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SELF-SIMILARITY AND SYMMETRIES OF PASCAL'S TRIANGLES AND SIMPLICES MOD p

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1. INTRODUCTION

When one sees a printout of Pascal's triangle of binomial coefficients computed modulo the prime 2 (Figure 1, where all zeros have been replaced by blanks), one is immediately struck by the pleasing self-similarity in the picture. Modulo a prime p which is different from 2 (Figure 2), the self-similarity is a little more complicated, if no less striking.

Another striking feature of the mod 2 triangle (Figure 1) is its abundant symmetry. Specifically, each 2^{k+1}-rowed triangular array of integers Δ^k is invariant under reflections across its axes of symmetry and, hence, under rotations of 120° or 240°, since these are compositions of reflections. Modulo a general

```
               1
              1 1
             1   1
            1 1 1 1
           1       1
          1 1     1 1
         1   1   1   1
        1 1 1 1 1 1 1 1
       1               1
      1 1             1 1
     1   1           1   1
    1 1 1 1         1 1 1 1
   1       1       1       1
  1 1     1 1     1 1     1 1
 1   1   1   1   1   1   1   1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

Figure 1: Pascal's triangle mod 2.

```
        1
       1 1
      1 2 1
     1     1
    1 1   1 1
   1 2 1 1 2 1
  1     2     1
 1 1   2 2   1 1
1 2 1 2 1 2 1 2 1
```

Figure 2: Pascal's triangle mod 3.

prime p, we no longer have this symmetry, but it turns out that the reflected and rotated versions of the triangles Δ^k do still enjoy the same kind of self-similarity that Δ^k itself does.

In this paper we will investigate the relationship between self-similarity and the symmetries of Pascal's triangle mod p. In the process, we will see that the same relationship holds when we look at arbitrary triangular arrays that are *defined* by a mod n self-similarity procedure. Moreover, we will also see that the same analysis can be applied to higher-dimensional arrays of multinomial coefficients, with similar results obtaining. And, indeed, the generalization that takes us from Pascal's triangle mod p to arbitrary self-similar triangular arrays will also succeed in the higher-dimensional setting.

Self-Similarity Procedure: For a fixed prime p, let Δ^0 be the triangular array consisting of the p rows at the top of the mod p Pascal's triangle. (For example, the top three rows in Figure 2, where $p = 3$.) If we replace each entry δ in Δ^0 by δ *times* Δ^0–a copy of Δ^0 all of whose entries have been multiplied modulo p by δ–we get Δ^1, the triangle composed of the first p^2 rows of Pascal's triangle mod p. Continuing, the replacement of each entry δ in Δ^0 by δ times a copy of Δ^k gives Δ^{k+1}, which is composed of the first p^{k+2} rows of the infinite triangle. On the other hand, had we replaced each entry δ in Δ^k by a copy of δ times Δ^0, we would again have gotten Δ^{k+1}. Since these two procedures yield the same result, we will simply refer to *the mod p self-similarity procedure* in what follows. Notice that for $p = 2$, Δ^3 yields Figure 1 and for $p = 3$, Δ^1 yields Figure 2. This mod p self-similarity of Pascal's triangle, which has been frequently noted–see [5], for example–is a feature of binomial coefficients and is easily verified using the result of Lucas for computing binomial coefficients modulo primes (Lemma 1 below).

The basic relationship between the symmetries of Pascal's triangle and mod p self-similarity alluded to above is given by the following

Proposition 1: *Let our modulus p be prime. If we denote by Θ^k the mod p Pascal's triangle Δ^k reflected across one its axes of symmetry, then Θ^k can also be obtained by applying the mod p self-similarity procedure to the seed triangle Θ^0 gotten by subjecting Δ^0 to the same reflection.*

The proof depends heavily on the following [2, pp. 417-420].

Lemma 1 (Lucas): *Fix a prime p. If $a = a_0 + a_1p + \cdots + a_kp^k$ and $b = b_0 + b_1p + \cdots + b_kp^k$ with $0 \le a_i, b_i < p$, then*

$$\binom{a}{b} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \mod p. \tag{1}$$

Proof of Proposition 1: Let Θ^k be obtained by reflecting Δ^k across its axis of symmetry that contains its lower left vertex. (The case where the lower right vertex is fixed is similar, while the case where the apex is fixed is, of course, trivial.) If we denote by $(\Theta^k)_{ab}$ the entry in the a^{th} row and the b^{th} column of Θ^k, then we can verify that

$$(\Theta^k)_{ab} = \binom{p^{k+1}-1-b}{p^{k+1}-1-a} \equiv \prod_{i=0}^{k} \binom{(p^{k+1}-1-b)_i}{(p^{k+1}-1-a)_i} \mod p. \tag{2}$$

[Note that we number all rows and all columns of our triangles starting with 0 and that the "columns" of a triangle are taken to be parallel to its left-hand edge.]

The first equation in (2) is obtained by inspecting the effect of the reflection on Δ^k and the second follows from Lemma 1. (Here x_i denotes the coefficient of p^i in the p-adic expansion of x.) But $(p^{k+1}-1)_i = p-1$ for $i = 0, \ldots, k$, and so for any $c < p^{k+1}$ we obtain $(p^{k+1}-1-c)_i = (p-1)-c_i$, since there is no borrowing p-adically in the subtraction of c from $p^{k+1}-1$. As a result, we now have

$$(\Theta^k)_{ab} \equiv \prod_{i=0}^{k} \binom{(p-1)-b_i}{(p-1)-a_i} \equiv \prod_{i=0}^{k} (\Theta^0)_{a_i b_i} \quad \bmod p, \tag{3}$$

where the last equality comes from an inspection of the effect of the reflection on Δ^0.

It now remains to show that the right-hand quantity in (3) is actually the $(a,b)^{th}$ entry of the triangle Φ^k that results when we apply the mod p self-similarity construction to Θ^0. Perhaps the most elegant way of doing this would be to point out that the self-similarity procedure is really a disguised variant of the Kronecker tensor product of matrices modulo p. (See [4, p. 8].) However, it is easy enough to prove the result inductively without this fact. We first define $\hat{a} = a - a_k p^k$ and $\hat{b} = b - b_k p^k$ when $0 \le a, b < p^{k+1}$. (So $a \equiv \hat{a} \bmod p^k$ and $b \equiv \hat{b} \bmod p^k$.) We can then show that if $\hat{a} \ge \hat{b}$, $(\Phi^k)_{ab}$ is the $(\hat{a},\hat{b})^{th}$ entry of the $(a_k,b_k)^{th}$ triangular block of Φ^k, and so it equals $(\Theta^0)_{a_k b_k}(\Phi^{k-1})_{\hat{a}\hat{b}}$. We now employ the inductive hypothesis and (3) to conclude that $(\Theta^k)_{ab} \equiv (\Phi^k)_{ab} \bmod p$. (Note that we are using the first of the two equivalent mod p self-similarity procedures.) If $\hat{a} < \hat{b}$ then we can show that $(\Phi^k)_{ab}$ lies in one of (Φ^k)'s largest inverted triangular *holes*, which we take to be filled with zeros. On the other hand, $\hat{a} < \hat{b}$ implies that $a_i < b_i$ for some $i < k$ and, therefore, that $(\Theta^0)_{a_i b_i} = 0$. This completes the proof of Proposition 1. □

2. TRIANGLES DEFINED BY MOD n SELF-SIMILARITY

We are so accustomed to the symmetry and grace of Pascal's triangle that we need little convincing to accept Proposition 1. What is perhaps surprising at first glance is the fact that it holds for *any* triangle Ψ^k obtained by applying the mod n self-similarity procedure to an arbitrary seed triangle Ψ^0. That is to say, instead of starting with a triangular array of numbers–like Pascal's triangle mod p–that possesses mod p self-similarity as one of its properties, we can take the mod p self-similarity procedure as our point of departure. Given any p-rowed triangle Ψ^0, we can use it as our seed to generate a family of triangles Ψ^k. (If we want to insure that $(\Psi^k)_{ab} = (\Psi^{k+1})_{ab} = \ldots$, and hence that Ψ^k embeds in Ψ^m for $k < m$–thereby justifying the term *self-similarity*–we do need to assume, however, that $(\Psi^0)_{00} = 1$.) For example, Figure 3 shows two seed triangles and the larger triangles they generate via the mod 3 self-similarity procedure. Note that the left-hand seed has 1 as the entry at its apex, but for the right-hand seed this is not the case. And, indeed, we see that the right-hand seed does not embed in the triangle it generates. Furthermore, the large triangle on the right-hand side of Figure 3 illustrates the fact that there is no unique seed that generates a given triangle: the three-rowed triangle at its top generates the entire triangle, but the given seed triangle does also.

In fact, we can carry out our self-similarity procedure with any modulus n–whether prime or not–and any n-rowed triangular seed. We must note, however, that if our modulus n is not prime, the n^{k+1}-rowed Pascal's triangle will no longer equal Δ^k, the triangular array generated modulo n by Δ^0.

```
        1                            2
       2 1                          1 2
      1 2 2                        1 2 1

        1                            1
       2 1                          2 1
      1 2 2                        2 1 2
     2     1                      2     1
    1 2   2 1                    1 2   2 1
   2 1 1 1 2 2                  1 2 1 2 1 2
  1     2     2                2     1     2
 2 1   1 2   1 2              1 2   2 1   1 2
1 2 2 2 1 1 2 1 1            1 2 1 2 1 2 1 2 1
```

Figure 3: two seeds and the triangles they generate mod 3

It turns out that in the context of triangular arrays generated by the mod n self-similarity procedure applied to arbitrary seed triangles, the conclusion of Proposition 1 still holds. Namely, we have

Proposition 2: *Fix n, not necessarily prime. Let Ψ^k be obtained by applying the mod n self-similarity procedure to the n-rowed seed triangle Ψ^0. If we denote by Υ^k the triangle obtained by subjecting Ψ^k to one of the symmetries of an equilateral triangle–a reflection or a suitable rotation–then Υ^k can also be constructed by applying the mod n self-similarity procedure to the seed triangle Υ^0 gotten by subjecting Ψ^0 to the same symmetry.*

Figure 3 provides an illustration of Proposition 2: rotating the left-hand seed by $+120°$ gives the right-hand seed, which, in turn, generates the larger triangle on the right; but that triangle is just the result of rotating the larger left-hand triangle by $+120°$.

The key fact used in establishing Proposition 2 is that the proof of Proposition 1 really does not rely on specific properties of binomial coefficients beyond the fact that we can use the lemma of Lucas (Lemma 1). But, in fact, having the appropriate Lucas-type lemma is equivalent to being generated from a seed triangle by what we have called the self-similarity process. That is, we have the following

Proposition 3: *Suppose Ψ^k is an n^{k+1}-rowed triangular array. Ψ^k is obtained by applying the mod n self-similarity procedure to the n-rowed seed triangle Ψ^0 if and only if*

$$(\Psi^k)_{ab} \equiv \prod_{i=0}^{k} (\Psi^0)_{a_i b_i} \mod n \tag{4}$$

for all a, b with $0 \le b \le a < n^{k+1}$. (Here $a = a_0 + a_1 n + \cdots + a_k n^k$ and $b = b_0 + b_1 n + \cdots + b_k n^k$ with $0 \le a_i, b_i < n$.)

Remark: Note that in Proposition 3 we do not require that $(\Psi^0)_{00} = 1$, i.e., that Ψ^0 embed in Ψ^k.

Proof: By induction on k. The demonstration that "self-similarity" implies "Lucas-type lemma" is given by second part of the proof of Proposition 1.

On the other hand, if we have a Ψ^k that satisfies a "Lucas-type lemma,"

$$(\Psi^k)_{ab} \equiv \prod_{i=0}^{k} T_{a_i b_i} \mod n$$

compare it with the Φ^k generated by the mod n self-similarity process from T and see that the triangles Ψ^k and Φ^k agree entry by entry. □

3. MULTINOMIAL COEFFICIENTS AND HIGHER DIMENSIONS

Let us now turn to higher dimensions. Putz points out in [3] that the *multinomial* coefficients, $\binom{a}{b_1,\ldots,b_m}$, can be arrayed in an m-dimensional figure, which he calls "Pascal's polytope." I prefer to use the geometrically more specific term *Pascal's simplex*. Nomenclature aside, as for Pascal's triangle it is true that a coefficient on the a-th level of this array is the sum of the appropriate ancestors on the $(a-1)$-st level above. For multinomial coefficients, this is just the familiar recurrence relation

$$\binom{a}{b_1,\ldots,b_m} = \binom{a-1}{b_1-1,b_2,\ldots,b_m} + \cdots + \binom{a-1}{b_1,\ldots,b_m-1}.$$

But Lucas's result (Lemma 1) has an m-dimensional analogue [1], so it turns out that Δ^k, the p^{k+1}-leveled m-dimensional Pascal's simplex modulo a prime p, can be obtained by applying the obvious extension of our mod p self-similarity procedure to the seed simplex Δ^0 consisting of the first p levels of Δ^k. Furthermore, the symmetry results we established in the two-dimensional case generalize easily to m dimensions, to wit:

Proposition 4: *If we denote by Φ^k the simplex obtained by subjecting Δ^k, the m-dimensional mod p Pascal's simplex, to one of the symmetries of an equilateral m-simplex–a reflection or a suitable rotation–then Φ^k can also be gotten by applying the mod p self-similarity procedure to the seed simplex Φ^0 obtained by subjecting Δ^0 to the same symmetry.*

Symmetries of an m-simplex correspond exactly to the permutations of their $m+1$ vertices. Since the symmetric group on $m+1$ letters is generated by simple transpositions, it is enough to prove our result for the symmetries corresponding to these transpositions, namely, reflections in hyperplanes that contain all but one of the vertices. And, indeed, for each transposition (ij) we can write down its reflection τ_{ij} for the p^k-leveled m-simplex explicitly:

$$\tau_{ij}\binom{a}{b_1,\ldots,b_i,\ldots,b_j,\ldots,b_m} = \binom{a}{b_1,\ldots,b_j,\ldots,b_i,\ldots,b_m}$$

for $0 < i < j \le m$ and, taking the 0-th vertex of the simplex to be its apex,

$$\tau_{0i}\binom{a}{b_1,\ldots,b_{i-1},b_i,b_{i+1},\ldots,b_m} = \binom{p^{k+1}-1-b_i}{b_1,\ldots,b_{i-1},p^{k+1}-1-a,b_{i+1},\ldots,b_m}$$

for $1 \le i \le m$. Given these formulae, the proof of Proposition 4 parallels that of Proposition 1.

Once again we can see that the symmetry results of Proposition 4 hold in general for the m-simplex Φ^k obtained by applying the self-similarity procedure to any seed simplex Φ^0:

Proposition 5: *Fix n, not necessarily prime. Let Ψ^k be the m-simplex obtained by applying the mod n self-similarity procedure to the n-leveled seed simplex Ψ^0. If we denote by Φ^k the simplex obtained by subjecting Ψ^k to one of the symmetries of an equilateral m-simplex–a reflection or a suitable rotation–then Φ^k can also be constructed by applying the mod n self-similarity procedure to the seed simplex Φ^0 gotten by subjecting Ψ^0 to the same symmetry.*

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