Heat Removal from Heat-sensitive Foods: an Economic Approach to the Transient Behaviour of Finned Surfaces

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ABSTRACT

The transient behaviour of a fin used to remove heat from the surface of a rectangular-shaped liquid food container is studied in dimensionless form leading to a single-term equation for the heat transferred in relation to time. For heat-sensitive foods the rate of heat removal is important so an economic value can be ascribed to such removal to balance the capital investment on finning the container. An optimal solution can be found in order to maximize the net profit involved using the fin length as the independent variable.

NOMENCLATURE

C_p specific heat capacity of the fin material ($J\ kg^{-1}\ K^{-1}$)
h heat transfer coefficient for convection from the fin ($W\ m^{-2}\ K^{-1}$)
k thermal conductivity of the fin material ($W\ m^{-1}\ K^{-1}$)
L_x thickness of the fin (m)
L_y width of the fin (m)
L_z length of the fin (m)
N number of times the fin is used to remove heat
Q total heat lost via the base of the fin (J)
$\dot{Q}$ heat flux via the base of the fin (W)
t elapsed time (s)
T uniform temperature in cross section of fin (K)

t^0 $=\dfrac{\rho C_p L_x^2}{k}$, reference time (s)

T_0 ambient temperature (K)
T_w temperature of the base of the fin for $t > 0$ (K)
V_h economic value of the heat withdrawn via the fin ($\$ \, J^{-1}$)
V_i capital investment cost of the fin material ($\$ \, m^{-3}$)
V_n net profit after having finned the surface (\$)
z linear dimension along which the conduction of heat is assumed to occur (m)

Dimensionless groups

$Bi = \dfrac{hL_x}{2k}$, Biot number

$V_0 = \dfrac{N\rho C_p V_h (T_w - T_0)}{V_i}$, dimensionless cost ratio

$Ge = 4\left(\dfrac{L_x}{L_y} + 1\right)$, geometrical shape factor

$Of = \dfrac{\pi^{1/2} V_i}{2N\rho C_p (T_w - T_0) V_h t^{*1/2}}$, dimensionless operating factor

$\dot{Q}^* = \dfrac{\dot{Q}}{kL_y (T_w - T_0)}$, dimensionless heat flux via the base of the fin

$\overline{Q^*} = \dfrac{\int_0^{t^*} \dot{Q}^* \, dt^*}{t^*}$, dimensionless average flux via the base of the fin

$\dot{Q}^{*\infty} = \lim_{t^* \to \infty} \dot{Q}^*$, dimensionless heat flux via the base of the fin in steady-state conditions

$t^* = \dfrac{kt}{\rho C_p L_x^2}$, dimensionless elapsed time

$T^* = \dfrac{T - T_0}{T_w - T_0}$, dimensionless temperature

$V_0 = \dfrac{N\rho C_p V_h (T_w - T_0)}{V_i}$, dimensionless cost ratio

V_n^* $= \dfrac{V_n}{L_x L_y L_z V_i}$, dimensionless net profit

z^* $= \dfrac{z}{L_x}$, dimensionless linear distance

Greek symbols

ρ absolute density of the fin material (kg m^{-3})

INTRODUCTION

Before a steady state can be reached in any food process some time must elapse after the process is initiated to allow the transient conditions to disappear. Such a time interval appears to be of relevance for heat transfer operations involving heat-sensitive systems so an unsteady state approach should be employed in such cases in order to be able to predict their thermal behaviour. For the case where batch operations are to take place in the cooling of heat-sensitive liquid foods placed inside containers (e.g. accumulation vessels linking continuous and batch operations in a food process), the heat removal may be of great economic importance. This is the case for a number of liquid foods such as milk after pasteurization or fruit juices after having been concentrated by evaporation. Instead of allowing the liquid to cool by natural convection between the container surface and the atmosphere an outer surface finning might be employed to speed up the cooling. Such a procedure requires a capital investment on fins which, however, can be recovered the more quickly the greater the economic value of the cooling process.

This paper deals mainly with the heat-removal characteristics of fins and the relationship between its size-dependent capital cost and the operating costs of removing the same heat by an alternative way, with the objective of maximizing the net profit achieved.

MATHEMATICAL ANALYSIS

A fin of length L_z is fitted to a given surface as shown in Fig. 1. The initial temperature along the fin is assumed to be uniform and equal to the ambient temperature T_0. At time $t=0$ the surface temperature changes suddenly to temperature T_w due to liquid contacting the surface. A heat balance to the fin gives (Eckert and Drake, 1959)

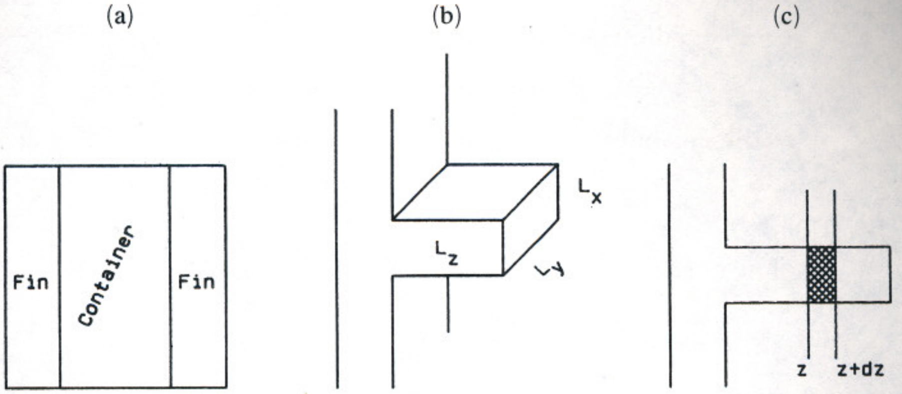

Fig. 1. Schematic of the finned liquid food container. (a) Top view of the finned liquid food container. (b) Cross-sectional view of a fin. (c) Elementary control volume over which enthalpy balance is made.

$$\begin{matrix}\text{heat input} & & \text{heat output} & & \text{heat output} & \\ \text{by} & = & \text{by} & + & \text{by} & +\text{heat accumulated} \\ \text{conduction} & & \text{conduction} & & \text{convection} & \end{matrix} \quad (1)$$

A reasonably good description of the system, may be obtained by approximating the true physical situation by a simplified model, assuming the temperature presents a unidimensional variation, no heat is lost from the end or from the edges of the fin and the heat flux at the surface is given by Newton's law for convective heat transfer with constant heat transfer coefficient (Bird *et al.*, 1960). The heat balance referred to above can then be applied to a volume element $L_x L_y \, dz$, where L_x and L_y denote the thickness and width of the fin, respectively, and z the linear dimension along which the conduction of heat is assumed to occur (see Fig. 1), to give (Jakob, 1949)

$$-kL_x L_y \left(\frac{\partial T}{\partial z}\right)_z = -kL_x L_y \left(\frac{\partial T}{\partial z}\right)_{z+dz} + 2h(L_x + L_y)(T - T_0)\,dz + L_x L_y \rho C_p \left(\frac{\partial T}{\partial t}\right)_z dz \quad (2)$$

k being the thermal conductivity of the fin material, h the heat transfer coefficient for convection from the fin surface, C_p the specific heat capacity of the fin material, ρ the absolute density of the fin material and T the uniform temperature of the cross section of the fin. Rearranging eqn (2) leads to

$$\frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial z^2}\right)-\left(\frac{\partial T}{\partial t}\right)-\frac{2h}{\rho C_p}\left(\frac{1}{L_x}+\frac{1}{L_y}\right)(T-T_0)=0 \tag{3}$$

Introducing dimensionless variables defined as

$$T^*=\frac{T-T_0}{T_w-T_0} \quad \text{a dimensionless temperature} \tag{4}$$

$$z^*=\frac{z}{L_x} \quad \text{a dimensionless linear distance} \tag{5}$$

$$t^*=\frac{t}{t^0} \quad \text{a dimensionless elapsed time} \tag{6}$$

t^0 being a reference time, and combining eqns (4)–(6) with eqn (3) leads to

$$\left(\frac{\partial^2 T^*}{\partial z^{*2}}\right)-\left(\frac{\partial T^*}{\partial t^*}\right)-Bi.Ge.T^*=0 \tag{7}$$

where *Bi* is a Biot number and *Ge* a geometrical shape factor defined as

$$Bi=\frac{h}{(2kL_x^{-1})} \tag{8}$$

and

$$Ge=4\left(\frac{L_x}{L_y}+1\right) \tag{9}$$

The reference time, t^0, is then given by

$$t^0=\frac{L_x L_y L_z \rho C_p (T_w-T_0)}{L_y L_z (T_w-T_0)/L_x} \tag{10}$$

t^0 being the time it would take to heat the fin from its initial temperature, T_0, to the wall temperature, T_w, if all the heat necessary entered by conduction via one of the horizontal surfaces of the fin, the remaining surfaces being perfectly insulated, and the temperature gradient were constant and given by $(T_w-T_0)L_x^{-1}$.

The following initial and boundary conditions are assumed for all real, positive values of z^* and t^*

$$\text{at } t^* = 0,\ T^* = 0 \tag{11}$$

$$\text{at } z^* = 0,\ T^* = 1 \tag{12}$$

when

$$z^* \to \infty,\ T^* \to 0 \tag{13}$$

Applying the Laplace transform to eqn (7) in order to eliminate the independent variable t^*, using eqns (11), (12) and (13), and then converting to the real domain (see Arpaci, 1966), eqn (7) leads to

$$T^*(z^*, t^*) = \frac{1}{2}\exp\{-(Bi\,.\,Ge)^{1/2} z^*\}\,\text{erfc}\left\{\frac{z^*}{2t^{*1/2}} - (Bi\,.\,Ge)^{1/2} t^{*1/2}\right\} + \frac{1}{2}\exp\{(Bi\,.\,Ge)^{1/2} z^*\}\,\text{erfc}\left\{\frac{z^*}{2t^{*1/2}} + (Bi\,.\,Ge)^{1/2} t^{*1/2}\right\} \tag{14}$$

where erfc is the complementary error function (Kreyszig, 1979). The heat flux via the base of the fin can then be expressed as

$$\dot{Q}^* = (Bi\,.\,Ge)^{1/2}\left\{\text{erf}[(Bi\,.\,Ge\,.\,t^*)^{1/2}] + \frac{\exp[-Bi\,.\,Ge\,.\,t^*]}{(\pi\,.\,Bi\,.\,Ge\,.\,t^*)^{1/2}}\right\} \tag{15}$$

$\dot{Q}^*$ being a dimensionless heat flux defined by

$$\dot{Q}^* = \frac{\dot{Q}}{kL_x L_y (T_w - T_0)/L_x} \tag{16}$$

where $\dot{Q}$ is the non-dimensionless counterpart of $\dot{Q}^*$.

When t^* tends to infinity then the operation tends to a steady state. In that case the dimensionless heat lost via the base of the fin is given by $\dot{Q}^{*\infty}$ defined as

$$\dot{Q}^{*\infty} = (Bi\,.\,Ge)^{1/2} \tag{17}$$

The ratio of $\dot{Q}^*$ to $\dot{Q}^{*\infty}$ can be observed in Fig. 2.

The average heat lost via the base of the fin, $\overline{Q}^*$, can then be written as

$$\overline{Q}^* = \frac{(Bi\,.\,Ge)^{1/2}}{Bi\,.\,Ge\,.\,t^*}\left\{\left(Bi\,.\,Ge\,.\,t^* + \frac{1}{2}\right)\text{erf}[(Bi\,.\,Ge\,.\,t^*)^{1/2}] + \left(\frac{Bi\,.\,Ge\,.\,t^*}{\pi}\right)^{1/2}\exp\{-Bi\,.\,Ge\,.\,t^*\}\right\} \tag{18}$$

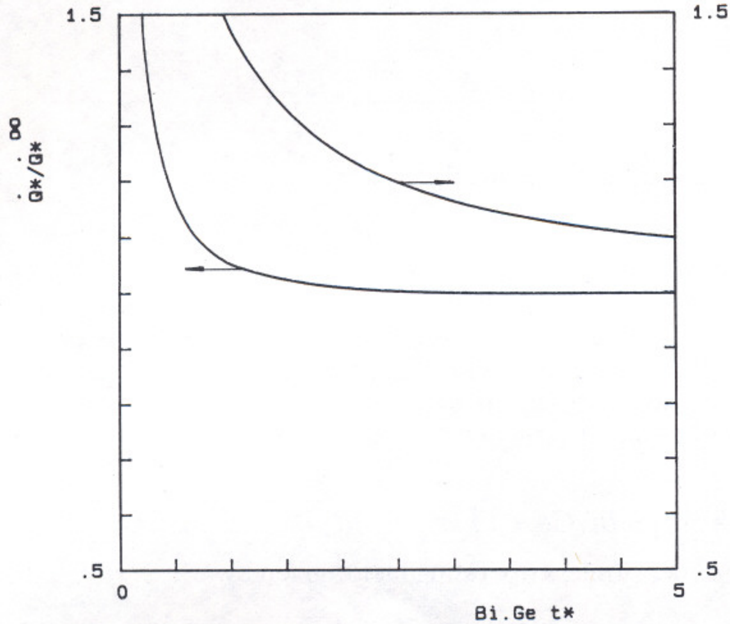

Fig. 2. Ratios of dimensionless heat lost via the base of the fin ($\dot{Q}^*$) and average dimensionless heat lost via the base of the fin ($\overline{Q}^*$) to the dimensionless heat lost under steady state operating conditions ($\dot{Q}^{*\infty}$).

The ratio of $\overline{Q}^*$ to $\dot{Q}^{*\infty}$ can also be observed in Fig. 2 (National Bureau of Standards, 1954).

When dealing with the option of either finning a given surface or not, a cost estimation problem then arises which usually has to be solved for the maximum net profit.

The net profit, V_n, may be expressed as the difference between the economic value of the heat withdrawn, V_h, and the capital investment, V_i, as follows

$$V_n = NQV_h - L_x L_y L_z V_i \tag{19}$$

N being the number of times the fin is to be used to remove heat from the stated initial conditions and Q the total heat lost via the base of the fin after time t. Using dimensionless variables in eqn (19) leads to

$$V_n^* = \frac{V_0}{(Bi\,.\,Ge)^{1/2}}\left[\left(Bi\,.\,Ge\,.\,t^* + \frac{1}{2}\right)\mathrm{erf}\{(Bi\,.\,Ge\,.\,t^*)^{1/2}\} + \left(\frac{Bi\,.\,Ge\,.\,t^*}{\pi}\right)^{1/2}\exp\{-Bi\,.\,Ge\,.\,t^*\}\right] - 1 \tag{20}$$

V_0 being a cost ratio defined by

$$V_0 = \frac{N\rho C_p(T_w - T_0)V_h}{V_i} \tag{21}$$

and V_n^* a dimensionless net profit given by

$$V_n^* = \frac{V_n}{L_x L_y L_z V_i} \tag{22}$$

The variation of V_n^* with $(Bi.Ge.t^*)$ can be observed in Fig. 3. Setting the fin length, L_z, as the independent variable during the course of net profit optimization, differentiating eqn (22) with respect to L_z, putting the resulting expression equal to zero and performing some algebraic work finally leads to

$$Of = \exp\{-Bi.Ge.t^*\} + (\pi.Bi.Ge.t^*)^{1/2}\,\mathrm{erf}\{(Bi.Ge.t^*)^{1/2}\} \tag{23}$$

Of being a dimensionless operating factor given by

$$Of = \frac{\pi^{1/2}}{2}\,\frac{V_i}{N\rho C_p(T_w - T_0)V_h\, t^{*1/2}} \tag{24}$$

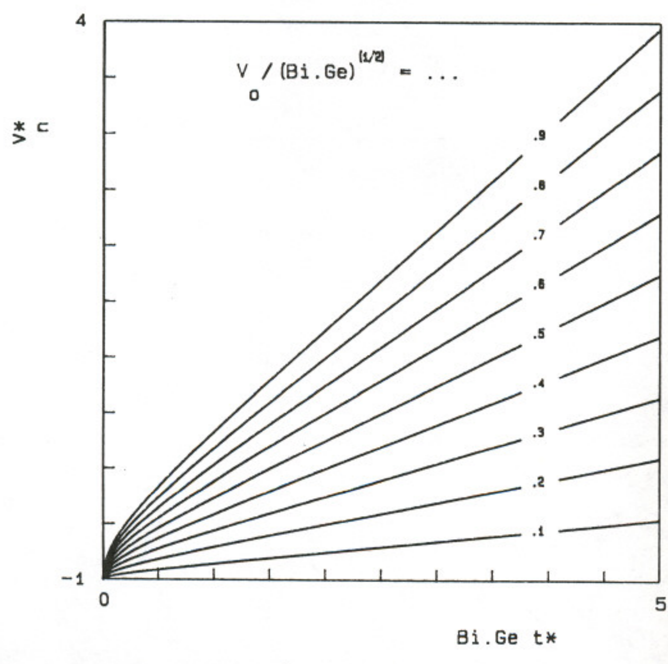

Fig. 3. Dimensionless net profit, V_n^*, versus factor $Bi.Ge.t^*$ for transient operating conditions.

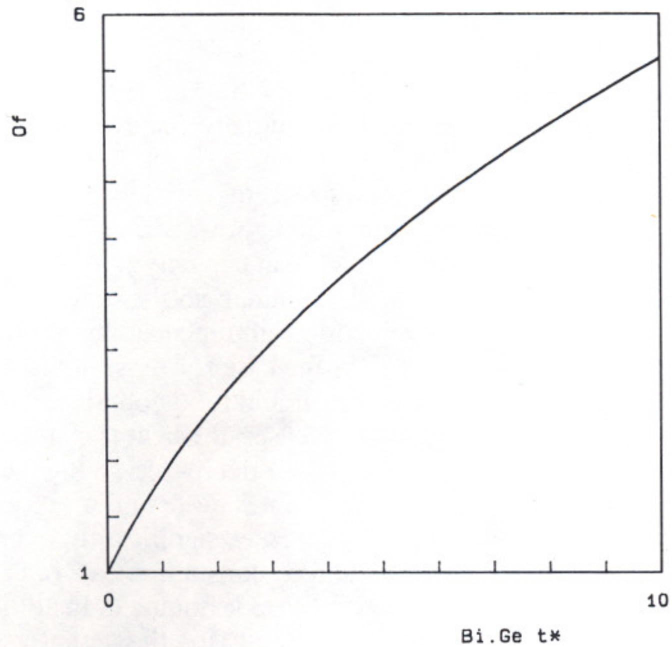

Fig. 4. Maximum net profit as a function of dimensionless operating factor (*Of*) versus factor $Bi.Ge.t^*$.

The optimum points for maximum net profit are shown in Fig. 4 as *Of* versus $Bi.Ge.t^*$.

DISCUSSION

If a steady state approach had been made the following result would have been obtained for the heat lost via the fin base

$$\dot{Q}^* = (Bi.Ge)^{1/2} \tanh\{(Bi.Ge)^{1/2}\} \tag{25}$$

The value for $\dot{Q}^*$ referred to above is equal to that stated by eqn (17) as long as the value of variable $Bi.Ge$ is large. This constraint is in good agreement with the validity of the currently accepted boundary condition which assumes no heat is lost by the tip of the fin so its temperature must be equal to the ambient temperature (Geankopolis, 1983).

When the heat removal process starts the temperature of the base increases from its initially ambient temperature to the wall actual temperature. So the initial rate of heat removal is extremely high, most of that heat being sensible heat acquired by the fin itself. This behaviour can be taken advantage of when designing the cooling fin system especially in cases where the liquid food container operates batchwise and it is frequently filled up with fresh hot fluid.

The analysis developed above assumes the wall temperature remains constant after the container has been initially filled up with fresh hot liquid food. However, the bulk temperature of the liquid decreases as time elapses, so the former analysis seems to depart from physical accuracy. One situation may occur which, however, ensures validity for the assumptions made: the liquid food heat capacity is far larger than the heat capacity of the metallic finned container, so the temperature of the base of the fin decreases relatively little with time leading to quasi-steady state conditions along the fin. This is the case for most liquid foods with high water contents and large capacity liquid food containers. Besides, as the temperature of the liquid food decreases, the heat transfer coefficient for natural convection inside the container decreases, so the quasi-steady state assumption referred to above, becomes more and more valid. If the temperature of the base of the fin decreases significantly over the time range chosen, V_0 and *Of* are no longer constant so an operating line crossing some of the lines sketched in Fig. 3 is obtained. In addition to t^*, the variation of $(T_w - T_0)$ then also accounts for the actual variation of *Of*.

In eqn (22) the economic value of the heat lost via the outer surface of the container if there were no fins should have been subtracted. This approximation is, however, in good agreement with physical evidence because the main resistance to heat transfer is on the outside, the cross-sectional area of each fin, $L_x L_y$, being usually negligible when compared to the actual area of the heat transfer surfaces of the fin, $2\, L_y L_z$.

A careful use of Fig. 4 enables the optimal fin length to be computed in order to achieve maximum net profit. After having defined all operating and geometrical conditions except the length of the fin a value for factor *Of* is obtained which corresponds to a certain value for $Bi \,.\, Ge \,.\, t^*$. The optimal length is then directly computed once t^* and *Bi* and *Ge* are known. For values of *Of* less than unity no optimization procedure is possible. For existing fin equipment the fin length cannot be changed, therefore the operating time for the cooling operation should be adjusted so that the working point is on the curve of Fig. 4, so maximizing the economic advantage of the existing fin.

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