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Noise-induced escape through a fractal basin boundary

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Abstract

We study noise-induced escape within a discrete dynamical system that has two co-existing chaotic attractors in phase space separated by a locally disconnected fractal basin boundary. It is shown that escape occurs via a unique accessible point on the fractal boundary. The structure of escape paths is determined by the original saddles forming the homoclinic structure of the system and by their hierarchical interrelations.

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1. Introduction

The problem of escape from a locally stable attractor is ubiquitous in the natural sciences, and arises most notably in chemical kinetics and in transport within nonlinear systems such as semiconductors. From the mathematical point of view, the presence of an attractor implies the existence of a corresponding basin of attraction, defined as the set of initial conditions in phase space from which the system approaches the attractor as time tends to infinity. The basin boundary is the set of points representing the limits of at least two different basins. In general, both an attractor and its basin boundary may have a complex geometrical structure. One of the most challenging cases to be considered, therefore, is the problem of escape from a chaotic attractor with fractal basin boundary.

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A promising approach to this problem is based on its analysis for very small noise intensities. In this limit, a stochastic dynamical system fluctuates to remote states along certain most probable deterministic paths [1,2], corresponding to rays in the WKB-like asymptotic solution of the Fokker–Planck equation [3]. The possibility of extending such an approach to chaotic systems was established earlier [4–7]. However, there are still no theoretical predictions about the mechanism of escape. In particular, the problems of the uniqueness of the escape path, and of the form of the boundary conditions, remain unsolved. An understanding of the escape mechanism can be expected to shed light on the complex dependence of the escape rate on the system parameters. Thus, in recent studies of escape in the presence of homoclinic tangencies, it was shown that they lead to a qualitative change in the escape mechanism that has the effect of decreasing the activation energy [8]. One can ask, what is the role of the saddle cycles arising at the tangencies of the stable and unstable manifolds and forming a homoclinic structure?

It was shown recently [9] that deep physical insight into this problem can be achieved through an analysis of actual fluctuational escape trajectories (see e.g. Ref. [10]). Moreover, such an analysis can be used to solve the problem of the energy-optimal control of switching from a chaotic attractor [11] in the *absence* of noise.

In this paper we apply a statistical analysis of the fluctuational paths to reveal a generic mechanism by which escape occurs from a chaotic attractor with a locally disconnected fractal basin boundary. As a model, we treat the two-dimensional map originally introduced by Holmes [12]:

$$x_{i+1} = y_i \ ,$$
$$y_{i+1} = -bx_i + ay_i - y_i^3 + \xi_i \ , \qquad (1)$$

where ξ_i is white Gaussian noise with $\langle \xi_i \rangle = 0$ and $\langle \xi_i, \xi_j \rangle = 2D\,\delta_{ij}$. The choice of (1) is justified by the fact that it demonstrates a generic type of locally disconnected fractal basin boundary (see Ref. [13]) and reproduces the more important features of the chaotic dynamics of the periodically driven Duffing oscillator [12]. The use of a discrete model brings the additional advantage of allowing us to speed up our numerical calculation for the case of low noise intensity. In Section 2 we present the results of numerical simulations allowing us to find a boundary point on the fractal basin boundary. Section 3 is devoted to a study of the homoclinic structure in (1) and its role in the formation of escape paths. Our conclusions are given in Section 4.

2. An accessible point as a boundary condition

We choose values of the control parameters b and a in (1) such that, in the purely deterministic case (when $D = 0$), there are two co-existing chaotic attractors in phase space separated by a locally disconnected fractal basin boundary (see Fig. 1(a)). We excite our system (1) with weak noise and collect the trajectories that include escape paths from one chaotic attractor to the other (see Fig. 1(b)). The noise intensity was chosen in such a way that the mean escape time was essentially large; the characteristic relaxation time of an invariant measure on the corresponding chaotic set was estimated

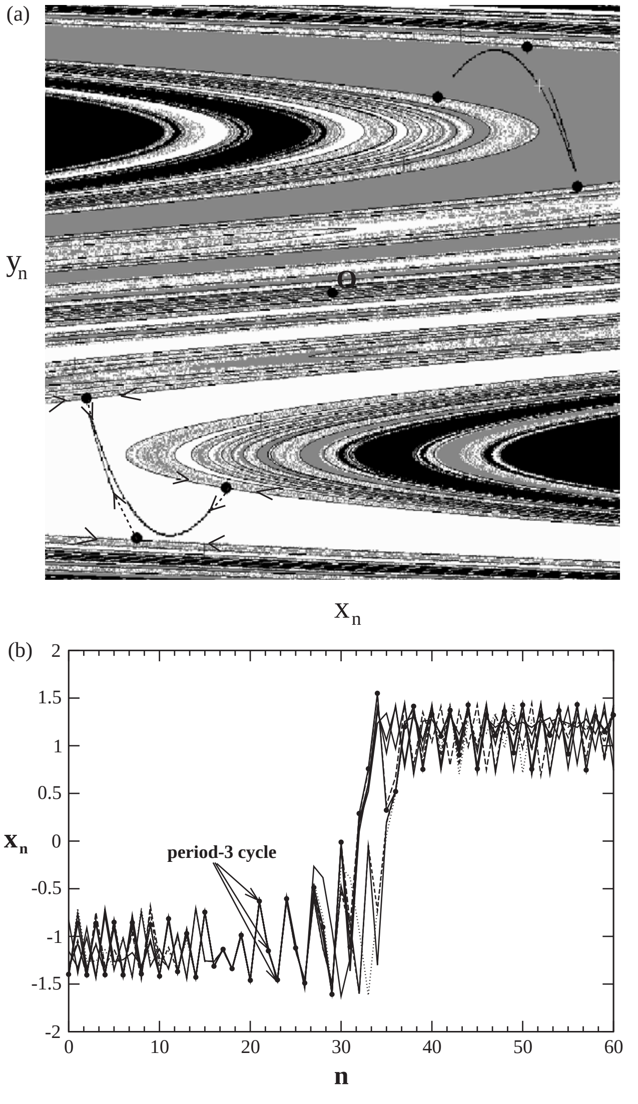

Fig. 1. (a) Two co-existing chaotic attractors (black lines) in (1) at $a = 2.7$, $b = 0.2$ and their basins of attraction illustrated in gray and white respectively. The points of the period-3 homoclinic saddles cycle are labeled by black dots, and the point O corresponds to the saddle at $(0, 0)$; (b) Some typical escape trajectories collected with $D = 10^{-5}$.

by us as 3×10^7 iterations. As can be clearly seen in Fig. 1(b), all escape trajectories have a part corresponding to the period-3 saddle cycle, apparently implying the presence of a boundary point located near, or directly on, the fractal boundary. Simple calculations have shown that period-3 saddle cycle does exist for the chosen parameter values and that it lies on the boundary. Moreover, its stable manifold (full black line) lying in the boundary detaches the open neighborhood including the chaotic attractor from the fractal basin boundary itself. One part of its unstable manifold belongs to the homoclinic structure forming the fractal boundary, whereas the other part (labeled by the dashed black line in Fig. 1(a)) approaches the attractor. Thus, we can classify this period-3 saddle point as an accessible boundary point. Indeed, by definition given in Ref. [14] a boundary point P is accessible from a given region if there is a curve of finite length connecting P to an attractor in the interior of the region, such that no point of the curve lies in the boundary except for P. In our case, the part of unstable manifold approaching the chaotic attractor plays the role of such a curve. Thus, in the present case, the period-3 saddle point plays the role of the boundary condition.

3. Homoclinic saddle cycles and their hierarchy

It is well known that the global behavior of a chaotic dynamical system is in many respects determined by a homoclinic structure, i.e., by a set consisting of homoclinic saddle cycles resulting at tangencies of the stable and unstable manifolds of a saddle point. In our case, we observe an infinite sequence of saddle-node bifurcations of period $3, 4, 5, 6, 7 \ldots$, which occur at parameter values $d_3 < d_4 < d_5 < d_6 < d_7 \ldots$ and are caused by the sequent tangencies of the stable and unstable manifolds of the saddle point O at $(0, 0)$. The homoclinic orbits appearing as the result of these bifurcations were classified earlier as *original saddles* [14]. It was also shown that their stable and unstable manifolds cross each other in the hierarchical sequence: the unstable manifold of the period-3 saddle crosses the stable manifold of the period-4 saddle, the unstable manifold of the period-4 saddle crosses the stable manifold of the period-5 saddle, etc. (see Fig. 3(a)). Our numerical calculations have shown that these original saddles play a key role in the escape through a fractal basin boundary and that their hierarchy defines the structure of the escape paths. A typical escape path obtained in numerical simulations is depicted in Fig. 2. As clearly seen from this figure, a phase trajectory leaving the chaotic attractor penetrates into the fractal basin boundary through a small neighborhood of the period-3 saddle cycle, makes a few turns, and then approaches another period-4 original saddle point. After that it moves to the basin of the other chaotic attractor, reaching it in the next two or three iterations. In fact, the heteroclinic structure formed by numerous sequent crossings of the stable and unstable manifolds of original homoclinic saddles plays the role of the "staircase" allowing a trajectory to pass over the fractal basin boundary and defining the structure of the escape paths. Moreover, a hierarchical relation between original saddles can be revealed if we characterize them by a parameter μ equal to the ratio $|\lambda_s(P)|/\lambda_u(P)$ of the stable and unstable eigenvalues of the linearized deterministic flow at a saddle point P [15]. Simple calculations show that, for the original saddles with periods $3, 4, 5, 6, 7, 8 \ldots$ in (1), the following hierarchical sequence of index μ values occurs: $\mu_3 = 3.339$, $\mu_4 = 3.08$, $\mu_5 = 2.999$, $\mu_6 = 2.339$, $\mu_7 = 1.958$, $\mu_8 = 1.539$. To estimate the influence of such a hierarchical relation on the escape paths, the probabilities of finding the original saddle of period N in the escape trajectory were computed for different values of noise

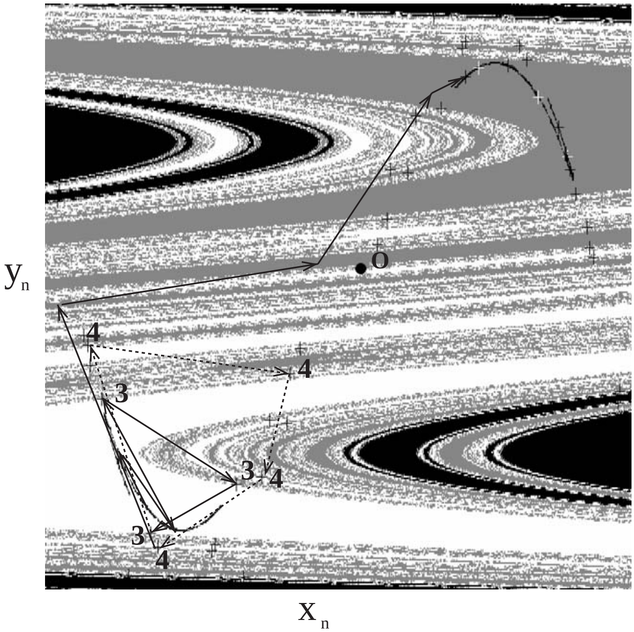

Fig. 2. The escape trajectory in (1) from one chaotic attractor (black lines) to the other passes through the period-3 (full black lines) and period-4 (dashed black lines) original saddle cycles. The values of the control parameters and noise intensity are the same as in Fig. 1.

intensity. Initial conditions for the iteration procedure were chosen randomly within a small neighborhood of the period-3 saddle cycle lying on the basin boundary. As seen from the results presented in Fig. 3(b), the probabilities of finding a fragment of the corresponding original saddle cycles in the escape trajectories are in a very good agreement with the hierarchical relation obtained above. In fact, the addition of noise reveals the hierarchy described, correcting slightly the absolute values of probability, but it does not change the structure of distribution qualitatively.

4. Conclusions

We have studied noise-induced escape from one chaotic attractor to another through a fractal basin boundary. It was shown that escape occurs through a unique point on the fractal boundary. The original saddles forming the homoclinic structure of system (1) play the key role in the formation of the escape paths, and the difference in their local stability defines the hierarchical relationship between them. Thus, we may claim that complicated structure of escape trajectories, caused by the thin homoclinic structure and their randomness, has in many respects a deterministic nature.

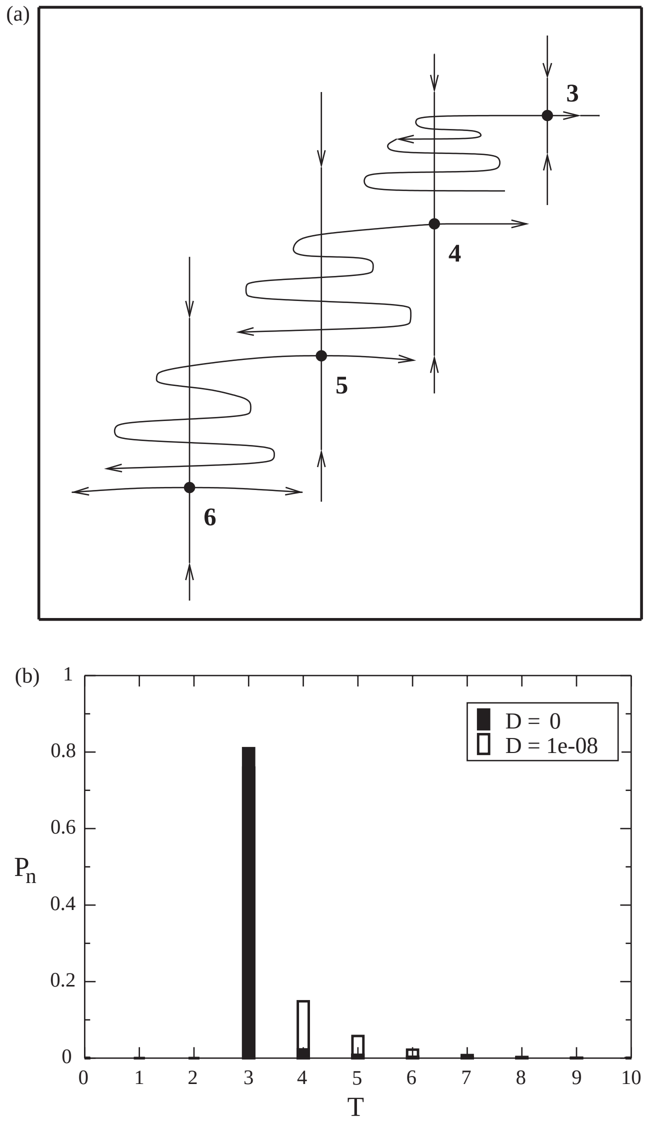

Fig. 3. (a) Heteroclinic crossings of the stable and unstable manifolds of different original saddles; (b) Probabilities of finding a fragment corresponding to the different period-T original saddle cycle in the collected escape trajectories.

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