Uniformity

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1 Introduction

Let us assume that the p-values $\{p_k\}_{k=1}^{n}$ are known from testing H_{0k} vs. H_{Ak}, $k = 1, \ldots, n$, in n independent studies on some common issue, and our aim is to achieve a decision on the overall question H_0^* : all the H_{0k} are true $vs.$ H_A^* : some of the H_{Ak} are true. As there are many different ways in which H_0^* can be false, selecting an appropriate test is in general unfeasible. On the other hand, combining the available p_k's so that $T(p_1, \ldots, p_n)$ is the observed value of a random variable whose sampling distribution under H_0^* is known is a simple issue, since under H_0^*, $p = (p_1, \ldots, p_n)$ is the observed value of a random sample $P = (P_1, \ldots P_n)$ from a standard uniform population. In fact, several different sensible combined testing procedures are often used [6, 11].

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Therefore an important issue is to test whether a given sequence $\{p_k\}_{k=1}^{n}$ is or is AQ2 not a sample from a standard uniform population. Recently Paul [10] discussed new characterizations of the uniform population, but as they are formulated in terms of expected values, they did not lead directly to new simple tests of uniformity. Gomes et al. [5] exploited the possibility of using computationally augmented samples to test uniformity, with the surprising result that power can decrease with sample augmentation in the class of alternatives they used. Sequeira [12] explains why this is so, and in Sect. 2 below we further discuss the question. In this chapter we use Sukhatme's transformation to suggest new ways of dealing with the matter.

Sukhatme's [13] transformation, from which Rényi's representation of exponential order statistics can easily be derived, appears in David and Nagaraja ([2], p. 17–18) and in Johnson et al. ([8], p. 305), with slightly different presentations, applied to the study of exponential and of uniform order statistics, respectively. Durbin [4] used ordered spacings of the uniform to investigate the construction of exact tests. In Sect. 3 we use a Sukhatme's like transformation to augment the set of order statistics from a uniform parent, and in Sect. 4 we investigate power issues when they are used in testing uniformity.

2 Uniformity Versus Mixtures of Uniform and Beta(1,2)

Gomes et al. [5] introduced the family $\{X_m\}_{m\in[-2,2]}$ of absolutely continuous random variables, with probability density function $f_{X_m}(x) = \left(mx - \frac{m-2}{2}\right)\mathrm{I}_{(0,1)}(x)$ (the uniform density corresponds to $m = 0$; for $m \in (0,2]$, X_m is a convex mixture of Beta(1,1) and Beta(2,1), and for $m \in [-2,0]$, X_m is a mixture of Beta(1,1) and Beta(1,2)). Observe that for all $m \in [-2,0)$, $\mathbb{P}[X_m \le x] - \mathbb{P}[U \le x] = \frac{m}{2}\,x\,(x-1) > 0$ for all $x \in (0,1)$, and thus pseudorandom numbers generated by X_m tend to be closer to 0 than pseudorandom numbers generated by a standard uniform random variable U. Thus this family can give important hints on nonuniformity of the set of p-values, cf. the concepts of random p-values in Kulinskaya et al. [9] and of generalized p-values in Hartung et al. [6].

Observe also that for $m \in (0,2]$, X_m tends to take values closer to 1 than the $X_0 \frown$ Uniform(0, 1) random variable, and hence in that range of values it provides a suitable alternative in the case of right one-tailed alternative tests. Moreover, the inverse of the corresponding distribution function is

$$F_{X_m}^{-1}(u) = \frac{\frac{m}{2} - 1 + \sqrt{\left(\frac{m}{2} - 1\right)^2 + 2mu}}{m}$$

and the generation of pseudo-random numbers from X_m for simulation studies is therefore straightforward.

Let U and X be two independent standard uniform random variables. The random variables $V = U + X - \mathrm{I}[U + X]$, where $\mathrm{I}[x]$ denotes the largest integer not greater than x, and $W = \min\left(\frac{U}{X}, \frac{1-U}{1-X}\right)$ are uniform and independent of X [3].

This fact was used by Gomes et al. [5] for computationally augmenting samples and to assess the power of detecting non-uniformity when the sample comes in fact from X_m, $m \in [-2, 0]$, with the strange result that power does not improve for increased samples.

The explanation is however simple: if X_m and X_p are two independent random variables, with $m, p \in [-2, 2]$, then $\min\left(\frac{X_m}{X_p}, \frac{1-X_m}{1-X_p}\right) \stackrel{d}{=} X_{\frac{mp}{6}}$ [1]. Hence, in case the algorithm uses uniform pseudorandom numbers to augment the sample, the augmented slice will in fact be a uniform subsample, and power decreases. Brilhante et al. [1] present better results using left-skewed parent pseudorandom numbers.

Still, the use of the family $\{X_m\}_{m \in [-2,2]}$ has many advantages, and instead of augmenting the sample *externally*, as in the above-mentioned papers, by using $V_m = U + X_m - \mathrm{I}[U + X_m]$ and $W_m = \min\left(\frac{U}{X_m}, \frac{1-U}{1-X_m}\right)$, with the spurious effect of always generating uniform pseudo p-values, we can use an alternative approach when the purpose is to test the null hypothesis of uniformity *vs.* X_m parent:

- Choose at random one $p_j \in \{p_k\}_{k=1}^n$.
- Generate $n-1$ pseudo p's of the form $\min\left(\frac{p_j}{p_k}, \frac{1-p_j}{1-p_k}\right)$, $k \neq j$.

3 Order Statistics, Spacings and Sukhatme's Transformation

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a random sample from the absolutely continuous positive random variable X with probability density function f_X and $(X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ the corresponding vector of ascending order statistics. For convenience we assume that left-endpoint $\alpha_X = 0$ and we define $X_{0:n} = \alpha_X = 0$.

The joint probability density function of the spacings $S_k = X_{k:n} - X_{k-1:n}$, $k = 1, \ldots, n$, is

$$f_{(S_1,S_2,\ldots,S_n)}(s_1, s_2, \ldots, s_n) = n!\, f_{(X_1,X_2,\ldots,X_n)}(s_1, s_1 + s_2, \ldots, s_1 + \cdots + s_n)$$

($s_k > 0$, $k = 1, \ldots, n$, and if the right-endpoint ω_X is finite, $\sum_{k=1}^n s_k < \omega_X$; in this case we can consider the rightmost spacing $S_{n+1} = \omega_X - X_{n:n}$, but this can be expressed as a function $\omega_X - \sum_{k=1}^n S_k$). Hence, the joint probability density function of the ascending reordering of those n spacings is

$$f_{(S_{1:n},S_{2:n},\ldots,S_{n:n})}(y_1, y_2, \ldots, y_n) = (n!)^2 f_{(X_1,X_2,\ldots,X_n)}(y_1, y_1 + y_2, \ldots, y_1 + \cdots + y_n)$$

where $0 < y_1 < \ldots < y_n$ and $\sum_{k=1}^n y_k < \omega_X$.

Now define

$$W_k = (n + 1 - k)(S_{k:n} - S_{k-1:n}), \quad k = 1, \ldots, n,$$

(similar to Sukhatme's transformation, as defined in David and Nagaraja [2], but applied to ascendingly ordered spacings), again with the convention $S_{0:n} = 0$.

The joint probability density function of $(W_1, W_2, \ldots, W_n)$ is

$$f_{(W_1,W_2,\ldots,W_n)}(w_1, w_2, \ldots, w_n) = n!\, f_{(X_1,X_2,\ldots,X_n)}\left(\tfrac{w_1}{n}, \tfrac{2w_1}{n} + \tfrac{w_2}{n-1}, \ldots, w_1 + \cdots + w_n\right)$$

$w_k > 0, k = 1, \ldots, n$, (observe that the k-th argument is

$$\frac{kw_1}{n} + \frac{(k-1)w_2}{n-1} + \cdots + \frac{(k+1-j)w_j}{n+1-j} + \cdots + \frac{w_k}{n+1-k}\,,\ \ k = 1, \ldots, n),$$

and the joint probability density function of the vector of partial sums $Y_k = \sum_{j=1}^{k} W_j$, $k = 1, \ldots, n$, is

$$f_{(Y_1,Y_2,\ldots,Y_n)}(y_1, y_2, \ldots, y_n) = n!\, f_{(X_1,X_2,\ldots,X_n)}\left(\tfrac{y_1}{n}, \ldots, \sum_{j=1}^{k} \tfrac{(k+1-j)(y_j-y_{j-1})}{n+1-j}, \ldots, y_n\right)$$

with $0 < y_1 < \ldots < y_n$ and the convention $y_0 = 0$.

If $X \frown \text{Uniform}(0, \omega_X)$, then

$$f_{(X_1,X_2,\ldots,X_n)}\left(\tfrac{y_1}{n}, \ldots, \sum_{j=1}^{k} \tfrac{(k+1-j)(y_j-y_{j-1})}{n+1-j}, \ldots, y_n\right) = \frac{1}{\omega_X^n} = f_{(X_1,X_2,\ldots,X_n)}(y_1, y_2, \ldots, y_n),$$

and hence $(Y_1, Y_2, \ldots, Y_n) \stackrel{d}{=} (X_{1:n}, X_{2:n}, \ldots, X_{n:n})$. [1]

This suggests that uniformity can be investigated testing whether $\{X_{k:n}\}_{k=1}^{n}$ and $\{Y_k\}_{k=1}^{n}$ can be considered samples from the same distribution. Unfortunately, under the null hypothesis that the parent distribution is standard uniform, the two vectors are not independent since we can re-express $Y_k = \sum_{j=1}^{k} S_{j:n} + (n-k)S_{k:n}$, and consequently $Y_n = X_{n:n}$. Thus, the Smirnov two-sample test is of no use in the present situation.

However, the observation of Fig. 1, where we compare the empirical distribution function (edf) corresponding to the order statistics $x_{k:n}$ (black) and the y_k (gray), in case of uniform and Beta(1,2) parents, suggests that $D_n^* = \sup_x |F_n^*(x) - G_n^*(x)|$, where F_n^* stands for the order statistics edf and G_n^* for the accumulated y_k edf, will be greater under the alternative $H_A : X$ nonuniform with support (0,1) than under the null hypothesis $H_0 : X \frown \text{Uniform}(0, 1)$.

[1] Observe that if $\omega_X < \infty$, we can consider $n+1$ spacings, with $S_{n+1} = \omega_X - X_{n:n}$; of course in this situation S_{n+1}, $S_{n+1:n+1}$ and W_{n+1} (where in this case it is convenient to use the transformation

$$W_k = (n+2-k)(S_{k:n+1} - S_{k-1:n+1}),$$

as in Johnson et al. [8], p. 305) can be expressed as simple functions of the predecessor members of the sequence. We still get the result that $(Y_1, Y_2, \ldots, Y_n) \stackrel{d}{=} (X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ in case of standard uniform parent X.

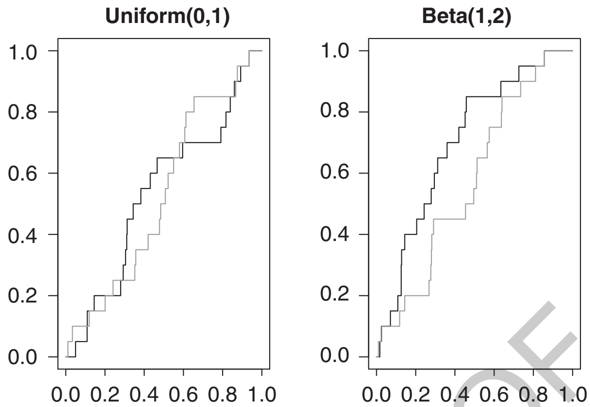

Fig. 1 Empirical distribution functions F_{20}^* and G_{20}^* for Uniform(0,1) and Beta(1,2) parents; this illustrates the general pattern

For uniformity testing purposes we present in Table 1 the upper critical points of D_n^*, $n = 3(1)30(5)100$, when the underlying parent is standard uniform ($U \stackrel{d}{=} X_0$). These points were obtained by generating 10,000 independent replicates of the sample $(D_{n,1}^*, D_{n,2}^*, \ldots, D_{n,50}^*)$ and defining the quantile of order p of D_n^* as the mean of the samples quantiles for $p = 0.9, 0.925, 0.95, 0.975, 0.99, 0.995, 0.999$.

We also performed a simulation study of the proportion of rejections of uniformity when the underlying parent was X_m, $m \in [-2, 0]$ and when making pairwise comparisons of the order statistics $\{x_{k:n}\}$ edf and the $\{y_k\}$ edf (the process of generating $\{y_k\}$ was iteratively repeated 10,000 times). Observe that the rationale for this procedure relies on the fact that if the original observations $\{p_k\}$ are indeed uniform, the "Sukhatme's" $\{y_k\}$ would be order statistics of standard uniform, and hence repeating Sukhatme's algorithm we would obtain again a set of order statistics of standard uniform.

From Fig. 2 we observe that the proportion of rejections of uniformity increases with n. However, the extended Sukhatme's like transformed data performs badly in detecting departures from uniformity when $n < 20$. This situation can obviously constitute a problem when combining p-values in meta-analytical syntheses since the number of available (reported) p-values is usually small.

Another way of assessing the usefulness of this extended Sukhatme's transformation in testing uniformity is by calculating the area limited by the edf's F_n^* and G_n^*, since under the validity of the null hypothesis $X \frown \text{Uniform}(0,1)$, the area between the two curves should be zero—big area values should indicate a departure from uniformity. In Table 2 we compare the areas obtained by simulation (10,000 runs) for some values of n when the underlying parents are standard uniform and Beta(1,2). Analyzing Table 2 we see that the area is indeed inferior for the standard uniform parent, except for some few cases. However, the differences between the two areas can be very small, which can difficult the task of testing uniformity with this procedure.

Table 1 Critical points of D_n^* when the underlying parent is Uniform (0,1)[a]

n	0.9	0.925	0.95	0.975	0.99	0.995	0.999
3	0.667	0.667	0.667	0.667	0.667	0.667	0.667
4	0.605	0.656	0.703	0.734	0.747	0.747	0.747
5	0.600	0.610	0.634	0.682	0.753	0.753	0.753
6	0.548	0.580	0.62	0.666	0.736	0.736	0.736
7	0.542	0.563	0.589	0.632	0.712	0.712	0.712
8	0.509	0.529	0.558	0.605	0.686	0.686	0.686
9	0.484	0.509	0.540	0.582	0.660	0.660	0.660
10	0.470	0.491	0.518	0.558	0.635	0.635	0.635
11	0.454	0.472	0.498	0.537	0.612	0.612	0.612
12	0.436	0.455	0.482	0.520	0.592	0.592	0.592
13	0.422	0.441	0.466	0.503	0.574	0.574	0.574
14	0.410	0.429	0.452	0.487	0.557	0.557	0.557
15	0.398	0.415	0.438	0.472	0.539	0.539	0.539
16	0.387	0.404	0.427	0.460	0.525	0.525	0.525
17	0.377	0.393	0.416	0.447	0.511	0.511	0.511
18	0.368	0.385	0.406	0.437	0.498	0.498	0.498
19	0.359	0.376	0.396	0.427	0.486	0.486	0.486
20	0.352	0.367	0.387	0.416	0.474	0.474	0.474
21	0.345	0.360	0.379	0.408	0.463	0.463	0.463
22	0.337	0.352	0.371	0.399	0.453	0.453	0.453
23	0.331	0.345	0.363	0.391	0.444	0.444	0.444
24	0.325	0.339	0.357	0.384	0.435	0.435	0.435
25	0.319	0.332	0.350	0.376	0.427	0.427	0.427
26	0.313	0.326	0.344	0.370	0.419	0.419	0.419
27	0.308	0.321	0.338	0.363	0.411	0.411	0.411
28	0.302	0.315	0.332	0.357	0.404	0.404	0.404
29	0.298	0.311	0.327	0.352	0.400	0.400	0.400
30	0.293	0.306	0.322	0.345	0.392	0.392	0.392
35	0.273	0.285	0.300	0.321	0.363	0.363	0.363
40	0.257	0.268	0.282	0.302	0.341	0.341	0.341
45	0.243	0.253	0.267	0.286	0.322	0.322	0.322
50	0.231	0.241	0.254	0.272	0.306	0.306	0.306
55	0.221	0.230	0.242	0.260	0.292	0.292	0.292
60	0.212	0.221	0.232	0.249	0.280	0.280	0.280
65	0.204	0.212	0.224	0.239	0.269	0.269	0.269
70	0.197	0.205	0.216	0.231	0.260	0.260	0.260
75	0.190	0.198	0.209	0.223	0.251	0.251	0.251
80	0.185	0.193	0.202	0.217	0.244	0.244	0.244
85	0.179	0.186	0.196	0.210	0.236	0.236	0.236
90	0.174	0.182	0.191	0.204	0.229	0.229	0.229
95	0.170	0.177	0.186	0.199	0.223	0.223	0.223
100	0.166	0.172	0.181	0.194	0.217	0.217	0.217

[a]The standard errors of the critical points are less than or equal to 0.001

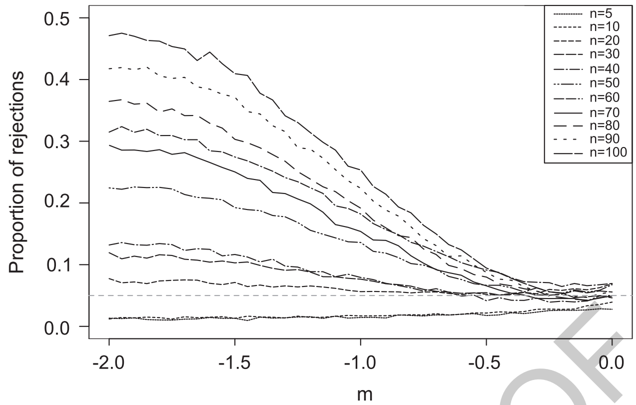

Fig. 2 Proportion of rejections of uniformity at level 0.05 using Sukhatme's like transformation when the underlying parent is X_m, $m \in [-2, 0]$

Table 2 Area limited by the functions F_n^* and G_n^* when the underlying parents are Uniform(0,1) and Beta(1,2)

	Beta(1,2)		Uniform(0,1)	
n	Area	s.e.	Area	s.e.
5	0.0366	0.00188	0.0333	0.00179
10	0.0848	0.00279	0.1027	0.00304
15	0.0794	0.00270	0.1216	0.00327
20	0.0860	0.00280	0.0820	0.00274
25	0.0620	0.00241	0.0608	0.00239
30	0.0823	0.00275	0.0495	0.00217
35	0.0699	0.00255	0.0526	0.00223
40	0.0742	0.00262	0.0411	0.00199
45	0.0665	0.00249	0.0450	0.00207
50	0.1005	0.00301	0.0319	0.00176
55	0.0927	0.00290	0.0370	0.00189
60	0.0774	0.00267	0.0376	0.00190
65	0.0830	0.00276	0.0247	0.00155
70	0.0648	0.00246	0.0425	0.00202
75	0.0371	0.00189	0.1369	0.00344
80	0.0682	0.00252	0.0388	0.00193
85	0.0702	0.00256	0.0403	0.00197
90	0.0901	0.00286	0.0395	0.00195
95	0.0701	0.00255	0.0358	0.00186
100	0.0730	0.00260	0.0498	0.00218

4 Conclusion

It seems worth to point out that the entropy of X_m, $m \in [-2, 2]$, is

$$H(X_m) = -\int_0^1 f_{X_m}(x)\ln(f_{X_m}(x))\mathrm{d}x = 0.5 + \ln(2) + \frac{\ln\left[\left(\frac{2-m}{2+m}\right)^m\right]}{8} - \frac{\ln(4-m^2)}{2} + \frac{\ln\left(\frac{2-m}{2+m}\right)}{2m},$$

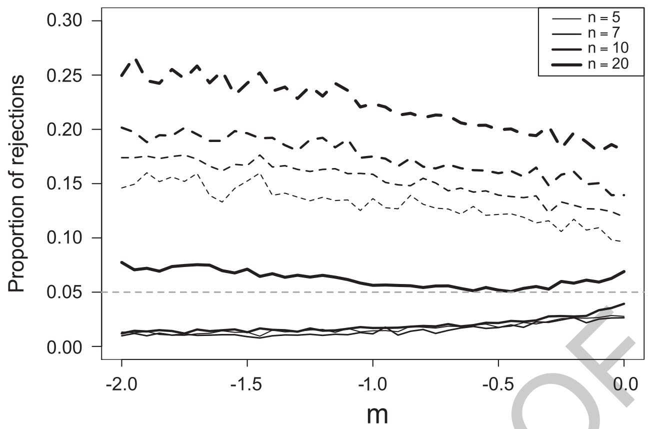

Fig. 3 Comparison of the proportion of rejections of uniformity using Sukhatme's like method and the method described in Sect. 2

(for a detailed study of entropy, cf. [7]), whose graph is concave, and hence the entropy of $\min\left(\frac{X_m}{X_p}, \frac{1-X_m}{1-X_p}\right) \stackrel{d}{=} X_{\frac{mp}{6}}$ is, for $m, p \in [-2, 2]$, nearer to the entropy of X_0 than to the entropy of X_m and X_p. We would thus expect that Sukhatme's like method of sample augmentation would provide better results than the method explained in Sect. 2. Observe however that further investigation of the matter seems to indicate the reverse, as shown in Fig. 3 (the solid lines correspond to Sukhatme's AQ3 like method and the dashed lines to the method described in Sect. 2). The general question of comparing analytically edfs of correlated samples remains unsolved, even for simple forms of weak dependence only simulation results in well-defined situations seem feasible.

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AUTHOR QUERIES

AQ1. First author has been treated as corresponding author. Please check.
AQ2. Please check if edit to sentence starting “Recently Paul [10] discussed....” is okay.
AQ3. Please check if edit to sentence starting “The general question...” is okay.
AQ4. Picture given in the “Acknowledgement” has been deleted. Please check.