# Fluctuational Transitions through a Fractal Basin Boundary

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Fluctuational transitions between two coexisting chaotic attractors, separated by a fractal basin boundary, are studied in a discrete dynamical system. It is shown that the transition mechanism is determined by a hierarchy of homoclinic points. The most probable escape path from a chaotic attractor to the fractal boundary is found using both statistical analyses of fluctuational trajectories and the Hamiltonian theory of fluctuations.

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The mechanism of fluctuational escape from a chaotic attractor (CA) across a fractal basin boundary (FBB) represents one of the most challenging unsolved problems in fluctuation theory [1–4]. The unpredictable and highly complex stochastic behavior of such systems arises in part from the presence of limit sets of complex geometrical structure, and in part from the fractality of the basin boundary [5,6]. For this reason, the central question—whether or not there exists a generic mechanism for fluctuational transitions across the FBB—has remained unanswered. It has been unclear (i) if boundary conditions can be found both on the CA and on the FBB, (ii) if there exits a unique escape path from the CA to the FBB, (iii) whether this path can be determined using the Hamiltonian theory of fluctuations, (iv) if there is any deterministic structure involved in the transition, and (v) what influence is exerted by the noise intensity. If transitions across FBBs are characterized by general features, a knowledge of them could considerably simplify analyses of both stability and control for chaotic dynamical systems, which are problems of broad interdisciplinary interest [7,8].

A promising approach to this problem is based on the analysis of fluctuations in the limit of small noise intensity: the system fluctuates to remote states along most probable deterministic paths [9–11] that correspond to rays in the WKB-like asymptotic solution of the Fokker-Planck equation [12]. The approach has been extended to chaotic systems, both continuous and discrete [1–4], and it was shown [13] that the homoclinic tangencies responsible for fractalization of the basins cause a decrease of the activation energy, i.e., of the minimum energy needed to escape from a basin.

In this Letter we demonstrate the existence of a mechanism of fluctuational transition between coexisting CAs separated by an FBB. It is determined by the hierarchy of original saddles forming the homoclinic structure, and it involves a unique most probable escape path (MPEP) leaving the CA and approaching an accessible orbit on the fractal boundary.

Of the several known types of FBBs [14,15], the locally disconnected kind is the most common, and it is the only FBB to have been observed experimentally [14,16]. As we will see, the mechanism of fluctuational transition across it is determined primarily by its deterministic structure, enabling us to argue that it is probably common to all systems with FBBs of this kind. To demonstrate that such a mechanism exists, we consider the two-dimensional map with locally disconnected FBBs, introduced by Holmes [17],

$$\begin{aligned} x_{n+1} &= f_1 = y_n, \\ y_{n+1} &= f_2 + \xi_n = -b\,x_n + d\,y_n - y_n^3 + \xi_n, \end{aligned} \tag{1}$$

where $\xi_n$ is zero mean, white, Gaussian noise of intensity $D$. In what follows we will adopt the notation $\mathbf{x_n} = \{x_n, y_n\}$, $\mathbf{f} = \{f_1, f_2\}$, and $\boldsymbol{\xi}_\mathbf{n} = \{0, \xi_n\}$. Because of symmetry, the noise-free system (1) with $b = 0.2$ and $2.0 \le d \le 2.745$ has pairs of coexisting attractors, the basins of which are separated by a boundary that may be either smooth or fractal, depending on the choice of parameter values. Our chosen values $b = 0.2$, $d = 2.7$, correspond to two coexisting CAs separated by a locally disconnected FBB (see Fig. 1). The fractal dimension of the boundary is 1.844 72. The chaotic attractors in (1) appear as the result of a period-doubling cascade and, for the parameters chosen, each consists of two disconnected parts.

We have modeled (1) numerically, exciting it with weak noise and collecting both the escape trajectories between the CAs and also the corresponding noise realizations that induced them. By ensemble averaging a few hundred such escape trajectories and noise realizations, we have obtained the optimal escape path and corresponding optimal force shown in Fig. 2. These results allow us to determine the boundary conditions near the CA and the FBB, and to demonstrate the uniqueness of the MPEP. In leaving the CA, the system falls into a small neighborhood of the saddle point of period 1 (S1) located between its two disconnected parts. Its stable manifolds separate the parts of the CA, while the unstable ones belong to the CA. The system makes a few iterations in the neighborhood of S1 [initial plateau in Fig. 2(a)] and then moves to the FBB in three steps, crossing it at a saddle point of

  

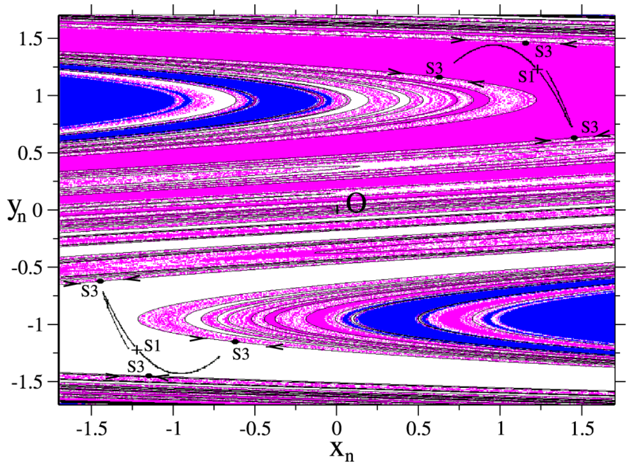

FIG. 1 (color online). The coexisting chaotic attractors (solid black regions) and their basins of attraction represented by gray and white, respectively. The accessible boundary saddle points of period 3 are shown by the small black circles labeled S3. Their stable manifolds are shown as solid black lines. The saddle points of period 1 are shown by the crosses labeled S1. The saddle point at the origin is labeled O.

period 3 (S3). Calculations show that, for the parameters chosen, S3 lies on the FBB and its stable manifold (solid black line in Fig. 1) is dense in the FBB and detaches the open neighborhood, including the attractor, from the FBB itself. This allows us to classify it as an accessible boundary point [18].

An analysis of the structure of escape paths inside the FBB has shown that the homoclinic saddle points play a key role in its formation. In the system (1), we observe an infinite sequence of saddle-node bifurcations of period $3, 4, 5, 6, 7, \ldots$, at parameter values $d_3 < d_4 < d_5 < d_6 < d_7 \ldots$, caused by tangencies of the stable and unstable manifolds of the saddle point O at the origin. The homoclinic orbits appearing as a result of these bifurcations were classified earlier as *original saddles*, and it was also shown that their stable and unstable manifolds cross each other in a hierarchical sequence [18]. It is this deterministic structure of the manifolds of the original saddles that determines the fluctuational escape mechanism across the FBB. Indeed, to escape from a CA, the system must first cross the stable manifold of the accessible orbit, and then the stable manifolds of the other original saddles in a predetermined hierarchical sequence. Once the system crosses the stable manifold of a saddle orbit it relaxes noise-free to the corresponding orbit, which it then leaves orbit along its unstable manifold. Therefore the hierarchical interrelation between original saddles involved in the escape has to be closely linked to eigenvalues of the Jacobian at these saddles, characterizing their local stability with respect to motion on the manifolds. To quantify this interrelation we introduce a parameter $\mu = |\lambda^{\rm st}(x_i)|/\lambda^{\rm un}(x_i)$, where $\lambda^{\rm st}(x_i)$ and $\lambda^{\rm un}(x_i)$ are the eigenvalues of the saddle point $x_i$ corresponding to the stable and unstable directions, respectively. This conclusion accords with the fact that the natural measure $\eta$ on a two-dimensional chaotic nonattracting set is concentrated along its unstable manifold and can be represented via unstable eigenvalues of unstable orbits: $\eta(C) = \sum 1/\lambda^{\rm un}(x_i)$, where $C$ is the region of phase space containing the chaotic saddle, $\lambda^{\rm un}(x_i)$ is the eigenvalue corresponding to the unstable manifold, and the summation is over all the unstable orbits $x_i$ in $C$ [19] (cf. [20]). Calculations have shown that, for the original saddles of period $3, 4, 5, 6, 7, 8, \ldots$ in (1), the following hierarchical sequence of index $\mu$ values occurs:

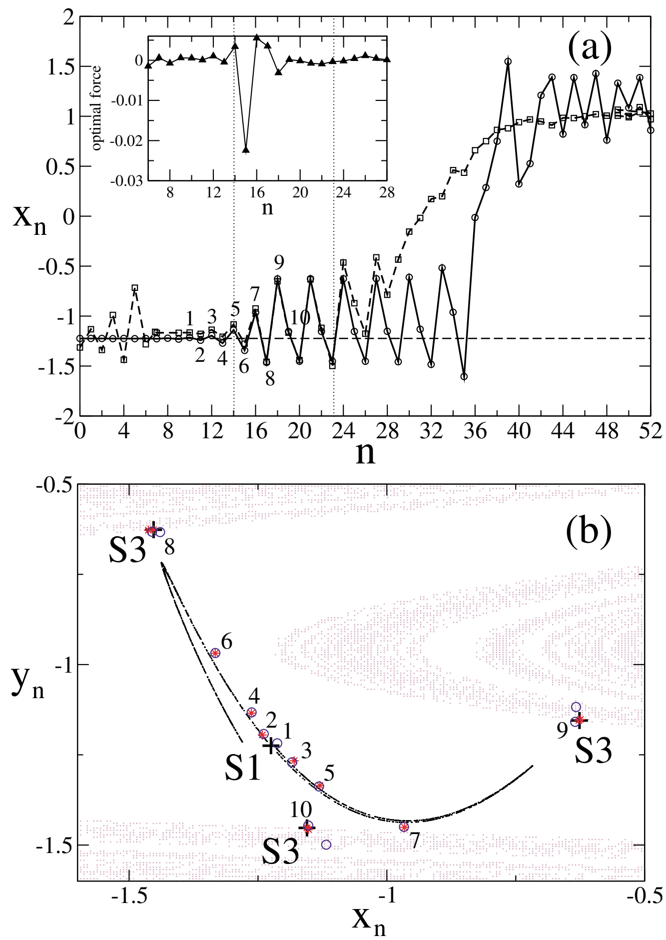

FIG. 2 (color online). (a) The most probable escape path (dashed line) connecting the CA with the period-3 saddle cycle lying on the fractal boundary, obtained from the Monte Carlo simulations with $D = 10^{-5}$. The optimal path found by the solution of the boundary-value problem (see text) is shown as a solid line. The $x$ coordinate of the saddle point S1 is shown by the horizontal dashed line. (b) A two-dimensional plot of the paths presented in (a) where the results obtained by solution of the boundary-value problem [consecutively numbered points indicated by circles, corresponding to the numbered points in (a)] coincide almost perfectly with those obtained by numerical simulation (stars). Inset in (a): the optimal force as determined in the numerical simulation.

$\mu_3 = 3.339$, $\mu_4 = 3.080$, $\mu_5 = 2.999$, $\mu_6 = 2.339$, $\mu_7 = 1.958$, $\mu_8 = 1.539$. Moreover, the values of $\mu$ corresponding to the other homoclinic saddle cycles are close to zero. Correspondingly, the probability of finding the system in their neighborhood tends to zero.

These results allow us to infer the features of fluctuational transitions through a locally disconnected FBB: (i) it always occurs through a unique accessible boundary point, and (ii) the original saddles forming the homoclinic structure of the system play a key role in the formation of the paths inside the FBB, the difference in their local stability defining the hierarchical relationship between them. It seems therefore that, as in simpler systems [9–12], fluctuation-driven escape across an FBB is in many respects deterministic in nature.

Our arguments about the structure of the escape path found in the numerical simulations are further supported by direct calculations of the MPEP using the Hamiltonian theory of fluctuations [1–4]. In this theory the MPEP is the path which minimizes the energy $S = \frac{1}{2}\sum_{n=1}^{N} \xi_n^2$ of the possible realizations of noise $\{\xi_n\}$ inducing a transition of the system (1) from the CA (with the initial condition on S1) to the FBB (with the final condition on the accessible orbit S3). The Lagrangian of the corresponding variational problem can be found following [3] (cf. [21]) in the form

$$L = \sum_{n=1}^{N}\left[\frac{1}{2}\boldsymbol{\xi}_{\mathbf{n}}^{\mathbf{T}}\boldsymbol{\xi}_{\mathbf{n}} + \boldsymbol{\lambda}_{\mathbf{n}}^{\mathbf{T}}(\mathbf{x}_{\mathbf{n+1}} - \mathbf{f}(x_n) - \boldsymbol{\xi}_{\mathbf{n}})\right],$$

where $\mathbf{x}_{\mathbf{n+1}}$, $\mathbf{f}(\mathbf{x}_{\mathbf{n}})$, and $\boldsymbol{\xi}_{\mathbf{n}}$ are the two-dimensional vectors defined in (1) and we introduce the two-dimensional vector $\boldsymbol{\lambda}_{\mathbf{n}} = \{\lambda_n^x, \lambda_n^y\}$ of Lagrange multipliers. Varying $L$ with respect to $\boldsymbol{\xi}_{\mathbf{n}}$, $\boldsymbol{\lambda}_{\mathbf{n}}$, and $\mathbf{x}_{\mathbf{n}}$, the following area-preserving map is obtained:

$$\begin{aligned} x_{n+1} &= y_n, \qquad y_{n+1} = -bx_n + dy_n - y_n^3 + \lambda_n^y, \\ \lambda_{n+1}^x &= (d - 3x_{n+1}^2)\lambda_n^x/b - \lambda_n^y/b, \qquad \lambda_{n+1}^y = \lambda_n^x. \end{aligned} \tag{2}$$

Equations (2) are supplemented by boundary conditions: $\lim_{n\to-\infty}\lambda_n^y = 0$, $(x_n^0, y_n^0) \in S1$, $(x_n^1, y_n^1) \in S3$. The MPEP is the minimum-energy heteroclinic trajectory connecting S1 to S3 in the phase space of (2). The solution of this boundary-value problem involves parametrizing the complex structure of the multiple local energy minima (see, e.g., [22,23] for a discussion) requiring, in turn, a proper parametrization of the unstable manifold in the vicinity of the initial conditions [24]. The resultant MPEP is shown in Fig. 2. It can be seen that, between the vertical dotted lines in Fig. 2(a), the theoretical MPEP closely coincides with the path obtained by statistical analysis of escape trajectories in the Monte Carlo simulations. Note that no further action is required to bring the system to the other attractor once it has reached the accessible orbit of the FBB, i.e., the points numbered 8 in Figs. 2(a) and 2(b); correspondingly, the optimal force measured in the numerical simulations (inset) falls back to zero.

To demonstrate that the mechanism of transition across the FBB is robust with respect to noise-induced perturbations and can indeed be characterized by the value of index $\mu$ defined above, we have used randomly chosen initial conditions in a very small neighborhood of the accessible point S3 through which escape occurs [see Fig. 2(b)]. By definition, any arbitrarily small neighborhood of S3 lies within the FBB, and must contain points belonging to the basins of *both* attractors. Therefore the system can cross the FBB starting from a very small neighborhood of S3, even in the absence of noise. By collecting all such successful escape paths, we have calculated the probabilities for the system to pass via small neighborhoods of different original saddle cycles during its escape, both in the presence and absence of noise. The corresponding probabilities, shown in Fig. 3, demonstrate the same hierarchical interrelationship in both cases, determined by the value of index $\mu$. This structure is evidently robust with respect to noise-induced perturbations. The addition of noise causes a slight broadening of the distribution in Fig. 3 and a small increase in the probability of escape via original saddles of larger period.

In conclusion, we have revealed the mechanism by which noise-induced escape occurs across a locally disconnected FBB. We have found the unique most probable escape path from the chaotic attractor to the fractal boundary, using both statistical analyses of fluctuational trajectories and Hamiltonian fluctuation theory. We have shown that the original saddles forming the homoclinic structure play a key role in effecting the transition across the FBB itself. Their local stability defines the hierarchical relationship between the probabilities for the system to pass via small neighborhoods of different original saddle cycles during escape, both in the presence and absence of noise. Our conjecture, that the mechanism is generic to the wide class of two-dimensional maps and flows [14,16,17] exhibiting the same type of FBB, has

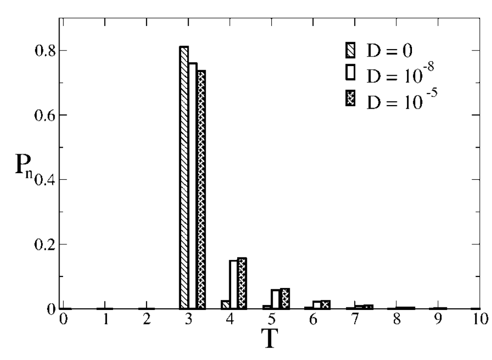


FIG. 3. Probabilities of finding a fragment corresponding to the different period-$T$ original saddle cycle in the collected escape trajectories calculated for the different values of the noise intensity $D$.

recently been confirmed for several systems of this kind including the Duffing oscillator, the Hénon map, and Goodwin's economic model [25]. Possible applications include the development of new energy-optimal control schemes, e.g., for the $CO_2$ laser, a discrete model which also exhibits this type of FBB [26].

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[1] R. L. Kautz, Phys. Lett. A **125**, 315 (1987).
[2] P. D. Beale, Phys. Rev. A **40**, 3998 (1989).
[3] P. Grassberger, J. Phys. A **22**, 3283 (1989).
[4] R. Graham, A. Hamm, and T. Tel, Phys. Rev. Lett. **66**, 3089 (1991).
[5] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1983).
[6] E. Ott, *Chaos in Dynamical Systems* (Cambridge University, Cambridge, England, 2002).
[7] A. L. Fradkov and A. Y. Pogromsky, *Introduction to Control of Oscillations and Chaos, Series on Nonlinear Science A* (World Scientific, Singapore, 1998), Vol. 35.
[8] S. Boccaletti, C. Grebogi, Y. C. Lai, H. Mancini, and D. Maza, Phys. Rep. **329**, 103 (2000).
[9] L. Onsager, and S. Machlup, Phys. Rev. **91**, 1505 (1953).
[10] M. I. Dykman, P. V. E. McClintock, V. N. Smelyanskiy, N. D. Stein, and N. G. Stocks, Phys. Rev. Lett. **68**, 2718 (1992).
[11] D. G. Luchinsky, J. Phys. A **30**, L577 (1997); D. G. Luchinsky and P. V. E. McClintock, Nature (London) **389**, 463 (1997).
[12] M. I. Freidlin and A. D. Wentzel, *Random Perturbations in Dynamical Systems* (Springer, New York, 1984).
[13] S. M. Soskin, M. Arrayás, R. Mannella, and A. N. Silchenko, Phys. Rev. E **63**, 051111 (2001).
[14] S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, Physica (Amsterdam) **17D**, 125 (1985).
[15] J. C. Sommerer and E. Ott, Nature (London) **365**, 138 (1993); H. E. Nusse and J. A. Yorke, Science **271**, 1376 (1996); B. R. Hunt, E. Ott, and E. Rosa, Phys. Rev. Lett. **82**, 3597 (1999).
[16] M. L. Cartwright and J. E. Littlewood, Ann. Math. **54**, 1 (1951); F. C. Moon and G.-X. Li, Phys. Rev. Lett. **55**, 1439 (1985).
[17] P. Holmes, Philos. Trans. R. Soc. London A **292**, 419 (1979).
[18] C. Grebogi, E. Ott, and J. A. Yorke, Physica (Amsterdam) **24D**, 243 (1987).
[19] M. Dhamala and Y. C. Lai, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 2991 (2002).
[20] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **37**, 1711 (1988).
[21] M. I. Dykman, Phys. Rev. A **42**, 2020 (1990).
[22] D. G. Luchinsky, S. Beri, R. Mannella, P. V. E. McClintock, and I. A. Khovanov, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 583 (2002).
[23] R. Graham and T. Tel, Phys. Rev. Lett. **52**, 9 (1984).
[24] S. Beri, D. G. Luchinsky, R. Mannella, P. V. E. McClintock (to be published).
[25] H. W. Lorenz and H. E. Nusse, Chaos Solitons Fractals **13**, 957 (2002).
[26] V. N. Chizhevsky, E. V. Grigorieva, and S. A. Kashchenko, Opt. Commun. **133**, 189 (1997).