ON THE COHOMOLOGY OF THE COMPLEMENT OF A TORAL ARRANGEMENT

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Let $(T, \mathcal{A})$ be a toral arrangement over $\mathbb{C}$, where T is a complex torus and $\mathcal{A}$ is a finite set of kernels of rational characters of T. The complement of the arrangement, M, is formed by taking the union of the kernels of the characters in $\mathcal{A}$ and deleting this from T. Let $\chi_1 - \mu_1, \ldots, \chi_s - \mu_s$ be the distinct irreducible factors of the characters minus 1 in $\mathcal{A}$, where χ_i is a character of T with connected kernel and μ_i is a n^{th} root of unity for some n. Let $M_j = \mathbb{C}^l \setminus \bigcup_{k=1}^{l} \ker(z_k) \cup \bigcup_{i=1}^{j} \ker(\chi_i - \mu_i)$. Using de Rham cohomology with complex coefficients, we show that if, for all $1 \leq r \leq s$, $H^*_{DR}(M_{r-1} \cap Z_r)$ is generated as a $\mathbb{C}$-algebra by the set

$$\left\{ \left[\frac{d_{\mathbb{C}} z_1|_{(M_{r-1}\cap Z_r)}}{z_1|_{(M_{r-1}\cap Z_r)}} \right], \ldots, \left[\frac{d_{\mathbb{C}} z_l|_{(M_{r-1}\cap Z_r)}}{z_l|_{(M_{r-1}\cap Z_r)}} \right], \left[\frac{d_{\mathbb{C}} \chi_1|_{(M_{r-1}\cap Z_r)}}{(\chi_1-\mu_1)|_{(M_{r-1}\cap Z_r)}} \right], \ldots, \left[\frac{d_{\mathbb{C}} \chi_{r-l}|_{(M_{r-1}\cap Z_r)}}{(\chi_{r-l}-\mu_{r-l})|_{(M_{r-1}\cap Z_r)}} \right] \right\},$$

then the cohomology $H^*_{DR}(M)$ is generated as a $\mathbb{C}$-algebra by the set

$$\{[\tfrac{d_{\mathbb{C}} z_1}{z_1}], \ldots, [\tfrac{d_{\mathbb{C}} z_l}{z_l}], [\tfrac{d_{\mathbb{C}} \chi_1}{\chi_1-\mu_1}], \ldots, [\tfrac{d_{\mathbb{C}} \chi_s}{\chi_s-\mu_s}]\}.$$

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Let k be a field, V be a l-dimensional vector space over k, and $\mathcal{A} = \{\ker(\alpha_1), \ldots, \ker(\alpha_s)\}$ be a a finite set of hyperplanes formed from the linear functionals $\alpha_i : V \to k$, $i = 1, \ldots, s$. Then recall that the pair $(\mathcal{A}, V)$ is called a *hyperplane arrangement* [OT]. The complement of the arrangement is

$$M = V \setminus \bigcup_{i=1}^{s} \ker(\alpha_i).$$

In 1971, E. Brieskorn proved that the cohomology ring of M with coefficients in $\mathbb{Z}$ is generated by s anticommuting elements, one for each hyperplane [Br, Lemma 3].

Let A be a connected, graded, skew-commutative k-algebra with identity, and let V be a r-dimensional k-vector space with basis $\{v_1, \ldots, v_r\}$. We say that A is generated by the set $\{a_1, \ldots, a_r\}$ if there exists a surjective k-algebra homomorphism $f : \bigwedge^* V \to A$ such that $f(v_i) = a_i$ for $1 \leq i \leq r$. In particular, $A^q = f(\bigwedge^q V)$ and A^q is spanned by the set $\{a_{i_1} \ldots a_{i_q} \mid 1 \leq i_1 < \cdots < i_q \leq r\}$.

In 1991, R. Jozsa and J. Rice [JR, Theorem 1] presented the following version of Brieskorn's result.

Theorem 1.1 *The cohomology of the complement of $\mathcal{A}$, $H^*(M)$, is generated by the cohomology classes of the differential forms $\frac{d\alpha_j|_M}{\alpha_j|_M}$ for $j = 1, \ldots, s$.*

In their proof Jozsa and Rice use de Rham cohomology with complex coefficients. However, their arguments are independent of this interpretation and apply in any cohomology theory once you replace $\frac{d\alpha_j|_M}{\alpha_j|_M}$ with $(\alpha_j|_M)^*(\sigma)$ for a non-zero element σ of $H^1(\mathbb{C}^*)$ where $\mathbb{C}^*$ represents the set of non-zero complex numbers.

Let $T = (\mathbb{C}^*)^l$ be a complex torus and let $\mathcal{A}' = \{\ker(\chi_1'), \ldots, \ker(\chi_s')\}$ be a finite set of kernels of rational characters of T. Recall that the pair $(T, \mathcal{A}')$ is called a *toral arrangement over* $\mathbb{C}$ [Dou]. We define the *complement of the arrangement* to be

$$M' = T \setminus \bigcup_{i=1}^{s} \ker(\chi_i').$$

The objective of this work is to prove the analog of Theorem 1.1 for toral arrangements. First we prove an extension of Jozsa and Rice's result that holds in any cohomology with field coefficients, and then we prove an analogous theorem for toral arrangements.

In Chapter 2 we emulate the method of Jozsa and Rice to extend Theorem 1.1 as follows. Consider a cohomology theory with coefficients in k that satisfies the Künneth Formula and has a ring structure that makes the cohomology groups into a skew-commutative graded k-algebra. For $q = 0, 1$, we know that $H^q(\mathbb{C}^*)$ is isomorphic to $\mathbb{C}$ as a k-vector space [Spa, Theorem 4.6.6]. We denote a non-zero element of $H^0(\mathbb{C}^*)$ by c_1 and a non-zero element of $H^1(\mathbb{C}^*)$ by σ. If X is a smooth real manifold, $f_1, \ldots, f_r$ are smooth maps from X to $\mathbb{C}$, and $Z_i = f_i^{-1}(0)$, we analyze the cohomology of the complements $M_j = X \setminus (\bigcup_{i=1}^{j} Z_i)$ for $1 \le j \le r$. The goal of Chapter 2 is a proof of the following theorem.

Theorem 1.2 *Assume that*

(1) $Z_r \not\subseteq \bigcup_{i=1}^{r-1} Z_i$,

(2) *0 is a regular value of* $f_r|_{M_{r-1}}$,

(3) *as a k-algebra $H^*(M_{r-1} \cap Z_r)$ is generated by the set*

$$\{(f_1|_{(M_{r-1}\cap Z_r)})^*(\sigma), \dots, (f_{r-1}|_{(M_{r-1}\cap Z_r)})^*(\sigma)\},$$

(4) *as a k-algebra $H^*(M_{r-1})$ is generated by the set*

$$\{(f_1|_{M_{r-1}})^*(\sigma), \dots, (f_{r-1}|_{M_{r-1}})^*(\sigma)\}.$$

Then as a k-algebra $H^*(M_r)$ is generated by the set $\{(f_1|_{M_r})^*(\sigma), \dots, (f_r|_{M_r})^*(\sigma)\}$.

In Chapter 3 we consider real and complex tangent spaces, their properties, and how they are related. We see that complex tangent space is simply the complexification of real tangent space, and then we use this relationship to define the complex differential $d_{\mathbb{C}}$. This leads to the definitions of real and complex q-forms and de Rham cohomology with real and complex coefficients. The conclusion of this chapter is the proof that de Rham cohomology with complex coefficients satisfies the Eilenberg-Steenrod Axioms of cohomology and the particular properties of a cohomology theory that we use to prove Theorem 1.2.

In order to apply Theorem 1.2 to toral arrangements, we need to know that 0 is a regular value of holomorphic functions defined on $\mathbb{C}^l$. We address this problem in Chapter 4. Showing that 0 is a regular value requires that the real differential of a holomorphic function be surjective which, in turn, involves taking partial derivatives with respect to the real coordinate functions. Since the function is defined on $\mathbb{C}^l$ it is much easier to take partial derivatives with respect to the complex coordinate functions. We define holomorphic tangent space and the holomorphic differential and then use them to show that for a holomorphic function the real differential is surjective if and only if the complex differential is surjective.

In Chapter 5 we finally address the application of finding the generators of the cohomology of the complement of a toral arrangement, where the cohomology is de Rham cohomology with complex coefficients. For $1 \leq k \leq s$ consider the irreducible factors of $\chi'_k - 1$. From all the irreducible factors, for all k, let $\chi_1 - \mu_1, \ldots, \chi_s - \mu_s$ be the distinct irreducible factors, where χ_i is a character of T with connected kernel and μ_i is a n^{th} root of unity for some n. We show that the de Rham cohomology of the complement of $\mathcal{A}'$, denoted by $H^*_{DR}(M')$, is generated as a $\mathbb{C}$-algebra by the set $\{[\frac{d_{\mathbb{C}} z_1}{z_1}], \ldots, [\frac{d_{\mathbb{C}} z_l}{z_l}], [\frac{d_{\mathbb{C}} \chi_1}{\chi_1 - \mu_1}], \ldots, [\frac{d_{\mathbb{C}} \chi_s}{\chi_s - \mu_s}]\}$ if, for all $l+1 \leq r \leq l+s$, $H^*_{DR}(M_{r-1} \cap Z_r)$ is generated as a $\mathbb{C}$-algebra by the set

$$\left\{ \left[\frac{d_{\mathbb{C}} z_1|_{(M_{r-1} \cap Z_r)}}{z_1|_{(M_{r-1} \cap Z_r)}} \right], \ldots, \left[\frac{d_{\mathbb{C}} z_l|_{(M_{r-1} \cap Z_r)}}{z_l|_{(M_{r-1} \cap Z_r)}} \right], \left[\frac{d_{\mathbb{C}} \chi_1|_{(M_{r-1} \cap Z_r)}}{(\chi_1 - \mu_1)|_{(M_{r-1} \cap Z_r)}} \right], \ldots, \left[\frac{d_{\mathbb{C}} \chi_{r-l}|_{(M_{r-1} \cap Z_r)}}{(\chi_{r-l} - \mu_{r-l})|_{(M_{r-1} \cap Z_r)}} \right] \right\}.$$

Unless otherwise noted the canonical references for this paper are [BT], [Spa], [Wa]. In Chapter 2 we will continue to use k to represent any field. We also use the real numbers, denoted by $\mathbb{R}$, and the complex numbers, denoted by $\mathbb{C}$. When either $\mathbb{R}$ or $\mathbb{C}$ is acceptable we will use the notation $\mathbb{F}$.

CHAPTER 2

EXTENSION OF BRIESKORN'S LEMMA

Before we state and prove the extension of Theorem 1.1 we first review the axioms of a cohomology theory and recall the definition of a tangent space of a manifold.

2.1 Axiomatic Cohomology Theory

Recall that a pair of topological spaces (X, A) is a topological space X and a subspace A of X. If $A = \emptyset$, then $(X, \emptyset)$ is usually abbreviated by X. A cohomology theory with coefficients in k on a category of topological pairs is a collection of three functions as follows. For each pair (X, A) of topological spaces, $H^*(X, A) = \{H^q(X, A)\}$ is a graded k-vector space. The function f^* is defined for each map $f : (X, A) \to (Y, B)$ of topological spaces, and its value is a homomorphism of graded vector spaces $f^* : H^*(Y, B) \to H^*(X, A)$. The third function is the coboundary operator, a linear transformation $\delta^* : H^q(A) \to H^{q+1}(X, A)$. These three functions satisfy the following properties known as the Eilenberg-Steenrod axioms [ES, Section I.3c].

(1) If f is the identity, then f^* is the identity.

(2) $(g \circ f)^* = f^* \circ g^*$

(3) **Naturality Axiom:** If $f : (X, A) \to (Y, B)$ is a map of topological spaces and

$f|_A : A \to B$, then the following diagram commutes

$$\begin{array}{ccc} H^{q+1}(Y,B) & \xrightarrow{f^*} & H^{q+1}(X,A) \\ \uparrow \delta^* & & \uparrow \delta^* \\ H^q(B) & \xrightarrow{(f|_A)^*} & H^q(A). \end{array}$$

(4) **Exactness Axiom:** For any pair (X, A) with inclusion maps $i : A \to X$ and $\beta : X \to (X, A)$, there is an exact sequence

$$\cdots \to H^q(X,A) \xrightarrow{\beta^*} H^q(X) \xrightarrow{i^*} H^q(A) \xrightarrow{\delta^*} H^{q+1}(X,A) \to \cdots. \qquad (2.1)$$

(5) **Homotopy Axiom:** If (X, A) and (Y, B) are pairs of topological spaces and $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then $f_0^* \equiv f_1^* : H^*(Y, B) \to H^*(X, A)$.

(6) **Excision Axiom:** For any pair (X, A), if W is an open subset of X such that the closure of W is contained in the interior of A, then the inclusion map $j : (X \setminus W, A \setminus W) \to (X, A)$ induces an isomorphism

$$j^* : H^*(X, A) \to H^*(X \setminus W, A \setminus W). \qquad (2.2)$$

(7) **Dimension Axiom:** If X is a one point space, then $H^q(X) = 0$ for $q \neq 0$ and $H^0(X) \cong k$.

When $i : A \to X$ is an inclusion, we will use the notation $H^*(i) = H^*(X, A)$.

A consequence of these axioms is an understanding of the cohomology of $\mathbb{C}$ and $\mathbb{C}^*$. Cohomology is homotopy invariant by the Homotopy Axiom and $\mathbb{C}$ is homotopic to a point; thus, by the Dimension Axiom the cohomology of $\mathbb{C}$ is k in dimension 0 and 0 in all other dimensions. A fixed generator of $H^0(\mathbb{C})$ is labeled c_1. The axioms also imply an exact Mayer-Vietoris cohomology sequence [ES, Theorem 15.3c]. Using this together with the cohomology of $\mathbb{C}$ by Theorem 4.6.6 in [Spa] we have that the cohomology of $\mathbb{C}^*$ is

$$H^q(\mathbb{C}^*) \cong \begin{cases} k & \text{if } q = 0, 1 \\ 0 & \text{if } q > 1. \end{cases}$$

We label a generator of $H^q(\mathbb{C}^*)$ by c_1 in degree 0 and by σ in degree 1.

For the proofs of this section we need the cohomology to have two additional attributes. First, it must satisfy the Künneth Formula as follows.

Formula 2.1 Künneth Formula *Suppose X and Y are topological spaces. Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be projections. There exists a unique map $K : H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ satisfying $K(\omega \otimes \phi) = p_1^*(\omega) \cup p_2^*(\phi)$. The map K is a graded k-algebra isomorphism. In particular, for all n*

$$K : \bigoplus_{p+q=n} H^p(X) \otimes H^q(Y) \to H^n(X \times Y),$$

is an isomorphism.

Secondly, we require that the cohomology has a ring structure that makes the cohomology groups into a skew-commutative graded k-algebra. When multiplying two elements of the cohomology we either use the notation $a \cup b$ or we leave the symbol out and denote it by ab if the meaning is clear. From now on we assume that we are working with cohomology that has coefficients in k and satisfies these two conditions.

There are many examples of cohomology theories that satisfy the Eilenberg-Steenrod axioms as well as the two additional properties given above. One example is singular cohomology theory on the category of topological pairs [Spa]. In Chapter 3 we will show that de Rham cohomology on the category of pairs consisting of smooth manifolds and submanifolds is another example.

2.2 Tangent Spaces

Let X be a real smooth manifold. For an open subset U of X, let $\mathcal{C}^\infty_{\mathbb{F}}(U)$ denote the ring of smooth functions from U to $\mathbb{F}$. Consider m an element of X. Let U and V be open subsets of X containing m. If f and g are smooth function on U and V respectively, we say that (U, f) is related to (V, g) if f and g agree on some open neighborhood of m. This is clearly an equivalence relation. Let $[U, f]$ denote the equivalence class of (U, f). Then $[U, f]$ is called a *germ of a smooth function on* X *near* m. We can add germs by $[U, f] + [V, g] = [U \cap V, f + g]$, multiply germs by $[U, f] \cdot [V, g] = [U \cap V, fg]$, and multiply by an element α in $\mathbb{F}$ by $\alpha[U, f] = [U, \alpha f]$. It is straightforward to see that these operations are well-defined. It is also easy to see that the set of germs of smooth functions on X near m forms a $\mathbb{F}$-algebra. We call it the *ring of germs of* $\mathcal{C}^\infty_{\mathbb{F}}$ *at* m and denote it by $\mathcal{C}^\infty_{\mathbb{F},m}$; that is

$$\mathcal{C}^\infty_{\mathbb{F},m} = \{[U, f] \mid m \in U \subseteq X, U \text{ open}, f \in \mathcal{C}^\infty_{\mathbb{F}}(U)\}.$$

If $[U, f]$ is in $\mathcal{C}^\infty_{\mathbb{F},m}$ we will say that f is in $\mathcal{C}^\infty_{\mathbb{F},m}$.

We can make $\mathbb{F}$ into a $\mathcal{C}^\infty_{\mathbb{F},m}$-module, denoted by $\mathbb{F}_m$, where the multiplication is defined by $[V, g] \cdot \alpha = g(m)\alpha$, for α in $\mathbb{F}$ and $[V, g]$ in $\mathcal{C}^\infty_{\mathbb{F},m}$. This is well-defined because if $[U, f]$ is equivalent to $[V, g]$, then f and g agree on an open set containing m, so $f(m) = g(m)$. Define the $\mathbb{F}$-*tangent space at* m *of* X, $T_{\mathbb{F},m}X$, to be the set of $\mathbb{F}$-linear derivations from $\mathcal{C}^\infty_{\mathbb{F},m}$ to $\mathbb{F}_m$. Thus

$$\begin{aligned} T_{\mathbb{F},m}X &= Der_{\mathbb{F}}(\mathcal{C}^\infty_{\mathbb{F},m}, \mathbb{F}_m) \\ &= \{\theta : \mathcal{C}^\infty_{\mathbb{F},m} \to \mathbb{F}_m \mid \theta(fg) = f(m)\theta(g) + g(m)\theta(f) \forall f, g \in \mathcal{C}^\infty_{\mathbb{F},m}\}. \end{aligned}$$

When the field $\mathbb{F}$ is the reals we denote the tangent space simply by T_mX. Define $(\theta_1 + \theta_2)(f) = \theta_1(f) + \theta_2(f)$ and $(\alpha\theta)(f) = \alpha(\theta(f))$ for θ_1, θ_2 in $T_{\mathbb{F},m}X$ and α in $\mathbb{F}$. It is easy to see that these definitions do not depend on the choice of equivalence class of the representative f. Then by following through the definitions we see that $\theta_1 + \theta_2$ and $\alpha\theta$ are again tangent vectors at m. In this way $T_{\mathbb{F},m}X$ is a $\mathbb{F}$-vector space.

When $\mathbb{F} = \mathbb{R}$, recall that the differential of a smooth map at m in X is a map on real tangent spaces. Suppose Y is a smooth real manifold and $\phi : X \to Y$ is a smooth mapping. Then the *differential of* ϕ *at* m is the map $(d\phi)_m : T_mX \to T_{\phi(m)}Y$ defined by $((d\phi)_m(\theta))(g) = \theta(g \circ \phi)$ for θ in T_mX and g in $\mathcal{C}^\infty_{\mathbb{R},\phi(m)}$.

2.3 Tubular Neighborhoods

Let X and Y be real manifolds and $f : X \to Y$ be a smooth map. The next Theorem, a version of the implicit function theorem, is proven in [Wa, Theorem 1.38].

Theorem 2.2 *If p is a point of Y such that $f^{-1}(p)$ is non-empty and the map of tangent spaces $(df)_x : T_xX \to T_{f(x)}Y$ is surjective for all x in $f^{-1}(p)$, then $f^{-1}(p)$ is a submanifold of X. Moreover, the real dimension of $f^{-1}(p)$ is $\dim X - \dim Y$.*

Given p an element of Y, we say that *p is a regular value of f* if the map of tangent spaces $(df)_x : T_xX \to T_{f(x)}Y$ is surjective for all x in $f^{-1}(p)$. Theorem 2.2 implies that when p is a regular value of f, $f^{-1}(p)$ is a submanifold of X of real dimension $\dim X - \dim Y$. A fact about regular values that we will find useful is given in the next lemma.

Lemma 2.3 *If p in Y is a regular value of f and V is an open subset of X, then p is a regular value of $f|_V$.*

Proof. The fact that V is open in X yields that for all x in V, the tangent spaces at x are equal; i.e. $T_xV = T_xX$. By assumption we have that $(df)_x : T_xX \to T_{f(x)}Y$ is surjective for all x in $f^{-1}(p)$. So $(df|_V)_x : T_xV \to T_{f(x)}Y$ is surjective for all x in $(f|_V)^{-1}(p)$. ∎

Let N be a n-dimensional submanifold of X. Recall that a *tubular neighborhood* of N is defined to be a subset U of X which has the structure of a $(l-n)$-dimensional real vector bundle over N with N as the zero section [Ko]. Now assume that $f : X \to \mathbb{C}$ is a smooth map and 0 is a regular value of f. Let $Z = f^{-1}(0)$. It follows from Theorem 2.2, that Z is a closed submanifold of X. Let $n = \dim X - \dim Z$. We will consider a tubular neighborhood of Z.

Before stating the next result we establish some notation. If $g : A \to B$ and $h : A \to C$ then the product map, $g \times h : A \to B \times C$, is the map that sends an element a in A to the element $(g(a), h(a))$ in $B \times C$.

Proposition 2.4 *There is a tubular neighborhood U of Z and a smooth projection map, $\pi : U \to Z$, such that $(\pi \times f|_U)^* : H^q(Z \times \mathbb{C}) \to H^q(U)$ is an isomorphism.*

Proof. Since Z is closed submanifold the existence of a tubular neighborhood is guaranteed [Ko, Corollary 2.3]. Let U be a tubular neighborhood of Z. Then

U has the structure of a real vector bundle over Z and so there exists a smooth projection map $\pi : U \to Z$. This implies that Z is a deformation retract of U with retraction π, and so π is a homotopy equivalence with inverse i where $i : Z \to U$ is the inclusion map. Now we have the composition $(\pi \times f|_U) \circ i : Z \to Z \times \mathbb{C}$ with image $Z \times \{0\}$. If we compose this map with the restriction map $\rho : Z \times \mathbb{C} \to Z \times \{0\}$ defined by $\rho(z, \alpha) = (z, 0)$ for α in $\mathbb{C}$, we have a diffeomorphism, and thus a homotopy equivalence, from Z to $Z \times \{0\}$. Since $\mathbb{C}$ is homotopic to $\{0\}$ and Z is homotopic to itself, we also have that the product map $j : Z \times \{0\} \to Z \times \mathbb{C}$ is a homotopy equivalence. Combining this with the previous composition we have

$$(\pi \times f|_U) \circ i = j \circ \rho \circ (\pi \times f|_U) \circ i : Z \to Z \times \mathbb{C}$$

is a homotopy equivalence. By the Homotopy Axiom, it follows that the map on cohomology $((\pi \times f|_U) \circ i)^*$ is an isomorphism. The Homotopy Axiom also gives that π^* is an isomorphism with inverse i^*. Thus composing these two isomorphisms we have

$$\pi^* \circ ((\pi \times f|_U) \circ i)^* = \pi^* \circ i^* \circ (\pi \times f|_U)^* = (\pi \times f|_U)^*,$$

which is also an isomorphism. ∎

From Theorem 2.2 and the definition of a tubular neighborhood, we see that U has the structure of a 2-dimensional real vector bundle over Z. Thus as real vector spaces the fibers over Z are isomorphic to $\mathbb{C}$. We use this result in the next corollary.

Corollary 2.5 *When we restrict the product map $\pi \times f|_U$ to $U \setminus Z$, the map*

$$(\pi|_{U \setminus Z} \times f|_{U \setminus Z})^* : H^q(\mathbb{C}^*) \to H^q(U \setminus Z) \text{ is an isomorphism.}$$

Proof. Let $\widetilde{\pi} = \pi|_{U\setminus Z}$ and $\widetilde{f} = f|_{U\setminus Z}$. Then $\widetilde{\pi} : U \setminus Z \to Z$ is a fiber bundle, and the fibers are $\mathbb{C}^*$ because by definition of tubular neighborhood Z is the zero section of the vector bundle. Additionally we have the natural projection $p : Z \times \mathbb{C}^* \to Z$ which is also a bundle with fiber $\mathbb{C}^*$. We have the commutative diagram connecting these two bundles:

$$\begin{array}{ccc} U \setminus Z & \xrightarrow{\widetilde{\pi}\times\widetilde{f}} & Z \times \mathbb{C}^* \\ \big\downarrow \widetilde{\pi} & & \big\downarrow p \\ Z & \xrightarrow{\text{id}} & Z. \end{array}$$

Corollary 2.7.14 of [Spa] states that a fiber bundle is a fibration. By the results of Section 7.2 in [Spa] we see that a fibration is a weak fibration and that weak fibrations have long exact homotopy sequences where the $\overline{\partial}$ map is a natural transformation. Thus the vertical maps in the diagram each give rise to long exact homotopy sequences. Let z be in Z and let α be an element of the fiber $\mathbb{C}^*$. We have the following commutative diagram of long exact homotopy sequences

$$\begin{array}{ccccccccc} \cdots \to & \pi_n(\mathbb{C}^*, \alpha) & \overset{i'_\#}{\to} & \pi_n(U \setminus Z, \alpha) & \overset{\widetilde{\pi}_\#}{\to} & \pi_n(Z, z) & \overset{\bar{\partial}'}{\to} & \pi_{n-1}(\mathbb{C}^*, \alpha) & \to \cdots \\ & \big\downarrow = & & \big\downarrow & & \big\downarrow = & & \big\downarrow = & \\ \cdots \to & \pi_n(\mathbb{C}^*, \alpha) & \overset{i''_\#}{\to} & \pi_n(Z \times \mathbb{C}^*, \alpha) & \overset{p_\#}{\to} & \pi_n(Z, z) & \overset{\bar{\partial}''}{\to} & \pi_{n-1}(\mathbb{C}^*, \alpha) & \to \cdots, \end{array}$$

where $i' : (\mathbb{C}^*, \alpha) \to (U \setminus Z, \alpha)$ and $i'' : (\mathbb{C}^*, \alpha) \to (Z \times \mathbb{C}^*, \alpha)$ are inclusions. By the Five Lemma $U \setminus Z$ and $Z \times \mathbb{C}^*$ have isomorphic homotopy groups. The map $\widetilde{\pi} \times \widetilde{f} : U \setminus Z \to Z \times \mathbb{C}^*$ is a weak homotopy equivalence [Spa, Section 7.6].

An application of Morse Theory, Theorem 6.6 in [Mi], shows that a smooth manifold is homotopy equivalent to a CW complex. The sets $U \setminus Z$ and $Z \times \mathbb{C}^*$ are manifolds and therefore homotopy equivalent to CW complexes. Moreover, a weak homotopy equivalence between CW complexes is in fact a homotopy equivalence [Spa, Corollary 7.6.24]. Therefore, $\widetilde{\pi} \times \widetilde{f}$ is a homotopy equivalence and by the Homotopy Axiom the corresponding map in cohomology is an isomorphism. ∎

2.4 Extension of Theorem 1.1

Let X be a real smooth l-dimensional manifold, let $f_1, \ldots, f_r$ be smooth maps from X to $\mathbb{C}$, and let $Z_i = f_i^{-1}(0)$. Let $M = X \setminus (Z_1 \bigcup \ldots \bigcup Z_{r-1})$ and suppose that 0 is a regular value of $f_r|_M$. Then $(f_r|_M)^{-1}(0) = M \cap Z_r$. Proposition 2.4 and Corollary 2.5 yield the fact that $M \cap Z_r$ has a tubular neighborhood U in M such that $(\pi \times f)^*$ and $(\widetilde{\pi \times f})^* = (\pi|_{U \setminus Z_r} \times f_r|_{U \setminus Z_r})^*$ are isomorphisms, where π is the natural projection map from U into $M \cap Z_r$ and $f = f_r|_U$.

Consider the inclusions maps

$$i_U : U \setminus Z_r \to U \text{ and } i = \text{id} \times i_1 : (M \cap Z_r) \times \mathbb{C}^* \to (M \cap Z_r) \times \mathbb{C},$$

where $\text{id} : M \cap Z_r \to M \cap Z_r$ is the identity map and $i_1 : \mathbb{C}^* \to \mathbb{C}$ is the inclusion map. Notice that for i_U and i we have the commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\pi \times f} & (M \cap Z_r) \times \mathbb{C} \\ \uparrow i_U & & \uparrow i \\ U \setminus Z_r & \xrightarrow{\widetilde{\pi \times f}} & (M \cap Z_r) \times \mathbb{C}^*. \end{array}$$

Furthermore, by the Exactness Axiom of cohomology, each of the horizontal inclusion maps induces a long exact sequence in cohomology. Using the fact that $(\pi \times f)^*$ and $(\widetilde{\pi \times f})^*$ are isomorphisms, together with the Five Lemma, we conclude that these sequences are isomorphic. Accordingly, by functorality and the Naturality Axiom we have the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\delta_U^*} & H^q(i_U) & \xrightarrow{\beta_U^*} & H^q(U) & \xrightarrow{i_U^*} & H^q(U \setminus Z_r) & \xrightarrow{\delta_U^*} & \cdots \\ & & \downarrow\cong & & \uparrow(\pi \times f)^* & & \uparrow(\widetilde{\pi \times f})^* & & \\ \cdots & \xrightarrow{\delta^*} & H^q(i) & \xrightarrow{\beta^*} & H^q((M \cap Z_r) \times \mathbb{C}) & \xrightarrow{i^*} & H^q((M \cap Z_r) \times \mathbb{C}^*) & \xrightarrow{\delta^*} & \cdots. \end{array} \tag{2.3}$$

Our goal is to show that these long exact sequences split into short exact sequences. First we consider the bottom row of the diagram. We have two natural projection maps from $(M \cap Z_r) \times \mathbb{C}$. The projection onto $M \cap Z_r$ is denoted by p_1 and the projection onto $\mathbb{C}$ is denoted by p_2. Associated with these maps are the maps on cohomology,

$$p_1^* : H^*(M \cap Z_r) \to H^*((M \cap Z_r) \times \mathbb{C}) \quad \text{and} \quad p_2^* : H^*(\mathbb{C}) \to H^*((M \cap Z_r) \times \mathbb{C}).$$

Lemma 2.6 *The map $p_1^* : H^q(M \cap Z_r) \to H^q((M \cap Z_r) \times \mathbb{C})$ is an isomorphism.*

Proof. We know that $H^q(\mathbb{C}) = 0$ for $q \geq 1$. Thus $\bigoplus_{m+n=q} H^m(M \cap Z_r) \otimes H^n(\mathbb{C})$ reduces to $H^q(M \cap Z_r) \otimes H^0(\mathbb{C})$. The Künneth Formula 2.1 gives the isomorphism $K : H^q(M \cap Z_r) \otimes H^0(\mathbb{C}) \to H^q(M \cap Z_r \times \mathbb{C})$. The following diagram relates K and p_1^*:

$$\begin{array}{ccc} H^q(M \cap Z_r) \otimes H^0(\mathbb{C}) & \xrightarrow{K} & H^q(M \cap Z_r \times \mathbb{C}) \\ \nu\downarrow & \nearrow p_1^* & \\ H^q(M \cap Z_r), & & \end{array}$$

where ν is the natural isomorphism between $H^q(M \cap Z_r) \otimes H^0(\mathbb{C})$ and $H^q(M \cap Z_r)$. Since p_2^* is a k-algebra homomorphism, $p_2^*(c_1)$ is the identity in $H^q(M \cap Z_r \times \mathbb{C})$. So $K(\omega \otimes c_1) = p_1^*(\omega) \cup p_2^*(c_1) = p_1^*(\omega)$. Therefore, the diagram commutes and so p_1^* is an isomorphism. ∎

Using the result that p_1^* is an isomorphism yields the fact the cohomology map $\pi^* : H^q(M \cap Z_r) \longrightarrow H^q(U)$ is also an isomorphism. We will interrupt the proof that the bottom row of Diagram 2.3 is exact to prove this fact about π^*.

Lemma 2.7 *The map* $\pi^* = (\pi \times f)^* \circ p_1^* : H^q(M \cap Z_r) \to H^q(U)$ *is an isomorphism.*

Proof. Notice that the following diagram commutes

$$\begin{array}{ccc} U & \stackrel{\pi \times f}{\longrightarrow} & (M \cap Z_r) \times \mathbb{C} \\ \pi \downarrow & \swarrow p_1 & \\ M \cap Z_r . & & \end{array}$$

Thus the corresponding diagram with the induced cohomology maps also commutes. We have the following commutative diagram

$$\begin{array}{ccc} H^*(U) & \stackrel{(\pi \times f)^*}{\longleftarrow} & H^*((M \cap Z_r) \times \mathbb{C}) \\ \pi^* \uparrow & \nearrow p_1^* & \\ H^*(M \cap Z_r) . & & \end{array}$$

By Proposition 2.4 and Lemma 2.6 we know that $(\pi \times f)^*$ and p_1^* are isomorphisms, we conclude that $\pi^* = (\pi \times f)^* \circ p_1^*$ is an isomorphism. ∎

Now consider $H^q((M \cap Z_r) \times \mathbb{C}^*)$. We know that $H^q(\mathbb{C}^*) = 0$ for $q \geq 2$. Thus $\bigoplus_{m+n=q} H^m(M \cap Z_r) \otimes H^n(\mathbb{C}^*)$ reduces to

$$\left(H^q(M \cap Z_r) \otimes H^0(\mathbb{C}^*)\right) \oplus \left(H^{q-1}(M \cap Z_r) \otimes H^1(\mathbb{C}^*)\right).$$

The Künneth Formula 2.1 gives the isomorphism

$$K : \left(H^q(M \cap Z_r) \otimes H^0(\mathbb{C}^*)\right) \oplus \left(H^{q-1}(M \cap Z_r) \otimes H^1(\mathbb{C}^*)\right) \to H^q((M \cap Z_r) \times \mathbb{C}^*).$$

As in Lemma 2.6 we also have the natural isomorphism

$$\nu : H^q(M \cap Z_r) \otimes H^0(\mathbb{C}^*) \to H^q(M \cap Z_r).$$

It follows that the map

$$\begin{aligned} \nu^{-1} \oplus \mathrm{id} \quad : \quad & H^q(M \cap Z_r) \oplus \Big(H^{q-1}(M \cap Z_r) \otimes H^1(\mathbb{C}^*)\Big) \\ & \longrightarrow \Big(H^q(M \cap Z_r) \otimes H^0(\mathbb{C}^*)\Big) \oplus \Big(H^{q-1}(M \cap Z_r) \otimes H^1(\mathbb{C}^*)\Big) \end{aligned}$$

is an isomorphism. Let $P = K \circ (\nu^{-1} \oplus \mathrm{id})$. Thus we have proved that P is an isomorphism as stated in the next Lemma. In addition we will consider how elements are mapped under P. We use the notation $\tilde{p_1}$ to denote the projection map onto the first coordinate of $(M \cap Z_r) \times \mathbb{C}^*$ and $\tilde{p_2}$ to denote the projection map onto the second coordinate.

Lemma 2.8 *The map*

$$P : H^q(M \cap Z_r) \oplus \Big(H^{q-1}(M \cap Z_r) \otimes H^1(\mathbb{C}^*)\Big) \to H^q((M \cap Z_r) \times \mathbb{C}^*)$$

is an isomorphism. In particular, $H^q((M \cap Z_r) \times \mathbb{C}^*)$ *is spanned as a* k*-vector space by* $\{\tilde{p_1}^*(\omega) \mid \omega \in H^q(M \cap Z_r)\} \bigcup \{\tilde{p_1}^*(\theta) \cup \tilde{p_2}^*(\sigma) \mid \theta \in H^{q-1}(M \cap Z_r)\}$.

Proof. We saw that P is an isomorphism by the proceeding discussion. If ω is in $H^q(M \cap Z_r)$ and θ is in $H^{q-1}(M \cap Z_r)$, then

$$\begin{aligned} P(\omega, \theta \otimes \sigma) &= K \circ (\nu^{-1} \oplus \mathrm{id})(\omega, \theta \otimes \sigma) = K(\omega \otimes c_1, \theta \otimes \sigma) \\ &= \tilde{p_1}^*(\omega) \cup \tilde{p_2}^*(c_1) + \tilde{p_1}^*(\theta) \cup \tilde{p_2}^*(\sigma) = \tilde{p_1}^*(\omega) + \tilde{p_1}^*(\theta) \cup \tilde{p_2}^*(\sigma). \end{aligned}$$

Where the last equality holds since $\tilde{p_2}^*$ is a k-algebra homomorphism and so $\tilde{p_2}^*(c_1)$ is the identity in $H^q(M \cap Z_r \times \mathbb{C}^*)$. Thus we have the desired result. ∎

Now that we have described $H^q((M \cap Z_r) \times \mathbb{C})$ and $H^q((M \cap Z_r) \times \mathbb{C}^*)$ we can explore the map between them. Let $i = \mathrm{id} \times i_1 : (M \cap Z_r) \times \mathbb{C}^* \to (M \cap Z_r) \times \mathbb{C}$, where id is the identity map on $M \cap Z_r$ and i_1 is the inclusion from $\mathbb{C}^*$ into $\mathbb{C}$.

Lemma 2.9 *The cohomology map* $i^* : H^q((M \cap Z_r) \times \mathbb{C}) \to H^q((M \cap Z_r) \times \mathbb{C}^*)$ *is injective.*

Proof. Recall the projection map $p_1 : (M \cap Z_r) \times \mathbb{C} \to M \cap Z_r$. Also note that the composition $p_1 \circ i$ is the same map as the projection $\tilde{p_1} : (M \cap Z_r) \times \mathbb{C}^* \to M \cap Z_r$. If ω is in $H^q(M \cap Z_r)$ then $i^* \circ p_1^*(\omega) = (p_1 \circ i)^*(\omega) = \tilde{p_1}^*(\omega)$. Using this together with the isomorphisms given in Lemmas 2.6 and 2.8, we have that the following diagram commutes,

$$\begin{array}{ccc} H^q((M \cap Z_r) \times \mathbb{C}) & \xrightarrow{i^*} & H^q((M \cap Z_r) \times \mathbb{C}^*) \\ \uparrow p_1^* & & \uparrow P \\ H^q(M \cap Z_r) & \xrightarrow{I} & H^q(M \cap Z_r) \oplus (H^{q-1}(M \cap Z_r) \otimes H^1(\mathbb{C}^*)), \end{array}$$

where the map on the bottom row is the injection $I(\omega) = (\omega, 0)$. This diagram commutes since $i^* \circ p_1^*(\omega) = \tilde{p_1}^*(\omega)$ and

$$\begin{aligned} P \circ I(\omega) &= P(\omega, 0) = K \circ (\nu^{-1} \otimes \mathrm{id})(\omega, 0) \\ &= K(\omega \otimes c_1, 0) = \tilde{p_1}^*(\omega) \otimes \tilde{p_2}^*(c_1) = \tilde{p_1}^*(\omega). \end{aligned}$$

Thus the map i^* is injective since the two vertical maps are isomorphisms and the bottom map is an injection. ∎

We will use Lemmas 2.6 and 2.8 to show that the bottom row of the long exact sequence from Diagram 2.3 breaks into short exact sequences. Since the two rows are isomorphic this results in the top row splitting into short exact sequences.

Proposition 2.10 *The top row of Diagram 2.3 splits into short exact sequences of the form*

$$0 \to H^q(U) \xrightarrow{i_U^*} H^q(U \setminus Z_r) \xrightarrow{\delta_U^*} H^{q+1}(i_U) \to 0.$$

Proof. By Lemma 2.9, the map i^* (from the bottom row of Diagram 2.3) is injective, whence $\ker(i^*) = 0$. Since the sequence is exact we have that $\mathrm{im}(\beta^*) = \ker(i^*) = 0$, and so $\beta^* = 0$. It follows that $\ker(\beta^*) = H^{q+1}(i)$, and by exactness we have $\mathrm{im}(\delta^*) = \ker(\beta^*)$. Thus $\mathrm{im}(\delta^*) = H^{q+1}(i)$ and δ^* is surjective. This yields the fact that the bottom row breaks into short exact sequences. Moreover, we saw in Diagram 2.3 that the two rows are isomorphic, hence the top row also splits into short exact sequences. ∎

As a consequence of Proposition 2.10 we have that Diagram 2.3 breaks into a commutative diagram of short exact sequences. We can use this to investigate the make-up of $H^q(U)$ and $H^q(U \setminus Z_r)$.

Proposition 2.11 $H^q(U) = \{\pi^*(\omega) \mid \omega \in H^q(M \cap Z_r)\}$ *and* $H^q(U \setminus Z_r)$ *is spanned by* $\{\tilde{\pi}^*(\omega) \mid \omega \in H^q(M \cap Z_r)\} \bigcup \{\tilde{\pi}^*(\theta) \cup \tilde{f}^*(\sigma) \mid \theta \in H^{q-1}(M \cap Z_r)\}$.

Proof. We have the commutative diagram

$$\begin{array}{ccc}
H^q(U) & \xrightarrow{i_U^*} & H^q(U \setminus Z_r) \\
\big\uparrow (\pi \times f)^* & & \big\uparrow (\widetilde{\pi \times f})^* \\
H^q((M \cap Z_r) \times \mathbb{C}) & \xrightarrow{i^*} & H^q((M \cap Z_r) \times \mathbb{C}^*) \\
\big\uparrow p_1^* & & \big\uparrow P \\
H^q(M \cap Z_r) & \xrightarrow{I} & H^q(M \cap Z_r) \oplus (H^{q-1}(M \cap Z_r) \otimes H^1(\mathbb{C}^*)).
\end{array}$$

From Lemma 2.6 we have that

$$H^q((M \cap Z_r) \times \mathbb{C}) = \{p_1^*(\omega) \mid \omega \in H^q(M \cap Z_r)\}.$$

Now using Lemma 2.7, we obtain that $(\pi \times f)^* p_1^*(\omega) = \pi^*(\omega)$ is the image of $p_1^*(\omega)$ under this isomorphism. Hence, $H^q(U) = \{\pi^*(\omega) \mid \omega \in H^q(M \cap Z_r)\}$.

From Lemma 2.8 we have that $H^q((M \cap Z_r) \times \mathbb{C}^*)$ is spanned by

$$\{\tilde{p_1}^*(\omega) \mid \omega \in H^q(M \cap Z_r)\} \bigcup \{\tilde{p_1}^*(\theta) \cup \tilde{p_2}^*(\sigma) \mid \theta \in H^{q-1}(M \cap Z_r)\}.$$

Using the facts that $\tilde{p_1}$ is projection on the first coordinate, $\tilde{p_2}$ is projection on the second coordinate, and $\widetilde{\pi \times f} = \tilde{\pi} \times \tilde{f}$, we have the following results. First,

$$(\widetilde{\pi \times f})^* \left(\tilde{p_1}^*(\omega)\right) = (\tilde{p_1} \circ (\widetilde{\pi \times f}))^*(\omega) = \tilde{\pi}^*(\omega).$$

Second,

$$\begin{aligned} (\widetilde{\pi \times f})^* \left(\tilde{p_1}^*(\theta) \cup \tilde{p_2}^*(\sigma)\right) &= (\tilde{p_1} \circ (\widetilde{\pi \times f}))^*(\theta) \cup (\tilde{p_2} \circ (\widetilde{\pi \times f}))^*(\sigma) \\ &= \tilde{\pi}^*(\theta) \cup \tilde{f}^*(\sigma). \end{aligned}$$

Therefore, $H^q(U \setminus Z_r)$ is spanned by

$$\{\tilde{\pi}^*(\omega) \mid \omega \in H^q(M \cap Z_r)\} \cup \{\tilde{\pi}^*(\theta) \cup \tilde{f}^*(\sigma) \mid \theta \in H^{q-1}(M \cap Z_r)\}. \qquad \blacksquare$$

Before we continue, we shall require a deeper understanding of the action of $\tilde{\pi}^*$. We have the commutative diagram

$$\begin{array}{ccc} U \setminus Z_r & \xrightarrow{i_U} & U \\ \tilde{\pi}\downarrow & & \swarrow \pi \\ M \cap Z_r & & \end{array}$$

Thus,

$$\tilde{\pi}^* = i_U^* \circ \pi^*. \tag{2.4}$$

Now consider the inclusion map $i_M : M \setminus Z_r \to M$ and the corresponding long exact sequence,

$$\cdots \to H^q(i_M) \xrightarrow{\beta_M^*} H^q(M) \xrightarrow{i_M^*} H^q(M \setminus Z_r) \xrightarrow{\delta_M^*} H^{q+1}(i_M) \to \cdots.$$

Let j be the inclusion from U into M and $\tilde{j}$ be the restriction of j to $U \setminus Z_r$, that is $\tilde{j} : U \setminus Z_r \to M \setminus Z_r$. We have the following commutative diagram:

$$\begin{array}{ccc} M \setminus Z_r & \xrightarrow{i_M} & M \\ \uparrow \tilde{j} & & \uparrow j \\ U \setminus Z_r & \xrightarrow{i_U} & U. \end{array} \tag{2.5}$$

We also have the excision isomorphism in Equation 2.2 with $X = M$, $A = M \setminus Z_r$, and $W = M \setminus U$ yielding an isomorphism between $H^*(i_M)$ and $H^*(i_U)$. Thus we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\beta_M^*} & H^q(M) & \xrightarrow{i_M^*} & H^q(M \setminus Z_r) & \xrightarrow{\delta_M^*} & H^{q+1}(i_M) & \xrightarrow{\beta_M^*} & \cdots \\ & & \downarrow j^* & & \downarrow \tilde{j}^* & & \downarrow \cong & & \\ 0 & \longrightarrow & H^q(U) & \xrightarrow{i_U^*} & H^q(U \setminus Z_r) & \xrightarrow{\delta_U^*} & H^{q+1}(i_U) & \longrightarrow & 0. \end{array} \tag{2.6}$$

In the next theorem we will see that the top row of Diagram 2.6 actually splits into short exact sequences and we will prove that $H^0(M) = k$ and if $q \geq 1$ then $H^q(M)$ is spanned as a k-algebra by $\{f_{i_1}^*(\sigma) \cdots f_{i_q}^*(\sigma) | 1 \leq i_1 < \cdots < i_q \leq r\}$. We will use the following notation in the theorem: $M_0 = X$ and $M_j = X \setminus (\bigcup_{i=1}^{j} Z_i)$ for $1 \leq j \leq r$. So, in the theorem $M = M_{r-1}$ and $M \setminus Z_r = M_r$.

Theorem 2.12 *Suppose $f_1, \ldots, f_r$ are smooth maps from X to $\mathbb{C}$ such that*

(1) $Z_r \nsubseteq \bigcup_{i=1}^{r-1} Z_i$,

(2) *0 is a regular value of* $f_r|_{M_{r-1}}$ *and*

(3) *the map* $j^* : H^*(M_{r-1}) \to H^*(U)$ *is surjective.*

Then as a k-algebra $H^*(M_r)$ *is generated by* $i_M^*(H^*(M_{r-1})) \bigcup \{(f_r|_{M_r})^*(\sigma)\}$.

Proof. Let $M = M_{r-1}$. By assumption 0 is a regular value of $f_r|_M$. Using this together with the fact that $(f_r|_M)^{-1}(0) = M \cap Z_r$, we have that the previous results in this chapter hold and we will continue to use the preceding notation. In particular we have Diagram 2.6, a commutative diagram of exact sequences. Since $M_r = M \setminus Z_r$ we will first show that the top row of this diagram breaks into short exact sequences and then use this to determine k-algebra generators of $H^*(M_r)$. To show that the top row of this diagram splits into short exact sequences it is enough to show that δ_M^* is surjective, since using the fact that the sequence is exact implies that i_M^* is injective. To show that δ_M^* is surjective it suffices to show that $\tilde{\jmath}^*$ is surjective since the diagram commutes and δ_U^* is surjective.

From Proposition 2.11, $H^q(U \setminus Z_r)$ is spanned by elements of the form $\tilde{\pi}^*(\omega)$ and $\tilde{\pi}^*(\theta) \cup (f_r|_{U \setminus Z_r})^*(\sigma)$ where ω is in $H^q(M \cap Z_r)$ and θ is in $H^{q-1}(M \cap Z_r)$. To show that $\tilde{\jmath}^*$ is surjective we shall divide the argument.

Showing first that $\tilde{\pi}^*(\omega)$ is in the image of $\tilde{\jmath}^*$. Notice that since j^* is surjective there exists a ξ in $H^k(M)$ such that $j^*(\xi) = \pi^*(\omega)$. Now observe that

$$\begin{aligned}
\tilde{\pi}^*(\omega) &= (i_U^* \circ \pi^*)(\omega) \quad \text{(by Equation 2.4)}\\
&= (i_U^* \circ j^*)(\xi)\\
&= (\tilde{\jmath}^* \circ i_M^*)(\xi) \quad \text{(by Diagram 2.5)}\\
&= \tilde{\jmath}^*(i_M^*(\xi)).
\end{aligned}$$

Thus $\tilde{\pi}^*(\omega)$ is in the image of $\tilde{\jmath}^*$. Similarly,

$$\tilde{\pi}^*(\theta) = \tilde{\jmath}^*(i_M^*(\xi')), \tag{2.7}$$

where ξ' is in $H^{q-1}(M)$.

Next we will show that $(f_r|_{U\setminus Z_r})^*(\sigma)$ is in the image of $\tilde{\jmath}^*$. We have the commutative diagram

$$\begin{array}{ccc}
U \setminus Z_r & \xrightarrow{f_r|_{U\setminus Z_r}} & \mathbb{C}^* \\
& \tilde{\jmath} \searrow & \uparrow f_r|_{M\setminus Z_r} \\
& & M \setminus Z_r.
\end{array}$$

So $\tilde{\jmath}^* \circ (f_r|_{M\setminus Z_r})^* = (f_r|_{U\setminus Z_r})^*$. Combining this with Equation 2.7 we have

$$\tilde{\pi}^*(\theta) \cup (f_r|_{U\setminus Z_r})^*(\sigma) = \tilde{\jmath}^*\left(i_M^*(\xi') \cup (f_r|_{M\setminus Z_r})^*(\sigma)\right),$$

which is an element in the image of $\tilde{\jmath}^*$. Therefore, $\tilde{\jmath}^*$ is surjective since a spanning set of $H^q(U \setminus Z_r)$ is in the image of $\tilde{\jmath}^*$.

Therefore, we have the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^q(M) & \xrightarrow{i_M^*} & H^q(M \setminus Z_r) & \xrightarrow{\delta_M^*} & H^{q+1}(i_M) & \longrightarrow & 0 \\
& & \downarrow j^* & & \downarrow \tilde{\jmath}^* & & \downarrow \cong & & \\
0 & \longrightarrow & H^q(U) & \xrightarrow{i_U^*} & H^q(U \setminus Z_r) & \xrightarrow{\delta_U^*} & H^{q+1}(i_U) & \longrightarrow & 0.
\end{array}$$

Using this exact sequence, $H^q(M \setminus Z_r)$ is spanned by the images under i_M^* of the set that spans $H^q(M)$ and a set of elements whose image under δ_M^* span $H^{q+1}(i_M)$. For the second set we use the facts that we know the spanning set of $H^q(U \setminus Z_r)$ together with the fact that $\tilde{\jmath}^*$ and δ_U^* are surjective. We arrive at the fact that $H^{q+1}(i_M)$ is spanned by δ_M^* of the elements that map to the spanning set of $H^q(U \setminus Z_r)$ under $\tilde{\jmath}^*$. In our proof that $\tilde{\jmath}^*$ is surjective, we showed that these elements are of the form

$$i_M^*(\xi) \text{ and } \left(i_M^*(\xi') \cup (f_r|_{M\setminus Z_r})^*(\sigma)\right),$$

where ξ is in $H^q(M)$ and ξ' is in $H^{q-1}(M)$. Since $M \setminus Z_r = M_r$ we have that $H^q(M_r)$ is spanned by $\{i_M^*(\xi) \mid \xi \in H^q(M)\} \bigcup \{i_M^*(\xi') \cup (f_r|_{M_r})^*(\sigma) \mid \xi' \in H^{q-1}(M)\}$, and so $H^*(M_r)$ is generated as a k-algebra by $\{i_M^*(H^*(M)) \bigcup \{(f_r|_{M_r})^*(\sigma)\}$. ∎

We have two corollaries of this Theorem.

Corollary 2.13 *Suppose $f_1, \ldots, f_r$ are smooth maps from X to $\mathbb{C}$ such that*

(1) $Z_r \not\subseteq \bigcup_{i=1}^{r-1} Z_i$,

(2) *0 is a regular value of $f_r|_{M_{r-1}}$,*

(3) *as a k-algebra $H^*(M_{r-1} \cap Z_r)$ is generated by the set*

$$\{(f_1|_{(M_{r-1}\cap Z_r)})^*(\sigma), \ldots, (f_{r-1}|_{(M_{r-1}\cap Z_r)})^*(\sigma)\},$$

(4) *as a k-algebra $H^*(M_{r-1})$ is generated by the set*

$$\{(f_1|_{M_{r-1}})^*(\sigma), \ldots, (f_{r-1}|_{M_{r-1}})^*(\sigma)\}.$$

Then as a k-algebra $H^*(M_r)$ is generated by the set $\{(f_1|_{M_r})^*(\sigma), \ldots, (f_r|_{M_r})^*(\sigma)\}$.

Proof. Observe that conditions (3) and (4) imply that $H^0(M_{r-1} \cap Z_r)$ and $H^0(M_{r-1})$ are diffeomorphic to k, and therefore $M_{r-1} \cap Z_r$ and M_{r-1} must be connected.

Again we let $M = M_{r-1}$. By the third assumption, the generators of $i_M^*(H^*(M))$ are $\{(f_1|_{M_r})^*(\sigma), \ldots, (f_{r-1}|_{M_r})^*(\sigma)\}$. Therefore, by Theorem 2.12 the results will hold if j^* is surjective.

If $r = 1$ then by definition $M_0 = X$, and by assumption $H^0(X) \cong k$ and $H^q(X) = 0$ for $q \geq 1$. Also $M_0 \cap Z_1 = Z_1$ and $H^0(Z_1) \cong k$ and $H^q(Z_1) = 0$ for $q \geq 1$. By Lemma 2.7 we have that $\pi^* : H^q(Z_1) \to H^q(U)$ is an isomorphism. Since j^* is a k-algebra homomorphism, it maps the identity of $H^0(X)$ to the identity in $H^0(U)$. It follows that j^* is an isomorphism.

Assume $r > 1$. For $i = 1, \ldots, r-1$ the intersection $Z_i \cap M = Z_i \cap \left(X \setminus (\bigcup_{i=1}^{r-1} Z_i)\right)$ is empty, implying that $f_i(M) \subseteq \mathbb{C}^*$. We will consider $f_i|_M : M \to \mathbb{C}^*$. Now using the isomorphism π^* from Lemma 2.7 together with the cohomology map corresponding to $f_i|_M$ we obtain the following commutative diagram

$$\begin{array}{ccc} H^q(M) & \xrightarrow{j^*} & H^q(U) \\ \uparrow (f_i|_M)^* & & \cong \uparrow \pi^* \\ H^q(\mathbb{C}^*) & \xrightarrow{(f_i|_{(M \cap Z_r)})^*} & H^q(M \cap Z_r). \end{array}$$

Now let ω be in $H^q(M \cap Z_r)$. By the assumption ω is a k-linear combination of elements of the form

$$(f_{i_1}|_{(M \cap Z_r)})^*(\sigma) \cdots (f_{i_q}|_{(M \cap Z_r)})^*(\sigma) \text{ for } 1 \leq i_1 < \ldots < i_q \leq r-1.$$

By Proposition 2.11, every element of $H^q(U)$ is of the form $\pi^*(\omega)$. Moreover,

$$\pi^*((f_{i_1}|_{(M\cap Z_r)})^*(\sigma)\cdots(f_{i_q}|_{(M\cap Z_r)})^*(\sigma)) = j^*((f_{i_1}|_M)^*(\sigma)\cdots(f_{i_q}|_M)^*(\sigma)), \qquad (2.8)$$

the latter of which is an element in the image of j^*, since we assumed the elements of $H^*(M)$ have the form $(f_{i_1}|_M)^*(\sigma)\cdots(f_{i_q}|_M)^*(\sigma)$ for $1 \leq i_1 < \ldots < i_q \leq r-1$. Therefore, j^* is surjective. ∎

The next corollary follows from Corollary 2.13.

Corollary 2.14 *Suppose $f_1, \ldots, f_s$ are smooth maps from X to $\mathbb{C}$. Assume that $n \geq 1$ and that for all r such that $n \leq r \leq s$ the following conditions hold*

(1) $Z_r \not\subseteq \bigcup_{i=1}^{r-1} Z_i$,

(2) *0 is a regular value of $f_r|_{M_{r-1}}$,*

(3) *as a k-algebra $H^*(M_{r-1} \cap Z_r)$ is generated by the set*

$$\{(f_1|_{(M_{r-1}\cap Z_r)})^*(\sigma), \ldots, (f_{r-1}|_{(M_{r-1}\cap Z_r)})^*(\sigma)\},$$

(4) *as a k-algebra $H^*(M_{n-1})$ is generated by the set*

$$\{(f_1|_{M_{n-1}})^*(\sigma), \ldots, (f_{n-1}|_{M_{n-1}})^*(\sigma)\}.$$

Then as a k-algebra $H^*(M_s)$ is generated by the set $\{(f_1|_{M_s})^*(\sigma), \ldots, (f_s|_{M_s})^*(\sigma)\}$.

Proof. By conditions (1)-(3) and Corollary 2.13 we have that $H^*(M_n)$ is generated by the set $\{(f_1|_{M_n})^*(\sigma), \ldots, (f_n|_{M_n})^*(\sigma)\}$. Now apply Corollary 2.13 and we get the results for $H^*(M_{n+1})$. By recursion we get the desired result. ∎

CHAPTER 3

DE RHAM COHOMOLOGY

3.1 Properties of Real and Complex Tangent Space

Let X be a real l-dimensional smooth manifold with m an element of X. Let U be open in X. If f is in $\mathcal{C}^\infty_{\mathbb{C}}(U)$, then $f = u + iv$ where u, v are in $\mathcal{C}^\infty_{\mathbb{R}}(U)$. Define $\psi : \mathcal{C}^\infty_{\mathbb{C}}(U) \to \mathcal{C}^\infty_{\mathbb{R}}(U) \otimes_{\mathbb{R}} \mathbb{C}$ by $\psi(u + iv) = u \otimes 1 + v \otimes i$.

Proposition 3.1 *The map ψ gives an isomorphism of $\mathbb{C}$-algebras*

$$\mathcal{C}^\infty_{\mathbb{C}}(U) \cong \mathcal{C}^\infty_{\mathbb{R}}(U) \otimes_{\mathbb{R}} \mathbb{C}.$$

Proof. It follows from the definitions that ψ is a $\mathbb{C}$-linear map. In the other direction, there is the $\mathbb{C}$-linear map $g \otimes \alpha \mapsto \alpha g$ for g in $\mathcal{C}^\infty_{\mathbb{R}}(U)$ and α in $\mathbb{C}$. These two maps are inverses since their composition is the identity. The group $\mathcal{C}^\infty_{\mathbb{C}}(U)$ is a $\mathbb{C}$-algebra under pointwise multiplication, and $\mathcal{C}^\infty_{\mathbb{R}}(U) \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-algebra using the tensor and multiplying component-wise. Therefore, we have an isomorphism of $\mathbb{C}$-algebras. ∎

Recall the definition of the tangent space from Chapter 2. In the next proposition we show that the complexification of real tangent space is the complex tangent space. First we define maps between $T_{\mathbb{C},m}X$ and $T_m X \otimes_{\mathbb{R}} \mathbb{C}$. If θ is in $T_{\mathbb{C},m}X$ and f is in $\mathcal{C}^\infty_{\mathbb{C},m}$, then $\theta(f)$ is in $\mathbb{C}$, and so it is straightforward to check that there exists $\mathbb{R}$-linear derivations $\theta_1, \theta_2 : \mathcal{C}^\infty_{\mathbb{C}} \to \mathbb{R}$ where $\theta(f) = \theta_1(f) + i\theta_2(f)$. It follows that the restrictions of θ_1 and θ_2 to $\mathcal{C}^{\infty}_{\mathbb{R},m}$ are in T_mX; denote these restrictions by $\overline{\theta_1}$ and $\overline{\theta_2}$ respectively. Define

$$\Psi : T_{\mathbb{C},m}X \to T_mX \otimes \mathbb{C} \quad \text{by} \quad \Psi(\theta) = \overline{\theta_1} \otimes 1 + \overline{\theta_2} \otimes i.$$

For the map in the other direction we define $\widetilde{\alpha\theta}$ for θ in T_mX and $\alpha = a + ib$ in $\mathbb{C}$ as follows:

$$\widetilde{\alpha\theta}(f) = \alpha[\theta(\text{Re}f) + i\theta(\text{Im}f)] = [a\theta(\text{Re}f) - b\theta(\text{Im}f)] + i[b\theta(\text{Re}f) + a\theta(\text{Im}f)].$$

It follows from the definitions that $\tilde{\theta}$ is a $\mathbb{C}$-linear derivation and that $\widetilde{\alpha\theta} = \alpha\tilde{\theta}$. Thus it is readily seen that $\widetilde{\alpha\theta}$ is in $T_{\mathbb{C},m}X$. The mapping $(\theta, \alpha) \mapsto \widetilde{\alpha\theta}$ is bilinear, so there exists a unique

$$\Phi : T_mX \otimes \mathbb{C} \to T_{\mathbb{C},m}X \quad \text{with} \quad \Phi(\theta \otimes \alpha) = \widetilde{\alpha\theta}.$$

The next proposition will show that Ψ and Φ are inverse functions.

Proposition 3.2 *The map $\Psi : T_{\mathbb{C},m}X \to T_mX \otimes_{\mathbb{R}} \mathbb{C}$ is an isomorphism of $\mathbb{C}$-vector spaces with inverse Φ.*

Proof. By following the definition of Ψ and using the fact that restriction is $\mathbb{R}$-linear, it is easy to see that Ψ is a $\mathbb{C}$-linear mapping. It is straightforward to check that the mapping Φ is also $\mathbb{C}$-linear. To prove that these maps are isomorphisms we will show that their composition is the identity. First observe that for θ in $T_{\mathbb{C},m}X$ and f in $\mathcal{C}^{\infty}_{\mathbb{C},m}$,

$$\begin{aligned}
\theta(f) &= \theta(\text{Re}f + i\text{Im}f) \\
&= \theta(\text{Re}f) + i\theta(\text{Im}f) \\
&= [\overline{\theta_1}(\text{Re}f) - \overline{\theta_2}(\text{Im}f)] + i[\overline{\theta_2}(\text{Re}f) + \overline{\theta_1}(\text{Im}f)].
\end{aligned}$$

It follows that

$$\begin{aligned}
\Phi\circ\Psi(\theta)(f) &= \Phi(\overline{\theta_1}\otimes 1+\overline{\theta_2}\otimes i)(f)\\
&= \widetilde{\overline{\theta_1}}(f)+i\widetilde{\overline{\theta_2}}(f)\\
&= \overline{\theta_1}(\mathrm{Re}f)+i\overline{\theta_1}(\mathrm{Im}f)-\overline{\theta_2}(\mathrm{Im}f)+i\overline{\theta_2}(\mathrm{Re}f)\\
&= \theta(f).
\end{aligned}$$

Secondly, for θ in T_mX and $\alpha=a+ib$ in $\mathbb{C}$, we have

$$\begin{aligned}
\Psi\circ\Phi(\theta\otimes\alpha) &= \Psi(\widetilde{\alpha\theta})\\
&= (a\theta\circ\mathrm{Re}-b\theta\circ\mathrm{Im})\otimes 1+(b\theta\circ\mathrm{Re}+a\theta\circ\mathrm{Im})\otimes i\\
&= \theta\circ\mathrm{Re}\otimes(a+ib)+\theta\circ\mathrm{Im}\otimes(-b+ia)\\
&= \theta\otimes\alpha,
\end{aligned}$$

since this is applied to a real valued function f, so $\mathrm{Re}f=f$ and $\mathrm{Im}f=0$. Therefore, Φ and Ψ are isomorphisms. ∎

Since X is a real manifold, there exists a neighborhood of m and a set of coordinate functions $\{x_1,\ldots,x_l\}$ that form a coordinate system for X near m.

Theorem 3.3 *The set* $\{\frac{\partial}{\partial x_j}|_m \mid 1\le j\le l\}$ *is a basis of* T_mX.

Proof. This is Remark 1.20(a) in [Wa]. ∎

We have similar results for complex tangent space. By Theorem 3.2 we know that $\Phi: T_mX\otimes_{\mathbb{R}}\mathbb{C}\to T_{\mathbb{C},m}X$ is an isomorphism of complex vector spaces, with $\Phi(\frac{\partial}{\partial x_j}|_m\otimes 1)=\widetilde{\frac{\partial}{\partial x_j}}|_m$. Thus $\{\widetilde{\frac{\partial}{\partial x_j}}|_m \mid 1\le j\le l\}$ is a complex basis of $T_{\mathbb{C},m}X$.

When f is in $\mathcal{C}^\infty_{\mathbb{R},m}$ the differential of f is a mapping from T_mX into $T_{f(m)}\mathbb{R}$. We can identify $T_{f(m)}\mathbb{R}$ with $\mathbb{R}$ via the mapping $a\frac{d}{dt}|_{f(m)} \mapsto a$ for a in $\mathbb{R}$. Thus we can view $(df)_m$ as a mapping from T_mX into $\mathbb{R}$, and [Wa, 1.22(5)] shows that $(df)_m(\theta) = \theta(f)$ for θ in T_mX. In this way we can consider the differential of a smooth real-valued map as an element in the dual space of T_mX. We call the dual space of T_mX the *cotangent space of* X *at* m, and denote it by T^*_mX. Using the evaluation of a differential of a real-valued map together with Theorem 3.3 we see that

$$\{\frac{\partial}{\partial x_j}|_m \mid 1 \leq j \leq l\} \text{ and } \{(dx_j)_m \mid 1 \leq j \leq l\} \tag{3.1}$$

are dual bases of T_mX and T^*_mX, respectively.

Next we will define a map on complex tangent spaces, the complex differential of a smooth map at m in X. Suppose Y is a smooth real n-manifold and $\phi : X \to Y$ is a smooth map. The *complex differential of* ϕ *at* m is the map $\phi_* : T_{\mathbb{C},m}X \to T_{\mathbb{C},\phi(m)}Y$ defined to be the composition

$$T_{\mathbb{C},m}X \xrightarrow{\Psi} T_mX \otimes_{\mathbb{R}} \mathbb{C} \overset{(d\phi)_m \otimes 1}{\longrightarrow} T_{f(m)}Y \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\Phi} T_{\mathbb{C},m}Y. \tag{3.2}$$

If f is in $\mathcal{C}^\infty_{\mathbb{C},m}$ then the complex differential is $f_* : T_{\mathbb{C},m}X \to T_{\mathbb{C},f(m)}\mathbb{C}$. In analogy with the real case we want a linear functional associated with f that maps $T_{\mathbb{C},m}X$ to $\mathbb{C}$. When our manifold is $\mathbb{C}$ we denote the two coordinate functions by x and y, and so a complex basis of $T_{\mathbb{C},f(m)}\mathbb{C}$ is $\{\widetilde{\frac{\partial}{\partial x}}|_{f(m)}, \widetilde{\frac{\partial}{\partial y}}|_{f(m)}\}$. We can map $T_{\mathbb{C},f(m)}\mathbb{C}$ into $\mathbb{C}$ via the mapping

$$\Upsilon(\alpha\widetilde{\frac{\partial}{\partial x}}|_{f(m)} + \beta\widetilde{\frac{\partial}{\partial y}}|_{f(m)}) = \alpha + i\beta$$

for α, β in $\mathbb{C}$. For f in $\mathcal{C}^\infty_{\mathbb{C},m}$ we define

$$(d_{\mathbb{C}}f)_m = \Upsilon \circ f_* : T_{\mathbb{C},m}X \to \mathbb{C}.$$

The next Proposition simplifies the computation of $(d_{\mathbb{C}}f)_m$.

Proposition 3.4 *If f is an element of $\mathcal{C}^\infty_{\mathbb{C},m}$ and θ is an element of $T_{\mathbb{C},m}X$, then*

$$(d_{\mathbb{C}}f)_m(\theta) = \theta(f).$$

Proof. Let θ be in $T_{\mathbb{C},m}X$, so θ is a derivation from $\mathcal{C}^\infty_{\mathbb{C},m}$ to $\mathbb{C}_m$. There exists $\theta_1, \theta_2 : \mathcal{C}^\infty_{\mathbb{C}} \to \mathbb{R}$ with $\theta = \theta_1 + i\theta_2$. For each $j = 1, 2$ the derivation $(df)_m(\theta_j)$ is in $T_{f(m)}\mathbb{C}$, so we may express it in terms of the basis $\{\frac{\partial}{\partial x}|_{f(m)}, \frac{\partial}{\partial y}|_{f(m)}\}$. Let $f = u + iv$ where u, v are in $\mathcal{C}^\infty_{\mathbb{R},m}$. Since $x \circ f = u$ and $y \circ f = v$, we have

$$\begin{aligned}(df)_m(\overline{\theta_j}) &= (df)_m(\overline{\theta_j})(x)\tfrac{\partial}{\partial x}|_{f(m)} + (df)_m(\overline{\theta_j})(y)\tfrac{\partial}{\partial y}|_{f(m)} \\ &= \theta_j(u)\tfrac{\partial}{\partial x}|_{f(m)} + \theta_j(v)\tfrac{\partial}{\partial y}|_{f(m)},\end{aligned}$$

for $j = 1, 2$. Using this formula together with the definition of f_* and $\widetilde{\theta}$, if g is in $\mathcal{C}^\infty_{\mathbb{C},f(m)}$ we can evaluate $f_*(\theta)(g)$ as follows:

$$\begin{aligned}f_*(\theta)(g) &= \Phi \circ ((df)_m \otimes 1) \circ \Psi(\theta_1 + i\theta_2)(g) \\ &= \widetilde{(df)_m}(\overline{\theta_1})(g) + i\widetilde{(df)_m}(\overline{\theta_2})(g) \\ &= (df)_m(\overline{\theta_1})(\mathrm{Re}g) + i(df)_m(\overline{\theta_1})(\mathrm{Im}g) \\ &\quad + i\left((df)_m(\overline{\theta_2})(\mathrm{Re}g) + i(df)_m(\overline{\theta_2})(\mathrm{Im}g)\right) \\ &= \theta_1(u)\tfrac{\partial}{\partial x}|_{f(m)}(\mathrm{Re}g) + \theta_1(v)\tfrac{\partial}{\partial y}|_{f(m)}(\mathrm{Re}g) \\ &\quad + i\theta_1(u)\tfrac{\partial}{\partial x}|_{f(m)}(\mathrm{Im}g) + i\theta_1(v)\tfrac{\partial}{\partial y}|_{f(m)}(\mathrm{Im}g) \\ &\quad + i\theta_2(u)\tfrac{\partial}{\partial x}|_{f(m)}(\mathrm{Re}g) + i\theta_2(v)\tfrac{\partial}{\partial y}|_{f(m)}(\mathrm{Re}g) \\ &\quad - \theta_2(u)\tfrac{\partial}{\partial x}|_{f(m)}(\mathrm{Im}g) - \theta_2(v)\tfrac{\partial}{\partial y}|_{f(m)}(\mathrm{Im}g) \\ &= \theta(u)\tfrac{\partial}{\partial x}|_{f(m)}(\mathrm{Re}g) + \theta(v)\tfrac{\partial}{\partial y}|_{f(m)}(\mathrm{Re}g)\end{aligned}$$

$$+ i\theta(u)\tfrac{\partial}{\partial x}|_{f(m)}(\text{Im}g) + i\theta(v)\tfrac{\partial}{\partial y}|_{f(m)}(\text{Im}g)$$
$$= \quad \theta(u)\tfrac{\partial}{\partial x}|_{f(m)}(g) + \theta(v)\tfrac{\partial}{\partial y}|_{f(m)}(g).$$

Thus $f_*(\theta) = \theta(u)\frac{\partial}{\partial x}|_{f(m)} + \theta(v)\frac{\partial}{\partial y}|_{f(m)}$. Now compose with the mapping Υ of $T_{\mathbb{C},f(m)}\mathbb{C}$ into $\mathbb{C}$, and we have $(d_{\mathbb{C}}f)_m(\theta) = \theta(u) + i\theta(v) = \theta(f)$. ∎

Notice that for f in $\mathcal{C}^{\infty}_{\mathbb{C},m}$ the notion of $(d_{\mathbb{C}}f)_m : T_{\mathbb{C},f(m)}\mathbb{C} \to \mathbb{C}$ allows us to view $(d_{\mathbb{C}}f)_m$ as an element of the dual space of $T_{\mathbb{C},m}X$. We shall refer to the dual space of $T_{\mathbb{C},m}X$ as the *complex cotangent space of X at m* and shall denote it by $T^*_{\mathbb{C},m}X$. The next Proposition shows that the complexification of real cotangent space is complex cotangent space. We define a mapping from $T^*_{\mathbb{C},m}X$ to $T^*_m X \otimes_{\mathbb{R}} \mathbb{C}$ as follows. From Proposition 3.2 we have an isomorphism $\Phi : T_m X \otimes_{\mathbb{R}} \mathbb{C} \to T_{\mathbb{C},m}X$. There exists a natural isomorphism , $: \mathbb{C} \to \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ by , $(a + ib) = a \otimes 1 + b \otimes i$. We also have the natural isomorphism from $\text{Hom}_{\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}}(T_m X \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})$ to $\text{Hom}_{\mathbb{R}}(T_m X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Combining these maps we have:

$$\begin{aligned}
T^*_{\mathbb{C},m}X \quad &= \quad \text{Hom}_{\mathbb{C}}(T_{\mathbb{C},m}X, \mathbb{C}) \\
&\xrightarrow{\circ\Phi} \quad \text{Hom}_{\mathbb{C}}(T_m X \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}) \\
&\xrightarrow{\Gamma\circ} \quad \text{Hom}_{\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}}(T_m X \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}) \\
&\xrightarrow{\cong} \quad \text{Hom}_{\mathbb{R}}(T_m X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \\
&= \quad T^*_m X \otimes_{\mathbb{R}} \mathbb{C}.
\end{aligned}$$

Label the composition of these maps by Θ.

Proposition 3.5 *$\Theta : T^*_{\mathbb{C},m}X \to T^*_m X \otimes_{\mathbb{R}} \mathbb{C}$ is an isomorphism, with $\Theta((d_{\mathbb{C}}f)_m) = (du)_m \otimes 1 + (dv)_m \otimes i$ where $f = u + iv$ is in $\mathcal{C}^{\infty}_{\mathbb{C},m}$ and u, v are in $\mathcal{C}^{\infty}_{\mathbb{R},m}$.*

Proof. Since each of the maps is an isomorphism we have the desired result. Let θ be an element of T_mX and $\alpha = a + ib$ be in $\mathbb{C}$. Then

$$\begin{aligned} , \circ (d_{\mathbb{C}}f)_m \circ \Phi(\theta \otimes \alpha) &= \; , (\widetilde{\alpha\theta} f) \\ &= \; , (a\theta(u) - b\theta(v) + i(b\theta(u) + a\theta(v))) \\ &= a\theta(u) \otimes 1 - b\theta(v) \otimes 1 + b\theta(u) \otimes i + a\theta(v) \otimes i \\ &= (du)_m(\theta) \otimes a - (dv)_m(\theta) \otimes b \\ &\quad + (du)_m(\theta) \otimes ib + (dv)_m(\theta) \otimes ia \end{aligned}$$

and

$$\begin{aligned} ((du)_m \otimes 1 + (dv)_m \otimes i)(\theta \otimes \alpha) &= (du)_m(\theta) \otimes (a + ib) + (dv)_m(\theta) \otimes i(a + ib) \\ &= (du)_m(\theta) \otimes a - (dv)_m(\theta) \otimes b \\ &\quad + (du)_m(\theta) \otimes ib + (dv)_m(\theta) \otimes ia. \end{aligned}$$

Since these are the same we have the desired result. ∎

3.2 De Rham Cohomology

Consider the q^{th} exterior power of the cotangent bundle

$$\Lambda^q T^*_{\mathbb{F}} X = \bigcup_{m \in X} \Lambda^q T^*_{\mathbb{F},m} X,$$

with projection map $\pi : \Lambda^q T^*_{\mathbb{F}} X \to X$. Note that $\Lambda^0 T^*_{\mathbb{F}} X = X \times \mathbb{F}$ and $\Lambda^1 T^*_{\mathbb{F}} X$ is simply the cotangent bundle. The space $\Lambda^q T^*_{\mathbb{F}} X$ has a natural manifold structure such that π is smooth. Recall from Section 2.15 of [Wa] that a q-form on X is a smooth map $s : X \to \Lambda^q T^*_{\mathbb{F}} X$ such that $\pi \circ s = id_X$. Let $A^q_{\mathbb{F}}(X)$ be the set of q-forms on X.

Observe that $A^q_{\mathbb{F}}(X)$ forms an $\mathbb{F}$-vector space under pointwise operations. There is also a product (the wedge product) on forms. For ω_1 in $A^q_{\mathbb{F}}(X)$ and ω_2 in $A^r_{\mathbb{F}}(X)$, $\omega_1 \wedge \omega_2$ satisfies $(\omega_1 \wedge \omega_2)(m) = \omega_1(m) \wedge \omega_2(m)$. For f in $\mathcal{C}^\infty_{\mathbb{F}}(X)$ and ω in $A^q_{\mathbb{F}}(X)$, observe that $f \wedge \omega = f\omega$. It follows that $A^*_{\mathbb{F}}(X)$ has the structure of both a $\mathcal{C}^\infty_{\mathbb{F}}(X)$-module and of a graded algebra over $\mathbb{F}$ with wedge multiplication.

Next we consider maps on q-forms. Theorem 2.20 in [Wa] states that there exists a unique $\mathbb{R}$-linear anti-derivation $d : A^q(X) \to A^{q+1}(X)$ such that

(1) $d^2 = 0$

(2) $df(m) = (df)_m$ for f in $\mathcal{C}^\infty_{\mathbb{R}}(X)$.

This anti-derivation is called the *exterior derivative*. The definition of an anti-derivation is that $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$ for ω in $A^r(X)$ and η in $A^q(X)$.

Now we will define a $\mathbb{C}$-linear anti-derivation $d_{\mathbb{C}} : A^q_{\mathbb{C}}(X) \to A^{q+1}_{\mathbb{C}}(X)$ that satisfies the property that $d^2_{\mathbb{C}} = 0$. To define this anti-derivation we need the next Proposition which shows that the set of complex q-forms on X is isomorphic to the set of real q-forms on X with scalars extended to $\mathbb{C}$. For $q = 0$, this follows from Proposition 3.1, so we assume that $q > 0$. If ω is in $A^q_{\mathbb{C}}(X)$, there exists ω_1, ω_2 in $A^q(X)$ such that for each m in X, $\omega(m) = \omega_1(m) \otimes 1 + \omega_2(m) \otimes i$. Define $\Psi' : A^q_{\mathbb{C}}(X) \to A^q(X) \otimes_{\mathbb{R}} \mathbb{C}$ by the rule

$$\Psi'(\omega) = \omega_1 \otimes 1 + \omega_2 \otimes i.$$

For the map in the other direction, define $\widetilde{\alpha\omega}$ in $A^q_{\mathbb{C}}(X)$ by $\widetilde{\alpha\omega}(m) = \omega(m) \otimes \alpha$ for ω

in $A^q(X)$, α in $\mathbb{C}$, and m in X. The mapping $(\omega, \alpha) \mapsto \widetilde{\alpha\omega}$ is bilinear, so there exists a unique

$$\Phi' : A^q(X) \otimes_{\mathbb{R}} \mathbb{C} \to A^q_{\mathbb{C}}(X) \text{ with } \Phi'(\omega \otimes \alpha) = \widetilde{\alpha\omega}.$$

Proposition 3.6 *If $q > 0$ the mapping $\Psi' : A^q_{\mathbb{C}}(X) \to A^q(X) \otimes_{\mathbb{R}} \mathbb{C}$ is an isomorphism of $\mathcal{C}^\infty_{\mathbb{C}}(X)$-modules with inverse Φ'.*

Proof. To see that Ψ' is $\mathbb{C}$-linear, let $\alpha = a + bi$ be in $\mathbb{C}$. Then $\alpha\omega = a\omega_1 - b\omega_2 + i(b\omega_1 + a\omega_2)$, and so $\Psi'(\alpha\omega) = \omega_1 \otimes \alpha + \omega_2 \otimes i\alpha = \alpha\Psi'(\omega)$. It also follows from the definitions that $\Psi'(\omega + \omega') = \Psi'(\omega) + \Psi'(\omega')$ for ω, ω' in $A^q_{\mathbb{C}}(X)$. It is also straightforward to check that the mapping Φ' is $\mathbb{C}$-linear.

Next we show that the composition of these two maps is the identity. For ω in $A^q_{\mathbb{C}}(X)$, we have $\Phi' \circ \Psi'(\omega) = \Phi'(\omega_1 \otimes 1 + \omega_2 \otimes i) = \widetilde{\omega_1} + \widetilde{i\omega_2}$. For m in X notice $(\widetilde{\omega_1} + \widetilde{i\omega_2})(m) = \omega_1(m) \otimes 1 + \omega_2(m) \otimes i = \omega(m)$, so this composition is the identity. Now for the reverse composition, let ω be in $A^q(X)$ and $\alpha = a + ib$ be in $\mathbb{C}$. Then $\Psi' \circ \Phi'(\omega \otimes \alpha) = \Psi'(\widetilde{\alpha\omega}) = a\omega \otimes 1 + b\omega \otimes i = \omega \otimes a + \omega \otimes ib = \omega \otimes \alpha$. Thus both compositions are the identity, and therefore the mappings are isomorphisms.

Observe that this is a $\mathcal{C}^\infty_{\mathbb{C}}(X)$-module isomorphism. Indeed, if f is in $\mathcal{C}^\infty_{\mathbb{C}}(X)$, ω is in $A^q_{\mathbb{C}}(X)$, and m is in X then $(f\omega)(m) = f(m)\omega_1(m) \otimes 1 + f(m)\omega_2(m) \otimes i$ and so $\Psi'(f\omega) = f\omega_1 \otimes 1 + f\omega_2 \otimes i = f\Psi'(\omega)$. ∎

Now we define $d_{\mathbb{C}} : \mathcal{C}^\infty_{\mathbb{C}}(X) \to A^1_{\mathbb{C}}(X)$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}^\infty_{\mathbb{R}}(X) \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{d \otimes 1} & A^1(X) \otimes_{\mathbb{R}} \mathbb{C} \\ \uparrow \psi & & \downarrow \Phi' \\ \mathcal{C}^\infty_{\mathbb{C}}(X) & \xrightarrow{d_{\mathbb{C}}} & A^1_{\mathbb{C}}(X), \end{array}$$

where the left and right isomorphisms are given in Propositions 3.1 and 3.6, respectively. If we follow through these maps for f in $\mathcal{C}^\infty_{\mathbb{C}}(X)$ and m in X, we see that $d_{\mathbb{C}}(f)(m) = (d_{\mathbb{C}} f)_m$.

For $q > 0$ define $d_{\mathbb{C}} : A^q_{\mathbb{C}}(X) \to A^{q+1}_{\mathbb{C}}(X)$ to be the unique function for which the diagram commutes:

$$\begin{array}{ccc} A^q(X) \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{d\otimes 1} & A^{q+1}(X) \otimes_{\mathbb{R}} \mathbb{C} \\ \uparrow \Psi' & & \downarrow \Phi' \\ A^q_{\mathbb{C}}(X) & \xrightarrow{d_{\mathbb{C}}} & A^{q+1}_{\mathbb{C}}(X). \end{array} \tag{3.3}$$

In other words, $d_{\mathbb{C}}$ is the composition

$$A^q_{\mathbb{C}}(X) \xrightarrow{\Psi'} A^q(X) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{d\otimes 1} A^{q+1}(X) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\Phi'} A^{q+1}_{\mathbb{C}}(X).$$

Since $d^2 = 0$ it follows that $d^2_{\mathbb{C}} = 0$. Moreover, as d is an anti-derivation so is $d_{\mathbb{C}}$.

Now we use the set of q-forms together with the exterior derivative to form a complex. We will then use this complex to define the de Rham cohomology of X. Temporarily let d^q denote the exterior derivative from $A^q(X)$ to $A^{q+1}(X)$. Consider the complex

$$0 \to A^0(X) \xrightarrow{d^0} A^1(X) \xrightarrow{d^1} \cdots A^{q-1}(X) \xrightarrow{d^{q-1}} A^q(X) \xrightarrow{d^q} \cdots A^l(X) \to 0. \tag{3.4}$$

A q-form ω in $A^q(X)$ is called *closed* if $d^q\omega = 0$ and it is called *exact* if there exists a $(q-1)$-form η such that $\omega = d^{q-1}\eta$. Let $Z^q = \ker(d^q)$ denote the set of closed q-forms, and let $B^q = d^{q-1}(A^{q-1}(X))$ denote the set of exact q-forms. These sets are real vector spaces under pointwise operations. Moreover, as $d^2 = 0$ we have $B^q \subseteq Z^q$. The qth *de Rham cohomology group of* X with real coefficients is defined to be the quotient space

$$H^q_{\mathrm{DR}}(X; \mathbb{R}) = H^q(A^*(X), d) = Z^q/B^q.$$

Tensoring with $\mathbb{C}$ is exact, so applying $\cdot \otimes_{\mathbb{R}} \mathbb{C}$ to Complex 3.4 gives a new complex. Combining this sequence with Diagram 3.3, we get the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A^0(X) \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{d\otimes 1} & A^1(X) \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{d\otimes 1} & \cdots & \xrightarrow{d\otimes 1} & A^l(X) \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \\ 0 & \longrightarrow & A^0_{\mathbb{C}}(X) & \xrightarrow{d_{\mathbb{C}}} & A^1_{\mathbb{C}}(X) & \xrightarrow{d_{\mathbb{C}}} & \cdots & \xrightarrow{d_{\mathbb{C}}} & A^l_{\mathbb{C}}(X) & \longrightarrow & 0. \end{array}$$

The qth *de Rham cohomology group of* X with complex coefficients is

$$H^q_{\mathrm{DR}}(X;\mathbb{C}) = H^q(A^*_{\mathbb{C}}(X), d_{\mathbb{C}}) = \ker(d_{\mathbb{C}})/d_{\mathbb{C}}(A^{q-1}_{\mathbb{C}}(X)).$$

Let the qth cohomology group of the complex $A^*(X) \otimes \mathbb{C}$ be represented by

$$H^q(A^*(X) \otimes \mathbb{C}, d \otimes 1) = \ker(d \otimes 1)/(d \otimes 1)(A^{q-1}(X) \otimes \mathbb{C}).$$

Since the two complexes in the diagram are isomorphic, their cohomologies are isomorphic, thus $H^q(A^*_{\mathbb{C}}(X), d_{\mathbb{C}}) \cong H^q(A^*(X) \otimes \mathbb{C}, d \otimes 1)$. Moreover, the fact that tensoring with $\mathbb{C}$ is exact implies that $H^q(A^*(X), d) \otimes \mathbb{C} \cong H^q(A^*(X) \otimes \mathbb{C}, d \otimes 1)$. This leads to the fact that $H^q_{\mathrm{DR}}(X;\mathbb{C}) \cong H^q_{\mathrm{DR}}(X;\mathbb{R}) \otimes \mathbb{C}$, and so de Rham cohomology with complex coefficients is simply de Rham cohomology with real coefficients with scalars extended to the complexes.

An element of $H^*_{\mathrm{DR}}(X;\mathbb{F})$ is a coset, so for $\omega \in \ker(d_{\mathbb{F}}) \subseteq A^q_{\mathbb{F}}(X)$ we will represent the cohomology class $\omega + \mathrm{im}(d_{\mathbb{F}})$ by $[\omega]$. Define the wedge product of two elements in $H^*_{\mathrm{DR}}(X;\mathbb{F})$ with coefficients in $\mathbb{F}$ as follows: if $[\omega_1]$ is in $H^q_{\mathrm{DR}}(X;\mathbb{F})$ and $[\omega_2]$ is in $H^r_{\mathrm{DR}}(X;\mathbb{F})$ then define $[\omega_1] \wedge [\omega_2]$ in $H^{q+r}_{\mathrm{DR}}(X;\mathbb{F})$ to be $[\omega_1 \wedge \omega_2]$.

Proposition 3.7 *The wedge product on* $H^*_{\mathrm{DR}}(X;\mathbb{F})$ *is well-defined and* $H^*_{\mathrm{DR}}(X;\mathbb{F})$ *is a graded, skew-commutative,* $\mathbb{F}$*-algebra.*

Proof. The definition of the wedge product makes sense as follows. If $[\omega_1]$ is in $H^q_{\mathrm{DR}}(X;\mathbb{F})$ and $[\omega_2]$ is in $H^r_{\mathrm{DR}}(X;\mathbb{F})$ then $d_{\mathbb{F}}(\omega_1) = d_{\mathbb{F}}(\omega_2) = 0$. So $d_{\mathbb{F}}(\omega_1 \wedge \omega_2) = 0$ which implies that $[\omega_1 \wedge \omega_2]$ is in $H^{q+r}_{\mathrm{DR}}(X;\mathbb{F})$. Suppose $[\omega_1] = [\omega_3]$ and $[\omega_2] = [\omega_4]$. Then

$$\begin{aligned} [\omega_1 \wedge \omega_2] &= [\omega_1] \wedge [\omega_2] \\ &= (\omega_1 + \mathrm{im}(d_{\mathbb{F}})) \wedge (\omega_2 + \mathrm{im}(d_{\mathbb{F}})) \\ &= (\omega_3 + \mathrm{im}(d_{\mathbb{F}})) \wedge (\omega_4 + \mathrm{im}(d_{\mathbb{F}})) \\ &= [\omega_3 \wedge \omega_4], \end{aligned}$$

thus the wedge product is well-defined. Since the wedge product on $\Lambda^* T^*_{\mathbb{F},p} X$ is skew-commutative, the wedge product on the quotient space is skew-commutative. That is, for $[\omega_1]$ in $H^q_{\mathrm{DR}}(X;\mathbb{F})$ and $[\omega_2]$ in $H^r_{\mathrm{DR}}(X;\mathbb{F})$ we have $[\omega_1 \wedge \omega_2] = (-1)^q [\omega_2 \wedge \omega_1]$. Lastly, as $A^*_{\mathbb{F}}(X)$ is an $\mathbb{F}$-algebra, so is $H^*_{\mathrm{DR}}(X;\mathbb{F})$. ∎

Recall in Chapter 2 that $[\sigma]$ denoted the fixed generator of $H^1(\mathbb{C}^*)$. Next we will see that $[\sigma]$ corresponds to $\frac{d_{\mathbb{C}} z}{z}$ in $H^1_{\mathrm{DR}}(\mathbb{C}^*;\mathbb{C})$. First use the fact that $d_{\mathbb{C}}$ is a derivation to see that

$$d_{\mathbb{C}}\left(\frac{d_{\mathbb{C}} z}{z}\right) = d_{\mathbb{C}}\left(\frac{1}{z} \wedge d_{\mathbb{C}} z\right) = d_{\mathbb{C}}\left(\frac{1}{z}\right) \wedge d_{\mathbb{C}} z + \frac{1}{z} \wedge d_{\mathbb{C}}(d_{\mathbb{C}} z) = \frac{-1}{z^2} d_{\mathbb{C}} z \wedge d_{\mathbb{C}} z + 0 = 0.$$

Thus $[\frac{d_{\mathbb{C}} z}{z}]$ is in $H^1_{\mathrm{DR}}(\mathbb{C}^*;\mathbb{C})$. It follows that for all $\alpha \in \mathbb{C}$, we have $[\frac{d_{\mathbb{C}} \alpha z}{z}]$ is in $H^1_{\mathrm{DR}}(\mathbb{C}^*;\mathbb{C})$. Thus in de Rham cohomology with complex coefficients the generator of $H^1_{\mathrm{DR}}(\mathbb{C}^*;\mathbb{C})$ is $[\frac{d_{\mathbb{C}} z}{z}]$.

Now if X and Y are smooth real manifolds we want to define maps between $H^q_{\mathrm{DR}}(X;\mathbb{F})$ and $H^q_{\mathrm{DR}}(Y;\mathbb{F})$. To do this we use a map on q-forms that commutes with

$d_{\mathbb{F}}$. Again suppose that $\phi : X \to Y$ is a smooth mapping, then [Wa, Section 2.22] shows that ϕ determines a function, $\phi^* : A^q(Y) \to A^q(X)$. This function satisfies the following properties:

Proposition 3.8 *If ω is in $A^*(Y)$ and g is in $\mathcal{C}^\infty_{\mathbb{R}}(Y)$, then*

1. $\phi^*(g) = g \circ \phi$

2. $\phi^*(g\omega) = \phi^*(g) \cdot \phi^*(\omega) = g \circ \phi \cdot \phi^*(\omega)$,

3. *the map ϕ^* commutes with the exterior derivative, that is $d\phi^* = \phi^* d$.*

Proof. This is Proposition 2.23 of [Wa]. ∎

From this map on real q-forms we define $\phi^*_{\mathbb{C}} : A^q_{\mathbb{C}}(Y) \to A^q_{\mathbb{C}}(X)$ to be the unique function for which the following diagram commutes:

$$\begin{array}{ccc} A^q(Y) \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\phi^* \otimes 1} & A^q(X) \otimes_{\mathbb{R}} \mathbb{C} \\ \uparrow \Psi' & & \downarrow \Phi' \\ A^q_{\mathbb{C}}(Y) & \xrightarrow{\phi^*_{\mathbb{C}}} & A^q_{\mathbb{C}}(X), \end{array}$$

where the vertical isomorphism were given in Proposition 3.6.

Proposition 3.9 *If ω is in $A^q_{\mathbb{C}}(Y)$ and g is in $\mathcal{C}^\infty_{\mathbb{C}}(Y)$, then*

1. $\phi^*_{\mathbb{C}}(g) = g \circ \phi$,

2. $\phi^*_{\mathbb{C}}(g\omega) = \phi^*_{\mathbb{C}}(g) \cdot \phi^*_{\mathbb{C}}(\omega) = g \circ \phi \cdot \phi^*_{\mathbb{C}}(\omega)$,

3. $d_{\mathbb{C}}\phi^*_{\mathbb{C}} = \phi^*_{\mathbb{C}} d_{\mathbb{C}}$.

Proof. The proof of these facts follow immediately from Proposition 3.8 together with the definition of $\phi_{\mathbb{C}}^*$. ∎

Since $\phi_{\mathbb{C}}^*$ and $d_{\mathbb{C}}$ commute, $\phi_{\mathbb{C}}^*$ induces a map on cohomology,

$$\phi_{\mathbb{C}}^* : H^q_{\mathrm{DR}}(Y;\mathbb{C}) \to H^q_{\mathrm{DR}}(X;\mathbb{C}) \quad \text{by} \quad \phi_{\mathbb{C}}^*([\omega]) = [\phi_{\mathbb{C}}^*(\omega)].$$

We can use this to see how non-zero functions in $\mathcal{C}^\infty_{\mathbb{C}}$ act on the special element $[\frac{d_{\mathbb{C}}z}{z}]$ of $H^1_{\mathrm{DR}}(\mathbb{C}^*;\mathbb{C})$.

Proposition 3.10 *If g is in $\mathcal{C}^\infty_{\mathbb{C}}$ with $g(x) \neq 0$ for all x in X, then*

$$g^* : H^1_{\mathrm{DR}}(\mathbb{C}^*;\mathbb{C}) \to H^1_{\mathrm{DR}}(X;\mathbb{C}) \textit{ and } g^*([\tfrac{d_{\mathbb{C}}z}{z}]) = [\tfrac{d_{\mathbb{C}}g}{g}].$$

Proof. Using parts (1) and (2) of Proposition 3.9,

$$g^*\left(\tfrac{d_{\mathbb{C}}z}{z}\right) = g^*\left(\tfrac{1}{z}\right) g^*(d_{\mathbb{C}}z) = \tfrac{1}{z\circ g} d_{\mathbb{C}} g^*(z) = \tfrac{1}{g} d_{\mathbb{C}}(g),$$

since $z : \mathbb{C} \to \mathbb{C}$ is the identity and so $z \circ g = g$. The result now follows. ∎

3.3 Properties of de Rham Cohomology

In Chapter I, Bott and Tu [BT] show that real de Rham cohomology satisfies the Eilenberg-Steenrod Axioms except for the Excision Axiom (Example 1.6, Section 2, Proposition 2.1, Example 2.6, and Proposition 6.49). They also show that it satisfies the Künneth Formula (Equation 5.9). A special case of the Universal Coefficient Theorem [Dol, VI.7.1] states that the formal properties of cohomology carry over to arbitrary coefficients. So these properties that are true for real coefficients are also true for complex coefficients. In Proposition 3.7 we showed that de Rham cohomology with complex coefficients has a ring structure that makes it into a graded $\mathbb{C}$-algebra. Thus, in order to see that the results of Chapter 2 hold for de Rham cohomology with complex coefficients, we need only check that it satisfies the Excision Axiom. From now on we will assume that de Rham cohomology has complex coefficients, denoted simply $H^*_{\mathrm{DR}}(X)$.

Suppose that A is a closed submanifold of X with inclusion $i : A \to X$. Recall from Section I.6 of [BT] that $A^k_{\mathbb{C}}(i) = A^k_{\mathbb{C}}(X) \bigoplus A^{k-1}_{\mathbb{C}}(A)$ and the exterior derivative $d_{\mathbb{C}} : A^k_{\mathbb{C}}(i) \to A^{k+1}_{\mathbb{C}}(i)$ is defined by $d_{\mathbb{C}}(\omega, \theta) = (d_{\mathbb{C}}\omega, i^*\omega - d_{\mathbb{C}}\theta)$ for $\omega \in A^k_{\mathbb{C}}(X)$ and $\theta \in A^{k-1}_{\mathbb{C}}(A)$. It is a straightforward result of the definition to see that $d^2_{\mathbb{C}} = 0$. Suppose that U is an open subset of X such that $A \subseteq U \subseteq X$. We have the natural inclusion maps $i_X : X \setminus A \to X$, $i_U : U \setminus A \to U$, $j_1 : U \to X$, and $j_2 : U \setminus A \to X \setminus A$, giving the commutative diagram:

$$\begin{array}{ccc} U \setminus A & \xrightarrow{i_U} & U \\ \downarrow j_2 & & \downarrow j_1 \\ X \setminus A & \xrightarrow{i_X} & X. \end{array}$$

Each inclusion yields a complex as above. Combining these with the above diagram we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A^0_{\mathbb{C}}(X) & \xrightarrow{d_{\mathbb{C}}} & A^1_{\mathbb{C}}(X) \oplus A^0_{\mathbb{C}}(X \setminus A) & \xrightarrow{d_{\mathbb{C}}} & A^2_{\mathbb{C}}(X) \oplus A^1_{\mathbb{C}}(X \setminus A) & \longrightarrow & \cdots \\ & & \downarrow j_1^* & & \downarrow j_1^* + j_2^* & & \downarrow j_1^* + j_2^* & & \\ 0 & \to & A^0_{\mathbb{C}}(U) & \xrightarrow{d_{\mathbb{C}}} & A^1_{\mathbb{C}}(U) \oplus A^0_{\mathbb{C}}(U \setminus A) & \xrightarrow{d_{\mathbb{C}}} & A^2_{\mathbb{C}}(U) \oplus A^1_{\mathbb{C}}(U \setminus A) & \longrightarrow & \cdots. \end{array}$$

Lemma 3.11 *The relative cohomologies of the inclusion maps $i_X : X \setminus A \to X$ and $i_U : U \setminus A \to U$ are isomorphic. That is $H^k_{\mathrm{DR}}(i_X) \cong H^k_{\mathrm{DR}}(i_U)$ for all k.*

Proof. Let $[\omega, \theta]$ be in $H^k_{\mathrm{DR}}(i_X)$ so that (ω, θ) is an element of $A^k_{\mathbb{C}}(X) \oplus A^{k-1}_{\mathbb{C}}(X \setminus A)$ and $d_{\mathbb{C}}(\omega, \theta) = 0$. The image of $[\omega, \theta]$ under $(j_1^* + j_2^*)$ is $[\omega \circ j_1, \theta \circ j_2]$ in $H^k_{\mathrm{DR}}(i_U)$. We want to see that this is an isomorphism. To do so we will first show it is surjective and then that it is injective. For both arguments we shall require a partition of unity.

Since X is a manifold it is regular and paracompact. By regularity we can choose an open subset W such that $A \subseteq W \subseteq \overline{W} \subseteq U$, where $\overline{W}$ is the closure of W. Then the complement of $\overline{W}$, $\overline{W}^c$, and U form an open cover of X. By paracompactness there is a partition of unity for X subordinate to the open cover $\{\overline{W}^c, U\}$. So there exists smooth functions $\phi_1, \phi_2 : X \to \mathbb{R}$ such that the support of ϕ_1 is contained in U, the support of ϕ_2 is contained in $\overline{W}^c$, and $(\phi_1 + \phi_2)(x) = 1$ for all $x \in X$. Combining these we see that $\phi_1 \equiv 1$ on $\overline{W}$.

Now let $[\omega_1, \theta_1] \in H^k_{\mathrm{DR}}(i_U)$ so $(\omega_1, \theta_1) \in A^k_{\mathbb{C}}(U) \oplus A^{k-1}_{\mathbb{C}}(U \setminus A)$ and $d_{\mathbb{C}}(\omega_1, \theta_1) = (d_{\mathbb{C}}\omega_1, \omega_1|_{U\setminus A} - d_{\mathbb{C}}\theta_1) = 0$. This implies that $d_{\mathbb{C}}\omega_1 = 0$ and $\omega_1|_{U\setminus A} = d_{\mathbb{C}}\theta_1$. We want to show that this is in the image of the map $(j_1^* + j_2^*)$. Define $[\omega, \theta] \in H^k_{\mathrm{DR}}(i_X)$ as follows:

$$\omega = \begin{cases} 0 & \text{on } X \setminus U \\ d_{\mathbb{C}}(\phi_1\theta_1) & \text{on } U \setminus A \\ \omega_1 & \text{on } W \end{cases} \qquad \text{and} \qquad \theta = \begin{cases} 0 & \text{on } X \setminus U \\ \phi_1\theta_1 & \text{on } U \setminus A. \end{cases}$$

To see that this is well-defined we need to show that ω and θ are smooth, and that $d_{\mathbb{C}}(\omega, \theta) = 0$. To see that ω is smooth we notice that it agrees on the intersection of these sets in the definition. Observe first that $d_{\mathbb{C}}(\phi_1\theta_1) = \phi_1\omega_1 + d_{\mathbb{C}}\phi_1 \wedge \theta_1$, and so the support of $d_{\mathbb{C}}(\phi_1\theta_1)$ is contained in the support of ϕ_1, a subset of U. Outside the support of ϕ_1 the form $d_{\mathbb{C}}(\phi_1\theta_1)$ is 0. Also $d_{\mathbb{C}}(\phi_1\theta_1)|_{W\setminus A}$ equals ω_1, since $\phi_1 \equiv 1$ on W.

So the definitions agree on $(U \setminus A) \cap W = W \setminus A$. The form θ is smooth since if $x \notin U$ then $\phi_1 \equiv 0$ on some neighborhood of x, and so θ agrees on the intersection of the sets. Next, consider $d_{\mathbb{C}}(\omega, \theta) = (d_{\mathbb{C}}\omega, \omega|_{X\setminus A} - d_{\mathbb{C}}\theta)$. The first term is 0 since $d_{\mathbb{C}}\omega = 0$ for all x. The second term is 0 on all of X since $\omega = d_{\mathbb{C}}\theta$ as follows. If $x \in X \setminus U$ then $\omega = d_{\mathbb{C}}\theta = 0$ on a neighborhood of x, and if $x \in U \setminus A$ then $\omega = d_{\mathbb{C}}\theta = d_{\mathbb{C}}(\phi_1\theta_1)$ on a neighborhood of x. Thus $d_{\mathbb{C}}(\omega, \theta) = 0$.

Finally, we show that $(j_1^* + j_2^*)[\omega, \theta] = [\omega|_U, \theta|_{U\setminus A}] = [\omega_1, \theta_1]$. To do this we will show that $(\omega|_U, \theta|_{U\setminus A}) - (\omega_1, \theta_1) = (\omega|_U - \omega_1, \theta|_{U\setminus A} - \theta_1) = (\omega|_U - \omega_1, \phi_1\theta_1 - \theta_1)$ is in the image of $d_{\mathbb{C}}$. Define a new form, $\eta \in A_{\mathbb{C}}^{k-1}(U)$, by

$$\eta = \begin{cases} (\phi_1 - 1)\theta_1 & \text{on } U \setminus A \\ 0 & \text{on } W. \end{cases}$$

Note that η is smooth since $\phi_1 \equiv 1$ on W. We need to explore the definition of η and $d_{\mathbb{C}}\eta$ in terms of our original forms ω_1, ω, and θ_1. On $U \setminus A$, observe that

$$\eta = \phi_1\theta_1 - \theta_1 = \theta_1|_{U\setminus A} - \theta_1 \qquad \text{and}$$

$$d_{\mathbb{C}}\eta = d_{\mathbb{C}}(\phi_1\theta_1 - \theta_1) = \phi_1 d_{\mathbb{C}}\theta_1 + d_{\mathbb{C}}\phi_1 \wedge \theta_1\omega_1 - d_{\mathbb{C}}\theta_1 = \phi_1\omega_1 + d_{\mathbb{C}}\phi_1 \wedge \theta_1\omega_1 - \omega_1 = \omega|_U - \omega_1.$$

On W, $d_{\mathbb{C}}\eta = 0 = \omega|_U - \omega_1$. So we see that on all of U, $d_{\mathbb{C}}\eta = \omega|_U - \omega_1$. Using these equalities, we see that $d_{\mathbb{C}}(\eta, 0) = (d_{\mathbb{C}}\eta, \eta|_{U\setminus A}) = (\omega|_U - \omega_1, \theta|_{U\setminus A} - \theta_1) = (\omega|_U, \theta|_{U\setminus A}) - (\omega_1, \theta_1)$. Thus $(\omega|_U, \theta|_{U\setminus A})$ and (ω_1, θ_1) are in the same coset, and so $(j_1^* + j_2^*)$ maps $[\omega, \theta]$ to $[\omega_1, \theta_1]$. Thus $(j_1^* + j_2^*)$ is surjective.

Now we show that $(j_1^* + j_2^*)$ is an injection. Suppose that $(j_1^* + j_2^*)[\omega, \theta] = [\omega|_U, \theta|_{U\setminus A}] = 0$. Then $d_{\mathbb{C}}(\omega, \theta) = 0$ and there exists $(\omega_1, \theta_1) \in A_{\mathbb{C}}^k(U) \oplus A_{\mathbb{C}}^{k-1}(U \setminus A)$ such that $(\omega_1|_U, \theta_1|_{U\setminus A}) = d_{\mathbb{C}}(\omega_1, \theta_1) = (d_{\mathbb{C}}\omega_1, \omega_1|_{U\setminus A} - d_{\mathbb{C}}\theta_1)$. This implies that

$d_{\mathbb{C}}\omega_1 = \omega|_U$ and $\omega_1|_{U\setminus A} = \theta_1|_{U\setminus A} + d_{\mathbb{C}}\theta_1$. Our goal is to show that $[\omega, \theta] = 0$, or equivalently, to show that $(\omega, \theta) = d_{\mathbb{C}}(\omega', \theta')$ where $(\omega', \theta') \in A^{k-1}_{\mathbb{C}}(X) \oplus A^{k-2}_{\mathbb{C}}(X \setminus A)$. We will now define ω', θ', and $\eta \in A^{k-2}_{\mathbb{C}}(U)$, a form needed in the definition of ω',

$$\theta' = \begin{cases} 0 & \text{on } X \setminus U \\ \phi_1\theta_1 & \text{on } U \setminus A, \end{cases} \quad \eta = \begin{cases} -\phi_2\theta_1 & \text{on } U \setminus A \\ 0 & \text{on } W, \end{cases} \quad \text{and } \omega' = \begin{cases} \theta + d_{\mathbb{C}}\theta' & \text{on } X \setminus A \\ \omega_1 + d_{\mathbb{C}}\eta & \text{on } U. \end{cases}$$

Since each of their parts are smooth, to see that these forms are all well-defined and smooth we only need to show that they agree on the intersection of the spaces in their definitions. If $x \notin U$ then $\phi_1\theta_1 \equiv 0$ on a neighborhood of x as $\phi_1 \equiv 0$ since the support of ϕ_1 is contained in U, and so θ' is well-defined. If $x \in W \setminus A$ then $\phi_2\theta_1 \equiv 0$ on a neighborhood of x as $\phi_2 \equiv 0$ since the support of ϕ_2 is contained in $X \setminus \overline{W}$. Thus η is well-defined. Thus ω' is well-defined because on $U \setminus A$, $\omega_1 + d_{\mathbb{C}}\eta = \theta + d_{\mathbb{C}}\theta'$ as follows:

$$\begin{aligned} (\omega_1 + d_{\mathbb{C}}\eta)|_{U\setminus A} &= \omega_1|_{U\setminus A} + d_{\mathbb{C}}\eta|_{U\setminus A} \\ &= \theta|_{U\setminus A} + d_{\mathbb{C}}\theta_1 - d_{\mathbb{C}}(\phi_2\theta_1) \\ &= \theta|_{U\setminus A} + d_{\mathbb{C}}(\theta_1 - \phi_2\theta_1) \\ &= \theta|_{U\setminus A} + d_{\mathbb{C}}((1 - \phi_2)\theta_1) \\ &= \theta|_{U\setminus A} + d_{\mathbb{C}}(\phi_1\theta_1) \\ &= \theta|_{U\setminus A} + d_{\mathbb{C}}\theta'|_{U\setminus A} \\ &= (\theta + d_{\mathbb{C}}\theta')|_{U\setminus A}. \end{aligned}$$

Using the fact that $d_{\mathbb{C}}(\omega, \theta) = (d_{\mathbb{C}}\omega, \omega|_{X\setminus A} - d_{\mathbb{C}}\theta) = 0$, observe that

$$d_{\mathbb{C}}\omega' = \begin{cases} d_{\mathbb{C}}\theta & \text{on } X \setminus A \\ d_{\mathbb{C}}\omega_1 & \text{on } U. \end{cases} = \begin{cases} \omega|_{X\setminus A} & \text{on } X \setminus A \\ \omega|_U & \text{on } U. \end{cases} = \omega.$$

Thus $d_{\mathbb{C}}(\omega', \theta') = (d_{\mathbb{C}}\omega', \omega'|_{X\setminus A} - d_{\mathbb{C}}\theta') = (\omega, \theta + d_{\mathbb{C}}\theta' - d_{\mathbb{C}}\theta') = (\omega, \theta)$ and so the map $(j_1^* + j_2^*)$ is injective. Thus the mapping $H^k_{\mathrm{DR}}(i_X) \to H^k_{\mathrm{DR}}(i_U)$ is an isomorphism. ■

CHAPTER 4

COMPLEX MANIFOLDS AND HOLOMORPHIC MAPS

4.1 Complex Tangent and Cotangent Spaces

In this chapter we will suppose that X is a complex l-manifold. This implies that X is a smooth real $2l$-manifold. All the results of Chapter 3 hold with dimension $2l$.

Let m be an element of X with U a neighborhood of m. Let $\{z_1, \ldots, z_l\}$ be a set of local holomorphic coordinates for X at m on U, with $z_j = x_j + iy_j$ where x_j and y_j are real valued maps. Thus $\{x_1, y_1, \ldots, x_l, y_l\}$ are local coordinates for the real manifold structure of X at m. Recall that by Equation 3.1 we have the bases $\{\frac{\partial}{\partial x_j}|_m, \frac{\partial}{\partial y_j}|_m \mid 1 \leq j \leq l\}$ of $T_m X$ and $\{(dx_j)_m, (dy_j)_m \mid 1 \leq j \leq l\}$ of $T_m^* X$.

Lemma 4.1 *If X is a complex l-manifold and m in X, let*

$$\beta = \left\{ \tfrac{1}{2}(\tfrac{\partial}{\partial x_j}|_m \otimes 1 - \tfrac{\partial}{\partial y_j}|_m \otimes i),\ \tfrac{1}{2}(\tfrac{\partial}{\partial x_j}|_m \otimes 1 + \tfrac{\partial}{\partial y_j}|_m \otimes i) \mid 1 \leq j \leq l \right\} \text{ and}$$

$$\beta^* = \{((dx_j)_m \otimes 1 + (dy_j)_m \otimes i)\,,\ ((dx_j)_m \otimes 1 - (dy_j)_m \otimes i)\ \mid 1 \leq j \leq l\}\,.$$

Then β and β^* are dual bases of the complex vector spaces $T_m X \otimes_{\mathbb{R}} \mathbb{C}$ and $T_m^* X \otimes_{\mathbb{R}} \mathbb{C}$, respectively.

Proof. First we will see that β^* is a complex basis of $T_m^* X \otimes_{\mathbb{R}} \mathbb{C}$. To see that β^* spans, it is sufficient to notice that we can obtain $(dx_j)_m \otimes 1$, $(dx_j)_m \otimes i$, $(dy_j)_m \otimes 1$, and $(dy_j)_m \otimes i$ for all $1 \leq j \leq l$ from linear combinations of the elements in β^*. To see that β^* is linearly independent, use the fact that $\{(dx_j)_m, (dy_j)_m \mid 1 \leq j \leq l\}$ is a real basis of $T_m^* X$.

Now to see that β and β^* are dual bases:

$$
\begin{aligned}
&((dx_j)_m \otimes 1 + (dy_j)_m \otimes i)(\tfrac{1}{2}(\tfrac{\partial}{\partial x_k}|_m \otimes 1 - \tfrac{\partial}{\partial y_k}|_m \otimes i)) \\
&= \tfrac{1}{2}(dx_j)_m(\tfrac{\partial}{\partial x_k}|_m) \otimes 1 - \tfrac{1}{2}(dx_j)_m(\tfrac{\partial}{\partial y_k}|_m) \otimes i + \\
&\qquad \tfrac{1}{2}(dy_j)_m(\tfrac{\partial}{\partial x_k}|_m) \otimes i + \tfrac{1}{2}(dy_j)_m(\tfrac{\partial}{\partial y_k}|_m) \otimes 1 \\
&= \tfrac{1}{2}\delta_{jk} \otimes 1 + \tfrac{1}{2}\delta_{jk} \otimes 1 \\
&= \delta_{jk} \otimes 1 = \delta_{jk}.
\end{aligned}
$$

Similar arguments show that

$$((dx_j)_m \otimes 1 - (dy_j)_m \otimes i)(\tfrac{1}{2}(\tfrac{\partial}{\partial x_k}|_m \otimes 1 + \tfrac{\partial}{\partial y_k}|_m \otimes i)) = \delta_{jk},$$

$$((dx_j)_m \otimes 1 + (dy_j)_m \otimes i)(\tfrac{1}{2}(\tfrac{\partial}{\partial x_k}|_m \otimes 1 + \tfrac{\partial}{\partial y_k}|_m \otimes i)) = 0,$$

$$((dx_j)_m \otimes 1 - (dy_j)_m \otimes i)(\tfrac{1}{2}(\tfrac{\partial}{\partial x_k}|_m \otimes 1 - \tfrac{\partial}{\partial y_k}|_m \otimes i)) = 0.$$

Therefore, β and β^* are dual bases. ∎

We will use the isomorphisms $T_m^*X \otimes_{\mathbb{R}} \mathbb{C} \cong T_{\mathbb{C},m}^*X$ and $T_mX \otimes_{\mathbb{R}} \mathbb{C} \cong T_{\mathbb{C},m}X$ to find bases of the complex tangent and cotangent spaces that correspond to β and β^*. The conjugate of z_j, $\overline{z_j} : U \to \mathbb{C}$, is defined by $\overline{z_j} = x_j - iy_j$. It follows from Proposition 3.5 that under the isomorphism $\Theta : T_{\mathbb{C},m}^*X \to T_m^*X \otimes_{\mathbb{R}} \mathbb{C}$,

$$\Theta((d_{\mathbb{C}}z_j)_m) = ((dx_j)_m \otimes 1 + (dy_j)_m \otimes i)$$

and

$$\Theta((d_{\mathbb{C}}\overline{z_j})_m) = ((dx_j)_m \otimes 1 - (dy_j)_m \otimes i).$$

We also have the isomorphism $\Phi : T_mX \otimes_{\mathbb{R}} \mathbb{C} \to T_{\mathbb{C},m}X$ given in Proposition 3.2. Define the *partial with respect to* z_j *at* m, $\frac{\partial}{\partial z_j}|_m$, to be the image under the map Φ of

$\frac{1}{2}(\frac{\partial}{\partial x_j}|_m \otimes 1 - \frac{\partial}{\partial y_j}|_m \otimes i)$;

$$\frac{\partial}{\partial z_j}|_m = \frac{1}{2}(\widetilde{\frac{\partial}{\partial x_j}}|_m - i\widetilde{\frac{\partial}{\partial y_j}}|_m). \tag{4.1}$$

Define the *partial with respect to* $\overline{z_j}$ *at* m, $\frac{\partial}{\partial \overline{z_j}}|_m$, to be the image under the map Φ of $\frac{1}{2}(\frac{\partial}{\partial x_j}|_m \otimes 1 + \frac{\partial}{\partial y_j}|_m \otimes i)$;

$$\frac{\partial}{\partial \overline{z_j}}|_m = \frac{1}{2}(\widetilde{\frac{\partial}{\partial x_j}}|_m + i\widetilde{\frac{\partial}{\partial y_j}}|_m). \tag{4.2}$$

Thus $\frac{\partial}{\partial z_j}|_m$ and $\frac{\partial}{\partial \overline{z_j}}|_m$ are $\mathbb{C}$-linear derivations.

We can now restate Lemma 4.1 for complex tangent and cotangent spaces.

Proposition 4.2 *If X is a complex l-manifold and m is in X, then the sets*

$$\{\frac{\partial}{\partial z_j}|_m, \frac{\partial}{\partial \overline{z_j}}|_m \mid 1 \leq j \leq l\} \text{ and } \{(d_{\mathbb{C}} z_j)_m, (d_{\mathbb{C}} \overline{z_j})_m \mid 1 \leq j \leq l\}$$

are dual bases of $T_{\mathbb{C},m}X$ and $T^*_{\mathbb{C},m}X$, respectively.

Proof. The proof follows from the preceding definitions and Lemma 4.1. ∎

Before we continue, observe that we can use the fact that $\frac{\partial}{\partial z_j}|_m$ is a $\mathbb{C}$-linear derivation to see that on rational functions in $z_1, \ldots, z_l$, where the functions are defined, $\frac{\partial}{\partial z_j}|_m$ is actually a partial derivative.

Proposition 4.3 *Let m be in X and n_j be in $\mathbb{Z}$ for $j \in \{1, \ldots, l\}$. If $z_1^{n_1} \cdots z_l^{n_l}(m)$ is defined, then*

$$\frac{\partial}{\partial z_j}|_m(z_1^{n_1} \cdots z_l^{n_l}) = n_j z_1^{n_1} \cdots z_j^{n_j - 1} \cdots z_l^{n_l}(m).$$

Proof. Recall that $z_k = x_k + iy_k$ for $1 \le k \le l$, so

$$\begin{aligned}
\frac{\partial}{\partial z_j}|_m(z_k) &= \tfrac{1}{2}(\widetilde{\tfrac{\partial}{\partial x_j}|_m} - i\widetilde{\tfrac{\partial}{\partial y_j}|_m})(x_k + iy_k) \\
&= \tfrac{1}{2}\left(\tfrac{\partial}{\partial x_j}|_m(x_k) + i\tfrac{\partial}{\partial x_j}|_m(y_k) - i\tfrac{\partial}{\partial y_j}|_m(x_k) + \tfrac{\partial}{\partial y_j}|_m(y_k)\right) \\
&= \tfrac{1}{2}(\delta_{jk} + \delta_{jk}) \\
&= \delta_{jk}.
\end{aligned}$$

Now use the fact that $\frac{\partial}{\partial z_j}|_m$ is a $\mathbb{C}$-linear derivation to see that

$$\frac{\partial}{\partial z_j}|_m(z_k^{n_k}) = n_k z_k^{n_k - 1}(m)\delta_{jk}$$

and that

$$\begin{aligned}
\frac{\partial}{\partial z_j}|_m(z_1^{n_1}\cdots z_l^{n_l}) &= \sum_{k=1}^{l} n_k z_1^{n_1}\cdots z_k^{n_k-1}\cdots z_l^{n_l}(m)\delta_{jk} \\
&= n_j z_1^{n_1}\cdots z_j^{n_j-1}\cdots z_l^{n_l}(m).
\end{aligned}$$

This is the desired result. ∎

4.2 Holomorphic Tangent Space

Recall that U is a neighborhood of m and that $\{z_1, \ldots, z_l\}$ is a set of local holomorphic coordinates for X at m on U. For f in $\mathcal{C}^\infty_{\mathbb{C}}(U)$, f is said to be *holomorphic on U* if $\frac{\partial f}{\partial \bar{z}_j}|_m = 0$ for all $j \in \{1, \ldots, l\}$ and for all m in U. The fact that $\frac{\partial}{\partial \bar{z}_j}|_m$ is a $\mathbb{C}$-linear derivation implies that the sum, difference, product, and quotient (when it is defined) of holomorphic functions are holomorphic. The following computation shows that the function z_k is holomorphic for all $1 \le k \le l$:

$$\frac{\partial}{\partial \bar{z}_j}|_m(z_k) = \tfrac{1}{2}\left(\widetilde{\tfrac{\partial}{\partial x_j}|_m} + i\widetilde{\tfrac{\partial}{\partial y_j}|_m}\right)(z_k)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x_j}|_m(x_k) + i\frac{\partial}{\partial x_j}|_m(y_k) + i\frac{\partial}{\partial y_j}|_m(x_k) - \frac{\partial}{\partial y_j}|_m(y_k)\right)$$

$$= \frac{1}{2}(\delta_{jk} - \delta_{jk}) = 0.$$

Thus polynomial and rational functions on U are holomorphic. The next Proposition gives another method of determining whether a function is holomorphic.

Proposition 4.4 *Let f be in $\mathcal{C}^\infty_{\mathbb{C}}(U)$ with $f = u + iv$. Then f is holomorphic if and only if f satisfies the generalized Cauchy-Riemann equations,*

$$\frac{\partial u}{\partial x_j}|_m = \frac{\partial v}{\partial y_j}|_m \text{ and } \frac{\partial u}{\partial y_j}|_m = -\frac{\partial v}{\partial x_j}|_m,$$

for all $j \in \{1, \ldots, l\}$ and all m in U.

Proof. First let m be in U and notice that

$$\frac{\partial}{\partial \overline{z_j}}|_m(f) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}|_m(u) + i\frac{\partial}{\partial x_j}|_m(v) + i\frac{\partial}{\partial y_j}|_m(u) - \frac{\partial}{\partial y_j}|_m(v)\right)$$

for all $1 \leq j \leq l$ and m in U. Thus

$$\frac{\partial f}{\partial \overline{z_j}}|_m = \frac{1}{2}\left(\frac{\partial u}{\partial x_j}|_m + i\frac{\partial v}{\partial x_j}|_m + i\frac{\partial u}{\partial y_j}|_m - \frac{\partial v}{\partial y_j}|_m\right) \tag{4.3}$$

for all $1 \leq j \leq l$. Now if f is holomorphic, Equation 4.3 implies that

$$0 = \frac{\partial u}{\partial x_j}|_m + i\frac{\partial v}{\partial x_j}|_m + i\frac{\partial u}{\partial y_j}|_m - \frac{\partial v}{\partial y_j}|_m.$$

So the real part is 0 and the imaginary part is 0, and this results in the Cauchy-Riemann equations. For the other direction, assume that the Cauchy-Riemann equations hold for all j, then make substitutions into Equation 4.3 to obtain

$$\frac{\partial f}{\partial \overline{z_j}}|_m = \frac{1}{2}\left(\frac{\partial u}{\partial x_j}|_m + i\frac{\partial v}{\partial x_j}|_m - i\frac{\partial v}{\partial x_j}|_m - \frac{\partial u}{\partial x_j}|_m\right) = 0.$$

Thus we have the desired result. ∎

The complex differential of a holomorphic function has an especially nice form in local coordinates since all the partials with respect to $\overline{z_j}$ are 0. Using this fact together with Proposition 4.3, we obtain the following result.

Corollary 4.5 *Let m be in X and n_j be in $\mathbb{Z}$ for $j \in \{1,\ldots,l\}$. If $z_1^{n_1}\cdots z_l^{n_l}(m)$ is defined, then*

$$(d_{\mathbb{C}}(z_1^{n_1}\cdots z_l^{n_l}))_m = \sum_{j=1}^{l} \tfrac{n_j z_1^{n_1}\cdots z_l^{n_l}(m)}{z_j(m)}(d_{\mathbb{C}} z_j)_m.$$

Proof. If f is in $\mathcal{C}^\infty_{\mathbb{C}}(U)$ and f is holomorphic then $(d_{\mathbb{C}} f)_m = \sum_{j=1}^{l} \frac{\partial f}{\partial z_j}|_m (d_{\mathbb{C}} z_j)_m$ for all m in U. The proof now follows from Proposition 4.3 and the fact that $z_1^{n_1}\cdots z_l^{n_l}$ is holomorphic. ∎

In Proposition 4.2 we saw that $\{\frac{\partial}{\partial z_j}|_m, \frac{\partial}{\partial \overline{z_j}}|_m \mid 1 \le j \le l\}$ is a basis of the complex tangent space $T_{\mathbb{C},m}X$. This basis and the definition of a holomorphic function lead to two new notions of a tangent space that are each subspaces of $T_{\mathbb{C},m}X$. The subspace of $T_{\mathbb{C},m}X$ spanned by $\{\frac{\partial}{\partial z_j}|_m \mid 1 \le j \le l\}$ will be denoted by $T_{H,m}X$ and is called the *holomorphic tangent space at m of X*. Similarly the subspace of $T_{\mathbb{C},m}X$ spanned by $\{\frac{\partial}{\partial \overline{z_j}}|_m \mid 1 \le j \le l\}$ will be denoted by $T_{A,m}X$ and is called the *antiholomorphic tangent space at m of X*. Evidently

$$T_{\mathbb{C},m}X = T_{H,m}X \oplus T_{A,m}X.$$

In the next Proposition we prove that the differential of a holomorphic function maps holomorphic tangent spaces to holomorphic tangent spaces and antiholomorphic tangent spaces to antiholomorphic tangent spaces. To do this we first need the following lemma.

Lemma 4.6 *If f is in $\mathcal{C}^\infty_{\mathbb{C}}(U)$ and f is holomorphic, then* $\frac{\partial}{\partial z_j}|_m(\overline{z} \circ f) = \frac{\partial}{\partial \overline{z_j}}|_m(f)$.

Proof. Using the fact that $z = x + iy$ and $\overline{z} = x - iy$ as well as the definitions of $\frac{\partial}{\partial z_j}|_m$ and $\frac{\partial}{\partial \overline{z_j}}|_m$ given in Equations 4.1 and 4.2, we can expand both sides of the equality. We see that

$$\begin{aligned} \frac{\partial}{\partial z_j}|_m(\overline{z} \circ f) &= \frac{1}{2}\left[\widetilde{\frac{\partial}{\partial x_j}}|_m(\overline{z} \circ f) - i\widetilde{\frac{\partial}{\partial y_j}}|_m(\overline{z} \circ f)\right] \\ &= \frac{1}{2}\left[\frac{\partial}{\partial x_j}|_m(x \circ f) - i\frac{\partial}{\partial x_j}|_m(y \circ f) - i\frac{\partial}{\partial y_j}|_m(x \circ f) - \frac{\partial}{\partial y_j}|_m(y \circ f)\right] \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \overline{z_j}}|_m(f) &= \frac{\partial}{\partial \overline{z_j}}|_m(z \circ f) = \frac{1}{2}\left[\widetilde{\frac{\partial}{\partial x_j}}|_m(z \circ f) + i\widetilde{\frac{\partial}{\partial y_j}}|_m(z \circ f)\right] \\ &= \frac{1}{2}\left[\frac{\partial}{\partial x_j}|_m(x \circ f) + i\frac{\partial}{\partial x_j}|_m(y \circ f) + i\frac{\partial}{\partial y_j}|_m(x \circ f) - \frac{\partial}{\partial y_j}|_m(y \circ f)\right]. \end{aligned}$$

Since f is holomorphic, it satisfies the Cauchy-Riemann Equation $\frac{\partial x \circ f}{\partial y_j}|_m = -\frac{\partial y \circ f}{\partial x_j}|_m$ by Proposition 4.4. Using this substitution in the middle two terms, we see that these two equations are equivalent. ∎

Proposition 4.7 *If f is in $\mathcal{C}^\infty_{\mathbb{C}}(U)$ and f is holomorphic, then*

$$f_*(T_{H,m}X) \subseteq T_{H,f(m)}\mathbb{C} \text{ and } f_*(T_{A,m}X) \subseteq T_{A,f(m)}\mathbb{C}$$

for all m in U.

Proof. Since $\{\frac{\partial}{\partial z_j}|_m, \frac{\partial}{\partial \overline{z_j}}|_m \mid 1 \leq j \leq l\}$ is a basis of $T_{\mathbb{C},m}X$ and $\{\frac{\partial}{\partial z}|_{f(m)}, \frac{\partial}{\partial \overline{z}}|_{f(m)}\}$ is a basis of $T_{\mathbb{C},f(m)}\mathbb{C}$, by the definition of holomorphic and antiholomorphic tangent space it is sufficient to show that

$$f_*\left(\frac{\partial}{\partial z_j}|_m\right) = \alpha\frac{\partial}{\partial z}|_{f(m)} \text{ and } f_*\left(\frac{\partial}{\partial \overline{z_j}}|_m\right) = \beta\frac{\partial}{\partial \overline{z}}|_{f(m)}$$

for some α, β in $\mathbb{C}$. Since $f_*(\frac{\partial}{\partial z_j}|_m)$ is in $T_{\mathbb{C},f(m)}\mathbb{C}$ we can expand it in terms of the basis and use Lemma 4.6 together with the fact that f is holomorphic to obtain

$$\begin{aligned} f_*\left(\frac{\partial}{\partial z_j}|_m\right) &= f_*\left(\frac{\partial}{\partial z_j}|_m\right)(z)\frac{\partial}{\partial z}|_{f(m)} + f_*\left(\frac{\partial}{\partial z_j}|_m\right)(\overline{z})\frac{\partial}{\partial \overline{z}}|_{f(m)} \\ &= \frac{\partial}{\partial z_j}|_m(z\circ f)\frac{\partial}{\partial z}|_{f(m)} + \frac{\partial}{\partial z_j}|_m(\overline{z}\circ f)\frac{\partial}{\partial \overline{z}}|_{f(m)} \\ &= \frac{\partial}{\partial z_j}|_m(f)\frac{\partial}{\partial z}|_{f(m)} + \frac{\partial}{\partial \overline{z_j}}|_m(f)\frac{\partial}{\partial \overline{z}}|_{f(m)} \\ &= \frac{\partial}{\partial z_j}|_m(f)\frac{\partial}{\partial z}|_{f(m)}. \end{aligned}$$

Similarly $f_*(\frac{\partial}{\partial \overline{z_j}}|_m) = \frac{\partial}{\partial \overline{z_j}}|_m(f)\frac{\partial}{\partial z}|_{f(m)} + \frac{\partial}{\partial \overline{z_j}}|_m(\overline{z_j}\circ f)\frac{\partial}{\partial \overline{z}}|_{f(m)} = \frac{\partial}{\partial \overline{z_j}}|_m(\overline{z_j}\circ f)\frac{\partial}{\partial \overline{z}}|_{f(m)}$. ∎

If f is in $\mathcal{C}^\infty_{\mathbb{C}}(U)$ and f is holomorphic, Proposition 4.7 implies that we have another notion of differential,

$$\overline{f_*} : T_{H,m}X \to T_{H,f(m)}\mathbb{C},$$

called the *holomorphic differential*. We now have several notions of differentials. The real differential $(df)_m : T_mX \to T_{f(m)}\mathbb{C}$, the real differential with scalars extended to the complexes $(df)_m \otimes 1 : T_mX \otimes_{\mathbb{R}} \mathbb{C} \to T_{f(m)}\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, the complex differential $f_* : T_{\mathbb{C},m}X \to T_{\mathbb{C},f(m)}\mathbb{C}$, and now the holomorphic differential $\overline{f_*} : T_{H,m}X \to T_{H,f(m)}\mathbb{C}$. We also have maps that relate the domains and ranges of these maps. We have the natural inclusion map $\iota : T_mX \to T_mX \otimes_{\mathbb{R}} \mathbb{C}$ that maps θ to $\theta \otimes 1$, the isomorphism $\Phi : T_mX \otimes_{\mathbb{R}} \mathbb{C} \to T_{\mathbb{C},m}X$ that maps $\theta \otimes \alpha$ to $\widetilde{\alpha\theta}$ given in Proposition 3.2, and the projection map $\rho : T_{\mathbb{C},m}X \to T_{H,m}X$ that sends $\sum_{j=1}^l \alpha_j \frac{\partial}{\partial z_j}|_m + \sum_{j=1}^l \beta_j \frac{\partial}{\partial \overline{z_j}}|_m$ to $\sum_{j=1}^l \alpha_j \frac{\partial f}{\partial z_j}|_m$. In addition, we have the map $\Upsilon : T_{\mathbb{C},f(m)}\mathbb{C} \to \mathbb{C}$ and its restriction to holomorphic tangent space $\overline{\Upsilon} : T_{H,f(m)}\mathbb{C} \to \mathbb{C}$. Combining all these maps yields the following diagram:

$$\begin{array}{ccccccc} T_mX & \xrightarrow{\iota} & T_mX\otimes_{\mathbb{R}}\mathbb{C} & \xrightarrow{\Phi} & T_{\mathbb{C},m}X & \xrightarrow{\rho} & T_{H,m}X \\ \big\downarrow (df)_m & & \big\downarrow (df)_m\otimes 1 & & \big\downarrow f_* & & \big\downarrow \overline{f_*} \\ T_{f(m)}\mathbb{C} & \xrightarrow{\iota} & T_{f(m)}\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} & \xrightarrow{\Phi} & T_{\mathbb{C},f(m)}\mathbb{C} & \xrightarrow{\rho} & T_{H,f(m)}\mathbb{C} \\ & & & & \big\downarrow \Upsilon & & \big\downarrow \overline{\Upsilon} \\ & & & & \mathbb{C} & \xrightarrow{id} & \mathbb{C}. \end{array} \tag{4.4}$$

Lemma 4.8 *Diagram 4.4 commutes.*

Proof. By the definitions, the first and last squares on the top row as well as the bottom square commute. The middle square commutes by the definition of the complex differential given in Equation 3.2. ∎

By composing the maps across the top of the diagram we get a mapping that we will label $\Lambda = \rho\circ\Phi\circ\iota : T_mX \to T_{H,m}X$. We will show in the next lemma that Λ is a real vector space isomorphism.

Lemma 4.9 *The map $\Lambda : T_mX \to T_{H,m}X$ is an isomorphism of real vector spaces.*

Proof. By Proposition 3.3 the set $\{\frac{\partial}{\partial x_j}|_m, \frac{\partial}{\partial y_j}|_m \mid 1\le j\le l\}$ is a real basis of T_mX. The image of these basis elements under $\Phi\circ\iota$ is $\widetilde{\frac{\partial}{\partial x_j}}|_m$ and $\widetilde{\frac{\partial}{\partial y_j}}|_m$. Notice that we can use Equations 4.1 and 4.2 to solve for $\widetilde{\frac{\partial}{\partial x_j}}|_m$ and $\widetilde{\frac{\partial}{\partial y_j}}|_m$ in terms of $\frac{\partial}{\partial z_j}|_m$ and $\frac{\partial}{\partial \overline{z_j}}|_m$. As a result we see that

$$\widetilde{\frac{\partial}{\partial x_j}}|_m = \frac{\partial}{\partial z_j}|_m + \frac{\partial}{\partial \overline{z_j}}|_m \text{ and } \widetilde{\frac{\partial}{\partial y_j}}|_m = i\frac{\partial}{\partial z_j}|_m - i\frac{\partial}{\partial \overline{z_j}}|_m.$$

Consider how Λ acts on these basis elements:

$$\Lambda\left(\frac{\partial}{\partial x_j}|_m\right) \;=\; \rho\circ\Phi\circ\iota\left(\frac{\partial}{\partial x_j}|_m\right) = \rho\circ\Phi\left(\frac{\partial}{\partial x_j}|_m\otimes 1\right)$$

$$= \rho\left(\widetilde{\frac{\partial}{\partial x_j}|_m}\right) = \rho\left(\frac{\partial}{\partial z_j}|_m + \frac{\partial}{\partial \overline{z_j}}|_m\right) = \frac{\partial}{\partial z_j}|_m$$

and

$$\begin{aligned}\Lambda\left(\frac{\partial}{\partial y_j}|_m\right) &= \rho \circ \Phi \circ \iota\left(\frac{\partial}{\partial y_j}|_m\right) = \rho \circ \Phi\left(\frac{\partial}{\partial y_j}|_m \otimes 1\right) \\ &= \rho\left(\widetilde{\frac{\partial}{\partial y_j}|_m}\right) = \rho\left(i\frac{\partial}{\partial z_j}|_m - i\frac{\partial}{\partial \overline{z_j}}|_m\right) = i\frac{\partial}{\partial z_j}|_m.\end{aligned}$$

Now by definition, $T_{H,m}X$ has real basis $\{\frac{\partial}{\partial z_j}|_m, i\frac{\partial}{\partial z_j}|_m \mid 1 \leq j \leq l\}$. Thus Λ maps a real basis to a real basis and is therefore an isomorphism. ∎

We can use this fact that the maps across the top and middle rows of Diagram 4.4 are vector space isomorphisms together with the fact that Diagram 4.4 commutes to establish relationships between the maps in Diagram 4.4. In particular, for the proof of the extension of Brieskorn's Lemma we need to show that $(df)_m$ is surjective.

Proposition 4.10 *If f is a holomorphic function in $\mathcal{C}^\infty_{\mathbb{C}}(U)$ and m is in U then $(df)_m$ is surjective if and only if $(d_{\mathbb{C}}f)_m$ is surjective.*

Proof. Using Lemmas 4.8 and 4.9 we see that $(df)_m$ is surjective if and only if $\overline{f_*}$ is surjective. In Chapter 3 we defined $(d_{\mathbb{C}}f)_m$ to be the composition of the maps $f_* : T_{\mathbb{C},m}X \to T_{\mathbb{C},f(m)}\mathbb{C}$ and $\Upsilon : T_{\mathbb{C},f(m)}\mathbb{C} \to \mathbb{C}$, where Υ was defined on the basis $\{\widetilde{\frac{\partial}{\partial x}}|_{f(m)}, \widetilde{\frac{\partial}{\partial y}}|_{f(m)}\}$ of $T_{\mathbb{C},f(m)}\mathbb{C}$ as follows: $\Upsilon(\widetilde{\frac{\partial}{\partial x}}|_{f(m)}) = 1$ and $\Upsilon(\widetilde{\frac{\partial}{\partial y}}|_{f(m)}) = i$. By Proposition 4.2 we now know that $\{\frac{\partial}{\partial z}|_f(m), \frac{\partial}{\partial \overline{z}}|_f(m)\}$ is another basis of $T_{\mathbb{C},f(m)}\mathbb{C}$. Using the definition of Υ and the definition of the partial derivatives we see that

$$\Upsilon(\frac{\partial}{\partial z}|_m) = 1 \text{ and } \Upsilon(\frac{\partial}{\partial \overline{z}}|_m) = 0.$$

Thus the mapping Υ restricted to holomorphic tangent space gives a complex vector space isomorphism onto $\mathbb{C}$. This action of Υ on the basis of $T_{\mathbb{C},f(m)}$ together with the fact from Proposition 4.7 that the differential of a holomorphic function maps holomorphic tangent space into holomorphic tangent space implies that $\overline{f_*}$ is surjective if and only if $(d_{\mathbb{C}}f)_m$ is surjective. This completes the proof. ∎

CHAPTER 5

GENERATORS OF TORAL ARRANGEMENTS

We will define a toral arrangement and then use Corollary 2.14 to arrive at the generators of the cohomology of the complement of a toral arrangement in certain cases. We will conclude by looking at some examples. For this chapter we will assume that our field is $\mathbb{C}$ and that the cohomology is any cohomology that satisfies the conditions given in Chapter 2 with coefficients in $\mathbb{C}$.

5.1 Theorem 1.1 for Certain Toral Arrangements

Assume that $\{z_1, \ldots, z_l\}$ are the coordinate functions on $\mathbb{C}^l$. Let $T = (\mathbb{C}^*)^l$, then T is a complex l-dimensional torus. By [Spr, Chapter 2] we see that T is an algebraic group under multiplication. A *rational character of* T is an algebraic group homomorphism $\chi : T \to \mathbb{C}^*$. Let $\mathcal{A} = \{\ker(\chi'_1), \ldots, \ker(\chi'_t)\}$ be a finite set of kernels of characters of T. The pair $(T, \mathcal{A})$ is called a *toral arrangement over* $\mathbb{C}$. The complement of the arrangement is

$$M = T \setminus \bigcup_{i=1}^{t} \ker(\chi'_i).$$

For a character χ of T if we let the *zero set of* $\chi - 1$ be the set $Z(\chi-1) = (\chi-1)^{-1}(0)$, then $Z(\chi - 1) = \ker \chi$. So

$$M = T \setminus \bigcup_{i=1}^{t} Z(\chi'_i - 1).$$

The set of regular functions on T, $\mathbb{C}[T]$, is the localization of $\mathbb{C}[z_1, \ldots, z_l]$ at the function $z_1 \cdots z_l$ [Spr, Theorem 2.5.2]. Thus

$$\mathbb{C}[T] = \mathbb{C}[z_1, \ldots, z_l]_{z_1 \cdots z_l} = \mathbb{C}[z_1, \ldots, z_l, z_1^{-1}, \ldots, z_l^{-1}].$$

By Exercise III.4.6 in [Hu] the localization of a unique factorization domain (UFD) is a UFD, and so $\mathbb{C}[T]$ is a UFD. Thus we can talk about elements of $\mathbb{C}[T]$ being irreducible. By Exercise 2.5.12 in [Spr] a character χ of T is of the form $\chi = z_1^{n_1} \ldots z_l^{n_l}$ where n_i is an integer (positive, negative, or 0). Therefore, every character of T is a regular function of T and is consequently in $\mathbb{C}[T]$. We will consider the characters from our arrangement $\mathcal{A}$. Proposition 3.4 of [Dou] implies that for each k, there is a factorization of $\chi'_k - 1$ into a product of irreducible factors $\chi''_k - \mu$ where χ''_k is a character of T with a connected kernel and μ is a n^{th} root of unity for some n. These may not all be distinct, so we let $\chi_1 - \mu_1, \ldots, \chi_s - \mu_s$ be the distinct functions that come from the factors of the $(\chi'_k - 1)$'s. We can use this and the definition of zero set to simplify M. We have

$$\begin{aligned} M &= T \setminus \bigcup_{k=1}^{t} Z(\chi'_k - 1) \\ &= T \setminus \bigcup_{k=1}^{t} Z\left((\chi''_{k_1} - \mu_{k_1}) \ldots (\chi''_{k_{n_k}} - \mu_{k_{n_k}})\right) \\ &= T \setminus \bigcup_{k=1}^{t} \left(\bigcup_{j=1}^{n_k} Z(\chi''_{k_j} - \mu_{k_j}) \right) \\ &= T \setminus \bigcup_{i=1}^{s} Z(\chi_i - \mu_i). \end{aligned}$$

Thus in considering the complement of the arrangement it is sufficient to use the complement of the zero sets of the distinct irreducible factors of $\chi'_1 - 1, \ldots, \chi'_t - 1$. We can view T as $\mathbb{C}^l \setminus \bigcup_{i=1}^{l} Z(z_i)$. So,

$$M = \mathbb{C}^l \setminus \left(\bigcup_{i=1}^{l} Z(z_i) \cup \bigcup_{i=1}^{s} Z(\chi_i - \mu_i) \right).$$

Set $f_i = z_i$ for $1 \leq i \leq l$ and $f_{l+k} = \chi_k - \mu_k$ for $1 \leq k \leq s$. The zero set of f_j will be labeled by Z_j for $1 \leq j \leq l+s$. Then for $0 \leq k \leq l+s$ define M_k as follows:

$$M_k = \begin{cases} \mathbb{C}^l & \text{if } k = 0 \\ \mathbb{C}^l \setminus (\bigcup_{i=1}^{k} Z_i) & \text{if } 1 \leq k \leq l+s \ . \end{cases}$$

Note that $M_{l+s} = M$ the complement of the toral arrangement.

Next we want to apply Corollary 2.14, so we must first show that 0 is a regular value of $f_r|_{M_{r-1}}$ for $1 \leq r \leq l+s$. By Exercise 2.5.12 in [Spr] a character χ of T is of the form $\chi = z_1^{n_1} \dots z_l^{n_l}$ where n_i is an integer (positive, negative, or 0).

Lemma 5.1 *If* $1 \leq r \leq l+s$ *then 0 is a regular value of* $f_r|_{M_{r-1}}$.

Proof. Since M_{r-1} is an open subset of $\mathbb{C}^l$, by Lemma 2.3 it suffices to show that 0 is a regular value of f_r. Let m be an element of $\mathbb{C}^l$ such that $f_r(m) = 0$. We need to show that $(df_r)_m : T_m\mathbb{C}^l \to T_0\mathbb{C}$ is surjective. According to Proposition 4.10, this holds if and only if $(d_{\mathbb{C}} f_r)_m : T_{\mathbb{C},m}\mathbb{C}^l \to \mathbb{C}$ is surjective. We divide the argument into two cases dependent on r.

If $1 \leq r \leq l$, then $f_r = z_r$. In this case we have that $(d_{\mathbb{C}} z_r)_m : T_{\mathbb{C},m}\mathbb{C}^l \to \mathbb{C}$ is a non-zero linear functional, and therefore the mapping is surjective.

On the other hand, suppose $l+1 \leq r \leq l+s$. Then as $f_r = \chi_{r-l} - \mu_{r-l}$ (a rational character of T minus a root of unity), we can write f_r as $z_1^{n_1} \dots z_l^{n_l} - \mu_{r-l}$ where n_j is in $\mathbb{Z}$ for $j \in \{1, \dots, l\}$. Let $z_j(m) = m_j$ for $1 \leq j \leq l$. The fact that $f_r(m) = 0$ results in $z_1^{n_1} \dots z_l^{n_l}(m) = m_1^{n_1} \dots m_l^{n_l} = \mu_{r-l}$, which implies that m_j does not equal 0 for all j. Now using Corollary 4.5 we see that

$$(d_{\mathbb{C}} f_r)_m = (d_{\mathbb{C}}(z_1^{n_1} \cdots z_l^{n_l} - \mu_{r-l}))_m = \sum_{j=1}^{l} \tfrac{n_j z_1^{n_1} \cdots z_l^{n_l}(m)}{z_j(m)} (d_{\mathbb{C}} z_j)_m = \textstyle\sum_{j=1}^{l} \frac{n_j \mu_{r-l}}{m_j} (d_{\mathbb{C}} z_j)_m.$$

This is not zero since at least one of the n_j's is non-zero. Moreover since $(d_{\mathbb{C}} f_r)_m$ is $\mathbb{C}$-linear, we have that it is surjective. ∎

Now we will prove the main theorem.

Theorem 5.2 *If, for all $l+1 \leq r \leq l+s$, $H^*(M_{r-1} \cap Z_r)$ is generated as a $\mathbb{C}$-algebra by the set*

$$\{(z_1|_{(M_{r-1}\cap Z_r)})^*(\sigma), \quad \ldots, \quad (z_l|_{(M_{r-1}\cap Z_r)})^*(\sigma),$$
$$(\chi_1|_{(M_{r-1}\cap Z_r)} - \mu_1)^*(\sigma), \ldots, (\chi_{r-l}|_{(M_{r-1}\cap Z_r)} - \mu_{r-l})^*(\sigma)\},$$

then as a $\mathbb{C}$-algebra $H^*(M)$ is generated by the set

$$\{(z_1|_M)^*(\sigma), \ldots, (z_l|_M)^*(\sigma), (\chi_1|_M - \mu_1)^*(\sigma), \ldots, (\chi_s|_M - \mu_s)^*(\sigma)\}.$$

Proof. The coordinate functions are $\{z_1, \ldots, z_l\}$. So

$$f_1 = z_1, \ldots, f_l = z_l, f_{l+1} = z_1^{n_{1,1}} \cdots z_l^{n_{1,l}} - \mu_1, \ldots, f_{l+s} = z_1^{n_{s,1}} \cdots z_l^{n_{s,l}} - \mu_s,$$

where $s \geq 1$. By Lemma 5.1 we know that 0 is a regular value of $f_r|_{M_{r-1}}$ for all r. The fact that $\chi_1 - \mu_1, \ldots, \chi_s - \mu_s$ are distinct and irreducible implies that $Z_r \not\subseteq \bigcup_{i=1}^{r-1} Z_i$ for all r. It is well known that $H^*(M_l)$ is generated by the set $\{(z_1|_M)^*(\sigma), \ldots, (z_l|_M)^*(\sigma)\}$ (this is a simple corollary of Theorem 1.1). Therefore, by Corollary 2.14 we know $H^*(M)$ is generated by the set

$$\{(z_1|_M)^*(\sigma), \ldots, (z_l|_M)^*(\sigma), (\chi_1|_M - \mu_1)^*(\sigma), \ldots, (\chi_s|_M - \mu_s)^*(\sigma)\}.$$

This is the desired result. ∎

If we use de Rham cohomology with complex coefficients we can rewrite the results of Theorem 5.2. Recall that in de Rham cohomology the fixed generator of $H^1_{\mathrm{DR}}(\mathbb{C}^*)$

is $[\frac{d_{\mathbb{C}}z}{z}]$, so we can use this in place of σ in Theorem 5.2. Moreover by Proposition 3.10 we also know that $g^*[\frac{d_{\mathbb{C}}z}{z}] = [\frac{d_{\mathbb{C}}g}{g}]$. Therefore, we can rewrite the results of Theorem 5.2 as follows.

Theorem 5.3 *If, for all $l+1 \leq r \leq l+s$, $H^*_{\mathrm{DR}}(M_{r-1} \cap Z_r)$ is generated as a $\mathbb{C}$-algebra by the set*

$$\left\{\left[\frac{d_{\mathbb{C}}z_1|_{(M_{r-1}\cap Z_r)}}{z_1|_{(M_{r-1}\cap Z_r)}}\right], \ldots, \left[\frac{d_{\mathbb{C}}z_l|_{(M_{r-1}\cap Z_r)}}{z_l|_{(M_{r-1}\cap Z_r)}}\right], \left[\frac{d_{\mathbb{C}}\chi_1|_{(M_{r-1}\cap Z_r)}}{(\chi_1-\mu_1)|_{(M_{r-1}\cap Z_r)}}\right], \ldots, \left[\frac{d_{\mathbb{C}}\chi_{r-l}|_{(M_{r-1}\cap Z_r)}}{(\chi_{r-l}-\mu_{r-l})|_{(M_{r-1}\cap Z_r)}}\right]\right\},$$

then $H^*_{\mathrm{DR}}(M;\mathbb{C})$ is generated as a $\mathbb{C}$-algebra by the set

$$\left\{\left[\frac{d_{\mathbb{C}}z_1|_M}{z_1|_M}\right], \ldots, \left[\frac{d_{\mathbb{C}}z_l|_M}{z_l|_M}\right], \left[\frac{d_{\mathbb{C}}\chi_1|_M}{(\chi_1-\mu_1)|_M}\right], \ldots, \left[\frac{d_{\mathbb{C}}\chi_s|_M}{(\chi_s-\mu_s)|_M}\right]\right\}.$$

5.2 Examples

In the first example we will show a case where Theorem 5.2 applies, and in the second example we will show a case where it does not apply. The crucial observation is that in the first example the $M_{r-1} \cap Z_r$ is isomorphic to a complement of $r-1$ irreducible hypersurfaces in a smaller rank torus and so by induction on the rank of the torus we get the desired result. In the second example this does not occur.

Example 1: Let $T = (\mathbb{C}^*)^2$ and $\mathcal{A} = \{\ker(z_1^2), \ker(z_2^2), \ker(z_1 z_2)\}$. Translating to zero sets we have that $\ker(z_1^2) = Z(z_1^2 - 1)$ and $z_1^2 - 1 = (z_1-1)(z_1+1)$ and similarly for z_2^2, but $z_1 z_2 - 1$ is already irreducible. So the complement is

$$\begin{aligned} M = \mathbb{C}^2 \quad \setminus \quad & (Z(z_1) \cup Z(z_2) \cup Z(z_1-1) \cup Z(z_1+1) \\ & \cup Z(z_2-1) \cup Z(z_2+1) \cup Z(z_1 z_2)) \end{aligned}$$

Let $f_1 = z_1$, $f_2 = z_2$, $f_3 = z_1 - 1$, $f_4 = z_1 + 1$, $f_5 = z_2 - 1$, $f_6 = z_2 + 1$, and $f_7 = z_1 z_2$. We will use Corollary 2.13 to find the generators of $H^*(M_7)$. By Lemma 5.1 we already know that 0 is a regular value for $f_r|_{M_{r-1}}$ for each r. So at each stage it is enough to know the generators of $H^*(M_{r-1})$ and the generators of $H^*(M_{r-1} \cap Z_r)$.

By Theorem 1.1 $H^*(M_2)$ is generated by the set $\{z_1|^*_{M_2}(\sigma), z_2|^*_{M_2}(\sigma)\}$ where $M_2 = \{(\alpha, \beta) \mid \alpha, \beta \neq 0\}$. Now consider M_3. The set $M_2 \cap Z_3 = \{(1, \beta) \mid \beta \neq 0\}$ which is isomorphic to $\mathbb{C}^*$ under the projection map $z_2|_{M_2 \cap Z_3}$. So $H^*(M_2 \cap Z_3)$ is generated by $\{z_2|^*_{M_2 \cap Z_3}(\sigma)\}$. Thus $H^*(M_3)$ is generated by $\{z_1|^*_{M_3}(\sigma), z_2|^*_{M_3}(\sigma), (z_1|_{M_3} - 1)^*(\sigma)\}$.

Next we consider M_4. The set $M_3 \cap Z_4 = \{(-1, \beta) \mid \beta \neq 0\}$ which is isomorphic to $\mathbb{C}^*$ under the projection map $z_2|_{M_3 \cap Z_4}$. So $H^*(M_3 \cap Z_4)$ is generated by $\{z_2|^*_{M_3 \cap Z_4}(\sigma)\}$. Thus $H^*(M_4)$ is generated by $\{\{z_1|^*_{M_4}(\sigma), z_2|^*_{M_4}(\sigma), (z_1|_{M_4} - 1)^*(\sigma), (z_1|_{M_4} + 1)^*(\sigma)\}$.

For M_5 we have the set $M_4 \cap Z_5 = \{(\alpha, 1) \mid \alpha \neq -1, 0, 1,\}$ which is isomorphic to $\mathbb{C} \setminus (Z(z) \bigcup Z(z-1) \bigcup Z(z+1))$ by the projection map $z_1|_{M_4 \cap Z_5}$. Now let $f'_1 = z$, $f'_2 = z - 1$, and $f'_3 = z + 1$, let $Z'_j = Z(f'_j)$, and let $M'_j = \mathbb{C} \setminus \bigcup_{i=1}^{j} Z'_i$. Again by Theorem 1.1 $H^*(M'_1)$ is generated by $\{z|^*_{M'_1}(\sigma)\}$. The set $M'_1 \cap Z'_2 = \{1\}$ and so $H^*(M'_1 \cap Z'_2) \cong \mathbb{C}$ and then by Corollary 2.13 $H^*(M'_2)$ is generated by $\{z|^*_{M'_2}(\sigma)(z|_{M'_2} - 1)^*(\sigma)\}$. The set $M'_2 \cap Z'_3 = \{-1\}$ and so $H^*(M'_2 \cap Z'_3) \cong \mathbb{C}$ and then by Corollary 2.13 $H^*(M'_3)$ is generated by $\{z|^*_{M'_3}(\sigma)(z|_{M'_3} - 1)^*(\sigma), (z|_{M'_3} + 1)^*(\sigma)\}$. The set $M'_3 = \mathbb{C} \setminus (Z(z) \bigcup Z(z-1) \bigcup Z(z+1))$ and so by the pullback $(z_1|_{M_4 \cap Z_5})^*$ the set $H^*(M_4 \cap Z_5)$ is generated by

$$\{\{z_1|^*_{M_4 \cap Z_5}(\sigma), (z_1|_{M_4 \cap Z_5} - 1)^*(\sigma), (z_1|_{M_4 \cap Z_5} + 1)^*(\sigma)\}.$$

Thus $H^*(M_5)$ is generated by

$$\{\{z_1|^*_{M_5}(\sigma), z_2|^*_{M_5}(\sigma), (z_1|_{M_5} - 1)^*(\sigma), (z_1|_{M_5} + 1)^*(\sigma), (z_2|_{M_5} - 1)^*(\sigma)\}.$$

For M_6 we have the set $M_5 \cap Z_6 = \{(\alpha, -1) \mid \alpha \neq -1, 0, 1, \}$. So as in $M_4 \cap Z_5$ we have that $H^*(M_5 \cap Z_6)$ is generated by

$$\{\{z_1|^*_{M_5 \cap Z_6}(\sigma), (z_1|_{M_5 \cap Z_6} - 1)^*(\sigma), (z_1|_{M_5 \cap Z_6} + 1)^*(\sigma)\}.$$

Thus $H^*(M_6)$ is generated by

$$\{\{z_1|^*_{M_6}(\sigma), z_2|^*_{M_6}(\sigma), (z_1|_{M_6} - 1)^*(\sigma), (z_1|_{M_6} + 1)^*(\sigma), (z_2|_{M_6} - 1)^*(\sigma), (z_2|_{M_6} + 1)^*(\sigma)\}.$$

Lastly for M_7 we have the set $M_6 \cap Z_7 = \{(\alpha, \frac{1}{\alpha}) \mid \alpha \neq -1, 0, 1, \}$. So as in $M_4 \cap Z_5$ we have that $H^*(M_6 \cap Z_7)$ is generated by

$$\{\{z_1|^*_{M_6 \cap Z_7}(\sigma), (z_1|_{M_6 \cap Z_7} - 1)^*(\sigma), (z_1|_{M_6 \cap Z_7} + 1)^*(\sigma)\}.$$

Thus $H^*(M_7)$ is generated by

$$\{\{z_1|^*_{M_7}(\sigma), z_2|^*_{M_7}(\sigma), \quad (z_1|_{M_7} - 1)^*(\sigma), \quad (z_1|_{M_7} + 1)^*(\sigma), (z_2|_{M_7} - 1)^*(\sigma),$$
$$(z_2|_{M_7} + 1)^*(\sigma), \quad (z_1 z_2)|^*_{M_7}(\sigma)\}.$$

Example 2: Let $T = (C^*)^2$ and $\mathcal{A} = \{\ker(z_1), \ker(z_1^2 z_2^3)\}$. Notice that $z_1^2 z_2^3 - 1$ is irreducible, so the complement is

$$M = \mathbb{C}^2 \setminus \left(Z(z_1) \cup Z(z_2) \cup Z(z_1 - 1) \cup Z(z_1^2 z_2^3 - 1) \right).$$

We will try to use Corollary 2.13 to find the generators of $H^*(M_4)$. By Lemma 5.1 we already know that 0 is a regular value for $f_r|_{M_{r-1}}$ for each r. So at each stage it is enough to know the generators of $H^*(M_{r-1})$ and the generators of $H^*(M_{r-1} \cap Z_r)$. We will try the problem both ways, with $z_1^2 z_2^3 - 1$ first after the hyperplanes z_1 and z_2 and with it second. We will see that regardless of the order the hypotheses of Theorem 5.2 do not apply.

First let $f_1 = z_1$, $f_2 = z_2$, $f_3 = z_1^2 z_2^3 - 1$, $f_4 = z_1 - 1$. Here it is easier to find M_3 directly than to use Corollary 2.13. Since $\ker z_1^2 z_2^3$ is connected and one dimensional it is isomorphic to $\mathbb{C}^*$, thus with out loss of generality we can change the coordinates so that $z_1' = z_1^2 z_2^3$. Then $M_3 = \{(\alpha, \beta) \mid \alpha, \beta \neq 0, \alpha \neq 1\} \cong (\mathbb{C} \setminus \{0,1\}) \times \mathbb{C}^*$. So by the Künneth Formula 2.1 $H^*(M_3) = H^*(\mathbb{C} \setminus \{0,1\}) \otimes H^*(\mathbb{C}^*)$. Thus $H^1(M_3) = H^0(\mathbb{C} \setminus \{0,1\}) \otimes H^1(\mathbb{C}^*) + H^1(\mathbb{C} \setminus \{0,1\}) \otimes H^0(\mathbb{C}^*) \cong \mathbb{C} \otimes \mathbb{C} + \mathbb{C}^2 \otimes \mathbb{C} \cong \mathbb{C}^3$, and so $H^*(M_3)$ has 3 generators. Notice that in the original coordinates, $M_3 = \{(\alpha, \beta) \mid \alpha, \beta \neq 0, \alpha^2\beta^3 \neq 1\}$ and $Z_4 = \{(1, \beta)\}$. Thus the intersection $M_3 \cap Z_4 = \{(1, \beta) \mid \beta \neq 0, \beta^3 \neq 1\}$, which is isomorphic to $\mathbb{C}$ minus 4 points (0, 1, ω, and ω^2 where ω and ω^2 are primitive 3^{rd} roots of unity). As in Example 1, $H^*(M_3 \cap Z_4)$ is generated by $\{z_2|^*_{M_3 \cap Z_4}(\sigma), (z_2|_{M_3 \cap Z_4} - 1)^*(\sigma), (z_2|_{M_3 \cap Z_4} - \omega)^*(\sigma), (z_2|_{M_3 \cap Z_4} - \omega^2)^*(\sigma)\}$. However in order for j^* from Theorem 2.12 to be surjective there should be at most three generators (since $\dim H^1(M_3) = 3$). Therefore, the hypotheses of Theorem 5.2 do not apply.

Second, let $f_1 = z_1$, $f_2 = z_2$, $f_3 = z_1 - 1$, $f_4 = z_1^2 z_2^3 - 1$. To make $M_3 \cap Z_4$ and M_4 easier to find we will change coordinates. Let $z_1' = z_1^2 z_2^3$ then $z_2' = z_1 z_2$. We solve this to see that in these new coordinates $z_1 = (z_1')^{-1}(z_2')^3$ and $z_2 = z_1'(z_2')^{-2}$. In the original coordinates $M_3 = \{(\alpha, \beta) \mid \alpha, \beta \neq 0, \alpha \neq 1\} \cong (\mathbb{C} \setminus \{0,1\}) \times \mathbb{C}^*$, and we saw above that it has 3 generators. In the new coordinates $M_3 = \{(\alpha', \beta') \mid \alpha', \beta' \neq 0, \frac{\beta'^3}{\alpha'} \neq 1\}$

and $Z_4 = \{(1, \beta')\}$. So $M_3 \cap Z_4 = \{(1, \beta') \mid \beta' \neq 0, {\beta'}^3 \neq 0\}$ which is isomorphic to $\mathbb{C}$ minus 4 points. As before $H^*(M_3 \cap Z_4)$ has 4 generators. So once again j^* cannot be surjective. Therefore, the hypotheses of Theorem 5.2 do not apply to this case.

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