho)^2(\ell d)^{\rho + j\rho} + k(\rho + j\rho)$$

and

$$\sum_{v \in B(\psi(w),w)} \mathrm{Ex}^-(v) < -2(\rho + j\rho)^2(\ell d)^{\rho + j\rho} + k(\rho + j\rho).$$

C_m was chosen so that

$$\sum_{v \in C_m} \mathrm{Ex}^-(v) < -6(\rho + j\rho)^2(\ell d)^{(\rho + j\rho)} + k(\rho + j\rho),$$

so there exists a vertex x so that

$$\sum_{v \in B(u,x)} \mathrm{Ex}^-(v) > -2(\rho + j\rho)^2(\ell d)^{\rho + j\rho} + k(\rho + j\rho)$$

and

$$\sum_{v \in B(x,v)} \mathrm{Ex}^-(v) > -2(\rho + j\rho)^2(\ell d)^{\rho + j\rho} + k(\rho + j\rho).$$

Note that either x is between $\psi(p)$ and $\psi(u)$ in the counterclockwise direction along C_m or x is between $\psi(w)$ and $\psi(p)$. in the counterclockwise direction along C_m. ψ

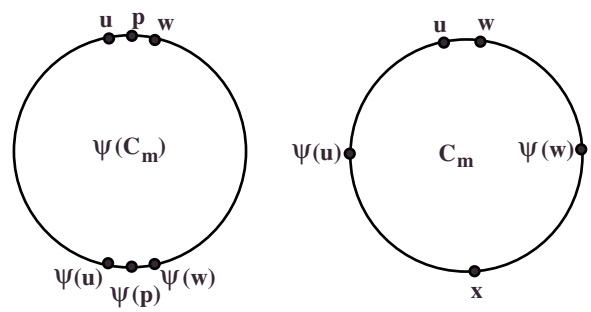

Figure 5.3: The relation of vertices u and w in $\psi(C_m)$ and C_m.

is a continuous orientation reversing map, so $\psi(x)$ must be between w and u in the counterclockwise direction along C_m. Since ψ maps vertices to vertices, $\psi(x) = u$ or $\psi(x) = w$ as illustrated in Figure 5.3. Thus, $d(u, x) \leq \rho + j\rho$ or $d(w, x) \leq \rho + j\rho$. This contradicts Lemma 3.11.

Thus ϕ cannot be orientation reversing ∎

One consequence of an orientation preserving automorphism is that if Γ has two vertices with an edge incident to both of the vertices that are fixed under ϕ, then ϕ is the identity automorphism.

Lemma 5.4 *Let Γ be a 1-ended, 3-connected, planar graph with finite degree and co-degree. Let ϕ be an orientation preserving automorphism such that two adjacent vertices are fixed under ϕ. Then ϕ is the identity automorphism.*

Proof. Let Γ be as above with an edge e conisting of two fixed vertices under an orientation preserving automorphism ϕ

Let $z \in V(e)$ and construct the Bilinski Map centered at z. Consider the vertices that are labelled 1 in the Bilinski Map. Since $z \in V(e)$, it is adjacent to a vertex

$y_1 \in V(e)$ where y_1 is labelled 1 in the Bilinski Map. Thus both z and y_1 are fixed vertices under ϕ. Thus the edge $(z, y_1) \in E(\Gamma)$ is also fixed under ϕ. Since ϕ is orientation preserving, any face incident with (z, y_1) is also fixed under ϕ. Thus all vertices incident to those faces are also fixed under ϕ. Similarily any faces incident to an edge containing z are also fixed under ϕ. Thus, every face containing z is fixed under ϕ and every vertex labelled 1 in the Bilinski Map centered at z is also fixed under ϕ.

Now suppose every vertex and face labelled $n-1$ for $n \geq 1$ is a fixed vertex under ϕ. Consider a vertex y_n labelled n in the Bilinski Map centered at z. Note that y_n is contained in a face F_n labelled n in the Bilinski map. Thus F_n has at least one vertex y_{n-1} labelled $n-1$. Note that y_{n-1} is contained by at least one face labelled F_{n-1} labelled $n-1$ in the Bilinski Map. Thus every face incident with y_{n-1} is fixed since this face contains a path of fixed vertices containing y_{n-1}. Thus F_n and in turn y_n are fixed under ϕ.

So by induction, every vertex is fixed under ϕ, thus ϕ is the identity automorphism.

∎

Now consider the case where ϕ is a bounded automorphism and has a fixed vertex in Γ under the automorphism. An example of this case is illustrated by the Poincare plane. Note that for the graph in Figure 5.4, If the center vertex is a fixed point, then every orientation preserving automorphism of the graph is a rotation. A rotation requires that a vertex close to the line at infinity will have a large distance from its image under the automorphism. The general proof of this case uses the Bilinski Map and a geodesic ray through the fixed point to show that vertices far from the

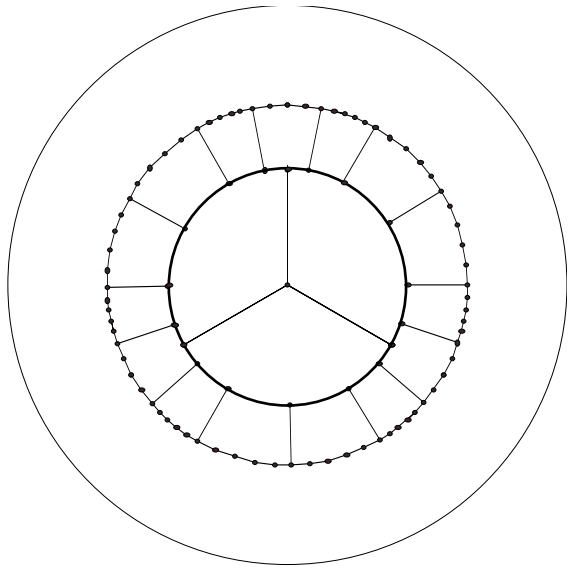

Figure 5.4: A graph with positive excess at each point as illustrated in the Poincare Plane.

fixed point on the geodesic ray are a large distance from their images under any automorphism other than the identity.

Lemma 5.5 *Let Γ be a 1-ended, 3-connected, planar graph with bounded degree and codegree. Suppose that ϕ is a bounded automorphism with fixed vertex x and the excess is positive at each vertex except for possibly x and the neighbors of x. Then ϕ is the identity automorphism.*

Proof. Note that since there are only a finite number of vertices with negative excess, let $-n$ for some positive integer n be a lower bound on the sum of their excess. Let x be a fixed vertex of ϕ. Construct a geodesic ray $R = x, x_1, x_2, ...$ starting from x. Consider its image $\phi(R)$. First suppose that R does not intersect $\phi(R)$ except at x. Since ϕ is a bounded automorphism, it has a maximum distance ρ between a vertex and its image under ϕ. Let $y = x_j$ where $j \geq 903(\rho + n + 5)$. Construct a

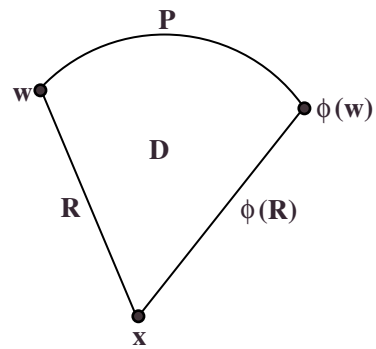

Figure 5.5: The case where R and $\phi(R)$ do not intersect.

geodesic path Q from y to $\phi(y)$.

Note that if Q intersects R or $\phi(R)$ on any vertices between x and y on R or between x and $\phi(y)$ on $\phi(R)$, a geodesic path P so that $|P| \leq \rho$ can be created from a vertex $w = x_i$ where $i \geq 903(\rho + n + 4)$. This is done by looking at the vertex $w = x_i$ where i is the minimum integer where Q intersects R or $\phi(R)$. Note that $i \geq 903(\rho + n + 4)$ since if it were not, it would contradict the fact that R and $\phi(R)$ are geodesic rays. Construct P by using the shortest path from w to $\phi(w)$ along R, then Q, then $\phi(R)$. Note that $|P| \leq |Q|$ or there would be a contradiction to Q being a geodesic ray. Create a disk D by starting at x, going along R to w, then to $\phi(w)$ along P, then back to x along $\phi(R)$, as illustrated in Figure 5.5.

Summing up the excess of D,

$$\begin{aligned} -2 &= \sum_{v \in B(w,x)} \mathrm{Ex}^-(v) + \sum_{v \in B(x,\phi(w))} \mathrm{Ex}^-(v) + \sum_{v \in B(\phi(w),w)} \mathrm{Ex}^-(v) \\ &\quad + Ex^-(x) + Ex^-(w) + Ex^-(\phi(w)) + \sum_{v \in D_{int}} \mathrm{Ex}(v). \end{aligned}$$

Since ϕ is orientation preserving and since there is positive excess at least $\frac{1}{903}$ at

each vertex by Theorem 2.2 except for possibly x and its neighbors,

$$\begin{aligned}\sum_{v\in B(w,x)} \mathrm{Ex}^-(v) + \sum_{v\in B(x,\phi(w))} \mathrm{Ex}^-(v) &= \sum_{v\in B(w,x)} \mathrm{Ex}(v) \\ &\geq \frac{1}{903}(903(\rho+n+4)-1) + \mathrm{Ex}(x_1) \\ &> \rho+n+1\end{aligned}$$

Thus,

$$\begin{aligned}-2 &= \sum_{v\in B(w,x)} \mathrm{Ex}^-(v) + \sum_{v\in B(x,\phi(w))} \mathrm{Ex}^-(v) + \sum_{v\in B(\phi(w),w)} \mathrm{Ex}^-(v) \\ &\quad + Ex^-(x) + Ex^-(w) + Ex^-(\phi(w)) + \sum_{v\in D_{int}} \mathrm{Ex}(v) \\ &\geq \rho+n+1+\sum_{v\in B(\phi(w),w)} \mathrm{Ex}^-(v) - 3\left(\frac{2}{3}\right) - n \\ &= \rho-1+\sum_{v\in B(\phi(w),w)} \mathrm{Ex}^-(v)\end{aligned}$$

This yields that $-(\rho+1) \geq \sum_{v\in B(\phi(w),w)} \mathrm{Ex}^-(v)$ and since the least excess a vertex can have is $-\frac{2}{3}$, $|P| > \rho$. This is a contradiction because P is geodetic and the maximum distance between w and $\phi(w)$ is ρ. Thus R intersects $\phi(R)$

Now suppose R and $\phi(R)$ intersect each other at a vertex other than x. Let $x_i \in R$ be the first intersection vertex of R and $\phi(R)$ after x. Note that $x_1 = \phi(x_1)$ since if not, then both R and $\phi(R)$ could not be geodesic. Also note that if $x_i = x_1$ then by Lemma 5.4 ϕ is the identity automorphism. Let x_i for some $i \geq 2$ be the first intersection vertex and consider $R' = x_i, x_{i+1}, ...$ as a geodesic subray of R. Every vertex of R' has positive excess since R' cannot contain a neighbor of x. Note that x_i is a fixed vertex under ϕ and that R' is a geodesic path starting with x_i. By the same argument as the first part of the proof R' intersects $\phi(R')$. Let w be the first vertex along R' after x_i where they intersect. Note that $\phi(w) = w$ since if not, then both

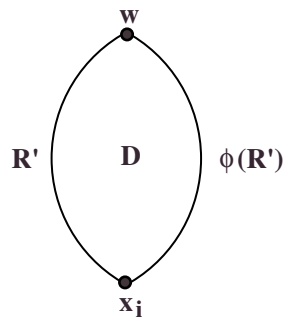

Figure 5.6: The case where R' intersects $\phi(R')$.

R' and $\phi(R')$ would not be geodesic. Consider the sum of the excess of the disk D created by R' and $\phi(R')$ when they intersect at w as illustrated in Figure 5.6. Note that the interior of D could not contain x or any neighbors of x or it would contradict that the original R is a geodesic. Since ϕ is orientation preserving by Lemma 5.3, along the boundary of D

$$\begin{aligned} 0 &< \sum_{v \in B(w,x')} \mathrm{Ex}(v) \\ &= \sum_{v \in B(w,x')} \mathrm{Ex}^-(v) + \sum_{v \in B(x',w)} \mathrm{Ex}^-(v) \end{aligned}$$

Thus the sum of the excess of D is

$$\begin{aligned} -2 &= \sum_{v \in B(w,x')} \mathrm{Ex}^-(v) + \sum_{v \in B(x',w)} \mathrm{Ex}^-(v) + \sum_{v \in D_{int}} \mathrm{Ex}(v) + \mathrm{Ex}^-(x') + \mathrm{Ex}^-(w) \\ &= \sum_{v \in B(w,x')} \mathrm{Ex}(v) + \mathrm{Ex}^-(x') + \mathrm{Ex}^-(w) \\ &\geq 0 - 2\left(\frac{2}{3}\right) \\ &= -\frac{4}{3} \end{aligned}$$

which is a contradiction.

Thus, every geodesic ray is fixed under ϕ and since ϕ is orientation preserving by

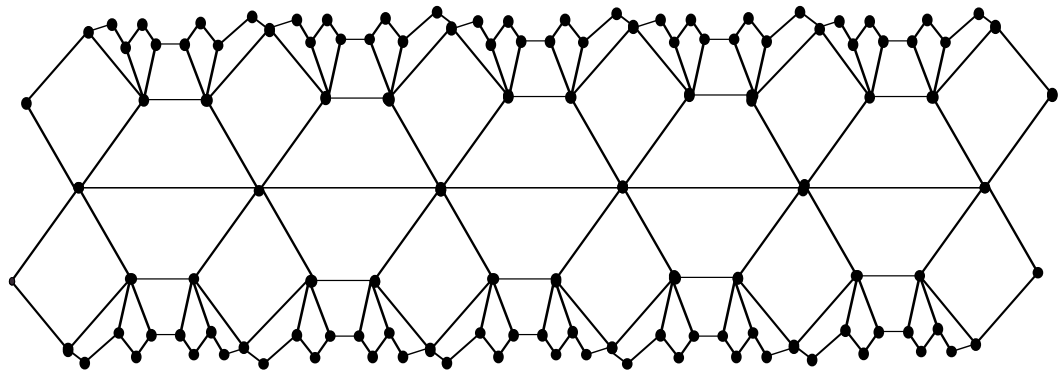

Figure 5.7: A graph with positive excess at each vertex and a geodesic translation invariant double ray.

Lemma 5.4, ϕ is the identity automorphism. ∎

Next consider the case where ϕ has no fixed points. A double ray L is *translation invariant under an automorphism* ϕ if $L = \phi(L)$. In Figure 5.7, an example is given of a graph with a translation invariant geodesic double ray. The next lemma shows that the farther you travel away from the geodesic translation invariant double ray to a vertex x, the farther the distance between x and $\phi(x)$.

Lemma 5.6 *Suppose that Γ is a 1-ended, 3-connected, planar graph with positive excess at each vertex and with bounded degree and codegree. Suppose also that there is a geodesic translation invariant double ray L under an automorphism ϕ on Γ. If ϕ is a bounded automorphism on Γ, then ϕ has a fixed vertex.*

Proof. Let ϕ be a bounded automorphism with bound ρ, and let L be a geodesic translation invariant double ray in Γ. Since each vertex has positive excess, at least one side is such that $\sum_{v \in L} \mathrm{Ex}^-(v) > 0$. Since L is geodesic, construct Bilinski Lines using L on the side with positve excess as illustrated in Theorem 3.12.

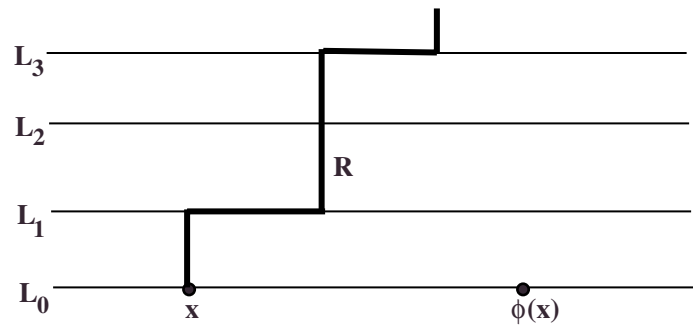

Figure 5.8: The Construction of Ray R.

Choose vertices x and $\phi(x)$ on L so that both are adjacent to L_1. Since L is translation invariant and the $\sum_{v \in L} \mathrm{Ex}^-(v) > 0$, each pair of vertices x and $\phi(x)$ on L are such that $\sum_{v \in B(\phi(x),x)} \mathrm{Ex}^-(v) > 0$. It is assumed that $\phi(x)$ is on the right of x (the case where $\phi(x)$ is on the left is similar).

Construct a ray R by starting at x, following along an edge to L_1. If this vertex is adjacent to L_2, continue along this edge to L_2. If it is not adjacent to L_2 continue along L_1 to the right until reaching the first vertex adjacent to L_2 then following the edge to L_2. Continue this pattern as illustrated in Figure 5.8.

Consider the image of R under ϕ. If R and $\phi(R)$ intersect, then the last vertex of R along L_k has a preimage of ϕ on R. Call these vertices z and $\phi(z)$ and note $\phi(z)$ is adjacent to L_{k+1}. Since ϕ is an automorphism, y is also adjacent to L_{k+1}. However, this contradicts the construction of R since R would follow the edge connecting z to L_{k+1} instead of continuing along L_k to $\phi(z)$. Thus, $R \cap \phi(R) = \emptyset$. This contradiction is illustrated in Figure 5.9.

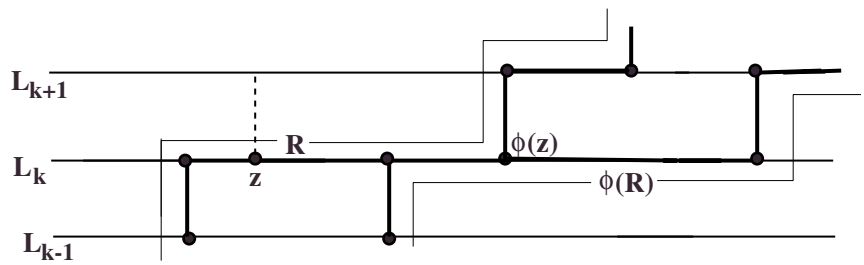

Figure 5.9: The Reason R and $\phi(R)$ May Not Intersect.

Since ϕ is a bounded automorphism, it has a maximum distance ρ between a vertex and its image under ϕ. Let $y \in R$ where $y \in L_t$ and $t > 903(\rho+2)$. Construct a geodesic path Q from y to $\phi(y)$.

Note that if Q intersects R or $\phi(R)$ on any vertices between x and y on R or between x and $\phi(y)$ on $\phi(R)$, a path P so that $|P| \leq \rho$ can be created from a vertex $w \in R \bigcap L_s$ where $s > 903(\rho+1)$ to $\phi(w)$. Since Q is geodesic and of length at most ρ, Q cannot intersect $L_{903(\rho+1)}$ so choose the smallest index s so that Q intersects L_s on either R or $\phi(R)$. Call this vertex w. Create the shortest path P following w along R to Q, then along Q to $\phi(R)$, then along $\phi(R)$ to $\phi(w)$. Note that $|P| \leq |Q|$ or it would contradict the fact that Q is geodesic.

Consider the disk D created by starting at x and following R to w, then along P to $\phi(w)$, then along $\phi(R)$ to $\phi(x)$, then finally along L back to x. This constuction is illustrated in Figure 5.10.

Note that along the disk D, $\sum_{v \in B(\phi(x),x)} \mathrm{Ex}^-(v) > 0$. Also, since ϕ is orientation preserving,

$$\sum_{v \in B(w,x)} \mathrm{Ex}(v) = \sum_{v \in B(w,x)} \mathrm{Ex}^-(v) + \sum_{v \in B(\phi(x),\phi(w))} \mathrm{Ex}^-(v).$$

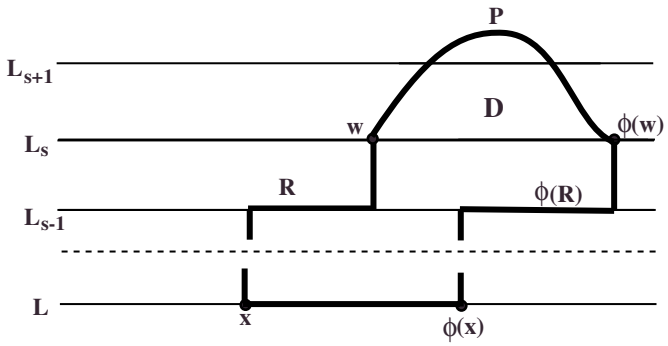

Figure 5.10: The construction of disk D.

Note that since the length of the disk along R is at least $903(\rho+1)$ by theorem 2.2,

$$\sum_{v\in B(w,x)} \mathrm{Ex}(v) > \left(\frac{1}{903}\right) 903(\rho+1) = \rho+1.$$

Thus, summing the excess of the disk D,

$$\begin{aligned}
-2 &= \sum_{v\in B(w,x)} \mathrm{Ex}^-(v) + \sum_{v\in B(\phi(w),w)} \mathrm{Ex}^-(v) + \sum_{v\in B(\phi(x),\phi(w))} \mathrm{Ex}^-(v) \\
&\quad + \sum_{v\in B(x,\phi(x))} \mathrm{Ex}^-(v) + \sum_{v\in D_{int}} \mathrm{Ex}(v) \\
&\quad + \mathrm{Ex}^-(x) + \mathrm{Ex}^-(w) + \mathrm{Ex}^-(\phi(w)) + \mathrm{Ex}^-(\phi(x)) \\
&> \sum_{v\in B(w,x)} \mathrm{Ex}(v) + \sum_{v\in B(\phi(w),w)} \mathrm{Ex}^-(v) + 4\left(\frac{-2}{3}\right) \\
&= \rho+1+\frac{-8}{3} + \sum_{v\in B(\phi(w),w)} \mathrm{Ex}^-(v)
\end{aligned}$$

This yields that $-\rho > \sum_{v\in B(\phi(w),w)} \mathrm{Ex}^-(v)$ and since $\mathrm{Ex}^-(v) \geq -\frac{2}{3}$ for all vertices v, $|P| > \rho$. This is a contradiction because P is geodetic and the maximum distance between w and $\phi(w)$ is ρ. Thus ϕ is not bounded. ∎

The last four lemmas are now combined to extend the Neimayer and Watkins result.

Theorem 5.7 *Suppose that Γ is a 1-ended, 3-connected, planar graph with positive excess at each vertex with bounded degree and codegree. If ϕ is a bounded automor-*

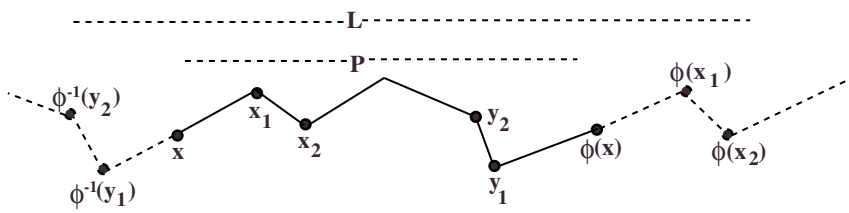

Figure 5.11: The extension of disk P to L.

phism on Γ*, then it is the identity automorphism.*

Proof. Let Γ be as stated above. Suppose that ϕ is a bounded automorphism and that ϕ is not the identity automorphism. By Lemma 5.3, ϕ is orientation preserving and by Lemma 5.5, Γ does not have a fixed vertex under ϕ. Let x be a vertex so that $d(x, \phi(x))$ is minimal. Construct a geodesic path P between x and $\phi(x)$. Let x_1 be the vertex on P adjacent to x. Note that $\phi(x_1)$ cannot be on P since if it were $d(x_1, \phi(x_1)) < d(x, \phi(x))$ which would contradict the minimality of $d(x, \phi(x))$. Since ϕ is an automorphism, $\phi(x)$ and $\phi(x_1)$ are adjacent since x and x_1 are adjacent. Extend P to the vertex $\phi(x_1)$ by adding the edge $(\phi(x), \phi(x_1))$. Continue this extension by using the next vertex in the path adjacent to x_1, call it x_2 and extend it to $\phi(x_2)$ in a similar manner. Continue this process. Also, extend P (in the other direction) by looking at the vertex adjacent to $\phi(x)$ on the original P, call it y_1 and extending P to $\phi^{-1}(y_1)$ similarly. This construction is illustrated in Figure 5.11. This process, defines a doubly infinite walk L.

There are only two possibilities for L. L is either a cycle or a geodesic double ray. If L intersects itself, then $x_i = x_{i+j}$ for some positive integers i and j,

$$x_{i+1} = x_{i+j+1}, x_{i+2} = x_{i+j+2}, ...x_{i+j} = x_{i+2j} = x_i.$$

Thus, L is a cycle. If L does not intersect itself it is a translation invariant geodesic double ray.

If L is a translation invariant geodesic double ray then it is the situation in Lemma 5.6 exists, giving a contradiction.

If L is a cycle, create a new graph Γ' by deleting the inside of the cycle and putting in a vertex x adjacent to every vertex of L. Note that in Γ, $\phi(L) = L$. So for Γ', define an automorphism ψ so that $\psi(v) = \phi(v)$ for every vertex that is in both Γ and Γ' and let $\psi(x) = x$. Note that ψ is an automorphism on Γ'. Lemma 5.5 gives that ψ is the identity automorphism. Every vertex that is in both Γ and Γ' is fixed by ϕ which means Γ has a fixed path under ϕ. Thus, by Lemma 5.4, ϕ is the identity automorphism. ∎

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