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Foucault's Pendulum, a Classical Analog for the Electron Spin State

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FOUCAULT'S PENDULUM, A CLASSICAL ANALOG
FOR THE ELECTRON SPIN STATE

A Thesis

Presented to

The Faculty of the Department of Physics & Astronomy
San José State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Rebecca A. Linck

August 2013

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The Designated Thesis Committee Approves the Thesis Titled

FOUCAULT'S PENDULUM, A CLASSICAL ANALOG
FOR THE ELECTRON SPIN STATE

by

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APPROVED FOR THE DEPARTMENT OF PHYSICS & ASTRONOMY

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August 2013

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ABSTRACT

FOUCAULT'S PENDULUM, A CLASSICAL ANALOG FOR THE ELECTRON SPIN STATE

by Rebecca A. Linck

Spin has long been regarded as a fundamentally quantum phenomena that is incapable of being described classically. To bridge the gap and show that aspects of spin's quantum nature can be described classically, this work uses a classical Lagrangian based on the coupled oscillations of Foucault's pendulum as an analog for the electron spin state in an external magnetic field. With this analog it is possible to demonstrate that Foucault's pendulum not only serves as a basis for explaining geometric phase, but is also a basis for reproducing a broad range of behavior from Zeeman-like frequency splitting to precession of the spin state. By demonstrating that unmeasured electron spin states can be fully described in classical terms, this research opens the door to using the tools of classical physics to examine an inherently quantum phenomenon.

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CHAPTER 1

INTRODUCTION

Since its introduction into the framework of quantum mechanics by Wolfgang Pauli in 1927, spin has been a source of mystery and confusion. Initially physicists attempted to explain spin in terms of physical rotation. They determined that the angular momentum of an electron as it orbits a proton is greater than the value predicted by orbital motion alone. They reasoned that, like the earth, the electron both revolves around the proton and rotates about a fixed internal axis. However, when they calculated the electron's spin angular momentum, they encountered a problem. According to Powell and Crasemann [Pow61, Ch. 10], the required spin angular momentum of an electron is so great that the electron's required rotational speed, "... contradicts the requirements of special relativity." Realizing they needed a new picture, physicists adopted the idea that spin angular momentum (or simply spin), like mass and charge is an intrinsic property of a particle that is independent of its motion.

The assertion that spin is an intrinsic property answered one question while raising others. If spin is not associated with physical motion then, what is its source? As physicists attempted to answer this and other questions, they discovered an array of properties that seemed both strange and counterintuitive in light of their classical understanding of physics.

One of their first discoveries was the determination that all elementary particles have a constant value of spin. According to Griffiths [Grif05, Ch. 4], all electrons have a spin of $s = \frac{1}{2}$, while all photons have a spin of $s = 1$, and all

gravitons have a spin of $s = 2$. For a given particle, measurements of the particle's spin can result in multiple outcomes. As an example, when a measurement of an electron's spin (m_s) is made, the result is either $m_s = \frac{1}{2}\hbar$ or $m_s = -\frac{1}{2}\hbar$. The combination of the particle's intrinsic spin value with its associated measurement outcome forms a spin state.

Building on the quantized nature of spin, physicists went on to discover something unexpected about the electron's gyromagnetic ratio. In addition to having an angular momentum, spinning charges have a magnetic dipole moment. In SI units, a fundamental particle with charge q and mass m , has an intrinsic magnetic dipole moment given by, $\mu = \frac{q\hbar}{2m}$. The ratio of a particle's magnetic dipole moment to its spin angular momentum is called the gyromagnetic ratio. For an electron with a spin angular momentum of $\frac{1}{2}\hbar$, the gyromagnetic ratio is, $\gamma = -\frac{e}{m}$. It is interesting to note that the classical gyromagnetic ratio of a uniform spinning sphere with the charge and mass of an electron is given by, $\gamma = -\frac{e}{2m}$. As a result, the electron has a gyromagnetic ratio that is double its classically predicted value.

With these and other properties in mind, physicists set about the task of explaining spin using the framework of classical mechanics. They soon determined that this was not possible. Using his observations as justification, Pauli stated that spin is "... not describable in classical terms" [Park64, Ch. 6]. Building on this, Dirac [Dirac58, Ch. 6] later claimed that, "spin does not correspond very closely to anything in classical mechanics so the method of classical analogy is not suitable for studying it." Eventually, these ideas so permeated quantum theory that they became part of the core concepts taught to students. In their texts on quantum mechanics, Landau and Lifshitz [Land58, Ch. 8] claim that spin is, "... essentially incapable of classical interpretation," while Liboff [Lib98, Ch. 9] claims that spin, "... has no classical counterpart."

To support the claim that spin (and the associated spin state) cannot be described classically, physicists compiled a list of properties that seem to show that spin cannot be described using the framework of classical mechanics. These properties are intended to show that spin is a fundamentally quantum phenomena. The following is a list of properties that have been used to defend this assertion:

- (1) The spin state must be described using complex numbers.
- (2) When rotated, the spin state must undergo a 4π (as opposed to the classical 2π) rotation to return to its original state.
- (3) The electron's gyromagnetic ratio is double the classically predicted value.
- (4) Measurements of the spin state have discrete outcomes that are predicted by associated probabilities.

In the years since these claims were first made, many have attempted to show that the spin state can be described classically. One such method involves the polarization of classical plane electromagnetic waves. As is explained by Klyshko [Kly93] and referenced by Wharton, Linck and Salazar-Lazaro [WLS11], both the doubled rotation of the spin state required to return it to its original state and the electron's doubled gyromagnetic ratio can be explained using this method. However, attempts to extend this analogy to explanations of the other aspects of the spin state's decidedly quantum nature have met with incomplete success. In particular, the extensions needed to describe the interaction between the spin state and an arbitrary magnetic field become increasingly abstract making them harder to understand within the classical framework.

Desiring to find a better classical analogy for electron spin, this work turns to a tool that is used successfully in both classical mechanics and in quantum theory.

This tool is the Lagrangian. In classical mechanics the Lagrangian for a system of particles is the difference between a system's kinetic and potential energies. So, for a single particle of mass m moving with speed v and with potential energy V , the Lagrangian can be expressed as: $L = \frac{1}{2}mv^2 - V$. In relativistic quantum theory the Lagrangian (and its extension the Lagrangian density), are used extensively. As a result, finding a classical Lagrangian that describes spin fits nicely into the already established frameworks on both sides of the problem.

As will be shown in the following work, the Lagrangian that describes the dynamics of Foucault's pendulum serves as a useful vehicle for describing many of the decidedly quantum properties of the electron spin state. To that end:

- We will begin by explaining what Foucault's pendulum is and then we will derive the Lagrangian that describes its associated dynamics.
- We will then show that the Lagrangian that describes Foucault's pendulum is capable of describing the dynamics of an unmeasured electron spin state in a constant one dimensional magnetic field.
- With an elegant modification, we will show that a modified Foucault's pendulum Lagrangian is capable of describing the dynamics of an unmeasured electron spin state in a time varying one dimensional magnetic field.
- Using Foucault's pendulum as a conceptual analog, we will show that many of the electron spin state's supposedly quantum characteristics listed above can be explained from a classical framework.

- Modifying the Lagrangian once more, we will show that the modified Lagrangian is capable of describing the dynamics of an unmeasured electron spin state in a time varying three dimensional magnetic field.
- We will then propose an alternate Lagrangian for electron spin that is based on the Schrödinger Equation and compare this new Lagrangian with the Lagrangian based on Foucault's pendulum.
- Finally, we will review our results and propose a possible extension.

CHAPTER 2

FOUCAULT'S PENDULUM

2.1 Foucault and His Pendulum

Foucault's pendulum is named for French inventor Jean Bernard Léon Foucault (1819 - 1868). According to Baker and Blackburn [Bak05, Ch. 4], though Foucault initially studied medicine, he ended up spending the majority of his life studying the physical sciences. His interests in optics led him to study interference, chromatic polarization, and the speed of light through various media. To facilitate his studies, Foucault invented and/or improved upon many devices. Among these, he invented a precision gyroscope and developed techniques which improved the manufacture of telescopes.

In 1851, Foucault was working on a telescope when he made an unexpected discovery. According to Baker and Blackburn [Bak05, Ch. 4], he noticed that “the plane of oscillation of a vibrating rod, that was fixed in a lathe chuck, remained fixed in orientation even as he slowly turned the chuck.” This observation led Foucault to wonder if the orientation of the plane of a simple pendulum's oscillation might similarly be affected by the earth's rotation.

In January of 1851, Foucault tested his idea. He set up a 2 meter long pendulum in the basement of his mother's home. As he watched the pendulum's 5 kilogram bob sweep back and forth in typical pendulum fashion, he noticed that the pendulum's plane of oscillation slowly precessed. With this observation, Foucault proved that it is possible to observe the affects of the earth's rotation. Desiring to improve upon his results, Foucault went on to build a number larger pendula. The

most famous of these was 68 meters long, had a 28 kilogram bob, and was suspended from the dome of the Pantheon in Paris. With these larger pendula, Foucault successfully demonstrated the earth's rotation for the general public.

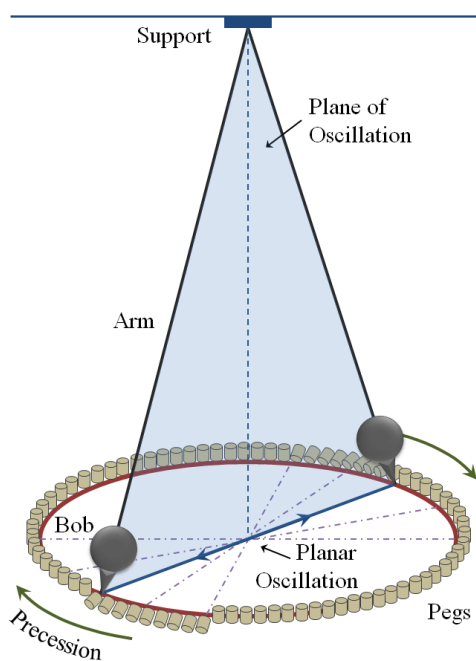


Figure 2.1: Foucault's Pendulum

Today, examples of Foucault's pendulum can be found in museums and science centers throughout the world. Though any planar pendulum setup in the manner shown in Fig. 2.1 can exhibit Foucault-like behavior, small pendula are rarely used for demonstrations. According to Hamill [Ham10, Ch. 13], Foucault pendula generally have very massive bobs and very long arms. This arrangement serves to maximize the effect. In addition, most exhibits employ the use of a device that is placed in the pendulum's support that periodically gives the pendulum a kick. Without this additional input of energy the effects of friction and air resistance would cause the pendulum to gradually come to rest. Since the rate of the pendulum's precession is constant (depending only on the pendulum's latitude

and the earth's rate of rotation), pegs are often placed along the perimeter of the pendulum's planar path. As the pendulum oscillates, its precession causes it to knock the pegs over, which in turn helps the observer to see the pendulum's slow rotation.

2.2 The Classical Lagrangian

Our study begins with the Lagrangian that describes the motion of Foucault's pendulum. Because the motion of the pendulum is observed in a non inertial reference frame, the Lagrangian that describes it is somewhat more complicated than the standard single particle Lagrangian. According to Landau and Lifshitz [Land76, Ch. 6], the Lagrangian for a particle in a non inertial reference frame can be expressed as

$$L = \frac{1}{2}mv^2 + m\mathbf{v} \cdot (\boldsymbol{\Omega} \times \mathbf{r}) + \frac{1}{2}m(\boldsymbol{\Omega} \times \mathbf{r})^2 - m\mathbf{W} \cdot \mathbf{r} - U. \quad (2.1)$$

Where in the non inertial frame \mathbf{r} and \mathbf{v} are the position and velocity of the particle and U is the potential energy of the particle. In addition, $\boldsymbol{\Omega}$ is the angular velocity and \mathbf{W} is the translational acceleration of the non inertial frame. By comparing this Lagrangian (2.1) with the single particle Lagrangian ($L = \frac{1}{2}mv^2 - U$), it is evident that observing the particle in a non inertial frame results in the addition of three new terms to the Lagrangian. Of these additional terms, two terms have special significance. The second term in equation (2.1) relates the energy imparted to the particle by the Coriolis force, while the third term in this equation relates the energy imparted to the particle by the centrifugal force.

To arrive at the Lagrangian that describes the motion of Foucault's pendulum, we begin by eliminating the parts of the generalized Lagrangian (2.1) that do not pertain to the pendulum. Measurements of the pendulum bob's position

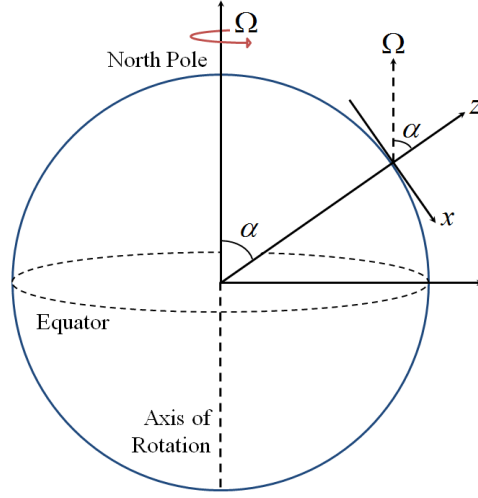


Figure 2.2: A Coordinate System Affixed to the Surface of the Rotating Earth

are taken with respect to a coordinate system that is affixed to the surface of the Earth. This coordinate system (indicated in Fig. 2.2), remains at a fixed distance from the center of the Earth. Therefore, ignoring the Earth's orbital motion, this coordinate system experiences no translational acceleration. As a result, $\mathbf{W} = 0$. Noting that the potential energy of the particle is due to gravity ($U = -m\mathbf{g} \cdot \mathbf{r}$), we can re-express the Lagrangian as

$$L = \frac{1}{2}mv^2 + m\mathbf{v} \cdot (\boldsymbol{\Omega} \times \mathbf{r}) + \frac{1}{2}m(\boldsymbol{\Omega} \times \mathbf{r})^2 + m\mathbf{g} \cdot \mathbf{r}. \quad (2.2)$$

Next, we express the position and velocity of the pendulum's bob in terms of their Cartesian components (as indicated in Fig. 2.2). We then break the Earth's rotation frequency vector ($\boldsymbol{\Omega}$) into components parallel and perpendicular to the plane of the Earth's surface. Plugging these expressions into equation (2.2) yields the following Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m\Omega [(y\dot{x} - \dot{x}y) \cos \alpha + (y\dot{z} - \dot{z}y) \sin \alpha] + \frac{1}{2}m\Omega^2 [y^2 + (x \cos \alpha + z \sin \alpha)^2] - mgz. \quad (2.3)$$

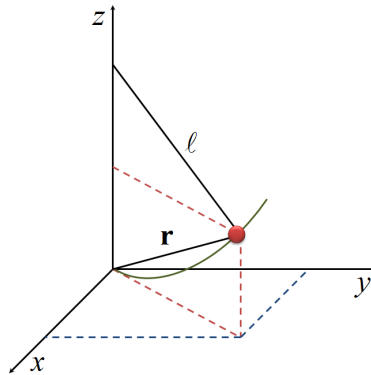


Figure 2.3: The Position of the Pendulum Bob Relative to the Earth's Surface

Now, we use Fig. 2.3 to re-express the pendulum bob's vertical motion in terms of the length of the pendulum arm, ($z = \frac{1}{2\ell}(x^2 + y^2 + z^2)$). In the small angle limit, the motion of the pendulum is restricted to the xy -plane, (so $z \approx 0$ and $\dot{z} \approx 0$). Using this approximation and defining a new variable $\beta = \Omega \cos \alpha$, the Lagrangian becomes

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + m\beta(\dot{y}x - \dot{x}y) + \frac{1}{2}m(\beta^2 - \omega_0^2)(x^2 + y^2) + \frac{1}{2}m\beta^2 y^2 \tan^2 \alpha. \quad (2.4)$$

Note that $\omega_0 = \sqrt{\frac{g}{\ell}}$ is the oscillation frequency of a simple planar pendulum.

Because the rotation rate of the Earth on its axis is relatively small (2π radians per day or 0.000073 radians per second), all terms that go as Ω^2 are generally dropped from this expression. Doing this, the Lagrangian becomes

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + m\beta(\dot{y}x - \dot{x}y) - \frac{1}{2}m\omega_0^2(x^2 + y^2). \quad (2.5)$$

This Lagrangian describes the dynamics of Foucault's pendulum in terms of a pair of coupled harmonic oscillators. If x and y are taken to be the positions of the two oscillators then β serves as the coupling parameter. Note that, when β is set to zero the Lagrangian in equation (2.5) describes two un-coupled harmonic oscillators.

CHAPTER 3

2D LAGRANGIAN

3.1 Electron Spin State in a Constant 1D Magnetic Field

Now, performing a change of variables, let x be x_1 and y be x_2 . Doing this, the Lagrangian in equation (2.5) becomes:

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + m\beta(\dot{x}_2x_1 - \dot{x}_1x_2) - \frac{1}{2}m\omega_0^2(x_1^2 + x_2^2). \quad (3.1)$$

Using Lagrange's Equations $\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0\right)$, it is possible to arrive at the following pair of coupled equations of motion:

$$\begin{aligned} \ddot{x}_1 &= 2\beta\dot{x}_2 - \omega_0^2x_1 \\ \ddot{x}_2 &= -2\beta\dot{x}_1 - \omega_0^2x_2 \end{aligned} \quad (3.2)$$

These equations, once solved, yield the real part of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-i\omega_+t} + b \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-i\omega_-t}. \quad (3.3)$$

Where a and b are arbitrary complex quantities and $\omega_{\pm} = \sqrt{\omega_0^2 + \beta^2} \pm \beta$.

Now, suppose an electron is placed in a magnetic field. According to Griffiths [Grif05, Ch. 4], the interaction between the electron's spin state $|\chi\rangle$ and an external magnetic field is described by the time dependent Schrödinger Equation $(i\hbar\frac{\partial}{\partial t}|\chi\rangle = H|\chi\rangle)$. The Hamiltonian (H) generally used to describe this interaction is $H = -\frac{1}{2}\gamma\hbar\mathbf{B} \cdot \boldsymbol{\sigma}$. Where, $\gamma = -\frac{e}{m}$, is the electron's gyromagnetic ratio, \mathbf{B} is the magnetic field vector and $\boldsymbol{\sigma}$ is the standard vector of Pauli spin matrices.

It is a standard practice in quantum mechanics to disregard global phase. Suppose instead that the global phase of the electron's spin state is included. Accounting for global phase due to rest mass oscillation at the Compton frequency ($\omega_0 = \frac{mc^2}{\hbar}$), the system's Hamiltonian becomes: $H = \hbar(\omega_0 I - \frac{1}{2}\gamma \mathbf{B} \cdot \boldsymbol{\sigma})$. Re-expressing the magnetic field vector so that $\boldsymbol{\beta} = -\frac{1}{2}\gamma \mathbf{B}$, the Schrödinger Equation for this system becomes:

$$i\hbar \frac{\partial}{\partial t} |\chi\rangle = \hbar(\omega_0 I + \boldsymbol{\beta} \cdot \boldsymbol{\sigma}) |\chi\rangle \quad (3.4)$$

Now, suppose that the electron is placed in a constant y -directed magnetic field. The solution to the Schrödinger Equation for this system can be expressed as:

$$\begin{bmatrix} \chi_+ \\ \chi_- \end{bmatrix} = a \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-i(\omega_0 + \beta)t} + b \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-i(\omega_0 - \beta)t}. \quad (3.5)$$

Written this way, a and b are complex constants and χ_+ and χ_- are the complex parts of the spin state $|\chi\rangle$. Note that the system's two eigenfrequencies ($\omega_{\pm} = \omega_0 \pm \beta$) correspond to a Zeeman energy splitting of equal amount above and below the ground state.

Comparing the solution for Foucault's pendulum in equation (3.3) to the solution to the Schrödinger equation in (3.5), the two solutions appear to be almost identical. If the difference in their associated eigenfrequencies could be resolved then the two solutions would have an identical form. Since the value of ω_0 for the pendulum is generally much larger than the value of β , the normal modes of the pendulum's oscillation are basically ($\omega_{\pm} \approx \omega_0 \pm \beta$). In this limit, the standard Lagrangian for Foucault's pendulum (3.1) maps onto the dynamics of an unmeasured spin state in a constant one-dimensional magnetic field.

3.2 An Alternate Lagrangian

Though the Lagrangian described in equation (3.1) does a passable job of mapping onto the spin state when $\omega_0 \gg \beta$, it would be nice to find a Lagrangian that yields the correct eigenfrequencies ($\omega_{\pm} = \omega_0 \pm \beta$) without putting a constraint on the relative strengths of ω_0 and β .

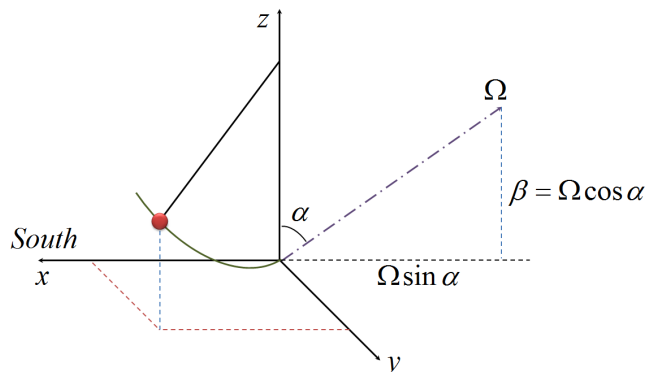


Figure 3.1: Components of the Earth's Rotation Frequency Vector with Respect to the Surface of the Earth

To do this we will begin by returning to the more general expression for the Foucault's pendulum Lagrangian found in equation (2.4). After performing a change of variables (x becomes x_1 and y becomes x_2), this Lagrangian becomes:

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + m\beta(\dot{x}_2x_1 - \dot{x}_1x_2) + \frac{1}{2}m(\beta^2 - \omega_0^2)(x_1^2 + x_2^2) + \frac{1}{2}m\beta^2x_2^2 \tan^2 \alpha. \quad (3.6)$$

Notice that the influence of the last term in this expression is dependent on x_2 , but is not dependent on x_1 . Since x_1 points in the direction of changing latitude, while x_2 points in the direction of changing longitude, the last term in equation (3.6) serves to increase the coupling between the pendulum's longitude and β . As there is no similar increase in coupling between the pendulum's latitude and β , equation (3.6) lacks symmetry. The last term, which arises from the influence of the centrifugal force, serves to modify (if only slightly) the pendulum's oscillation.

Unfortunately, the break in symmetry that is caused by the retention of the final term in equation (3.6) presents a problem when β is viewed as the outside influence of the magnetic field on an electron spin state.

To resolve this problem, suppose instead that the x_1 and x_2 directions represent the positions of two coupled simple harmonic oscillators. In this case, the appropriate Lagrangian would show no greater dependence on the motion of particle x_1 than it would show for particle x_2 . Since the last term in equation (3.6) breaks the required symmetry it is not allowed. As a result, the Lagrangian that describes Foucault's pendulum when it is represented as a pair of coupled harmonic oscillators is:

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + m\beta(\dot{x}_2x_1 - \dot{x}_1x_2) + \frac{1}{2}m(\beta^2 - \omega_0^2)(x_1^2 + x_2^2). \quad (3.7)$$

Notice that this Lagrangian only differs slightly from the Lagrangian listed in equation (3.1). Both equations describe the motion of two coupled harmonic oscillators. However, the addition of $\frac{1}{2}m\beta^2(x_1^2 + x_2^2)$ in equation (3.7), uniformly increases the influence of the coupling parameter β .

Now, performing the same analysis on the Lagrangian in equation (3.7) as was performed on the Lagrangian in equation (3.1), it is again possible to arrive at a pair of coupled equations of motion.

$$\begin{aligned} \ddot{x}_1 &= 2\beta\dot{x}_2 + (\beta^2 - \omega_0^2)x_1 \\ \ddot{x}_2 &= -2\beta\dot{x}_1 + (\beta^2 - \omega_0^2)x_2 \end{aligned} \quad (3.8)$$

These equations, once solved yield the real part of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-i(\omega_0+\beta)t} + b \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-i(\omega_0-\beta)t}. \quad (3.9)$$

Notice that this solution has a form that is identical to the solution to Schrödinger's Equation found in (3.5). Unlike the solution found equation (3.3), the eigenfrequencies for this solution have exactly the same form as the eigenfrequencies for the solution to the Schrödinger Equation ($\omega_{\pm} = \omega_0 \pm \beta$).

From the correspondence between the pendulum solution in equation (3.9) and the spin state solution in equation (3.5), two things can be concluded. First, retaining the β^2 term in equation (3.7) that arose from the influence of the centrifugal force and was dropped from equation (3.1) strengthened the map between the pendulum and the spin state. Second, because the two solutions have an identical form it is reasonable to conclude that the Lagrangian in equation (3.7) describes the dynamics of an unmeasured electron spin state in a one-dimensional constant magnetic field.

3.3 Electron Spin State in a Time Dependent 1D Magnetic Field

Building on the success of the last section, the next step involves testing the limits of the correspondence between the pendulum Lagrangian and the electron spin state. Taking this in small steps, we begin by testing to see if the Lagrangian in equation (3.7) yields the correct results for a time-varying magnetic field. Since β corresponds to the strength of the magnetic field for the spin state, testing the correspondence for a time-varying magnetic field requires letting $\beta \rightarrow \beta(t)$.

Before performing this analysis, let us first re-express the Lagrangian in equation (3.7) in a more compact form. To do this we will follow the method used by Wharton, Linck and Salazar-Lazaro [WLS11] and begin by defining a new quantity:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \equiv \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (3.10)$$

Expressed in this manner, p_1 and p_2 represent the pendulum's conjugate momenta. Using vector notation, it is also possible to express the conjugate momentum vector: $\mathbf{p} \equiv \dot{\mathbf{x}} + \mathbf{B}\mathbf{x}$. In this notation, \mathbf{x} and $\dot{\mathbf{x}}$ are the position and velocity vectors while \mathbf{B} is the β matrix in equation (3.10). Using these definitions the Lagrangian in equation (3.7) can be expressed as:

$$L_1 = \frac{1}{2}m (\mathbf{p} \cdot \mathbf{p} - \omega_0^2 \mathbf{x} \cdot \mathbf{x}). \quad (3.11)$$

Now, with this new expression for the pendulum Lagrangian, we return to the correspondence test for a time-varying magnetic field. As before, we begin by using Lagrange's equation to express a pair of coupled equations of motion based on the Lagrangian in (3.11). When β is taken to be time-dependent, the resulting equations of motion are:

$$\begin{aligned} \ddot{x}_1 - (\beta^2 - \omega_0^2)x_1 &= 2\beta\dot{x}_2 + \dot{\beta}x_2 \\ \ddot{x}_2 - (\beta^2 - \omega_0^2)x_2 &= -2\beta\dot{x}_1 - \dot{\beta}x_1 \end{aligned} \quad (3.12)$$

Returning to the Schrödinger Equation in (3.4), let the spin state vector $|\chi\rangle$ be expressed in terms of the complex parameters $a(t)$ and $b(t)$ such that $|\chi\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$. For a time dependent y-directed magnetic field, the Schrödinger Equation can then be broken into these two complex equations:

$$\begin{aligned} \dot{a} + i\omega_0 a &= -\beta b \\ \dot{b} + i\omega_0 b &= \beta a \end{aligned} \quad (3.13)$$

Expressed in this manner, it does not initially appear that the pendulum equations of motion (3.12) are consistent with the spin state equations (3.13). However, if a map exists between these two sets of equations then the Lagrangian in equation (3.11) can be used to describe the dynamics of an unmeasured electron spin state in a time-varying y-directed magnetic field.

We begin the process of finding a map by taking the time derivative of each of equations (3.13). After performing a series of substitutions using equations (3.13), it is possible to arrive at the following set of coupled second order equations:

$$\begin{aligned}\ddot{a} - (\beta^2 - w_0^2)a &= -2\beta\dot{b} - \dot{\beta}b \\ \ddot{b} - (\beta^2 - w_0^2)b &= 2\beta\dot{a} + \dot{\beta}a\end{aligned}\quad (3.14)$$

Comparing equations (3.14) to equations (3.12), it initially appears that the two sets of equations have nearly the same form. However, equations (3.12) are purely real while equations (3.14) are complex.

To resolve the issue of equations (3.14) being complex we employ the fact that any complex quantity can be expressed in terms of two real quantities. As a result, the complex quantity a can be expressed as $a = a_R + ia_I$, (where a_R is the real part of a and a_I is the imaginary part of a). Using this notation, we can then break equations (3.14) into four real second order equations. When combined, it is then possible to arrive at the following pair of real equations for the spin state:

$$\begin{aligned}(\ddot{a}_R + \ddot{b}_I) - (\beta^2 - w_0^2)(a_R + b_I) &= 2\beta(\dot{a}_I - \dot{b}_R) + \dot{\beta}(a_I - b_R) \\ (\ddot{a}_I - \ddot{b}_R) - (\beta^2 - w_0^2)(a_I - b_R) &= -2\beta(\dot{a}_R + \dot{b}_I) - \dot{\beta}(a_R + b_I)\end{aligned}\quad (3.15)$$

Comparing equations (3.15) with equations (3.12), it is evident that the two sets of equations are equivalent when

$$x_1 = \text{Re}(a - ib) \qquad x_2 = \text{Im}(a - ib). \quad (3.16)$$

As a result, equations (3.16) represent a map between the pendulum equations of motion (3.12) and the spin state equations (3.13).

Further comparison of the pendulum equations (3.12) with the spin state equations (3.13) yields an apparent problem with the scope of map (3.16). The

pendulum equations of motion are dependent on two parameters (x_1 and x_2), while the spin state equations are dependent on four parameters (a_R, a_I, b_R and b_I). As a result, it seems that the number of parameters needed to solve the spin state equations should be twice the number of parameters needed to solve the pendulum equations. This observation, however, misses an important fact about second order differential equations. To solve a second order differential equation requires both an initial position (x_0) and an initial velocity (\dot{x}_0). As a result, solving the two pendulum equations in (3.12) requires four parameters.

Returning to the spin state equations in (3.13), someone familiar with the standard practices in quantum theory might object to the assertion that four parameters are required to solve the Schrödinger Equation for this system. In quantum theory it is a common practice to disregard both the state's amplitude and the state's global phase. As a result, the solution to the Schrödinger Equation in (3.13) generally requires two parameters. If instead, the state's amplitude and global phase are retained then solving the Schrödinger Equation requires four parameters.

With this in mind, it is now apparent that solutions to both the pendulum equations (3.12) and the spin state equations (3.13) have four real parameters. So, under the map in equations (3.16), the solutions to the pendulum equations of motion (3.12) represent the real part of the solutions to the spin state equations (3.13). As a result, the Lagrangian in equation (3.11) accurately describes the dynamics of an unmeasured electron spin state in a time-varying y-directed magnetic field.

3.4 Conceptual Analog

Thus far, we have restricted our discussion of the connection between Foucault's pendulum and the electron spin state to a mathematical comparison of their associated dynamics. Before moving on with this mathematical comparison it is valuable to stop and delve into the conceptual connections between these two systems. As was discussed in the introduction, the spin state has been widely regarded as fundamentally non-classical. However, in addition to serving as a mathematical analog, Foucault's pendulum can serve as a conceptual framework for describing many of the spin state's non-classical characteristics.

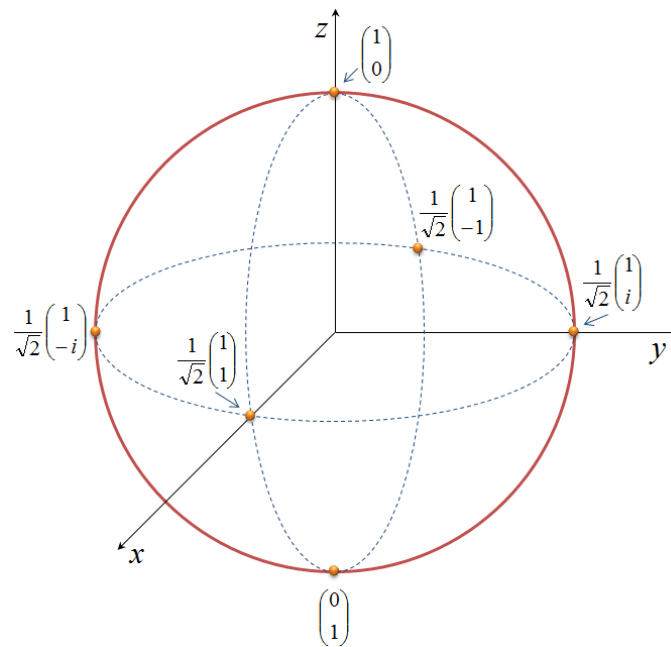


Figure 3.2: Bloch Sphere for the Electron Spin State

The Bloch sphere is used in quantum theory to describe the electron spin state. In general, a Bloch sphere (or Poincaré sphere in optics) is a graphical representation of a two level system. Such a system can be described in terms of a pair of states that form an orthonormal basis. This pair of states is mapped onto

the sphere so that one state is positioned directly opposite to the other state. Using the fact that a two level system can be described in terms of more than one orthonormal basis, the rest of the sphere is constructed so that any two opposite points on the sphere represent a different orthonormal basis that can be used to describe the system. Using complex superposition, it is then possible to express every state on the sphere in terms of any pair of opposing states on the sphere.

For the electron spin state, the spin-up ($\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$) and spin-down ($\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$) states form an orthonormal basis. It is standard practice to map these states onto the unit sphere at the points where the sphere crosses the z-axis, (as shown in Fig. 3.2). Using one of the properties of Bloch spheres, it is then possible to use superposition to describe any point on the spin state Bloch sphere in terms of these two states.

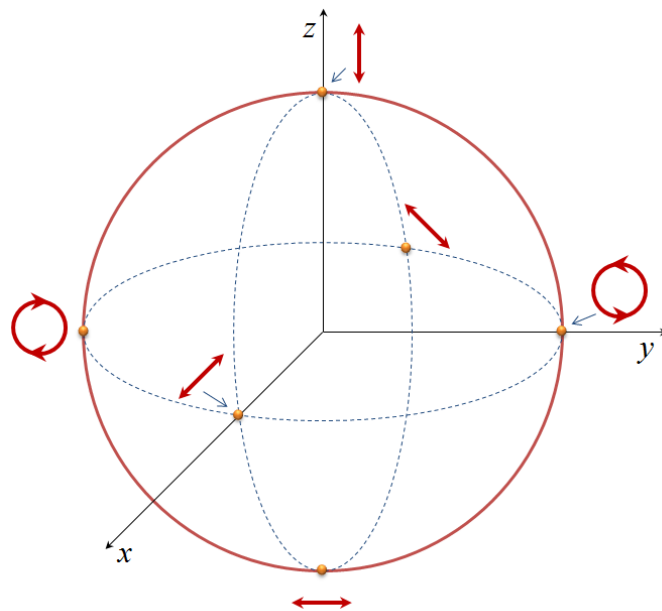


Figure 3.3: Bloch Sphere for Foucault's Pendulum

In addition to describing the electron spin state, the Bloch sphere can also be used to describe the states of the pendulum's oscillation, (shown in Fig. 3.3). For

the pendulum the two perpendicular directions of the pendulum's planar oscillation, (toward-away from the observer (\updownarrow) and right-left perpendicular to the observer (\leftrightarrow)), form an orthonormal basis. For comparison, these states have been mapped onto the z-axis in Fig. 3.3. As was the case for the spin state, all other points on the sphere correspond to a superposition any two opposing states on the sphere. As an example, it is possible to express counter-clockwise rotation as

$$\circlearrowleft = \updownarrow + e^{i\pi/2} \leftrightarrow \quad (3.17)$$

Note that expressed in this manner, complex numbers are used to encode the relative phase between two of the pendulum's orthonormal basis states. Complex numbers are used throughout classical physics for similar purposes. As an example, in circuit theory complex numbers are used to encode the amplitude and relative phase of the voltages and currents in AC circuits. Since it is possible to describe the pendulum's classical state using complex numbers, the fact that spin states are described using complex numbers does not necessarily imply that the states are inherently non-classical.

Now suppose that at some instant in time an observer stands in front of a Foucault pendulum and notes that when looking straight forward the pendulum is oscillating back and forth along the observer's sight line. In the notation used in Fig. 3.3, the pendulum is in the (\updownarrow) state. The observer then leaves and returns sometime later to find that the pendulum is again in the (\updownarrow) state. During the time that has passed between observations the plane of the pendulum's oscillation has slowly precessed. However, without additional information, the observer has no means for determining whether the pendulum has gone through a π rotation, a 2π rotation or some other integer multiple of π rotation.

Now suppose that the observer used the Bloch sphere in Fig. 3.3 to determine

the state of the pendulum at the time of the second observation. Starting from the top of the sphere and proceeding clockwise, the observer notes that the following progression of states can be used to complete a full rotation on the sphere: (\uparrow) to (\nearrow) to (\leftrightarrow) to (\searrow) to (\uparrow). While moving through this progression of states, the pendulum has experienced a 2π rotation with respect to the Bloch sphere. However, upon closer study it is apparent that during this rotation the pendulum has only experienced a π rotation in physical space. As a result, the pendulum is now out of phase with respect to its original state. To return the pendulum to its true original state requires an additional 2π rotation with respect to the Bloch sphere. From this it is evident that the pendulum Bloch sphere does not correspond to physical space.

With this in mind, it is now evident that the Bloch sphere in Fig. 3.3 cannot be used to determine the true state of the pendulum at the time of the second observation. The failure of the Bloch sphere to predict the true state of the pendulum is the consequence of a common practice in quantum theory. As was indicated earlier, the amplitude and global phase of a quantum state are generally disregarded. This bias is employed in the design of the Bloch sphere. As a result, there is no way to accurately use the Bloch sphere to determine the global phase of a state mapped onto the sphere.

In the world of quantum mechanics it is common knowledge that in order to return an electron spin state to its original orientation takes a 4π rotation. This fact has been verified through the interaction of rotated and unrotated spin states. Suppose the spin state of one electron is rotated through 2π while the spin state of a second electron is held fixed. When these two spin states interact they destructively interfere. From this we know that the spin states of the two electrons are out of phase. To bring the rotated spin state back into alignment with the unrotated spin state requires an additional 2π rotation.

Let us now return to the spin state Bloch sphere in Fig. 3.2. According to the sphere, the electron spin state requires a 2π rotation to return to its original orientation. However, we know that it actually requires a 4π rotation. By comparing this situation with the pendulum it is apparent that the Bloch spheres for these two systems may have the same flaw. In both cases the state requires a 4π rotation with respect to the Bloch sphere to perform a rotation that should only take 2π to complete. The doubled rotation in each case is caused by the lack of a means for tracking the global phase of the state being described by the sphere. Taking this into consideration, it now seems plausible that both the pendulum Bloch sphere and the spin state Bloch sphere do not correspond to physical space. As a result, the doubled rotation of the spin state may simply be a consequence of disregarding the state's global phase and interpreting rotations with respect to the Bloch sphere as physical rotations.

Now, quantum mechanics tells us that when an electron is placed in a magnetic field its spin state will precess with a frequency given by, $\omega = \gamma B$, where γ is the electron's gyromagnetic ratio and B is the strength of the external magnetic field. Taking the strength of the magnetic field to be a constant, it is interesting to note that this relationship establishes a direct connection between the electron's gyromagnetic ratio and the precession frequency of its spin state. Suppose the precession frequency is determined to be double its actual value. Since the electron's gyromagnetic ratio is the only thing that can change, its determined value must also be double its actual value.

Turning again to the Bloch sphere, note that since the standard use of the Bloch sphere results in the interpretation that the spin state must undergo a doubled rotation, it also results in the interpretation that the spin state's frequency of rotation must be doubled. As was indicated in the introduction, the electron's

gyromagnetic ratio is equal to double the classically determined value. Perhaps the analysis of the pendulum Bloch sphere holds the key to explaining this discrepancy. The gyromagnetic ratio is doubled because the information about the spin state's global phase has been disregarded resulting in the interpretation that the spin state's frequency of rotation is double its actual value.

CHAPTER 4

4D LAGRANGIAN

4.1 Electron Spin State in a Constant 3D Magnetic Field

In the last chapter it was shown that the Lagrangian in equation (3.11), which can be used to describe the dynamics of Foucault's pendulum, can also be used to describe the dynamics of an unmeasured electron spin state in a time-varying one-dimensional magnetic field. Unfortunately, the scope of the Lagrangian in equation (3.11) is limited to descriptions of spin state interactions with one-dimensional magnetic fields. To describe interactions with more complicated magnetic field arrangements will require the development of another Lagrangian.

Recall from the development of the Lagrangian in equation (3.11) that a restriction was placed on the Lagrangian. Equation (3.11) is the Lagrangian for Foucault's pendulum when the pendulum is represented as a pair of coupled harmonic oscillators. Suppose the number of oscillators is doubled while the basic form of the Lagrangian is maintained. To achieve this, a new definition of the conjugate momentum vector, \mathbf{p} is required. Suppose \mathbf{p} is defined as

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} 0 & -\beta_z & \beta_y & -\beta_x \\ \beta_z & 0 & \beta_x & \beta_y \\ -\beta_y & -\beta_x & 0 & \beta_z \\ \beta_x & -\beta_y & -\beta_z & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \quad (4.1)$$

As before, \mathbf{p} can be expressed in a more compact form as $\mathbf{p} \equiv \dot{\mathbf{x}} + \mathbf{B}\mathbf{x}$. Note that by comparing this result with the result following equation (3.10) it is apparent that

the form of \mathbf{p} has been maintained while the underlying definitions of \mathbf{p} , \mathbf{x} , $\dot{\mathbf{x}}$ and \mathbf{B} have changed. Using this definition of \mathbf{p} a new four oscillator Lagrangian can be defined as:

$$L_2 = \frac{1}{2}m (\mathbf{p} \cdot \mathbf{p} - \omega_0^2 \mathbf{x} \cdot \mathbf{x}). \quad (4.2)$$

When the magnetic field is taken to be constant, it is then possible to use Lagrange's equation to arrive at a set of four coupled equations of motion. When expressed in vector notation, these equations become:

$$\left[2\mathbf{B} \frac{\partial}{\partial t} + \mathbf{I} \left(\omega_0^2 - \beta^2 + \frac{\partial^2}{\partial t^2} \right) \right] \mathbf{x} = 0. \quad (4.3)$$

Where, $\beta \equiv \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}$ and \mathbf{B} is the 4×4 matrix of β components found in equation (4.1).

Following the method used for the two oscillator Lagrangian (3.11), it is possible to express a set of solutions to this system of equations. To do this we begin by converting the Cartesian vector $\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_x)$ into spherical coordinates $\boldsymbol{\beta} = (\beta, \theta, \phi)$. Returning to the Bloch sphere in Fig. 3.2, we then define the intersections of the sphere with the y-axis as $|y_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $|y_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Using this notation, the resulting solution to equation (4.3) can be expressed as the real part of:

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} &= a \begin{bmatrix} \cos(\theta/2)|y_-\rangle \\ \sin(\theta/2)e^{i\phi}|y_-\rangle \end{bmatrix} e^{-i(\omega_0+\beta)t} + b \begin{bmatrix} \sin(\theta/2)|y_-\rangle \\ -\cos(\theta/2)e^{i\phi}|y_-\rangle \end{bmatrix} e^{-i(\omega_0-\beta)t} \\ &+ c \begin{bmatrix} -\sin(\theta/2)e^{i\phi}|y_+\rangle \\ \cos(\theta/2)|y_+\rangle \end{bmatrix} e^{-i(\omega_0+\beta)t} + d \begin{bmatrix} \cos(\theta/2)e^{i\phi}|y_+\rangle \\ \sin(\theta/2)|y_+\rangle \end{bmatrix} e^{-i(\omega_0-\beta)t}. \end{aligned} \quad (4.4)$$

Expressed in this manner, a , b , c and d are arbitrary complex quantities.

Returning to the Schrödinger Equation in (3.4) and following the method of expressing the vector β in terms of spherical coordinates, it is possible to arrive at a set of solutions for the case where the electron is placed in a constant three dimensional magnetic field. The resulting solutions can be expressed as:

$$\begin{bmatrix} \chi_+(t) \\ \chi_-(t) \end{bmatrix} = f \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{bmatrix} e^{-i(w_0-\beta)t} + g \begin{bmatrix} \sin(\theta/2) \\ -\cos(\theta/2)e^{i\phi} \end{bmatrix} e^{-i(w_0+\beta)t}. \quad (4.5)$$

Where f and g are complex quantities that are subject to the normalization condition $|f|^2 + |g|^2 = 1$.

As was established in section 3.3, the next step in showing that the Lagrangian in equation (4.2) can be used to describe the spin state involves finding a map under which the solutions in equation (4.4) are equivalent to the solutions in equation (4.5). As demonstrated by Wharton, Linck and Salazar-Lazaro [WLS11], there are a number of maps between these two systems, (each of which work under a different set of imposed conditions). These maps can generally be expressed as:

$$\begin{aligned} \chi_+ &= \frac{(x_1 + ix_2)A^* + (x_3 - ix_4)B^*}{|A|^2 + |B|^2} \\ \chi_- &= \frac{(-x_1 + ix_2)B^* + (x_3 + ix_4)A^*}{|A|^2 + |B|^2} \end{aligned} \quad (4.6)$$

Where A and B are new complex quantities that are subject to the normalization condition $|A|^2 + |B|^2 = 1$. In addition, the map in equations (4.6) requires that $a = \sqrt{2}Af$, $b = \sqrt{2}Ag$, $c = \sqrt{2}Bf$ and $d = \sqrt{2}Bg$. Using these definitions serves to impose the condition: $ad = bc$.

Looking at the map in equation (4.6) it is important to note that the complex quantities A and B have arbitrary values. Any pair of values that satisfy the given definitions of a , b , c and d are allowed. As a result, the map in equations (4.6) is a

many to one map. This means that any coupled oscillator solution that satisfies the condition $ad = bc$ maps directly onto the single spin state solution. Interestingly, the condition that $ad = bc$ happens to mean that the coupled oscillator Lagrangian always equals zero ($L_2 = 0$).

Before moving on, we must address an apparent problem. The coupled oscillator solution depends on eight parameters, (the real and imaginary parts of a , b , c and d), while the spin state solution depends on four parameters, (the real and imaginary parts of f and g). To address this problem we begin by looking at the effect constraints have on the number of free parameters needed to describe each of these solutions.

Table 4.1: Effect of Constraints on Number of Required Free Parameters

Equation	Listed Parameters	Constraint	Resulting Free Parameters
4.4	8	$L_2 = 0$	6
4.5	4	$ f ^2 + g ^2 = 1$	3
4.6	4	$ A ^2 + B ^2 = 1$	3

Note that since the condition $ad = bc$ (which results in $L_2 = 0$), is a complex expression it actually imposes two constraints.

From this table it is apparent that equation (4.4) requires double the number of free parameters than are required by equation (4.5). To bring the two into balance requires the addition of three free parameters to the spin state solution. Recall that the coupled oscillator parameters (a , b , c and d) can be expressed in terms of both the spin state parameters (f and g) and the map parameters (A and

B). This means that the six free parameters for the coupled oscillator solution can be divided between the spin state solution and the map. Doing this treats A and B as hidden variables that are needed to fully describe the spin state. As a result, the map in equations (4.6) makes it possible for the Lagrangian in equation (4.2) to accurately describe the dynamics of an unmeasured electron spin state in a constant three dimensional magnetic field.

4.2 Time Varying 3D Magnetic Field - Part I (Vectors)

Building on the success of the last section, the next step involves showing that the Lagrangian in equation (4.2) can be used to describe the dynamics of an unmeasured electron spin state in a time varying three dimensional magnetic field. To do this we will again use Lagrange's equation to arrive at a set of four coupled equations of motion. When expressed in vector notation these equation are:

$$\left[2\mathbf{B} \frac{\partial}{\partial t} + \frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}^2 + \mathbf{I} \left(\omega_0^2 + \frac{\partial^2}{\partial t^2} \right) \right] \mathbf{x} = 0. \quad (4.7)$$

Where \mathbf{B} is a matrix of time dependent β components with exactly the same form as the matrix of constant β components listed in equation (4.1).

Following the methodology used thus far, the next step involves finding a map between equation (4.7) and the Schrödinger Equation in (3.4). Though it is possible to do this analysis using vectors, it is significantly more elegant if this analysis is done using quaternions. However, though quaternions were once widely used in physics, today their use is far from common. As a result, it is helpful to take a break from the analysis to discuss the algebra of quaternions along with some of their more relevant characteristics.

4.3 An Introduction to Quaternions

Quaternions were first introduced in 1842 by Irish mathematician Sir William Rowan Hamilton. According to Hanson [Hans06, Ch. 1], Hamilton was seeking to extend complex numbers so that they would be capable of describing three dimensional space. To achieve this, Hamilton developed the four part quaternion with one real component and three imaginary components. Though a quaternion can be expressed in a variety of ways, the form that most closely resembles a complex number is:

$$\mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3. \quad (4.8)$$

Expressed in this manner, q_0 is the real part of the quaternion while q_1 , q_2 and q_3 are the imaginary parts of the quaternion.

Complex numbers are built around the relationship, $i = \sqrt{-1}$. Hamilton built quaternions around an extension to this rule. As the story goes, one evening in October of 1842 Hamilton went for a walk along the Royal Canal in Dublin with his wife. As they reached the Broome bridge, Hamilton suddenly realized that the extension he was looking for could be summed up in the following fundamental quaternion rule:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \quad (4.9)$$

Where \mathbf{i} , \mathbf{j} and \mathbf{k} are unit quaternions. This so impressed him that Hamilton decided on the spot to use a knife and engrave this relationship into the side of the bridge. Apparently he was afraid that he might die before he would have the chance to share his discovery with the world. Building on this fundamental rule, Hamilton and others went on to describe the algebra of quaternions.

In the years following Hamilton's discovery, both mathematicians and physicists found uses for the algebra of quaternions. As an example, Maxwell wrote

his fundamental work in electromagnetic theory using quaternions. Unfortunately, quaternions often proved to be cumbersome and non-intuitive. In light of these issues, quaternions eventually fell from favor and were replaced by vectors. Today quaternions are only used to describe a small subset of phenomena.

Before discussing the properties of quaternions, it is important to note that the notation used to describe quaternions often resembles the notation used to describe vectors. This connection has a historical reason. To facilitate the transition from quaternions to vectors, quaternionic notation was incorporated into vector notation. One example of this is shown in the notation used to express the quaternion listed in equation (4.8). If $q_0 = 0$, this quaternion would appear to be a Cartesian vector. This is the case because the notation for the Cartesian unit vectors \hat{i} , \hat{j} and \hat{k} were originally borrowed from quaternionic notation.

With this in mind, let us now discuss some of the properties of quaternions. Suppose that a second quaternion is given by, $\mathbf{p} = p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3$. The following is a list of properties of quaternions compiled from Hanson [Hans06, Ch. 4, Ch. 7] and Kuipers [Kuip99, Ch. 5, Ch. 7]. It is important to note that this is not an all inclusive list. Instead, only those properties of quaternions which have been useful during the course of this work have been included.

(1) Quaternion addition is commutative. As a result, $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$.

(2) The sum of two quaternions is a quaternion such that,

$$\mathbf{p} + \mathbf{q} = (p_0 + q_0) + \mathbf{i}(p_1 + q_1) + \mathbf{j}(p_2 + q_2) + \mathbf{k}(p_3 + q_3). \quad (4.10)$$

(3) Quaternion multiplication is not commutative. So in general $\mathbf{pq} \neq \mathbf{qp}$.

(4) The product of unit quaternions is defined as:

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji} \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj} \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik} \quad (4.11)$$

(5) There are two ways to decompose quaternion multiplication using matrices:

(a) Left Multiplication:

$$\mathbf{pq} = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}. \quad (4.12)$$

(b) Right Multiplication:

$$\mathbf{pq} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}. \quad (4.13)$$

(6) The conjugate of a quaternion is:

$$\mathbf{q}^* = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3. \quad (4.14)$$

(7) The conjugate of a quaternion product is:

$$(\mathbf{qp})^* = \mathbf{p}^* \mathbf{q}^*. \quad (4.15)$$

(8) The norm of a quaternion is a scalar. The square of the quaternion norm is:

$$|\mathbf{q}|^2 = \mathbf{q}^* \mathbf{q} = \mathbf{q} \mathbf{q}^* = q_0^2 + q_1^2 + q_2^2 + q_3^2. \quad (4.16)$$

(9) The inverse of a quaternion is:

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{|\mathbf{q}|^2}. \quad (4.17)$$

4.4 Time Varying 3D Magnetic Field - Part II (Quaternions)

Now, it is time to return to the problem of finding a map between the coupled oscillator equations of motion in (4.7) and the Schrödinger Equation in (3.4). To do this using the algebra of quaternions we will begin by defining two quaternions:

$$\begin{aligned}\mathbf{q} &= x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \\ \mathbf{b} &= 0 + \mathbf{i}\beta_z - \mathbf{j}\beta_y + \mathbf{k}\beta_x\end{aligned}\tag{4.18}$$

Note that when \mathbf{b} is expressed using the matrix notation of equation (4.13) it is equivalent to the time dependent version of the matrix of β components in equation (4.1). From this it is apparent that \mathbf{b} takes the place of \mathbf{B} for right quaternionic multiplications.

By rearranging and changing notation it is possible to express the coupled oscillator equations of motion in (4.7) as,

$$\ddot{\mathbf{x}} + 2\mathbf{B}\dot{\mathbf{x}} + \left(\mathbf{B}^2 + \dot{\mathbf{B}} + \omega_0^2\right)\mathbf{x} = 0.\tag{4.19}$$

Re-expressing this result in quaternions where \mathbf{q} takes the place of \mathbf{x} and \mathbf{b} takes the place of \mathbf{B} (for right multiplication), yields the quaternionic equation of motion for the coupled oscillator system.

$$\ddot{\mathbf{q}} + 2\dot{\mathbf{q}}\mathbf{b} + \mathbf{q}\left(\mathbf{b}^2 + \dot{\mathbf{b}} + \omega_0^2\right) = 0.\tag{4.20}$$

To arrive at a quaternionic representation of the Schrödinger Equation, we follow a similar procedure. As was done with the coupled oscillator equation, we begin by rearranging the Schrödinger Equation in (3.4). Doing this yields,

$$\frac{\partial}{\partial t}|\chi\rangle + i\left(\vec{\beta} \cdot \vec{\sigma}\right)|\chi\rangle = -i\omega_0|\chi\rangle.\tag{4.21}$$

Now, suppose that \mathbf{s} is the quaternionic representation of the spin state vector $|\chi\rangle$. Since $|\chi\rangle = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$, the spin state quaternion can be expressed as,

$$\mathbf{s} = \text{Re}(\chi_+) + \mathbf{i}\text{Im}(\chi_+) + \mathbf{j}\text{Re}(\chi_-) + \mathbf{k}\text{Im}(\chi_-). \quad (4.22)$$

Using \mathbf{s} in the place of $|\chi\rangle$ brings us most of the way to finding a quaternionic representation of equation (4.21). All that is left is to find a quaternionic representation for the second term in equation (4.21). To do this, we will utilize the fact that $i \left(\vec{\beta} \cdot \vec{\sigma} \right) |\chi\rangle$ is a complex vector. As a result, it can generally be expressed as, $i \left(\vec{\beta} \cdot \vec{\sigma} \right) |\chi\rangle = \begin{pmatrix} f_0 + if_1 \\ f_2 + if_3 \end{pmatrix}$. Where f_0, f_1, f_2 and f_3 are purely real quantities.

Now suppose that the vector quantity $i \left(\vec{\beta} \cdot \vec{\sigma} \right) |\chi\rangle$ can be represented by the quaternionic quantity \mathbf{sb} . According to a property of quaternionic multiplication the quantity \mathbf{sb} is a quaternion. As a result, it can be generally represented as, $\mathbf{sb} = g_0 + \mathbf{i}g_1 + \mathbf{j}g_2 + \mathbf{k}g_3$. If \mathbf{sb} is the quaternionic representation of $i \left(\vec{\beta} \cdot \vec{\sigma} \right) |\chi\rangle$, then to be consistent with the definition of \mathbf{s} in equation (4.22) it must also be true that $f_0 = g_0, f_1 = g_1, f_2 = g_2$ and $f_3 = g_3$. From here it is a simple matter of multiplication to show that in fact the respective parts of $i \left(\vec{\beta} \cdot \vec{\sigma} \right) |\chi\rangle$ can be represented by the corresponding parts of \mathbf{sb} .

Using the correspondence between $|\chi\rangle$ and \mathbf{s} along with the correspondence between $i \left(\vec{\beta} \cdot \vec{\sigma} \right) |\chi\rangle$ and \mathbf{sb} , it is now possible to express a quaternionic representation of equation (4.21). The resulting quaternionic Schrödinger Equation is,

$$\dot{\mathbf{s}} + \mathbf{sb} = -i\omega_0\mathbf{s}. \quad (4.23)$$

Where \mathbf{i} is the unit quaternion.

With this equation we can now set about the work of determining if there is correspondence between the spin state and the coupled oscillator solution. To do

this, we will begin by taking the time derivative of equation (4.23). Solving equation (4.23) for $\dot{\mathbf{s}}$ and then substituting it into the time derivative expression, it is possible to arrive at the following equation,

$$\ddot{\mathbf{s}} + 2\dot{\mathbf{s}}\mathbf{b} + \mathbf{s}(\mathbf{b}^2 + \dot{\mathbf{b}} + \omega_0^2) = 0. \quad (4.24)$$

Notice that this equation has an identical form to the quaternionic equation of motion for the coupled oscillator system in equation (4.20).

To show that the coupled oscillator equation in (4.20) is equivalent to the spin state equation (4.24) we now need to find a map between the quaternions \mathbf{q} and \mathbf{s} . Recall the map in equation (4.6). Using this map it is possible to express \mathbf{q} in terms of the parts of the spin state vector. Doing this we find that,

$$\mathbf{q} = (\chi_+ A - B^* \chi_-^*) + (\chi_- A + B^* \chi_+^*)\mathbf{j}. \quad (4.25)$$

Using this expression it can be shown that $\mathbf{q} = \mathbf{u}\mathbf{s}$, where \mathbf{u} is a constant unit quaternion that is given by the expression:

$$\mathbf{u} = \text{Re}(A) + \mathbf{i}\text{Im}(A) + \mathbf{j}\text{Re}(B) - \mathbf{k}\text{Im}(B). \quad (4.26)$$

With this result we now have a map from the coupled oscillator solution onto the spin state. Solving for \mathbf{s} we find that $\mathbf{s} = \mathbf{u}^{-1}\mathbf{q}$. However, according to the definition of the quaternion inverse listed in equation (4.17), $\mathbf{u}^{-1} = \frac{\mathbf{u}^*}{|\mathbf{u}|^2}$. Since $|\mathbf{u}|^2$ is equivalent to the normalization condition $|A|^2 + |B|^2$, it is evident that $|\mathbf{u}|^2 = 1$. As a result, the map from the coupled oscillator solution onto the spin state can be expressed as,

$$\mathbf{s} = \mathbf{u}^* \mathbf{q} \quad (4.27)$$

Using this map, with the associated definition of \mathbf{u} , the coupled oscillator equation in (4.20) is equivalent to the spin state equation (4.24). As a result, it is now clear

that the Lagrangian in equation (4.2) is capable of describing the dynamics of an unmeasured electron spin state in a time varying three dimensional magnetic field.

Before moving on, let us return to the Lagrangian in equation (4.2). Using the notation introduced in this section it is possible to express this Lagrangian in terms of quaternions. To do this, we begin by expressing the conjugate momentum vector \mathbf{p} from equation (4.1) as a quaternion. Based the methodology used to express equation (4.20), the conjugate momentum quaternion can be expressed as,

$$\mathbf{p} = \dot{\mathbf{q}} + \mathbf{q}\mathbf{b} \quad (4.28)$$

Using this expression for \mathbf{p} while replacing \mathbf{x} with \mathbf{q} yields the following quaternionic Lagrangian,

$$L_3 = \frac{1}{2}m (|\mathbf{p}|^2 - \omega_0|\mathbf{q}|^2) \quad (4.29)$$

Now, suppose that the conjugate momentum quaternion is expressed in terms of \mathbf{q} and \mathbf{u} . To do this, we begin by using $\mathbf{q} = \mathbf{u}\mathbf{s}$ to express \mathbf{p} in terms of \mathbf{u} and \mathbf{s} . Then using equation (4.23) and the map in equation (4.27) it is possible to show that the conjugate momentum quaternion can be expressed as: $\mathbf{p} = -\mathbf{u}i\omega_0\mathbf{u}^*\mathbf{q}$.

This new expression for the conjugate momentum quaternion makes it possible to re-express the quaternionic Lagrangian. Using the definitions of the quaternion norm and the conjugate of a quaternion product it is possible to show that $|\mathbf{p}|^2 = \omega_0|\mathbf{u}|^4|\mathbf{q}|^2$. Substituting this expression into the Lagrangian, while noting that the quaternion norm is a scalar, yields: $L_3 = \frac{1}{2}m\omega_0 (|\mathbf{u}|^4 - 1) |\mathbf{q}|^2$. Since \mathbf{u} is a unit quaternion, its norm is equal to one. As a result, $L_3 = 0$.

Recall from the earlier discussion that $L_1 = 0$ for both the constant and time dependent one dimensional magnetic field arrangements while $L_2 = 0$ for the constant three dimensional magnetic field arrangement. The $L_3 = 0$ result shows that the Lagrangian for the time dependent three dimensional magnetic field

arrangement is consistent with the other Lagrangians. Since none of these conditions were imposed upon the system and instead came about as a result of the map needed in each case to show correspondence with the spin state, the $L = 0$ result may point to some deeper underlying truth about these systems.

CHAPTER 5

FIRST ORDER LAGRANGIAN FOR SPIN

5.1 The Lagrangian for Spin

Based on the work described so far, it is apparent that the Lagrangian in equation (4.2) along with its quaternion counterpart in equation (4.29) are capable of describing the dynamics of an unmeasured electron spin state in a time varying three dimensional magnetic field. It is, however, important to note that the equations of motion derived from these Lagrangians, which are listed in equations (4.3), (4.7), and (4.20), are second order differential equations. By comparison, the Schrödinger equation (3.4) is a first order differential equation. As a result, one might argue that it is unnecessary to go to second order to describe the dynamics of the electron's spin state. To address this issue involves developing a Lagrangian for electron spin that yields first order equations of motion.

Recall from the original discussion of the Schrödinger equation in (3.4), that the Hamiltonian for spin can be expressed as, $H = \hbar(\omega_0 I + \boldsymbol{\beta} \cdot \boldsymbol{\sigma})$. A Lagrangian that incorporates this Hamiltonian is:

$$L = \hbar [\langle \chi | \omega_0 I + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} | \chi \rangle - \text{Im} \langle \dot{\chi} | \chi \rangle]. \quad (5.1)$$

Note that when this Lagrangian is stated in terms of the spin Hamiltonian, H , it can also be expressed as, $L = \langle \chi | H | \chi \rangle - \hbar \text{Im} \langle \dot{\chi} | \chi \rangle$.

The process of testing to see if this Lagrangian correctly describes electron spin begins by using Lagrange's equation to express a set of equations of motion. To do this, we first need to express the spin state in terms of a set of real quantities.

Recall from section 3.3 that the spin state can be expressed as $|\chi\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$, where $a = a_R + ia_I$ and $b = b_R + ib_I$. Using this notation it is then possible to arrive at the following set of coupled equations of motion:

$$\begin{aligned} \dot{a}_R &= (\omega_0 + \beta_z)a_I - \beta_y b_R + \beta_x b_I & \dot{b}_R &= \beta_y a_R + \beta_x a_I + (\omega_0 - \beta_z)b_I \\ \dot{a}_I &= -(\omega_0 + \beta_z)a_R - \beta_x b_R - \beta_y b_I & \dot{b}_I &= -\beta_x a_R + \beta_y a_I - (\omega_0 - \beta_z)b_R \end{aligned} \quad (5.2)$$

Note that these four first order differential equations are real. As a result, verifying that they are equivalent to the Schrödinger equation (3.4) requires breaking the Schrödinger equation into a set of four real equations. Following the same method just used to express the spin state $|\chi\rangle$ in terms of real quantities, it can easily be shown that the Schrödinger equation can be expressed in an identical manner to equations (5.2). Therefore, the Lagrangian in equation (5.1) can be used to describe the interaction of an electron's spin state with an external magnetic field.

5.2 Comparison of First and Second Order Lagrangians

Building on the result from the last section, let us now compare the coupled oscillator Lagrangian in equation (4.2) with the new Lagrangian in equation (5.1). To aid in this comparison, recall that the two Lagrangians can be expressed as,

$$\begin{aligned} \text{First Order Lagrangian (eq. 5.1):} & \quad L = \hbar [\langle \chi | \omega_0 I + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} | \chi \rangle - Im \langle \dot{\chi} | \chi \rangle] \\ \text{Second Order Lagrangian (eq. 4.2):} & \quad L = \frac{1}{2} m (\mathbf{p} \cdot \mathbf{p} - \omega_0^2 \mathbf{x} \cdot \mathbf{x}) \end{aligned}$$

Where the terms first order and second order refer to the order of the equation's associated Lagrange's equations.

By comparing the form of these two equations and their associated Lagrange's equations (with regard to their solutions), a number of differences present themselves.

- (1) Though both equations are real, the first order Lagrangian is expressed in terms of complex quantities while the second order Lagrangian is expressed in terms of real quantities. Note that it is possible to express the first order Lagrangian in terms of purely real quantities. However doing so results in a Lagrangian whose form is complicated and lacks a method of obvious simplification.
- (2) As was mentioned in the introduction, the classical Lagrangian for a system of particles can be defined as, $L = T - V$, where T and V are the system's kinetic and potential energies. However, this is not the only definition of a classical Lagrangian. In general, the classical Lagrangian is defined as an equation that yields the correct equations of motion. As a result, both the first and second order Lagrangians are classical Lagrangians. However, where the second order Lagrangian can be interpreted classically the first order Lagrangian has no obvious classical analog.
- (3) To map the solutions to the equations of motion associated with the second order Lagrangian onto the spin state requires the introduction of additional (hidden) parameters. In comparison, the solutions to the equations of motion associated with the first order Lagrangian are spin state solutions and therefore require no additional parameters.
- (4) According to Goldstein [Gold02, Ch. 10], the Lagrangian for a one dimensional harmonic oscillator is given by: $L = \frac{1}{2m} (p^2 - m^2\omega^2q^2)$. The fact that the second order Lagrangian has a similar form means that the harmonic oscillator serves as a framework for understanding the underlying dynamics of the system. In contrast, the complex form of the first order Lagrangian lacks an obvious classical analog. As a result, there does not

appear to be a classical framework for understanding its associated dynamics.

- (5) According to Goldstein [Gold02, Ch. 13], the Klein-Gordon Lagrangian density (in the limit where $c = 1$), can be expressed as:

$\mathcal{L} = \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - (\nabla \phi)^2 - \mu^2 \phi^2 \right]$. Disregarding the middle term in this equation, which is due to the spatial component of the Lagrangian density, the second order Lagrangian has a form that is similar to this equation.

This fact may suggest a natural connection between the second order Lagrangian and relativistic quantum theory. Since there does not appear to be an equation in relativistic quantum theory that has a form similar to the first order Lagrangian, it is unclear how the first order Lagrangian may fit into the framework of relativistic quantum theory.

- (6) Building on the work done here, Wharton [Whar13] has shown that it is possible to use a Lagrangian that is based on the second order Lagrangian to extend the classical analogy for the electron spin state to encompass more of the state's quantum characteristics while expanding the system to include states with arbitrary values of spin. Unfortunately, a similar analysis cannot be done using the first order Lagrangian.

Taking these differences into account, it is evident that both Lagrangians have benefits. Most notably, the first order Lagrangian in equation (5.1) can be used to derive the Schrödinger Equation without the need for a map. As a result, the first order Lagrangian is a significant result. However, the form of the second order Lagrangian in equation (4.2) is significantly easier to interpret using the framework of classical physics. Since the goal of this work is to describe the electron spin state

using the framework of classical physics, this difference makes the second order Lagrangian a better fit for this work.

CHAPTER 6

SUMMARY AND CONCLUSION

The Lagrangian is a tool used extensively in both classical and quantum mechanics. Building on the framework of this natural bridge, we have used the classical Lagrangian that describes the dynamics of Foucault's pendulum as a basis for describing the dynamics of the electron spin state in an arbitrary magnetic field. Before discussing the significance of our results, let us first review what we have managed to show so far.

We began in Chapter 2 by deriving the classical Lagrangian in equation (3.1) which describes the dynamics of Foucault's pendulum. Then, in Chapter 3, we used this Lagrangian to express a set of coupled equations of motion which we then solved. Using the Schrödinger Equation in (3.4) we expressed a spin state solution for the case when an electron is placed in a constant one dimensional magnetic field. Direct comparison showed limited correspondence between the pendulum solutions and the spin state.

Desiring to find a better correspondence, we returned to the original derivation of the pendulum Lagrangian. Modeling the pendulum as a pair of coupled oscillators, we expressed the Lagrangian in equation (3.11). This Lagrangian was then shown to yield pendulum solutions with exactly the desired form. Testing the limits of this correspondence we performed the analysis with the one dimensional magnetic field now time dependent. Under the map in equation (3.16), we found that the equations of motion for the pendulum are equivalent to the Schrödinger Equation.

Building on the success found using the Lagrangian for two coupled oscillators, in Chapter 4 we expanded the number of oscillators to four and increased the number of coupling parameters to three. Using this Lagrangian, which is found in equation (4.2), we expressed a set of four coupled equations of motion which we then solved. Using the map in equations (4.6), we determined that the four oscillator Lagrangian is capable of describing the dynamics of an unmeasured electron spin state in a constant three dimensional magnetic field.

The final step with the four oscillator Lagrangian involved converting our system over into quaternions. Once there, we showed that the quaternionic equations of motion for the four oscillator system in equation (4.20) is equivalent to the quaternionic Schrödinger equation in (4.23) under the map in equation (4.27). With this, we showed that the Lagrangian in equation (4.2) and its quaternionic equivalent in equation (4.29), are capable of describing the dynamics of an unmeasured electron spin state in a time varying three dimensional magnetic field.

In addition to showing that Foucault's pendulum can be used as a basis for describing the dynamics of the electron spin state, we also showed that Foucault's pendulum can serve as a conceptual tool for understanding some of the spin state's quantum behavior. Recall from the introduction that we listed a set of properties that have been used to prove that the electron spin state is inherently non-classical. The following is a restatement of that list:

- (1) The spin state must be described using complex numbers.
- (2) When rotated, the spin state must undergo a 4π (as opposed to the classical 2π) rotation to return to its original state.
- (3) The electron's gyromagnetic ratio is double the classically predicted value.

- (4) Measurements of the spin state have discrete outcomes that are predicted by associated probabilities.

Of these properties, we managed to show in Chapter 3 that three can be explained classically using Foucault's pendulum. To do this, we began by introducing the Bloch sphere and the manner in which it is used in quantum theory to describe the spin state. We then showed that the Bloch sphere can also be used to describe the pendulum's purely real states of oscillation. Since complex numbers can be used to encode the relative phase between the pendulum's basis states, we then pointed out that the use of complex numbers to describe the spin state does not necessarily imply that the spin state is non-classical.

We then returned to the pendulum Bloch sphere and showed that a 2π rotation of the pendulum in physical space requires a 4π rotation of the pendulum with respect to the Bloch sphere. This fact means that the pendulum Bloch sphere does not correspond to physical space. In quantum theory the Bloch sphere is treated as a direct representation of physical space. Building on the similarities between the pendulum Bloch sphere and the spin state Bloch sphere, we then proposed that the spin state Bloch sphere also does not correspond to physical space. Based on this, we then showed that both the spin state's doubled rotation to return to its original state and the electron's doubled gyromagnetic ratio can be understood as consequences of the wrong impression that the spin state Bloch sphere corresponds to physical space.

Returning to the list of quantum properties, of the original list we managed to classically explain three properties using Foucault's pendulum. Unfortunately, we did not succeed in classically explaining the spin state's discrete measurement outcomes and their associated probabilities. For that explanation we need to turn to

an extension to this work. In that extension, Wharton [Whar13] proposes using a modified version of the Feynman Path Integral.

To explain Wharton’s approach to using the Feynman Path Integral we must first discuss the action. Both classical mechanics and quantum mechanics use the action. However, each uses the action for a decidedly different purpose.

In classical mechanics the action is used to determine the path of a particle. To do this, we begin by describing all of the possible paths a particle might take from point x_a to point x_b . We then define the classical action as,

$$S = \int_{t_a}^{t_b} L(\dot{x}, x, t) dt \quad (6.1)$$

Where L is the classical Lagrangian. Then using the principle of least action ($\delta S = 0$), we derive Lagrange’s equation, $\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0\right)$. Using this equation, along with an appropriate set of initial conditions, it is then possible to compute the particle’s actual path.

In quantum mechanics, the Heisenberg Uncertainty Principle tells us that we cannot know the exact path that a particle takes from point x_a to point x_b . As a result, the classical approach to using the action cannot be used in quantum mechanics. Instead, quantum mechanics relies on determining the probability that a particle at point x_a at time t_a will end up at point x_b at time t_b . To compute this probability Feynman [Feyn05] suggests performing the following calculation:

$$P(b, a) = \left| \sum_{\substack{\text{paths from} \\ \text{a to b}}} C e^{iS/\hbar} \right|^2 \quad (6.2)$$

Where C is a constant and S is the action. Expressed in this manner, the probability is dependent on contributions from all the possible paths that the particle might take getting from point x_a to point x_b . In this sense, it is a sum over all possible “histories” of the particle.

Looking at the probability calculation in equation (6.2) there are a couple of interesting things to note. First, this calculation is not dependent on the classical least action principle ($\delta S = 0$). As a result, the paths for which $\delta S \neq 0$ contribute to the total probability. Without contributions from these additional non-classical paths the calculated probability would not agree with observed outcomes. Second, this calculation takes into account paths that are not classically allowed. This fact helps explain why a particle can be observed in a state that is not classically predicted.

In his extension to this work, Wharton [Whar13] proposes exchanging the classical least action principle ($\delta S = 0$) with a “Null Lagrangian Condition” ($L = 0$). Recall from the earlier discussion, that each of the Lagrangians introduced in this work have one thing in common. The conditions required to show correspondence between the Lagrangian and the spin state always result in the Lagrangian equaling zero ($L = 0$). Building on this observation, Wharton suggests imposing the $L = 0$ condition onto equation (6.2). Doing this serves to reduce the number of paths involved in calculating the probability while still allowing for non-classical paths. With this condition, Wharton shows that using the second-order Lagrangian (but not the first-order Lagrangian from Chapter 5), it is both possible to derive the Born rule and arrive at discrete measurement outcomes.

This work began with the desire to show that many of the electron spin state’s quantum characteristics can be described using the framework of classical mechanics. To do this, we proposed using Foucault’s pendulum as a classical analog for the electron spin state. Using this analog (with Wharton’s extension), we have shown that it is possible to classically explain each of our listed quantum characteristics. In addition, we have shown that a classical Lagrangian can be used to describe the dynamics of an unmeasured electron spin state in an arbitrary

magnetic field. With these results we have shown that the description of the electron spin state is more closely linked to classical physics than has previously been demonstrated.

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