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Classical Models of the Spin 1/2 System

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CLASSICAL MODELS OF THE SPIN $\frac{1}{2}$ SYSTEM

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The Faculty of the Department of Physics
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Master of Science

by
Carlos H. Salazar-Lazaro
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CLASSICAL MODELS OF THE SPIN $\frac{1}{2}$ SYSTEM

by

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APPROVED FOR THE DEPARTMENT OF PHYSICS

SAN JOSÉ STATE UNIVERSITY

December 2012

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ABSTRACT

CLASSICAL MODELS OF THE SPIN $\frac{1}{2}$ SYSTEM

by Carlos H. Salazar-Lazaro

We proposed a Quaternionic mechanical system motivated by the Foucault pendulum as a classical model for the dynamics of the spin $\frac{1}{2}$ system. We showed that this mechanical system contains the dynamics of the spin state of the electron under a uniform magnetic field as it is given by the Schrodinger-Pauli-Equation (SPE). We closed with a characterization of the dynamics of this generalized classical system by showing that it is equivalent with the dynamics of the Schrodinger Pauli Equation as long as the solutions to the generalized classical system are roots of the Lagrangian, that is the condition $L = 0$ holds.

DEDICATION

To my family and friends at SJSU.

ACKNOWLEDGEMENTS

I would like to thank Prof. Wharton for his patience while writting this document and for his guidance and direction on the topics covered by the research exposed in this thesis.

TABLE OF CONTENTS

CHAPTER

LIST OF FIGURES

Figure

CHAPTER 1

INTRODUCTION

In spite of conventional wisdom that quantum spin is inherently non-classical, there is a well known classical analog to the two-level quantum system based on the classical polarization (CP) of a plane electro-magnetic (EM) wave. Such analogue comes with some limitations but nevertheless has been used to motivate introductory quantum mechanics texts like those of Baym [G69] and Sakurai [J.J94] to illustrate a classical system that has the Spinor-like properties of the spin $\frac{1}{2}$ system under a induced uniform magnetic field precession. Under the CP analogy, the well-known "Jones vector" and the Spinor $\mid \chi >$ that defines the spin $\frac{1}{2}$ state in quantum mechanics are correlated to explain analogous characteristics of both theories. For example, the quantum normalization condition $\langle \chi \mid \chi \rangle = 1$ corresponds to a normalization of the energy of the EM wave, and the global phase transformation $\mid \chi > \rightarrow \mid \chi > \exp(i\theta)$ is analogous to changing the phase of the EM wave. However, the power and depth of the CP analogy has not been widely appreciated as there are aspects of the analogy that have gone without appreciable mention in the literature. For example, the CP analogy contains a straightforward classical picture for a π geometric phase shift resulting from a full 2π rotation of the spin angular momentum. This fact has gone unnoticed in the literature with one possible exception by Klyshko [D.N93]. Nevertheless, the CP analogy breaks down when it is extended to the spin state of an electron under a spatially uniform time-varying magnetic field. This limitation, along with complications involving quantum measurement outcomes has prevented consensus on what makes quantum spin inherently non-classical.

1.1 New Results and Outline

In the following section, we will extend the CP analogy to two systems: the modified Foucault Pendulum (FP), which corresponds to two coupled classical oscillators, and the modified Quaternionic Foucault Pendulum (QFP), which corresponds to a system of 4 coupled classical oscillators. The modified Foucault pendulum will be defined to be an extension of the dynamics of the Foucault pendulum that includes a "natural" frequency term. The modified quaternionic Foucault pendulum will be defined as an extension of the dynamics of the modified Foucault pendulum from complex space to quaternionic space.

We will show that the dynamics of the modified Foucault pendulum reproduce the quantum dynamics of an unmeasured electron spin state in a spatially uniform time-varying magnetic field in the y-direction. Similarly, we will show that the modified quaternionic Foucault pendulum reproduces the quantum dynamics of an unmeasured electron spin state in a spatially uniform time constant magnetic field in an arbitrary direction. These results will show that if there is an inherent non-classical aspect to quantum spin, then such aspect cannot be part of the quantum dynamics. Further, in the process of showing the correspondence between the quaternionic Foucault pendulum and the quantum state, we will give an explicit many-to-one map from the classical system to the quantum system, which can be interpreted as the classical system having a natural set of "hidden variables" available to the classical analog but concealed to the complete specification of the quantum state.

The outline of the thesis is as follows:

- In Section (1.2) we give a short introduction to Quaternions to lay the ground work for subsequent sections

- In Section (2.1) we solve the Schrodinger-Pauli-Equation for the spin $\frac{1}{2}$ system under a uniform magnetic field in Spinor notation and quaternionic notation.

- In Section (2.2) we give an exposition of the Foucault pendulum. We solve the equations of motion of the Foucault pendulum and derive some of the associated constants of motion. We also draw analogues between the Foucault pendulum dynamics and the dynamics of the spin $\frac{1}{2}$ system.

- In Section (2.3) we show the special equivalence condition between the Foucault pendulum dynamics and the spin $\frac{1}{2}$ system for the special case of a time-varying magnetic field in the y direction. This result will establish precedence for the next section, as it will motivate the definition of the Quaternionic Foucault Pendulum to include a correpondence with a time constant magnetic field in arbitrary direction.

- In Section (2.4) we define the Quaternionic Foucault Pendulum and solve for the equations of motion and the constant motions that are derived from the quaternionic structure. We also give an interpretation of these constants of motion by drawing parallels to corresponding constants of motion for the Schrodinger-Pauli-Equation. We close this section by showing that an arbitrary quaternionic Foucault pendulum is equivalent to two identical Foucault pendulums at the same latitude.

- In Section (2.5) we consider the set of solutions to the quaternionic Foucault pendulum that are also roots of the Lagrangian, that is, solutions $\eta(t)$ that also satisfy $L(t, \eta(t), \dot{\eta}(t)) = 0$. We find conditions on $\eta(t)$ that are equivalent to the $L = 0$ constrain and we use these equivalent conditions to show the correspondence between the SPE and QFP. We show the derived correspondence to be a many-to-one map that relies on additional parameters that do not affect the quantum solution. Such parameters will be labeled "hidden variables" from a quantum perspective.
- In Section (2.6) we show a partial corresponding between the QFP and the SPE with a time-varying magnetic field.
- In Chapter (3) we close the discussion with a summary of the results exposed.

An appendix has been included to include more preliminary results used by the derivations of Chapter (2). These results were included in the appendix because they are too mathematical in nature and provide very little physical insight.

We close this chapter by introducing the notation used for Quaternions.

1.2 Preliminaries

The Quaternions were first discovered by the Irish mathematician Sir William Hamilton. Quaternions are a division ring of dimension 4 over the real numbers. That is, they are a vector space $\mathbb{R}^4$ with a non-commutative vector product for which every non-zero vector is a unit (that is, every non-zero element has a multiplicative inverse). The Quaternion algebra can be defined in different ways. We define it using the "scalar plus vector" notation.

Definition 1.2.1. The *Quaternion Algebra* is a free vector space with basis $1, \vec{i}, \vec{j}, \vec{k}$ equipped with a vector product. That is, $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}\vec{i} \oplus \mathbb{R}\vec{j} \oplus \mathbb{R}\vec{k}$ with a prescribed vector product that makes $\mathbb{H}$ into a division ring. A typical vector $v \in \mathbb{H}$ will be called a *Quaternion*. Using the coordinate representation, v can be represented as:

$$v = v_0 + v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

Given a Quaternion $v = v_0 + v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, we define the *scalar* or *real part* of v as v_0. And, we define the *vector* or *imaginary part* of v as $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Hence,

$$\begin{aligned} v &= v_0 + \vec{v}, \\ Re(v) &= v_0, \\ Im(v) &= \vec{v}. \end{aligned}$$

Using the scalar plus vector notation for Quaternions, we can define a product between Quaternions.

Definition 1.2.2. Let $v = v_0 + \vec{v}$ and $w = w_0 + \vec{w}$ be two Quaternions. Then, we define the *product* of v and w as:

$$\begin{aligned} vw &= (v_0 + \vec{v})(w_0 + \vec{w}) \\ &= v_0 w_0 - \langle \vec{v}, \vec{w} \rangle + v_0\vec{w} + w_0\vec{v} + \vec{v} \times \vec{w} \end{aligned}$$

Where $\langle , \rangle$ is the inner product of two vectors in $\mathbb{R}^3$ and $\times$ is the vector cross product between two vectors in $\mathbb{R}^3$.

Note that immediate consequences of the product are $\vec{k} = \vec{i}\vec{j}$, $-\vec{k} = \vec{j}\vec{i}$, $\vec{i}\vec{j} = -\vec{j}\vec{i}$. Alternatively, the Quaternion algebra can also be defined using the complexification construction. Recall that the real numbers $\mathbb{R}$ form a field. That is, an associative algebra with a commutative product where all non-zero elements have a multiplicative inverse. This field can be extended to the complex numbers by adjoining a square root of -1 called $\vec{i} = \sqrt{-1}$. This is done by considering the two dimensional real vector space $\mathbb{C} = \mathbb{R}1 \oplus \mathbb{R}\vec{i} \simeq \mathbb{R} \oplus \mathbb{R}$ spanned by the basis $1, \vec{i}$, and by defining a product between vectors as: let $a = a_0 + a_1\vec{i}$, and $b = b_0 + b_1\vec{i}$ be two complex numbers, then

$$\begin{aligned} ab &= (a_0 + a_1\vec{i})(b_0 + b_1\vec{i}) \\ &= (a_0b_0 - a_1b_1) + (a_1b_0 + b_1a_0)\vec{i} \end{aligned}$$

In tuple notation,

$$(a_0, a_1)(b_0, b_1) = (a_0b_0 - a_1b_1, a_1b_0 + b_1a_0) \tag{1.1}$$

It can be shown that this product makes $\mathbb{R} \oplus \mathbb{R} \simeq \mathbb{C}$ into a field. Note that by considering $\mathbb{C}$ acting on itself by $\rho(a)(b) = ab$, we can define a map of $\mathbb{C}$ into the general linear group $GL_2(\mathbb{R})$ (the group of 2×2 invertible matrices with real entries) via the use of the basis $1, i$ or $(1, 0), (0, 1)$. That is, by defining,

$$\begin{aligned} \rho(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \rho(\vec{i}) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \rho(a_0 + a_1\vec{i}) &= \begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix}. \end{aligned}$$

A similar construction applied to $\mathbb{C}$ will yield the Quaternion algebra $\mathbb{H}$. Recall that a typical Quaternion has representation $v = v_0 + v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Also, note that $\vec{k} = \vec{i}\vec{j}$ using the Quaternion product. Hence,

$$\begin{aligned} v &= v_0 + v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \\ &= v_0 + v_1\vec{i} + v_2\vec{j} + v_3\vec{i}\vec{j} \\ &= (v_0 + v_1\vec{i}) + (v_2 + v_3\vec{i})\vec{j} \\ &= v_{0,1} + v_{2,3}\vec{j} \end{aligned} \tag{1.2}$$

Where $v_{0,1}, v_{2,3}$ can be viewed as complex numbers because $\mathbb{R} \oplus \mathbb{R}\vec{i}$ is isomorphic to $\mathbb{C}$ as algebras. This expansion suggests that there is map between $\mathbb{H}$ and $\mathbb{C} \oplus \mathbb{C}\vec{j} \simeq \mathbb{C} \oplus \mathbb{C}$ where $\vec{j}$ is another square root of -1 different from $\vec{i}$. Let us consider the space $\mathbb{C} \oplus \mathbb{C}\vec{j}$ where $\vec{j}$ is a square root of -1 different form $\vec{i}$. Clearly, for two distinct vectors $c = c_0 + c_1\vec{j}$, $d = d_0 + d_1\vec{j}$ where $c_0, c_1, d_0, d_1 \in \mathbb{C}$,

$$\begin{aligned} cd &= (c_0 + c_1\vec{j})(d_0 + d_1\vec{j}) \\ &= (c_0d_0 + c_1\vec{j}d_1\vec{j}) + (c_0d_1\vec{j} + c_1\vec{j}d_0) \end{aligned}$$

Note that for any complex number $c = c_0 + c_1\vec{i}$,

$$\begin{aligned} c\vec{j} &= (c_0 + c_1\vec{i})\vec{j} \\ &= (c_0\vec{j} + c_1\vec{i}\vec{j}) \\ &= (c_0\vec{j} - c_1\vec{j}\vec{i}) \\ &= \vec{j}(c_0 - c_1\vec{i}) \\ &= \vec{j}\bar{c}. \end{aligned}$$

Where we have used $\vec{i}\vec{j} = -\vec{j}\vec{i}$ and $\bar{c}$ is the conjugate of the complex number c. Similarly, we can show $\vec{j}c = \bar{c}\vec{j}$. Hence, the product in $\mathbb{C} \oplus \mathbb{C}$ yields,

$$\begin{aligned} cd &= (c_0 d_0 + c_1 \bar{d}_1 \vec{j}^2) + (c_0 d_1 \vec{j} + c_1 \bar{d}_0 \vec{j}) \\ &= (c_0 d_0 - c_1 \bar{d}_1) + (c_0 d_1 + c_1 \bar{d}_0)\vec{j} \end{aligned}$$

Which yields the product in $\mathbb{C} \oplus \mathbb{C}$ as:

$$(c_0, c_1)(d_0, d_1) = (c_0 d_0 - c_1 \bar{d}_1, c_0 d_1 + c_1 \bar{d}_0) \tag{1.3}$$

It can be shown that $\mathbb{C} \oplus \mathbb{C}$ equipped with the above product makes $\mathbb{C} \oplus \mathbb{C}$ into an algebra that is isomorphic to the Quaternion algebra. We will call the above product the *right regular product* of Quaternions in $\mathbb{C} \oplus \mathbb{C}$. Note that by using the right regular representation $\rho_R : \mathbb{C} \oplus \mathbb{C} \to GL_2(\mathbb{C})$ defined by,

$$\begin{aligned} \rho_R((d_0, d_1))((c_0, c_1)) &= (c_0, c_1)(d_0, d_1) \\ \rho_R((d_0, d_1)) &= \begin{pmatrix} d_0 & -\bar{d}_1 \\ d_1 & \bar{d}_0 \end{pmatrix}. \end{aligned} \tag{1.4}$$

we can show that ρ_R maps $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}$ into $GL_2(\mathbb{C})$. Note that by choosing a slightly different expansion as Equation (1.2),

$$\begin{aligned} v &= v_0 + v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \\ &= v_0 + v_1\vec{i} + v_2\vec{j} - v_3\vec{j}\vec{i} \\ &= (v_0 + v_1\vec{i}) + \vec{j}(v_2 - v_3\vec{i}) \\ &= w_{0,1} + \vec{j}w_{2,3} \end{aligned}$$

We can deduce a relationship between $\mathbb{C} \oplus \vec{j}\mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$ and $\mathbb{H}$. This relationship can be inferred from:

$$\begin{aligned} cd &= (c_0 + \vec{j}c_1)(d_0 + \vec{j}d_1) \\ &= c_0d_0 + \vec{j}c_1d_0 + c_0\vec{j}d_1 + \vec{j}c_1\vec{j}d_1 \\ &= c_0d_0 + \vec{j}^2\bar{c_1}d_1 + \vec{j}c_1d_0 + \vec{j}\bar{c_0}d_1 \\ &= (c_0d_0 - \bar{c_1}d_1) + \vec{j}(c_1d_0 + \bar{c_0}d_1) \end{aligned}$$

Hence, we can define a product between vectors of $\mathbb{C} \oplus \mathbb{C}$ as

$$(c_0, c_1)(d_0, d_1) = (c_0d_0 - \bar{c_1}d_1, c_1d_0 + \bar{c_0}d_1)$$

It can be shown that $\mathbb{C} \oplus \mathbb{C}$ equipped with the above product makes $\mathbb{C} \oplus \mathbb{C}$ into an algebra that is isomorphic to the Quaternion algebra. We will call the above product the *left regular product* of Quaternions in $\mathbb{C} \oplus \mathbb{C}$. Note that by using the left regular representation $\rho_L : \mathbb{C} \oplus \mathbb{C} \to GL_2(\mathbb{C})$ defined by,

$$\begin{aligned} \rho_L((d_0, d_1))((c_0, c_1)) &= (d_0, d_1)(c_0, c_1) \\ \rho_L((d_0, d_1)) &= \begin{pmatrix} d_0 & -\bar{d_1} \\ d_1 & \bar{d_0} \end{pmatrix}. \end{aligned} \tag{1.5}$$

we can show that ρ_L maps $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}$ into $GL_2(\mathbb{C})$. This shows that ρ_R and ρ_L have the same matrix representation if we use different definitions for the Quaternion product on $\mathbb{C} \oplus \mathbb{C}$. Note that, if we were to identify a matrix that has the form of Equation (1.4) or Equation (1.5) acting on $\mathbb{C} \oplus \mathbb{C}$, we could identify $\mathbb{C} \oplus \mathbb{C}$ with $\mathbb{H}$ using the left regular product and view the matrix as the pre-image of a Quaternion under ρ_L. Alternatively, we could identify $\mathbb{C} \oplus \mathbb{C}$ with $\mathbb{H}$ using the right regular product and view the matrix as a pre-image under a Quaternion under ρ_R. This freedom in identifying $\mathbb{C} \oplus \mathbb{C}$ with $\mathbb{H}$ will help us deduce different but equivalent Spinor solutions to Spinor ODEs.

An important map on Quaternions is the *Conjugate* map.

Definition 1.2.3. Given a Quaternion $v = v_0 + \vec{v}$, the *Conjugate* of v is defined as:

$$\bar{v} \;=\; v_0 - \vec{v}$$

If we identify $\mathbb{C} \oplus \mathbb{C}$ with $\mathbb{H}$ using the right regular product, the $\overline{(c_0, c_1)} = (\overline{c_0}, -c_1)$. Similarly, if we identify $\mathbb{C} \oplus \mathbb{C}$ with $\mathbb{H}$ using the left regular product, the $\overline{(c_0, c_1)} = (\overline{c_0}, -c_1)$. The following proposition summarizes important properties of the Conjugate map.

Proposition 1.2.4. Let $v = v_0 + \vec{v} = c_1 + c_0\vec{j} = d_0 + \vec{j}d_1$ be a Quaternion, where c_0, c_1, d_0, d_1 are viewed as complex numbers. Then, $N(v)$ (the *Norm* of v) is defined as $v\overline{v}$, and,

$$\begin{aligned} Norm(v) &= v\overline{v} \\ &= \overline{v}v \\ &= v_0^2 + \langle \vec{v}, \vec{v} \rangle \\ &= c_0\overline{c_0} + c_1\overline{c_1} \\ &= d_0\overline{d_0} + d_1\overline{d_1} \end{aligned}$$

Let $w = w_0 + \vec{w}$ be another Quaternion. Then,

$$\begin{aligned} \overline{vw} &= \bar{w}\bar{v} \\ 2Re(v) &= 2v_0 \\ &= v + \overline{v} \\ 2Im(v) &= 2\vec{v} \\ &= v - \overline{v} \end{aligned}$$

Also,

$$\begin{aligned} N(vw) &= N(v)N(w) \\ &= N(w)N(v) \\ &= N(wv) \end{aligned}$$

CHAPTER 2

CLASSICAL MODELS OF THE SPIN $\frac{1}{2}$ SYSTEM

We will propose a classical system motivated by the Foucault pendulum via a generalization of the complex Lagrangian of the Foucault pendulum to Quaternions. This will yield a set of Euler-Lagrange equations based on 4-space which will be shown to contain the dynamics of the spin $\frac{1}{2}$ system subjected to a uniform magnetic field.

2.1 The Electron Spin State under a Uniform Magnetic Field

We will solve the Schrodinger-Pauli Equation (SPE) for the spin state of the electron χ under a uniform magnetic field and show how the resulting first order ODE can be mapped to a first order quaternionic differential equation. Let us consider the (SPE) for a spin $\frac{1}{2}$ particle (for instance, electrons) under a uniform magnetic field. Given a spin $\frac{1}{2}$ Spinor $\chi \in \mathbb{C} \oplus \mathbb{C}$ representing the spin state of the particle in the S_z eigenbasis. The SPE predicts the time evolution of χ by the following first order ODE.

$$i\hbar\frac{\partial \chi}{\partial t} = H\chi \tag{2.1}$$

Where H is the Hamiltonian of the system. For the spin $\frac{1}{2}$ particle, H is given as,

$$H = -\gamma\vec{B}\cdot\vec{S} + \hbar\omega_0$$

Where $\vec{B}$ is the magnetic field, γ is the gyromagnetic ratio, and $\vec{S}$ is the spin vector. The $-\gamma\vec{B}\cdot\vec{S}$ term is the energy of the spin vector in the magnetic field, the $\hbar\omega_0 I$ is the rest energy term that is introduced to make the correspondence between the SPE and the Foucault pendulum dynamics possible. One can interpret the rest energy as a rest mass by use of the equation $mc^2 = \hbar\omega_0$. In operator form, we have,

$$\begin{aligned} H &= -\gamma(B_x\langle\vec{i},\vec{S}\rangle + B_y\langle\vec{j},\vec{S}\rangle + B_k\langle\vec{k},\vec{S}\rangle) + \hbar\omega_0 I \\ &= -\gamma(B_xS_x + B_yS_y + B_zS_z) + \hbar\omega_0 I \end{aligned}$$

In the S_z eigenbasis, we have:

$$\begin{aligned} S_z &= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ S_x &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ S_y &= \frac{i\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Also, $\vec{B}$ in spherical coordinates is given by,

$$B(\sin(\phi)\cos(\theta)\vec{i} + \sin(\phi)\sin(\theta)\vec{j} + \cos(\phi)\vec{k}),$$

where B is the norm of $\vec{B}$. Hence, H is given in the S_z eigenbasis as,

$$\begin{aligned} H &= -\gamma B(\sin(\phi)\cos(\theta)S_x + \sin(\phi)\sin(\theta)S_y + \cos(\phi)S_z) + \hbar\omega_0 I \\ &= -\frac{\gamma B\hbar}{2}\begin{pmatrix} \cos(\phi) & \sin(\phi)\exp(-i\theta) \\ \sin(\phi)\exp(i\theta) & -\cos(\phi) \end{pmatrix} + \hbar\omega_0 I \end{aligned}$$

Thus, the SPE in operator form is given as,

$$i\hbar\frac{\partial\chi}{\partial t} = \left(-\frac{\gamma B\hbar}{2}\begin{pmatrix} \cos(\phi) & \sin(\phi)\exp(-i\theta) \\ \sin(\phi)\exp(i\theta) & -\cos(\phi) \end{pmatrix} + \hbar\omega_0 I\right)\chi$$

Hence, the SPE is equivalent to,

$$\begin{aligned} \frac{\partial\chi}{\partial t} &= \left(\frac{\gamma B}{2}\begin{pmatrix} i\cos(\phi) & i\sin(\phi)\exp(-i\theta) \\ i\sin(\phi)\exp(i\theta) & -i\cos(\phi) \end{pmatrix} - i\omega_0 I\right)\chi \\ &= \left(\frac{\gamma B}{2}\begin{pmatrix} i\cos(\phi) & -\overline{i\sin(\phi)\exp(i\theta)} \\ i\sin(\phi)\exp(i\theta) & \overline{i\cos(\phi)} \end{pmatrix} - i\omega_0 I\right)\chi \qquad (2.2) \end{aligned}$$

Where χ is a function $\chi : \mathbb{R} \to \mathbb{C} \oplus \mathbb{C}$, or a curve in $\mathbb{C} \oplus \mathbb{C}$. Note that the SPE in the form of Equation (2.2) has the form of the right regular or left regular quaternionic representation depending on the type of product that we define on $\mathbb{C} \oplus \mathbb{C}$. We will equip $\mathbb{C} \oplus \mathbb{C}$ with the right regular quaternionic product of Equation (1.3). Using this product, the SPE can be written as,

$$\frac{\partial\chi}{\partial t} = \left(\frac{\gamma B}{2}\rho_R(\cos(\phi)\vec{i} + \sin(\phi)exp(\vec{i}\theta)\vec{i}\vec{j}) - i\omega_0 I\right)\chi$$

Where ρ_R is the quaternionic right regular representation. Using the identification $\chi = (\chi_0, \chi_1) \in \mathbb{C} \oplus \mathbb{C}$ with the Quaternion $\eta = \chi_0 + \chi_1\vec{j}$, we can re-write the SPE as a Quaternion equation as,

$$\dot{\eta}(t) = \eta(t)(\frac{\gamma B}{2}\vec{\beta_0}) - (\vec{i}\omega_0)\eta(t) \qquad (2.3)$$

Where ω_0 is the rest energy term in H (a real number), and $\vec{\beta_0}$ is given by the purely imaginary unit Quaternion:

$$\vec{\beta_0} \;=\; \cos(\phi)\vec{i} - \sin(\phi)\sin(\theta)\vec{j} + \sin(\phi)\cos(\theta)\vec{k}$$

Equation (2.3) is the equivalent form of the SPE in quaternionic notation.

2.2 The Foucault Pendulum

We will introduce the Lagrangian of the Foucault pendulum and solve the Euler-Lagrange equations of motion using complex numbers. This will provide a motivation for the quaternionic Lagrangian of the generalized Foucault pendulum which we will call the Quaternionic Foucault Pendulum (QFP).

The Foucault pendulum or Foucault's pendulum, named after the French physicist Leon Foucault, is a simple device conceived as an experiment to demonstrate the rotation of the Earth. The experimental apparatus consists of a tall pendulum free to swing in a vertical plane. The actual plane of swing appears to rotate relative to the Earth; in fact, the plane is fixed in space while the Earth rotates under the pendulum once a sidereal day. Figure (2.1) shows a diagram of the Foucault pendulum on the surface of the Earth. In this figure, a pendulum of length l and mass m is located at latitude $\frac{\pi}{2} - \phi$. As the pendulum moves through the surface of the Earth, due to the rotation of the Earth, the motion of the pendulum precesses. The motion of the precession can be predicted in the small angle-limit approximation with respect to the vertical axis of the pendulum by making use of the β parameter which equals to $\Omega\cos(\phi)$ and the ω_0 parameter which equals to $\sqrt{\frac{g}{l}}$; where Ω is the angular velocity of the earth, l the length of the pendulum, and g is the acceleration due to gravity.

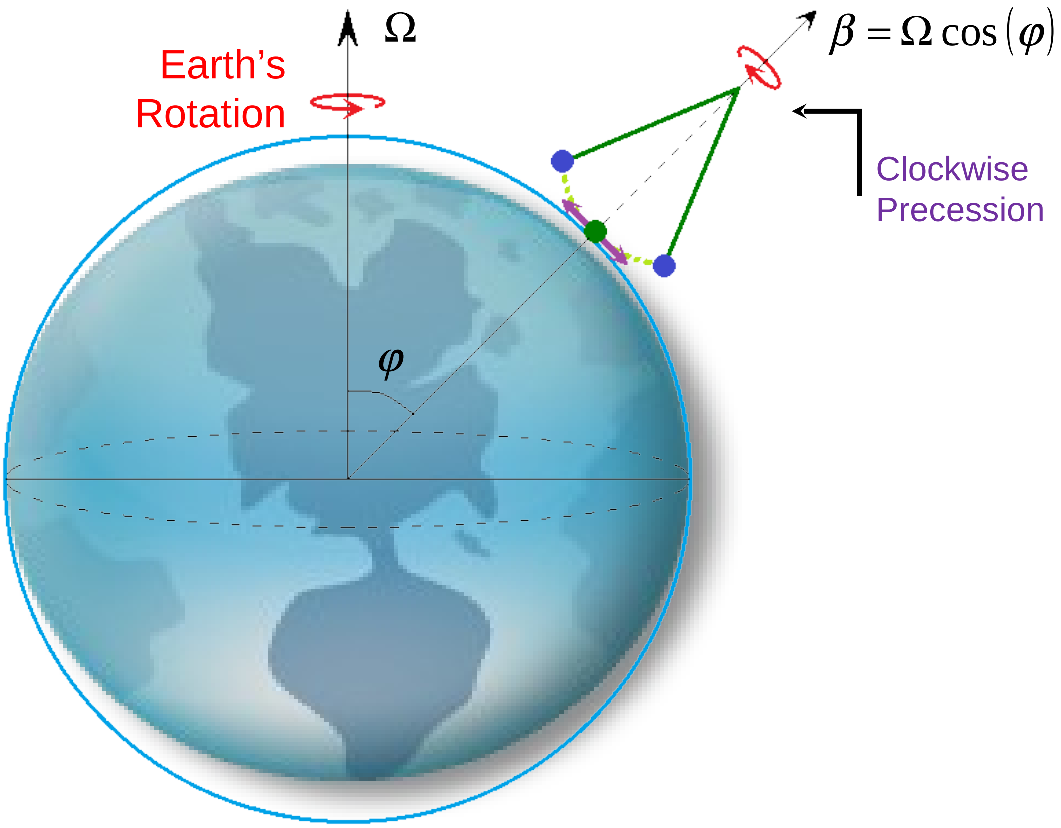

Figure 2.1: Depiction of a Foucault pendulum on the surface of the Earth.

The Lagrangian that describes the equations of motion of the Foucault pendulum (FP) in the small-angle limit approximation is given by

$$L = \frac{1}{2}\left\{\dot{x_1}(t)^2 + \dot{x_2}(t)^2\right\} - \frac{1}{2}\omega_0^2\left\{x_1(t)^2 + x_2(t)^2\right\} + \beta\left\{x_1(t)\dot{x_2}(t) - x_2(t)\dot{x_1}(t)\right\}$$

Where $\beta = \Omega\cos(\phi)$ is a real number, and $x_1(t), x_2(t)$ denote the position of the pendulum on the tangent plane (horizontal plane with orthogonal axes x_1, x_2) to the surface of the Earth at the location of the pendulum, and $\omega_0 = \sqrt{\frac{g}{l}}$ is the natural frequency of the pendulum.

We can write this equation in vector form, with $\vec{x} = [x_1(t)x_2(t)]^T$, and,

$$L = \frac{1}{2}\dot{\vec{x}}^T\dot{\vec{x}} - \frac{1}{2}\omega_0^2\vec{x}^T\vec{x} + \beta\dot{\vec{x}}^T\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{x}$$

Note that by mapping $\vec{x} \longrightarrow z = x_1(t) + ix_2(t) \in \mathbb{C}$, we can think of the trajectory of the pendulum given by $\vec{x}$ as a curve in the complex plane $\mathbb{C}$. Under this map, the Lagrangian takes the form,

$$\begin{aligned} L(t, z, \dot{z}) &= \frac{1}{2}\dot{\overline{z}}\dot{z} - \frac{1}{2}\omega_0^2\overline{z}z + Re(\dot{\overline{z}}(i\beta)z) \\ &= \frac{1}{2}\dot{\overline{z}}\dot{z} - \frac{1}{2}\omega_0^2\overline{z}z + \frac{1}{2}\left\{\dot{\overline{z}}(i\beta)z + \overline{z}\overline{(i\beta)}\dot{z}\right\} \end{aligned} \tag{2.4}$$

We note that without the $Re(\dot{\overline{z}}(i\beta)z)$ term, L is the Lagrangian of two independent oscillators with the same natural frequency ω_0. The term $Re(\dot{\overline{z}}(i\beta)z)$ introduces a coupling between the oscillators given by the x_1 and x_2 parameters that is also known as the Coriolis coupling given by the β parameter. Hence, the Foucault pendulum can be interpreted as two coupled harmonic oscillators with a Coriolis coupling.

The equations of motion can be deduced by calculating the Euler-Lagrange (E-L) equations. That is,

$$\frac{d}{dt}\left\{\frac{dL}{d\dot{z}}\right\} = \frac{dL}{dz}$$

For the Lagrangian given by Equation (2.4), we get,

$$\begin{aligned} \frac{dL}{d\dot{z}} &= \frac{1}{2}\dot{\overline{z}} + \frac{1}{2}\overline{z(i\beta)} \\ \frac{dL}{dz} &= -\frac{1}{2}\omega_0^2\overline{z} + \frac{1}{2}\dot{\overline{z}}(i\beta) \end{aligned}$$

Hence, the E-L equations give,

$$\ddot{z} + 2\beta i\dot{z} + \omega_0^2 z = 0$$

It can be shown that this equation has general solution,

$$z(t) = c_1 \exp(\beta_+ it) + c_2 \exp(\beta_- it)$$

Where,

$$\begin{aligned} \beta_+ &= -\beta + \sqrt{\beta^2 + \omega_0^2} \\ \beta_- &= -\beta - \sqrt{\beta^2 + \omega_0^2} \end{aligned}$$

And, c_1, c_2 are complex constants.

The solution space to the Euler-Lagrange equations of the Foucault pendulum deserves special attention because it has analogues in the solution space of the spin $\frac{1}{2}$ system. For example, the solution where $c_1 = 1, c_2 = 0$ ($z(t) = \exp(\beta_+ it)$) corresponds to a normal mode with clockwise rotation of the pendular plane of oscillation with frequency β_+. Similarly, the solution where $c_1 = 0, c_2 = 1$ ($z(t) = \exp(\beta_- it)$) corresponds to a normal mode with counterclockwise rotation of the pendular plane of oscillation with frequency β_-. We will see that both of these normal modes have analogues in the spin $\frac{1}{2}$ system by use of Proposition (2.3.1). It can be shown that the normal modes correspond to the $\mid y_+\rangle, \mid y_-\rangle$ states of the spin $\frac{1}{2}$ system of a negatively charged particle under a uniform magnetic field in the y-direction, where:

$$\begin{aligned} \mid y_+\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \\ \mid y_-\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \end{aligned}$$

With the clockwise precession corresponding to $\mid y_+\rangle$ and the counterclockwise precession corresponding to $\mid y_-\rangle$.

As supporting evidence of this correspondence, we note that the $\sqrt{\beta^2 + \omega_0^2}$ factor has the effect of producing two normal mode solutions of the Foucault pendulum that are equally spaced above and below a natural frequency $-\beta$ – just like the Zeeman splitting of the energy levels an electron in a uniform magnetic field. Also, we note that the precession of the normal modes give evidence of a Berry phase or geometric phase angle for the Foucault pendulum solutions – a phase already present in the spin $\frac{1}{2}$ system. As it is well known, a linear oscillation in the x_1 direction precesses into a linear oscillation in the x_2 direction and then back to the x_1 direction. However, this 2π rotation of $\vec{x} = (x_1, x_2)^T$ in the solution space corresponds to a π rotation of the pendular plane of oscillation in physical space. We note that a similar behavior is present in the spin $\frac{1}{2}$ system for a negatively charged particle under a uniform magnetic field in the y direction with the states,

$$\begin{aligned} | \, z_+ \rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ | \, z_- \rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Figure (2.2) illustrates the precession of the plane of oscillation of a Foucault pendulum at latitude 30^0 North. Notice the π rotation of the pendular plane of oscillation after the pendulum has been moved once around the earth.

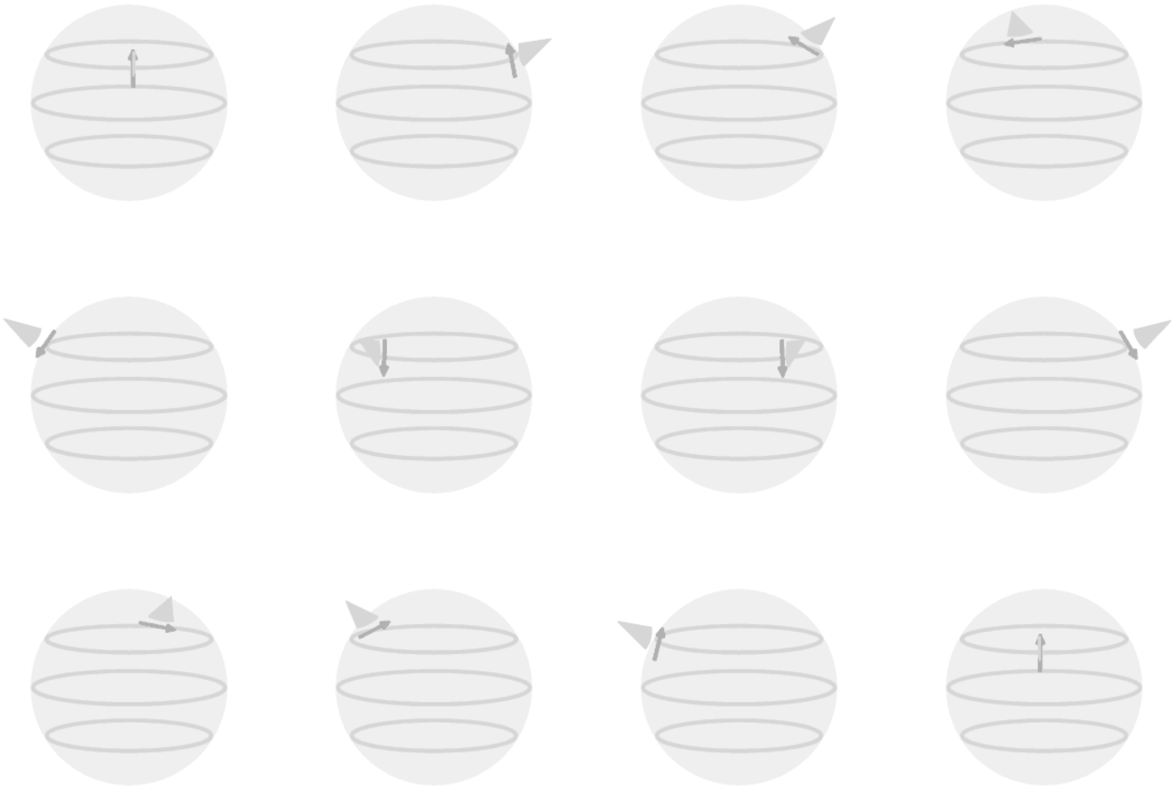

Figure 2.2: Precession of a Foucault pendulum at latitude 30^0 North.

Additionally, any solution of the E-L equation of the Foucault pendulum is a linear combination of the normal mode solutions. A property that has as analogue in the spin $\frac{1}{2}$ system the superposition principle of quantum mechanics. A more concise correspondence between the Foucault pendulum and the spin $\frac{1}{2}$ will be given in Section (2.3).

Now, we proceed to calculate some of the constants of motion of the Foucault pendulum. Note that $\mathbb{C}$ can be viewed as a Lie group under the right regular product of Equation (1.1). Also, note that for $\alpha \in \mathbb{C}$ of unit norm ($\overline{\alpha}\alpha = 1$),

$$L(\alpha \cdot t, \alpha z, \alpha \dot{z}) \quad = \quad L(t, z, \dot{z})$$

Hence, $G = \{\alpha \in \mathbb{C} \mid \overline{\alpha}\alpha = 1\}$ is a symmetry group of L. Clearly, G is a circle and hence G is a Lie group of dimension 1. Thus, by Proposition (A.3.2), there is exactly one linearly independent constant of motion. In order to calculate this constant, we first calculate the Lie algebra of G. Clearly, the Lie Algebra is given by $\mathbb{R}$, and the exponential map $\exp : \mathbb{R} \to G$ taking the Lie algebra to G is given by:

$$\exp(\theta) \quad = \quad \exp(i\theta) \in G \subset \mathbb{C}$$

Near the identity $1 \in G$, the elements of G are given by $\exp(id\theta)$ where $d\theta$ is a small number. Clearly,

$$\exp(id\theta) \quad = \quad 1 + id\theta + O(d\theta^2)$$

Hence, the infinitesimal generator of the Lie algebra is given by i. Note that $\xi_i(z) = zi$. Also, recall that $p = \frac{\partial L}{\partial \dot{z}} = \frac{1}{2}(\overline{\dot{z} + \beta i z})$. Hence, the constant of motion of this symmetry is given by,

$$\begin{aligned} S_i &= \langle p, \xi_i(z) \rangle \\ &= \frac{1}{2} \langle \overline{\dot{z} + \beta i z}, iz \rangle \\ &= \frac{1}{2} Re(\overline{\dot{z} + \beta i z} iz) \\ &= \frac{1}{2} \left\{ \frac{i}{2} (\dot{\overline{z}} z - \dot{z} \overline{z}) + \beta \overline{z} z \right\} \\ &= \frac{1}{2} \left\{ Im(\dot{z} \overline{z}) + \beta \overline{z} z \right\} \\ &= \frac{1}{2} \left\{ x_1 \dot{x_2} - x_2 \dot{x_1} + \beta (x_1^2 + x_2^2) \right\} \end{aligned}$$

It can be shown that if we let $x_1 = \rho\cos(\theta), x_2 = \rho\sin(\theta)$, then L becomes a function of $\rho, \dot{\rho}, \theta, \dot{\theta}$, and because L is cyclic in θ the canonical momentum $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \rho^2(\dot{\theta} + \beta)$ is a constant of motion. Also, by making the transformation $\rho^2 = x_1^2 + x_2^2$, $\theta = \arctan(\frac{x_1}{x_2})$, it can be shown that $S_i = \frac{p_\theta}{2}$. This verifies our result.

We note that the canonical momentum given by p_θ is not the same as the angular momentum because the latter is not a conserved quantity. Also, we point out energy as another conserved quantity corresponding to time translation symmetry in L.

2.3 A Special Equivalence Between the Foucault Pendulum and the Spin $\frac{1}{2}$ System

For the special case of a time-varying magnetic field in the y direction, one can show that the Foucault pendulum and the spin $\frac{1}{2}$ system have almost the same solutions provided that one allows the natural frequency of the Foucault pendulum to vary like $\sqrt{\omega_0^2 - \beta^2}$.

Proposition 2.3.1. Let X be the solution space of the E-L equations of the Foucault pendulum with parameters $\beta(t) = \frac{\gamma B(t)}{2}$ and natural frequency

$\omega_1 = \sqrt{\omega_0^2 - \beta(t)^2}$. Let Y be the solution space of the SPE with magnetic field $B(t) = \frac{2\beta(t)}{\gamma}\vec{j}$ and rest mass frequency ω_0. Let $z_1(t), z_2(t)$ be a basis for Y the solution space of the SPE. Then, $\{Re(z_1), Re(z_2), Im(z_1), Im(z_2)\}$ is a basis for X the solution space of the E-L of the Foucault pendulum. That is, $X = Re(Y) \oplus_{\mathbb{R}} Im(Y)$.

Proof. By considering a Foucault pendulum with a time-varying $\beta(t)$ and natural frequency ω_1, one can deduce the Euler-Lagrange equations as:

$$\ddot{z} + 2\dot{z}\beta i + z\dot{\beta} i + \omega_1^2 z = 0$$

Or, in coordinate notation by using the map,

$$z(t) = x_1(t) + ix_2(t) \quad \rightarrow \quad (x_1(t), x_2(t))^T,$$

We deduce that,

$$\begin{pmatrix} \ddot{x_1}(t) \\ \ddot{x_2}(t) \end{pmatrix} + 2\beta(t)J \begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} + (\omega_1^2 I + \dot{\beta}(t)J) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. On the other other hand, when we let the magnetic field be uniform in the y direction $B(t) = \frac{2\beta(t)}{\gamma}\vec{j}$, and let the rest mass be ω_0, then the SPE takes form:

$$\frac{\partial \chi}{\partial t} = \begin{pmatrix} -i\omega_0 & \frac{\gamma B(t)}{2} \\ -\frac{\gamma B(t)}{2} & -i\omega_0 \end{pmatrix} \chi$$

By letting $\chi(t) = (\chi_1(t), \chi_2(t))^T$ where χ_1, χ_2 are complex valued functions, we get the 2 dimensional linear ODE.

$$\begin{aligned}
\begin{pmatrix} \dot{\chi_1}(t) \\ \dot{\chi_2}(t) \end{pmatrix} &= \begin{pmatrix} -i\omega_0 & \beta(t) \\ -\beta(t) & -i\omega_0 \end{pmatrix} \begin{pmatrix} \chi_1(t) \\ \chi_2(t) \end{pmatrix} \\
&= (-i\omega_0 I - \beta(t)J) \begin{pmatrix} \chi_1(t) \\ \chi_2(t) \end{pmatrix}
\end{aligned}$$

Clearly, from this we deduce that:

$$\ddot{\vec{\chi}} = (-i\omega_0 I - \beta(t)J)\dot{\vec{\chi}} - \dot{\beta}(t)J\vec{\chi}$$

Where $\vec{\chi} = (\chi_1(t), \chi_2(t))^T$. Hence, when $\vec{\chi}$ is a solution to the SPE, we calculate,

$$\begin{aligned}
\ddot{\vec{\chi}} + 2\beta(t)J\dot{\vec{\chi}} + \dot{\beta(t)}J\vec{\chi} &= -(\omega_0^2 - \beta(t)^2)I\vec{\chi} \\
&= -\omega_1^2\vec{\chi}
\end{aligned}$$

Hence, $\vec{\chi}$ is a complex solution to the E-L of the Foucault pendulum with parameter $\beta(t)$ and natural frequency ω_1. Thus, the SPE yields complex solutions the E-L equation of the Foucault pendulum. We will use the following elementary claim to deduce a basis for the solution space X of the E-L equations of the Foucault pendulum using a basis of the solution space Y of the SPE.

Claim 2.3.2. Let X be a vector space of functions over the complex numbers with function basis given by $\{z_1(t), z_2(t)\}$. Assume further, that there are no complex linear combinations of $z_1(t), z_2(t)$ that yield a purely real function. Then, the set $\{Re(z_1), Re(z_2), Im(z_1), Im(z_2)\}$ is a linearly independent set of real functions where linear independence is taken over the real numbers instead of the complex numbers.

It can be verified that the SPE with rest mass ω_0 cannot admit purely real solutions. This is an elementary result in quantum mechanics. Hence, if the space Y of solutions to the SPE has basis $\{z_1(t), z_2(t)\}$ over the complex numbers. Then, the set $\Xi = \{Re(z_1), Re(z_2), Im(z_1), Im(z_2)\}$ is a linearly independent set of real functions with linear independence over the real numbers. Clearly, every complex function that satisfies the E-L of the Foucault pendulum must have its real and imaginary part also satisfy the E-L of the Foucault pendulum. Hence, every function of Ξ solves the E-L of the Foucault pendulum. In particular, Ξ generates a 4 dimensional vector subspace of the solution space X of the E-L of the Foucault pendulum. Clearly, this must yield that Ξ spans X because X is a 4 dimensional vector space over the real numbers as well.

□

We will seek to generalize the Foucault pendulum to 4 dimensions in such a way that Proposition (2.3.1) holds in some simpler form. We will do this for the case of a time independent uniform magnetic field.

2.4 The Quaternionic Foucault Pendulum (QFP)

In the previous Section (2.3), it was shown that the solution space X of the SPE with a special magnetic field B was related to the solution space of the E-L of the Foucault pendulum. We will seek to generalize this correspondence to an arbitrary uniform magnetic field. In order to do this, we propose extending the $\beta(t)$ parameter to an arbitrary purely imaginary Quaternion. Using the complex Foucault pendulum as motivation, we will propose a Quaternionic Foucault Pendulum (QFP). This quaternionic version will be shown to generalize Proposition (2.3.1) in the special case of an arbitrary magnetic field $B(t)$ that is time independent and uniform. The following diagram depicts the generalization hierarchy from the Foucault pendulum to the quaternionic Foucault pendulum along with their corresponding correspondences to the SPE.

$$\begin{tikzcd}
\mathrm{Q}FP & \mathrm{S}PE, \vec{B}(t) = \vec{B} \arrow[l, "L=0"'] \\
\mathrm{F}P \arrow[u, phantom, "\cup"] & \mathrm{S}PE, \vec{B}(t) = B_0(t)\vec{j} \arrow[l] \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

We will then solve the Euler-Lagrange equations for the quaternionic version and write the solution set in standard form.

The Lagrangian of the Foucault pendulum given by Equation (2.4) is defined over the complex numbers. We will generalize this Lagrangian to a function of the quaternionic variables $\eta(t), \dot{\eta}(t)$. That is,

$$\begin{aligned}
L(t, \eta, \dot{\eta}) &= \frac{1}{2}\dot{\overline{\eta}}\dot{\eta} - \frac{1}{2}\omega_0^2\overline{\eta}\eta + \frac{1}{2}\left\{\dot{\eta}\overline{\beta}\overline{\eta} + \eta\beta\dot{\overline{\eta}}\right\} \\
&= \frac{1}{2}\dot{\overline{\eta}}\dot{\eta} - \frac{1}{2}\omega_0^2\overline{\eta}\eta + Re(\dot{\eta}\overline{\beta}\overline{\eta})
\end{aligned}$$

Where in the above, β is a purely imaginary Quaternion and ω_0 is the natural frequency of the pendulum. As an observation, we note that when $\eta(t), \dot{\eta}(t), \beta$ are restricted to the complex numbers, L becomes the Lagrangian of the Foucault pendulum. Hence, it is justified that L generalizes the Foucault pendulum. We note that because $\beta = \vec{\beta}$ is a purely imaginary imaginary Quaternion, it has the property that $\beta^2 = -\|\vec{\beta}\|^2$.

The correspondence of Proposition (2.3.1) between the solution space of the FP and the solution space of the SPE can be made more direct if we substitute the natural frequency of the pendulum ω_0 with $\sqrt{\omega_0^2 - \|\beta(t)\|^2}$. We note that the E-L will keep their original forms even though this substitution for ω_0 makes ω_0 a function of t. This substitution amounts, to modifying the Lagrangian of the Foucault pendulum to:

$$L(t,\eta,\dot{\eta}) \quad = \quad \frac{1}{2}\dot{\overline{\eta}}\dot{\eta} - \frac{1}{2}(\omega_0^2 - \|\beta\|^2)\overline{\eta}\eta + Re(\dot{\eta}\overline{\beta}\overline{\eta}) \tag{2.5}$$

Thus, if we define the modified Foucault pendulum to be the dynamical system given by the solution space of the Euler-Lagrange equations of the Lagrangian given by Equation (2.5) where $\eta, \dot{\eta}, \beta$ are complex valued functions and β is purely imaginary. Then, we can rephrase Proposition (2.3.1) as,

Proposition 2.4.1. Let X be the solution space of the E-L equations of the modified Foucault pendulum with parameters $\beta(t) = \frac{\gamma B(t)}{2}$ and natural frequency ω_0. Let Y be the solution space of the SPE with magnetic field $B(t) = \frac{2\beta(t)}{\gamma}\vec{j}$ and rest mass frequency ω_0. Let $z_1(t), z_2(t)$ be a basis for Y the solution space of the SPE. Then, $\{Re(z_1), Re(z_2), Im(z_1), Im(z_2)\}$ is a basis for X the solution space of the E-L of the Foucault pendulum. That is, $X = Re(Y) \oplus_{\mathbb{R}} Im(Y)$.

We point out the E-L equations of the modified Foucault pendulum are,

$$\ddot{\eta} + \frac{d(\eta\beta)}{dt} + \dot{\eta}\beta + (\omega_0^2 - \|\beta\|^2)\eta \quad = \quad 0$$

We will take the Lagrangian given by Equation (2.5) as the Lagrangian of the Quaternionic Foucault Pendulum (QFP) by allowing $\beta, \eta(t), \dot{\eta}(t)$ to be quaternionic valued functions and forcing β to be a purely imaginary Quaternion.

Definition 2.4.2. The Quaternionic Foucault Pendulum (QFP) is the dynamical system given by the solution space of the Euler Lagrange equations of the Lagrangian defined by:

$$L(t,\eta,\dot{\eta}) \quad = \quad \frac{1}{2}\dot{\overline{\eta}}\dot{\eta} - \frac{1}{2}(\omega_0^2 - \|\vec{\beta}\|^2)\overline{\eta}\eta + Re(\dot{\eta}\overline{\beta}\overline{\eta})$$

Where L is defined on $\mathbb{R} \times \mathbb{H} \times \mathbb{H}$, $\eta(t), \dot{\eta}(t)$ are quaternionic functions, $\omega_0 \in \mathbb{R}$ is the natural frequency of the pendulum, and β is a purely imaginary Quaternion (i.e., $\beta = \vec{\beta}$).

We note that without the $Re(\dot{\eta}\overline{\beta}\overline{\eta})$ term, the Lagrangian of the QFP is nothing more that the Lagrangian of four independent oscillators with the same natural frequency ω_0. The term $Re(\dot{\eta}\overline{\beta}\overline{\eta})$ is a coupling term between the four oscillators that depends on three parameters that will correspond to the components of the magnetic field of the SPE.

By considering the map,

$$\eta(t) = \eta_0(t) + \eta_1(t)\vec{i} + \eta_2(t)\vec{j} + \eta_3(t)\vec{k} \quad \to \quad \vec{\eta}(t) = (\eta_0(t), \eta_1(t), \eta_2(t), \eta_3(t))^T,$$

we can re-write the Lagrangian of the QFP in 4-coordinate vector notation as:

$$L(t, \vec{\eta}(t), \dot{\vec{\eta}}(t)) \quad = \quad \frac{1}{2}\dot{\vec{\eta}}^T\vec{\eta} - \frac{1}{2}(\omega_0^2 - \|\vec{\beta}\|^2)\vec{\eta}^T\vec{\eta} + \dot{\vec{\eta}}^T\rho_R(\beta)\vec{\eta} \tag{2.6}$$

Where $\rho_R(\beta)$ is the right regular representation of Quaternions under the right regular product of $\mathbb{C} \oplus \mathbb{C}$. Since β is purely imaginary, $\beta = \vec{\beta} = \beta_x\vec{i} + \beta_y\vec{j} + \beta_z\vec{k}$. And, $\rho_R(\beta)$ is nothing more than the right isoclinic rotation corresponding to β. That is,

$$\rho_R(\beta) \quad = \quad \begin{pmatrix} 0 & -\beta_x & -\beta_y & -\beta_z \\ \beta_x & 0 & \beta_z & -\beta_y \\ \beta_y & -\beta_z & 0 & \beta_x \\ \beta_z & \beta_y & -\beta_x & 0 \end{pmatrix}$$

Where, we have identified $\mathbb{C} \oplus \mathbb{C}$ with $\mathbb{R}^4$ by using the map,

$$(a_0 + a_1 i, a_2 + a_3 i) \quad \to \quad (a_0, a_1, a_2, a_3).$$

Proposition 2.4.3. Any solution to the E-L equations of the QFP with time independent $\beta(t) = \beta$ parameter has form

$$\eta(t) \quad = \quad C_+ \exp(\beta_+ t) + C_- \exp(\beta_- t)$$

Where,

$$\begin{aligned} \beta_+ &= \frac{-\|\vec{\beta}\| + \omega_0}{\|\vec{\beta}\|}\vec{\beta} \\ \beta_- &= \frac{-\|\vec{\beta}\| - \omega_0}{\|\vec{\beta}\|}\vec{\beta} \end{aligned}$$

The function exp() is the exponential function defined over the Quaternions $\mathbb{H}$, and C_+, C_- are quaternionic constants.

Proof. Using the 4-coordinate vector notation for L, we can deduce the E-L equations as:

$$\ddot{\vec{\eta}} + \frac{d(\rho_R(\beta)\vec{\eta})}{dt} + \rho_R(\beta)\dot{\vec{\eta}} + (\omega_0^2 - \|\vec{\beta}\|^2)\vec{\eta} \quad = \quad \vec{0}$$

Or, in quaternionic notation,

$$\ddot{\eta} + \frac{d(\eta\beta)}{dt} + \dot{\eta}\beta + (\omega_0^2 - \|\vec{\beta}\|^2)\eta \quad = \quad 0$$

Because $\beta(t)$ is time independent, we can reduce the E-L equations to

$$\ddot{\eta} + 2\dot{\eta}\beta + (\omega_0^2 - \|\vec{\beta}\|^2)\eta \quad = \quad 0$$

The result now follows from Proposition (A.2.2).

□

2.4.1 Constants of Motion of the QFP

We will solve for the constants of motion that are generated by the symmetries induced by the quaternionic structure of L. For any unit Quaternion $a \in \mathbb{H}$, we consider the diffeomorphisms induced by the Lie structure of $\mathbb{H}^*$:

$$\begin{aligned} R_a(\eta) &= \eta a \\ L_a(\eta) &= a\eta \end{aligned}$$

Note that L is almost invariant under the action of R_a whenever a is a unit Quaternion. That is,

$$\begin{aligned} L(t, R_a(\eta), R_a(\dot{\eta})) &= \frac{1}{2}\overline{R_a(\dot{\eta})}R_a(\dot{\eta}) - \frac{1}{2}(\omega_0^2 - \|\vec{\beta}\|^2)\overline{R_a(\eta)}R_a(\eta) + Re(R_a(\dot{\eta})\bar{\beta}\overline{R_a(\eta)}) \\ &= \frac{1}{2}\dot{\overline{(\eta a)}}\dot{\eta}a - \frac{1}{2}(\omega_0^2 - \|\vec{\beta}\|^2)\overline{(\eta a)}\eta a + Re(\dot{\eta}a\bar{\beta}\overline{(\eta a)}) \\ &= \frac{1}{2}\bar{a}\dot{\overline{\eta}}\dot{\eta}a - \frac{1}{2}(\omega_0^2 - \|\vec{\beta}\|^2)\bar{a}\overline{\eta}\eta a + Re(\dot{\eta}a\bar{\beta}\bar{a}\overline{\eta}) \\ &= \frac{1}{2}\bar{a}a\dot{\overline{\eta}}\dot{\eta} - \frac{1}{2}(\omega_0^2 - \|\vec{\beta}\|^2)\bar{a}a\overline{\eta}\eta + Re(\dot{\eta}\overline{(\bar{a}\beta a)}\overline{\eta}) \\ &= L(t, \eta, \dot{\eta}) \text{ as long as } \beta = \overline{a}\beta a \end{aligned}$$

Where we have used the fact that $\overline{\eta}\eta, \dot{\overline{\eta}}\dot{\eta}$ are real numbers which commute with any Quaternion, and that $\bar{a}a = 1$ because a is a unit Quaternion. Hence, R_a is almost a symmetry as long as $\bar{a}\beta a = \beta$ or equivalently $\beta a = a\beta$. It can be shown that given a Quaternion $\beta = \beta_0 + \vec{\beta}$, the set of all Quaternions that commute with β is given by the set:

$$\begin{aligned} C_{\mathbb{H}}(\beta) &= \{\eta \in \mathbb{H} \mid \eta\beta = \beta\eta\} \\ &= \{\eta \in \mathbb{H} \mid \eta = a_0 + b_0\frac{\vec{\beta}}{\|\vec{\beta}\|}, a_0, b_0 \in \mathbb{R}\} \end{aligned}$$

Hence, R_a is a symmetry of L whenever $a = a_0 + b_0\frac{\vec{\beta}}{\|\vec{\beta}\|}$ where $a_0^2 + b_0^2 = 1$.

A similar calculation will yield that L_a is a symmetry of L for arbitrary unit Quaternion $a \in \mathbb{H}$.

Proposition 2.4.4. The diffeomorphisms L_a are symmetries of L for an arbitrary unit Quaternion a, as well as the diffeomorphisms R_a where $a = a_0 + b_0\frac{\vec{\beta}}{\|\vec{\beta}\|}$ and $a_0^2 + b_0^2 = 1$. These will be called the symmetries induced by the quaternionic structure of L. Also, these groups form a group of symmetries isomorphic to $S^1 \times S^3$ where S^n is the n-dimensional sphere.

Proof. As it was shown previously, the groups:

$$\begin{aligned} H_1 &= \{R_a \mid a\beta = \beta a, \overline{a}a = 1\} \\ H_2 &= \{L_a \mid \overline{a}a = 1\} \end{aligned}$$

Are symmetry groups of L. Hence, the group $\langle H_1, H_2 \rangle$ generated by H_1 and H_2 is a symmetry group of L. Note that by the associativity of Quaternion multiplication, it follows that every element of H_1 commutes with H_2. Hence, by the diamond theorem of group isomorphisms $\langle H_1, H_2 \rangle = H_1 \times H_2$ because $H_1 \cap H_2 = \{1\}$. Clearly, H_2 is isomorphic to the unit Quaternions as a group. This group is known to be isomorphic to S^3. Also,

$$H_1 = \{a \in \mathbb{H} \mid a = a_0 + b_0\frac{\vec{\beta}}{\|\vec{\beta}\|}, a_0^2 + b_0^2 = 1\}$$

By letting $\cos(\theta) = a_0, \sin(\theta) = b_0$, it follows that,

$$H_1 = \{a \in \mathbb{H} \mid a = \exp(\theta\frac{\vec{\beta}}{\|\vec{\beta}\|})\}$$

Clearly, under this representation of H_1, H_1 is isomorphic to S^1. The result follows. $\square$

Now, we proceed to calculate the constants of motion that correspond to these symmetries. We will do this by applying Proposition (A.3.2). As a first step, we calculate

$$\begin{aligned} \vec{p}(\vec{\eta},\dot{\vec{\eta}}) &= \frac{\partial L}{\partial \dot{\vec{\eta}}} \\ &= \dot{\vec{\eta}} + \rho_R(\beta)\vec{\eta} \end{aligned}$$

This calculation can be derived using Equation (2.6). Next, we calculate $\xi_g(\vec{\eta})$. Recall that in the notation of Proposition (A.3.2), ξ_g is the vector field generated by the diffeomorphism R_g which in our case is either R_a or L_a. It is a standard result in Lie theory that the exponential map maps the Lie algebra (tangent space of the Lie group at the identity) to the Lie group. Also, the exponential map can be used to identify the infinitesimal generators of R_a or L_a.

Proposition 2.4.5. Let $R_a(\eta) = \eta a$, $L_a(\eta) = a\eta$ where a is a unit Quaternion. Then, both R_a and L_a have the same infinitesimal generators, however, R_a corresponds to a left invariant vector field and L_a corresponds to a right invariant vector field. That is,

$$\begin{aligned} \xi_{R_a}(\vec{\eta}) &= \rho_L(\vec{a}_0)\vec{\eta} \\ \xi_{L_a}(\vec{\eta}) &= \rho_R(\vec{a}_0)\vec{\eta} \end{aligned}$$

Where $a = a_0 + \vec{a}$ and $\vec{a}_0 = \frac{a-a_0}{\sqrt{1-a_0^2}}$.

Proof. By assumption, a is a unit Quaternion. Hence, $a = a_0 + \vec{a}$, where $a_0^2 + \|\vec{a}\|^2 = 1$. Let θ be defined such that $\cos(\theta) = a_0$, $\sin(\theta) = \|\vec{a}\|$. Hence,

$$\begin{aligned} a &= a_0 + \|\vec{a}\|\frac{\vec{a}}{\|\vec{a}\|} \\ &= \cos(\theta) + \sin(\theta)\frac{a-a_0}{\sqrt{1-a_0^2}} \\ &= \exp(\theta\vec{a}_0) \end{aligned}$$

By Theorem (1.3.2) of Duistermaat and Kolk [JD99], $\vec{a}_0$ is the infinitesimal generator of the diffeomorphisms R_a and L_a. By Lemma (1.3.1) of Duistermaat and Kolk [JD99], R_a corresponds to the left invariant vector field generated by $\vec{a}_0$ and L_a corresponds to the right invariant vector field generated by $\vec{a}_0$. Hence, by the definition of left invariant and right invariant vector fields,

$$\begin{aligned}\xi_{R_a}(\vec{\eta}) &= \rho_L(\vec{a}_0)\vec{\eta}\\ \xi_{L_a}(\vec{\eta}) &= \rho_R(\vec{a}_0)\vec{\eta}\end{aligned}$$

$\square$

Now, we are ready to calculate the constants of motion induced by the quaternionic structure of L.

Proposition 2.4.6. Let $H_1 \times H_2$ be the symmetry group of the Quaternionic Foucault Pendulum (QFP) Lagrangian induced by the quaternionic structure of L as they are given in Proposition (2.4.4). Where,

$$\begin{aligned}H_1 &= \{R_a \mid a = \cos(\theta) + \sin(\theta)\vec{\beta}_0, \vec{\beta}_0 = \frac{\vec{\beta}}{\|\vec{\beta}\|}\}\\ H_2 &= \{L_a \mid a = \exp(\theta\vec{a}), \vec{a}\vec{a} = -1, \theta \in \mathbb{R}\}\end{aligned}$$

Let $\eta(t)$ be a solution to the Euler-Lagrange equations of the quaternionic Foucault pendulum. Then, the following are the constants of motion induced by $H_1 \times H_2$.

$$\begin{aligned}Re(\overline{\dot{\eta} + \eta\beta}\vec{\beta}_0\eta) &\qquad \text{Corresponding to } H_1\\ Im(\overline{\dot{\eta} + \eta\beta}\eta) &\qquad \text{Corresponding to } H_2\end{aligned}$$

Where $\vec{\beta}_0 = \frac{\vec{\beta}}{\|\vec{\beta}\|}$.

Proof. By Proposition (A.3.2), the constants of motion are:

$$\begin{aligned} S(R_a) &= \langle \vec{p}(\vec{\eta}, \dot{\vec{\eta}}), \xi_{R_a}(\vec{\eta}) \rangle \\ &= \langle \dot{\vec{\eta}} + \rho_R(\vec{\beta})\vec{\eta}, \rho_L(\vec{a}_0)\vec{\eta} \rangle \\ S(L_a) &= \langle \dot{\vec{\eta}} + \rho_R(\vec{\beta})\vec{\eta}, \rho_R(\vec{a}_0)\vec{\eta} \rangle \end{aligned}$$

We note that in quaternionic notation,

$$\begin{aligned} \xi_{R_a}(\eta) &= \vec{a}_0 \eta \\ \xi_{L_a}(\eta) &= \eta \vec{a}_0 \\ p(\vec{\eta}, \dot{\vec{\eta}}) &= \dot{\eta} + \eta\beta \end{aligned}$$

For the group H_1, the variable a can take on the $\vec{\beta}_0 = \frac{\vec{\beta}}{\|\vec{\beta}\|}$ value. Hence,

$$\begin{aligned} S(R_\beta) &= \langle \dot{\vec{\eta}} + \rho_R(\vec{\beta})\vec{\eta}, \rho_L(\vec{\beta}_0)\vec{\eta} \rangle \\ &= \langle \dot{\eta} + \eta\beta, \vec{\beta}_0\eta \rangle \\ &= Re(\overline{\dot{\eta} + \eta\beta}\vec{\beta}_0\eta) \end{aligned}$$

Where we have used the fact that $Re(\overline{\alpha}\gamma) = \langle \alpha, \gamma \rangle$ by using the definition of Quaternion multiplication.

For the group H_2, the variable a can take on an arbitrary unit Quaternion. Hence, $\vec{a}_0$ can take on an arbitrary purely imaginary unit Quaternion. In particular, the following quantities must be constants of motion.

$$\begin{aligned} \begin{pmatrix} S(L_{\vec{i}}) \\ S(L_{\vec{j}}) \\ S(L_{\vec{k}}) \end{pmatrix} &= \begin{pmatrix} Re(\overline{\dot{\eta} + \eta\beta}\eta\vec{i}) \\ Re(\overline{\dot{\eta} + \eta\beta}\eta\vec{j}) \\ Re(\overline{\dot{\eta} + \eta\beta}\eta\vec{k}) \end{pmatrix} \\ &= Im(\overline{\dot{\eta} + \eta\beta}\eta) \end{aligned}$$

Clearly, because $\vec{i}, \vec{j}, \vec{k}$ generate the Lie Algebra of H_2, any constant of motion corresponding to a $g \in H_2$ will be a linear combination of $S(L_{\vec{i}}), S(L_{\vec{j}}), S(L_{\vec{k}})$.

Similarly, because $\vec{\beta}_0$ is the generator of the Lie Algebra of H_1, any constant of motion corresponding to a $g \in H_1$ will be a constant multiple of $S(R_{\vec{\beta}_0})$.

□

We note that one can calculate these conserved quantities directly. These are given as,

$$\begin{aligned}
Im(\overline{\dot{\eta} + \eta\beta}\eta) &= \dot{\eta}_0\vec{\eta} - \eta_0\dot{\vec{\eta}} - \dot{\vec{\eta}} \times \vec{\eta} - \{(\eta_0^2 - \|\vec{\eta}\|^2)\vec{\beta} + 2\eta_0\vec{\beta} \times \vec{\eta} + 2\langle\vec{\beta}, \vec{\eta}\rangle\vec{\eta}\} \\
&= 2\omega_0\{\overline{C_-}\vec{\beta}_0 C_- - \overline{C_+}\vec{\beta}_0 C_+\} \\
Re(\overline{\dot{\eta} + \eta\beta}\vec{\beta}_0\eta) &= -\dot{\eta}_0\langle\vec{\beta}_0, \vec{\eta}\rangle + \langle\dot{\vec{\eta}}, \eta_0\vec{\beta}_0 + \vec{\beta}_0 \times \vec{\eta}\rangle + \|\vec{\beta}\|\{\|\vec{\eta}\|^2 + \eta_0^2\} \\
&= 2\omega_0\{\overline{C_+}C_+ - \overline{C_-}C_-\}
\end{aligned}$$

Where, C_+, C_- are quaternionic constants, and:

$$\begin{aligned}
\eta &= \eta_0 + \vec{\eta} \\
\vec{\beta}_0 &= \frac{\vec{\beta}}{\|\vec{\beta}\|} \\
\eta(t) &= C_+ \exp(\beta_+\vec{\beta}_0 t) + C_- \exp(\beta_-\vec{\beta}_0 t) \\
\beta_+ &= -\|\vec{\beta}\| + \omega_0 \\
\beta_- &= -\|\vec{\beta}\| - \omega_0
\end{aligned}$$

2.4.2 Interpretation of the Constants of Motion of the QFP

One can interpret the constants of motion of the QFP provided in the previous section by studying the constants of motion of the SPE. We note that for the following Lagrangian,

$$L_{SPE} = \frac{1}{2}\overline{\eta}\eta + \frac{1}{2}Re\left(\dot{\eta}\overline{\beta_1}\overline{\eta}\right) + \frac{1}{2}\omega_0 Re\left(\eta\overline{\beta_1}\overline{\eta}\vec{i}\right)$$

The E-L equations are those of the SPE, that is,

$$\dot{\eta} \;=\; \eta\beta - \vec{i}\omega_0\eta$$

Where ω_0 is a real number, β is a purely imaginary Quaternion, and β_1 is a purely imaginary Quaternion satisfying,

$$(1-\beta_1\vec{i}\omega_0)\left(\frac{-\beta_1}{\|\vec{\beta_1}\|}\right) \;=\; \beta$$

Note that for $\omega_0 = 0$, the Lagrangian of the SPE has the same group of symmetries as the Lagrangian of the QFP. Note that for L_{SPE}, $p = \eta\beta$. A direct calculation of the constants of motion for the groups H_1 and H_2 using L_{SPE} yields,

$$\begin{aligned}
S(R_{\beta_1}) &= \langle \eta\beta_1, \frac{\beta_1}{\|\beta_1\|}\eta \rangle \\
&= Re\left(\overline{\eta\beta_1}\frac{\beta_1}{\|\beta_1\|}\eta\right) \\
&= \frac{1}{\|\beta_1\|} Re\left(\bar{\beta}_1\overline{\eta}\beta_1\eta\right) \text{ corresponds to } H_1 \\
\begin{pmatrix} S(L_{\vec{i}}) \\ S(L_{\vec{j}}) \\ S(L_{\vec{k}}) \end{pmatrix} &= \begin{pmatrix} Re\left(\overline{\eta\beta_1}\eta\vec{i}\right) \\ Re\left(\overline{\eta\beta_1}\eta\vec{j}\right) \\ Re\left(\overline{\eta\beta_1}\eta\vec{k}\right) \end{pmatrix} \\
&= \overline{\eta}\eta \begin{pmatrix} Re\left(\overline{\beta_1}\vec{i}\right) \\ Re\left(\overline{\beta_1}\vec{j}\right) \\ Re\left(\overline{\beta_1}\vec{k}\right) \end{pmatrix} \\
&= \overline{\eta}\eta\beta_1 \text{ corresponds to } H_2
\end{aligned}$$

The above constants are the analogues of the constants of motion inhereted by the quaternionic structure of the QFP in the SPE when $\omega_0 = 0$. We can further calculate these constants explicitly by letting $\eta(t) = Ce^{\beta_1 t}$ giving,

$$\begin{aligned} S(R_{\beta_1}) &= Re\left(\beta_1\overline{C}\frac{\beta_1}{\|\beta_1\|}C\right) \text{ corresponds to } H_1 \\ \begin{pmatrix} S(L_{\vec{i}}) \\ S(L_{\vec{j}}) \\ S(L_{\vec{k}}) \end{pmatrix} &= \overline{C}C\beta_1 \text{ corresponds to } H_2 \end{aligned}$$

Hence for the H_2 group, the QFP constants have as analogues in the SPE constants of motion that are scalar multiples of the norm of the $\eta(t)$ state. Similarly, for the H_1 group, the QFP constant has analogue in the SPE the constant of motion given by $Re\left(\beta_1\overline{C}\frac{\beta_1}{\|\beta_1\|}C\right)$.

For $\omega_0 \neq 0$, we note that H_2 can consist only of unit quaternions that commute with $\vec{i}$. Hence, $S(L_{\vec{i}}) = \overline{\eta}\eta Re\left(\overline{\beta_1\vec{i}}\right)$ is the only constant of motin due to H_2. In which case, H_2 has as constant of motion a constant multiple of the norm of the $\eta(t)$ state. Thus a similar set of analogies that hold for the $\omega_0 = 0$ case also hold for the $\omega_0 \neq 0$ case.

2.4.3 A Canonical Reduction for the QFP

We will show how one can transform the solution space of the QFP into the solution space of a pair of independent Foucault pendulums at the same latitude using a right isoclinic rotation as long as the $\beta(t)$ parameter is time independent.

We note that given any solution $\eta(t)$ to the E-L equations of the QFP, we can consider the following transformations of functions,

$$\begin{aligned} R_\gamma(\eta(t)) &= \eta(t)\gamma \\ L_\gamma(\eta(t)) &= \gamma\eta(t) \end{aligned}$$

Where γ is a unit Quaternion.

Let $\mathcal{R}$ be the group of transformations generated by the R_γ and $\mathcal{L}$ be the group of transformations generated by the L_γ for arbitrary γ. As we know from the previous section, $L_\gamma(\eta(t))$ is always a solution of the E-L equations of the QFP as these transformations come from the symmetry group $\mathcal{L} = H_2$. We can view these transformations as gauge transformations because they leave the solution space of the E-L equations of the QFP invariant. Thus, the group $\mathcal{L}$ yields a 3 dimensional group of gauge transformations. We will see in Section (2.5.2) that these symmetries will correspond to hidden variables when mapping the solution space of the QFP to the solution space of the SPE.

On the other hand, $R_\gamma(\eta(t))$ is not always a solution of the E-L equations of the QFP unless γ commutes with β. The set of these γ is given by the group $H_1 \subset \mathcal{R}$. Thus, $\mathcal{R}$ has a subgroup of dimension 1 that leaves the solution space of the E-L equations of the QFP invariant. We may ask, what effect does the remaining transformations in $\mathcal{R}\backslash H_1$ have on the solution space of the E-L equations of the QFP? We will see in the next proposition that the remaining transformations in $\mathcal{R}\backslash H_1$ will yield a 2 dimensional orbit space that will make all QFP equivalent to the case when $\beta = \alpha\vec{k}$.

Proposition 2.4.7. Let $\eta(t)$ be the solution the E-L equations of the QFP with constant $\vec{\beta}(t) = \vec{\beta}$ parameter and natural frequency ω_0. Then, there exist a unit Quaternion γ independent of $\eta(t)$ but dependent of $\vec{\beta}$ such that $\eta(t)\gamma$ is the solution of the E-L equations of the QFP with constant $\vec{\beta}(t) = \alpha\vec{k}$ parameter for some $\alpha \in \mathbb{R}$ and natural frequency ω_0. In particular, the dynamics of any QFP with constant $\vec{\beta}$ parameter is equivalent to the dynamics of a QFP with constant $\vec{\beta}$ parameter a constant multiple of $\vec{k}$. We note that for $\vec{\beta}(t) = \alpha\vec{k}$, the corresponding magnetic field points in the x-direction not the z-direction, and the unit Quaternion γ corresponds to an orthogonal rotation of 3-space that maps $\vec{\beta}$ to $\alpha\vec{k}$ in 3-space.

Proof. By Proposition (A.1.3), there is a unit Quaternion γ and real number α such that:

$$\overline{\gamma}\vec{\beta}\gamma = \alpha\vec{k}$$

Recall the QFP Lagrangian,

$$L_{\vec{\beta}}(t, \eta(t), \dot{\eta}(t)) = \frac{1}{2}\dot{\overline{\eta}}\dot{\eta} - \frac{(\omega_0^2 - \|\vec{\beta}\|^2)}{2}\overline{\eta}\eta + Re\left(\dot{\eta}\vec{\beta}\overline{\eta}\right)$$

Note that by direct calculation, we can show that,

$$\begin{aligned} L_{\vec{\beta}}(t, \eta\gamma, \dot{\eta}\gamma) &= L_{\overline{\gamma}\vec{\beta}\gamma}(t, \eta, \dot{\eta}) \\ &= L_{\alpha\vec{k}}(t, \eta, \dot{\eta}) \end{aligned}$$

In particular, this shows that if $\eta'(t) = \eta(t)\gamma$ is a solution to the E-L equations of the QFP with $\vec{\beta}(t) = \vec{\beta}$ parameter. Then, $\eta(t) = \eta'(t)\overline{\gamma}$ is a solution to the E-L equations of the QFP with $\vec{\beta} = \alpha\vec{k}$ parameter. □

We note that the solution space of the E-L equations of the QFP with $\vec{\beta} = \alpha\vec{k}$ parameter is that of two independent Foucault pendulums with the same β parameter. This is because, in vector notation,

$$\begin{aligned} Re\left(\dot{\eta}\alpha\vec{k}\overline{\eta}\right) &= \alpha\dot{\vec{\eta}}^T\rho_R(\vec{k})\vec{\eta} \\ &= \dot{\vec{\eta}}^T \begin{pmatrix} 0 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix} \vec{\eta} \end{aligned}$$

Which decouples $L(t, \eta(t), \dot{\eta}(t))$ as,

$$L(t, \vec{\eta}(t), \dot{\vec{\eta}}(t)) \quad = \quad L_1(t, \vec{\psi_0}(t), \dot{\vec{\psi_0}}(t)) + L_1(t, \vec{\psi_1}(t), \dot{\vec{\psi_1}}(t))$$

Where $\vec{\psi_0}(t) = (\eta_0(t), \eta_1(t))^T$ and $\vec{\psi_1}(t) = (\eta_2(t), \eta_3(t))^T$, and L_1 is the Lagrangian of the modified complex Foucault pendulum of Equation (2.5) with $\beta = \Omega\cos(\phi)$ parameter equal to α. That is, L is the Lagrangian of two identical but independent Foucault pendulums that are at the same latitude as this guarantees the same ϕ and β.

2.5 Equivalence Conditions

We will find necessary and sufficient conditions on the solution set of the QFP, under the uniform field (constant β) assumption, that satisfy the condition $L = 0$. Then, we will show that the $L = 0$ condition is necessary and sufficient to establish a correspondence between the SPE and QFP. We start with a proposition that calculates the function $f(t) = \overline{\eta(t)}\eta(t)$ explicitly, where $\eta(t)$ is a solution to the E-L equations of the QFP.

Proposition 2.5.1. Let $f(t) = \overline{\eta(t)}\eta(t)$, where $\eta(t)$ is a solution to the Euler-Lagrange equations of the quaternionic Foucault pendulum Lagrangian L. Assume further, that $\beta(t) = \beta$ is a constant of time. Then,

1 The function $f'(t)$ is equal to,

$$f'(t) \quad = \quad \frac{1}{\|\beta\|^2} Re(\overline{\eta}\ddot{\eta}\beta)$$

2 The function $f''(t)$ is equal to,

$$f''(t) \quad = \quad 4\{\frac{1}{2}\dot{\overline{\eta}}\dot{\eta} - \frac{1}{2}(\omega_0^2 - \|\beta\|^2)\overline{\eta}\eta + Re(\beta\dot{\overline{\eta}}\eta)\}$$

3 The function $f'''(t) = -4\omega_0^2 f'(t)$.

4 The function,

$$f(t) \quad = \quad f(0) + \alpha \sin(2\omega_0 t) + \epsilon(\cos(2\omega_0 t) - 1)$$

For some real constants α, ϵ.

Proof. Clearly, $f'(t) = \dot{\overline{\eta(t)}}\eta(t) + \overline{\eta(t)}\dot{\eta}(t)$. Also, from the E-L equation,

$$\ddot{\eta}(t) + 2\dot{\eta}(t)\beta + (\omega_0^2 - \|\vec{\beta}\|^2)\eta(t) \quad = \quad 0$$

One can solve for the quantities,

$$\begin{aligned} \dot{\eta}(t) &= \frac{1}{2\|\vec{\beta}\|^2}\{\ddot{\eta}(t)\beta + (\omega_0^2 - \|\vec{\beta}\|^2)\eta(t)\beta\} \\ \dot{\overline{\eta}}(t) &= \frac{1}{2\|\vec{\beta}\|^2}\{-\beta\ddot{\overline{\eta}}(t) - (\omega_0^2 - \|\vec{\beta}\|^2)\beta\overline{\eta}(t)\} \end{aligned}$$

From these equations, it follows that,

$$\begin{aligned} \dot{\overline{\eta}}\eta + \overline{\eta}\dot{\eta} &= \frac{1}{2\|\vec{\beta}\|^2}\{-\beta\ddot{\overline{\eta}}\eta + \overline{\eta}\ddot{\eta}\beta\} \\ &= \frac{1}{2\|\vec{\beta}\|^2}Re(\overline{\eta}\ddot{\eta}\beta) \end{aligned}$$

Hence, part one follows.

For part two, note that a direct calculation yields,

$$f''(t) \quad = \quad 2\dot{\overline{\eta}}\dot{\eta} + \{\ddot{\overline{\eta}}\eta + \overline{\eta}\ddot{\eta}\}$$

Using the E-L equations, one can deduce that,

$$\ddot{\overline{\eta}}\eta + \overline{\eta}\ddot{\eta} \quad = \quad 4Re(\beta\dot{\overline{\eta}}\eta) - 2(\omega_0^2 - \|\vec{\beta}\|^2)\overline{\eta}\eta$$

Thus,

$$f'' = 4\{\frac{1}{2}\dot{\bar{\eta}}\dot{\eta} - \frac{1}{2}(\omega_0^2 - \|\vec{\beta}\|^2)\bar{\eta}\eta + Re(\beta\dot{\bar{\eta}}\eta)\}$$

Hence, part two follows. Now, we show part three. Note that a direct calculation yields,

$$\begin{aligned}
\frac{d\{\dot{\bar{\eta}}\dot{\eta}\}}{dt} &= \ddot{\bar{\eta}}\dot{\eta} + \dot{\bar{\eta}}\ddot{\eta} \\
&= (2\beta\dot{\bar{\eta}} - (\omega_0^2 - \|\vec{\beta}\|^2)\bar{\eta})\dot{\eta} + \dot{\bar{\eta}}(-2\dot{\eta}\beta - (\omega_0^2 - \|\vec{\beta}\|^2)\eta) \\
&= -(\omega_0^2 - \|\vec{\beta}\|^2)\{\bar{\eta}\dot{\eta} + \dot{\bar{\eta}}\eta\} \\
&= -(\omega_0^2 - \|\vec{\beta}\|^2)f'(t)
\end{aligned}$$

Also, note that,

$$\begin{aligned}
\frac{d\{Re\left(\beta\dot{\bar{\eta}}\eta\right)\}}{dt} &= Re\left(\beta(\ddot{\bar{\eta}}\eta + \beta\dot{\bar{\eta}}\dot{\eta})\right) \\
&= Re(\beta\ddot{\bar{\eta}}\eta) + Re(\beta\dot{\bar{\eta}}\dot{\eta}) \\
&= -Re(\bar{\eta}\ddot{\eta}\beta) + (\dot{\bar{\eta}}\dot{\eta})Re(\beta) \\
&= -\|\vec{\beta}\|^2 f'(t) + (\dot{\bar{\eta}}\dot{\eta}) * 0 \\
&= -\|\vec{\beta}\|^2 f'(t) \\
\frac{d\{\bar{\eta}\eta\}}{dt} &= f'(t)
\end{aligned}$$

Hence, it follows by using the formula for $f''(t)$ that,

$$f'''(t) = -4\omega_0^2 f'(t)$$

Thus part three follows. The previous equation shows that for $y(t) = f'(t)$, the function $y(t)$ satisfies the ODE $\ddot{y}(t) = -4\omega_0^2 y(t)$. Clearly, this ODE has solution,

$$\begin{aligned}
y(t) &= f'(t) \\
&= a\cos(2\omega_0 t) + b\sin(2\omega_0 t)
\end{aligned}$$

For some real constants a, b. Clearly, the integration of $y(t)$ yields the formula for $f(t)$. Hence part four follows.

□

The next proposition gives an explicit calculation of the constants α, ϵ of part 4 of Proposition (2.5.1).

Proposition 2.5.2. Let $\eta(t) = C_+ \exp(\beta_+ \vec{\beta}_0 t) + C_- \exp(\beta_- \vec{\beta}_0 t)$, where $\beta_+ = -\|\vec{\beta}\| + \omega_0, \beta_- = -\|\vec{\beta}\| - \omega_0$, be a solution to the Euler-Lagrange equations of the quaternionic Foucault pendulum. Then,

1 The function $\eta(t)\overline{\eta(t)}$ equals,

$$\eta(t)\overline{\eta(t)} \quad = \quad C_+\overline{C_+} + C_-\overline{C_-} + 2Re(C_+\overline{C_-})\cos(2\omega_0 t) - 2Re(C_+\vec{\beta}_0\overline{C_-})\sin(2\omega_0 t).$$

2 The function $\eta(t)\overline{\eta(t)}$ is a constant of t, if and only if

$$\begin{aligned} Re(C_+\overline{C_-}) &= 0 \\ Re(C_+\vec{\beta}_0\overline{C_-}) &= 0 \end{aligned}$$

3 If $\eta(t)\overline{\eta(t)}$ is a constant, then $\eta(t)\overline{\eta(t)} = C_+\overline{C_+} + C_-\overline{C_-}$.

Proof. Part one is a direct calculation that makes use of the formula for $\eta(t)$ and of,

$$\overline{\eta}(t) \quad = \quad \exp(-\beta_+\vec{\beta}_0 t)\overline{C_+} + \exp(-\beta_-\vec{\beta}_0 t)\overline{C_-}$$

Clearly,

$$\eta(t)\overline{\eta}(t) \quad = \quad C_+\overline{C_+} + C_-\overline{C_-} + C_+\exp((\beta_- - \beta_+)\vec{\beta}_0 t)\overline{C_-} + \overline{C_+}\exp(-(\beta_- - \beta_+)\vec{\beta}_0 t)C_-$$

Note that $\beta_- - \beta_+ = -2\omega_0$. Hence,

$$\eta(t)\overline{\eta}(t) \quad = \quad C_+\overline{C_+} + C_-\overline{C_-} + 2Re(C_+\exp(-2\omega_0\vec{\beta}_0 t)\overline{C_-})$$

A direct calculation of $C_+ \exp(-2\omega_0 \vec{\beta}_0 t)\overline{C_-}$ using the formula for Quaternion multiplication yields,

$$Re(C_+ \exp(-2\omega_0 \vec{\beta}_0 t)\overline{C_-}) \;=\; Re(C_+\overline{C_-})\cos(2\omega_0 t) - Re(C_+\vec{\beta}_0\overline{C_-})\sin(2\omega_0 t).$$

Thus, part one follows. Part two is a clear consequence of part 1 by using the linear independence of the set of functions $\{1, \cos(2\omega_0 t), \sin(2\omega_0 t)\}$ which imples the unique representation of the zero function as a linear combination of these functions,

$$0 * 1 + 0 * \cos(2\omega_0 t) + 0 * \sin(2\omega_0 t) \;=\; 0$$

By letting $\eta(t)\overline{\eta(t)} = E_0$ be a constant, we deduce that:

$$(C_+\overline{C_+} + C_-\overline{C_-} - E_0) * 1 + 2Re(C_+\overline{C_-})\cos(2\omega_0 t) - 2Re(C_+\vec{\beta}_0\overline{C_-})\sin(2\omega_0 t) \;=\; 0$$

Hence, $Re(C_+\overline{C_-}) = 0$ and $Re(C_+\vec{\beta}_0\overline{C_-}) = 0$ and part 2 follows. Part 3 is a clear consequence of parts 2 and 1. □

The next proposition characterizes the solutions $\eta(t)$ of the E-L equations of the QFP that satisfy the $L(t, \eta(t), \dot{\eta}(t)) = 0$ condition. This condition will be shown later to be necessary and sufficient to establish the correspondence of Proposition (2.4.1) between the SPE and QFP.

Proposition 2.5.3. Let $\eta(t)$ be a solution to the E-L equations of the QFP. Then,

1 The function $\eta(t)$ satisfies $L(t, \eta(t), \dot{\eta}(t)) = 0$ if and only if $\overline{\eta}(t)\eta(t)$ is a constant.

2 The following sets are the same,

$$\{\eta(t) = C_+ \exp(\beta_+\vec{\beta}_0 t) + C_- \exp(\beta_-\vec{\beta}_0 t) \;\;|\;\; L(t, \eta(t), \dot{\eta}(t)) = 0\}$$
$$\{\eta(t) = C_+ \exp(\beta_+\vec{\beta}_0 t) + C_- \exp(\beta_-\vec{\beta}_0 t) \;\;|\;\; Re(C_+\overline{C_-}) = 0, Re(C_+\vec{\beta}_0\overline{C_-}) = 0\}$$
$$\{\eta(t) = C_+ \exp(\beta_+\vec{\beta}_0 t) + C_- \exp(\beta_-\vec{\beta}_0 t) \;\;|\;\; \langle C_+, C_-\rangle = 0, \langle C_+, \rho_R(\vec{\beta}_0)C_-\rangle = 0\}$$

Proof. Recall that, by Proposition (2.5.1) part 2,

$$\frac{d^2\{\overline{\eta(t)}\eta(t)\}}{dt^2} \;=\; 4\{L - Re(\dot{\eta}\bar{\beta}\bar{\eta}) + Re(\beta\dot{\bar{\eta}}\eta)\}$$

Hence, if $\overline{\eta(t)}\eta(t)$ is a constant, then we must have,

$$L \;=\; Re(\dot{\eta}\bar{\beta}\bar{\eta}) - Re(\beta\dot{\bar{\eta}}\eta)$$

We will show that the right hand side of the above equation is zero. This will establish that $\overline{\eta(t)}\eta(t)$ is a constant implies $L(t, \eta(t), \dot{\eta(t)}) = 0$. Recall the E-L equations give,

$$\begin{aligned} \ddot{\overline{\eta}} + 2\overline{\beta}\dot{\overline{\eta}} + (\omega_0^2 - \|\vec{\beta}\|^2)\overline{\eta} &= 0 \\ \ddot{\eta} + 2\dot{\eta}\beta + (\omega_0^2 - \|\vec{\beta}\|^2)\eta &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} \ddot{\overline{\eta}}\eta - 2\beta\dot{\overline{\eta}}\eta + (\omega_0^2 - \|\vec{\beta}\|^2)\overline{\eta}\eta &= 0 \\ \ddot{\eta}\overline{\eta} - 2\dot{\eta}\overline{\beta}\overline{\eta} + (\omega_0^2 - \|\vec{\beta}\|^2)\eta\overline{\eta} &= 0 \end{aligned}$$

Hence, by taking the real part of both of the previous equations, we get:

$$\begin{aligned} Re(\ddot{\overline{\eta}}\eta) - 2Re(\beta\dot{\overline{\eta}}\eta) + (\omega_0^2 - \|\vec{\beta}\|^2)\overline{\eta}\eta &= 0 \\ Re(\ddot{\eta}\overline{\eta}) - 2Re(\dot{\eta}\overline{\beta}\overline{\eta}) + (\omega_0^2 - \|\vec{\beta}\|^2)\eta\overline{\eta} &= 0 \end{aligned}$$

Because $\overline{\eta}\eta = \eta\overline{\eta} = \langle\eta, \eta\rangle$, we can deduct the previous equations from each other to yield,

$$Re(\ddot{\overline{\eta}}\eta) - Re(\ddot{\eta}\overline{\eta}) - 2Re(\dot{\eta}\overline{\beta}\overline{\eta}) + 2Re(\beta\dot{\overline{\eta}}\eta) \;=\; 0$$

Hence,

$$Re(\dot{\eta}\overline{\beta}\overline{\eta}) - Re(\beta\dot{\overline{\eta}}\eta) \;=\; \frac{1}{2}\{Re(\ddot{\overline{\eta}}\eta) - Re(\ddot{\eta}\overline{\eta})\}$$

Note that for any two Quaternions α, γ, $Re(\alpha\overline{\gamma}) = Re(\overline{\alpha}\gamma) = \langle\alpha, \gamma\rangle$. Thus,

$$Re(\dot{\eta}\overline{\beta}\overline{\eta}) - Re(\beta\dot{\overline{\eta}}\eta) \;=\; 0$$

Establishing that $L = 0$.

Now, we proceed to show that if $\eta(t)$ satisfies $L(t, \eta(t), \dot{\eta}(t)) = 0$, then $\overline{\eta(t)}\eta(t)$ is a constant. We note that the previous calculation showed that for a general $\eta(t)$ that satisfies the E-L equations,

$$Re(\dot{\eta}\overline{\beta}\overline{\eta}) - Re(\beta\dot{\overline{\eta}}\eta) \;=\; 0$$

Hence, for any such $\eta(t)$,

$$\frac{d^2\{\overline{\eta}(t)\eta(t)\}}{dt^2} \;=\; 4L(t, \eta(t), \dot{\eta}(t))$$

In particular, this shows that the function $f(t) = \overline{\eta}(t)\eta(t)$ has zero second derivative. Hence, this shows that $f(t) = mt + b$ for some constants m, b. By Proposition (2.5.1), $f(t) = f(0) + \alpha\sin(2\omega_0 t) + \epsilon(\cos(2\omega_0 t) - 1)$. In order for $f(t)$ to satisfy both functional representations, $f(t)$ must be a constant function. This shows part 1. Part 2 is a clear consequence of Proposition (2.5.2). $\square$

2.5.1 Equivalent Equivalence Conditions

We note that one can identify $\mathbb{C} \oplus \mathbb{C}$ with $\mathbb{H}$ via the map,

$$(a_0 + ia_1, b_0 + ib_1) \to a_0 + a_1\vec{i} + (b_0 + b_1\vec{i})\vec{j}.$$

Using this map, one can solve the SPE and the E-L equations of the QFP in Spinor notation. That is, by viewing the solutions of these ODEs $\eta(t)$ as functions on $\mathbb{C} \oplus \mathbb{C}$ instead of $\mathbb{H}$ one can provide for solutions as functions on $\mathbb{C} \oplus \mathbb{C}$. One can then solve for an analogous result to Proposition (2.5.3) and Proposition (2.3.1).

Recall the Spinor form of the SPE as it was given by Equation (2.2),

$$\frac{\partial \chi}{\partial t} = \left(\frac{\gamma B}{2} \begin{pmatrix} i\cos(\phi) & -\overline{i\sin(\phi)\exp(i\theta)} \\ i\sin(\phi)\exp(i\theta) & \overline{i\cos(\phi)} \end{pmatrix} - i\omega_0 I \right) \chi$$

By viewing $\chi(t)$ as a function on $\mathbb{C} \oplus \mathbb{C}$, one can solve this ODE and find the general solution in Spinor notation as,

$$\begin{aligned} \chi(t) &= \begin{pmatrix} \chi_0(t) \\ \chi_1(t) \end{pmatrix} \\ &= f e^{-i\omega_0 t} e^{i\beta t} \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) e^{i\phi} \end{pmatrix} + g e^{-i\omega_0 t} e^{-i\beta t} \begin{pmatrix} \sin(\frac{\theta}{2}) \\ -\cos(\frac{\theta}{2}) e^{i\phi} \end{pmatrix} \end{aligned}$$

Where f, g are complex constants and $\beta = \frac{\gamma B}{2}$. We note however, that by mapping $\mathbb{C} \oplus \mathbb{C} \rightarrow \mathbb{H}$ using the map $(a, b) \rightarrow a + b\vec{j}$, one can map $\chi(t) \rightarrow \chi_0(t) + \chi_1(t)\vec{j}$ and transform $\chi(t)$ from Spinor notation to quaternionic notation as,

$$\chi(t) = e^{-\vec{i}\omega_0 t} e^{\vec{i}\beta t} f \beta_1 + e^{-\vec{i}\omega_0 t} e^{-\vec{i}\beta t} g \beta_2 \tag{2.7}$$

Where,

$$\begin{aligned} \beta_1 &= \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2}) e^{\vec{i}\phi} \vec{j} \\ \beta_2 &= \sin(\frac{\theta}{2}) - \cos(\frac{\theta}{2}) e^{\vec{i}\phi} \vec{j} \end{aligned}$$

One can also re-write the SPE ODE from Spinor notation to quaternionic notation as

$$\dot{\chi}(t) = \frac{\gamma B}{2} \chi(t) \beta_0 - \vec{i}\omega_0 \chi(t)$$

Where,

$$\vec{\beta_0} \;=\; \vec{i}(\cos(\theta) + \sin(\theta)e^{\vec{i}\phi}\vec{j})$$

By Proposition (A.2.1), this ODE has solution,

$$\chi(t) \;=\; e^{-\vec{i}\omega_0 t}Ce^{\frac{\gamma B}{2}\vec{\beta_0}t}$$

Where C is a quaternionic constant. The next proposition will provide a map between solutions to the SPE given by Proposition (A.2.1) and solutions given by the Spinor notation of Equation (2.7) with the $e^{-\vec{i}\omega_0 t}$ term omitted.

Proposition 2.5.4. Let η be the Quaternion valued function $\eta(t) = Ce^{\alpha\vec{\beta_0}t}$ where C is a quaternionic constant, $\alpha \in \mathbb{R}$, and $\vec{\beta_0} = \vec{i}(\cos(\theta) + \sin(\theta)e^{\vec{i}\phi}\vec{j})$. Assume further, that $\eta(t)$ can be written as,

$$\eta(t) \;=\; e^{\vec{i}\alpha t}f\beta_1 + e^{-\vec{i}\alpha t}g\beta_2$$

Where,

$$\begin{aligned}\beta_1 &= \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})e^{\vec{i}\phi}\vec{j}\\ \beta_2 &= \sin(\frac{\theta}{2}) - \cos(\frac{\theta}{2})e^{\vec{i}\phi}\vec{j}\end{aligned}$$

And f, g are complex constants. Then, the following must hold,

$$\begin{aligned}g + f\vec{j} &= e^{-\vec{i}\frac{\phi}{2}}Ce^{\vec{i}\frac{\phi}{2}}(\sin(\frac{\theta}{2}) + \cos(\frac{\theta}{2})\vec{j})\\ C &= e^{\vec{i}\frac{\phi}{2}}(g + f\vec{j})(\sin(\frac{\theta}{2}) - \cos(\frac{\theta}{2})\vec{j})e^{-\vec{i}\frac{\phi}{2}}\end{aligned}$$

Proof. A direct calculation yields,

$$\begin{aligned}\frac{1 - \vec{i}\vec{\beta_0}}{2} &= \cos(\frac{\theta}{2})\beta_1\\ \frac{1 + \vec{i}\vec{\beta_0}}{2} &= \sin(\frac{\theta}{2})\beta_2\end{aligned}$$

Also, one can re-write,

$$\begin{aligned} e^{\alpha\vec{\beta}_0 t} &= \cos(\alpha t) + \sin(\alpha t)\vec{\beta}_0 \\ &= \frac{e^{\vec{i}\alpha t} + e^{-\vec{i}\alpha t}}{2} + \frac{e^{\vec{i}\alpha t} - e^{-\vec{i}\alpha t}}{2\vec{i}}\vec{\beta}_0 \\ &= e^{\vec{i}\alpha t}\frac{1-\vec{i}\vec{\beta}_0}{2} + e^{-\vec{i}\alpha t}\frac{1+\vec{i}\vec{\beta}_0}{2} \\ &= e^{\vec{i}\alpha t}\cos(\frac{\theta}{2})\beta_1 + e^{-\vec{i}\alpha t}\sin(\frac{\theta}{2})\beta_2 \end{aligned}$$

Let $C = h + m\vec{j}$ where h, m are complex constants. Then, a direct calculation yields,

$$\begin{aligned} Ce^{\alpha\vec{\beta}_0 t} &= (h + m\vec{j})(e^{\vec{i}\alpha t}\cos(\frac{\theta}{2})\beta_1 + e^{-\vec{i}\alpha t}\sin(\frac{\theta}{2})\beta_2) \\ &= e^{\vec{i}\alpha t}\{h\cos(\frac{\theta}{2}) + m\sin(\frac{\theta}{2})e^{-\vec{i}\phi}\}\beta_1 + e^{-\vec{i}\alpha t}\{h\sin(\frac{\theta}{2}) - m\cos(\frac{\theta}{2})e^{-\vec{i}\phi}\}\beta_2 \end{aligned}$$

Hence,

$$\begin{aligned} f &= h\cos(\frac{\theta}{2}) + m\sin(\frac{\theta}{2})e^{-\vec{i}\phi} \\ g &= h\sin(\frac{\theta}{2}) - m\cos(\frac{\theta}{2})e^{-\vec{i}\phi} \end{aligned}$$

Or, in matrix notation,

$$\begin{pmatrix} g \\ f \end{pmatrix} = \begin{pmatrix} \sin(\frac{\theta}{2}) & -\cos(\frac{\theta}{2})e^{-i\phi} \\ \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2})e^{-i\phi} \end{pmatrix} \begin{pmatrix} h \\ m \end{pmatrix}$$

Hence,

$$\begin{aligned} \begin{pmatrix} ge^{i\frac{\phi}{2}} \\ fe^{i\frac{\phi}{2}} \end{pmatrix} &= \begin{pmatrix} \sin(\frac{\theta}{2})e^{i\frac{\phi}{2}} & -\cos(\frac{\theta}{2})e^{-i\frac{\phi}{2}} \\ \cos(\frac{\theta}{2})e^{i\frac{\phi}{2}} & \sin(\frac{\theta}{2})e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} h \\ m \end{pmatrix} \\ &= \rho_R(\sin(\frac{\theta}{2})e^{i\frac{\phi}{2}} + \cos(\frac{\theta}{2})e^{i\frac{\phi}{2}}\vec{j}) \begin{pmatrix} h \\ m \end{pmatrix} \end{aligned}$$

Thus, in quaternionic notation via the map $(a,b)^T \to a + b\vec{j}$.

$$e^{\vec{i}\frac{\phi}{2}}(g + f\vec{j}) \quad = \quad (h + m\vec{j})e^{\vec{i}\frac{\phi}{2}}(sin(\frac{\theta}{2}) + \cos(\frac{\theta}{2})\vec{j})$$

Clearly, from this equation, it follows that,

$$(g + f\vec{j}) \quad = \quad e^{-\vec{i}\frac{\phi}{2}}Ce^{\vec{i}\frac{\phi}{2}}(sin(\frac{\theta}{2}) + \cos(\frac{\theta}{2})\vec{j})$$

By using the fact that $(\sin(\frac{\theta}{2}) + \cos(\frac{\theta}{2})\vec{j})(\sin(\frac{\theta}{2}) - \cos(\frac{\theta}{2})\vec{j}) = 1$, we can solve for C in terms of $g + f\vec{j}$. $\square$

Corollary 2.5.5. *Let $\chi(t)$ be a solution to the SPE with constant magnetic field $\vec{B}$. In Spinor form, χ is given as,*

$$\chi(t) \quad = \quad e^{-i\omega_0 t}e^{i\beta t}f\beta_1 + e^{-i\omega_0 t}e^{-i\beta t}g\beta_2$$

Where $\beta = \frac{\gamma\|\vec{B}\|}{2}$ and γ is the Gyromagnetic ratio which can be approximated as $\gamma = \frac{q}{2m}$ where q is the charge of the particle, and m is the mass of the particle. We note that γ can be negative as q can be negative. Then, $\chi(t)$ can be written as $\chi(t) = e^{-\vec{i}\omega_0 t}Ce^{\beta\vec{\beta}_0 t}$ where C is given by Proposition (2.5.4)

Now, we can provide for a solution to the E-L equations in vector notation.

Proposition 2.5.6. Let $\eta(t) = (\eta_0(t), \eta_1(t), \eta_2(t), \eta_3(t))^T$ be a solution to the E-L equations in quaternionic form. Then, there are complex constants a, b, c, d such that,

$$\begin{aligned} \eta(t) \quad = \quad & Re\left(e^{-i\omega_0 t}\left\{a\begin{bmatrix} \cos(\frac{\theta}{2}) \mid y_-\rangle \\ \sin(\frac{\theta}{2})e^{i\phi} \mid y_-\rangle \end{bmatrix} e^{-i\|\vec{\beta}\|t} + b\begin{bmatrix} \sin(\frac{\theta}{2}) \mid y_-\rangle \\ -\cos(\frac{\theta}{2})e^{i\phi} \mid y_-\rangle \end{bmatrix} e^{i\|\vec{\beta}\|t}\right\}\right) + \\ & Re\left(e^{-i\omega_0 t}\left\{c\begin{bmatrix} -\sin(\frac{\theta}{2})e^{i\phi} \mid y_+\rangle \\ \cos(\frac{\theta}{2}) \mid y_+\rangle \end{bmatrix} e^{-i\|\vec{\beta}\|t} + d\begin{bmatrix} \cos(\frac{\theta}{2})e^{i\phi} \mid y_+\rangle \\ \sin(\frac{\theta}{2}) \mid y_+\rangle \end{bmatrix} e^{i\|\vec{\beta}\|t}\right\}\right) \end{aligned}$$

Where,

$$\begin{aligned} \mid y_+\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \\ \mid y_-\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \end{aligned}$$

Proof. Recall the solution to the E-L equations in quaternionic form.

$$\eta(t) = C_+ e^{(-\|\vec{\beta}\|+\omega_0)\vec{\beta}_0 t} + C_- e^{(-\|\vec{\beta}\|-\omega_0)\vec{\beta}_0 t}$$

Where C_+, C_- are quaternionic constants. By Proposition (2.5.4), we express,

$$\begin{aligned} C_+ e^{(-\|\vec{\beta}\|+\omega_0)\vec{\beta}_0 t} &= e^{(-\|\vec{\beta}\|+\omega_0)it} f_+\beta_1 + e^{-(-\|\vec{\beta}\|+\omega_0)it} g_+\beta_2 \\ C_- e^{(-\|\vec{\beta}\|-\omega_0)\vec{\beta}_0 t} &= e^{(-\|\vec{\beta}\|-\omega_0)it} f_-\beta_1 + e^{-(-\|\vec{\beta}\|-\omega_0)it} g_-\beta_2 \end{aligned}$$

For some complex constants f_+, g_+, f_-, g_-. By direct calculation these expressions yield,

$$\eta(t) = e^{-i\|\vec{\beta}\|t}\left\{e^{i\omega_0 t} f_+ + e^{-i\omega_0 t} f_-\right\}\beta_1 + e^{i\|\vec{\beta}\|t}\left\{e^{-i\omega_0 t} g_+ + e^{i\omega_0 t} g_-\right\}\beta_2$$

Now, define the complex constants a, b, c, d such that,[1]

$$\begin{aligned} f_+ &= \frac{\overline{d}e^{-i\phi}}{\sqrt{2}} \\ f_- &= \frac{a}{\sqrt{2}} \\ g_+ &= \frac{b}{\sqrt{2}} \\ g_- &= \frac{-\overline{c}e^{-i\phi}}{\sqrt{2}} \end{aligned}$$

Then, under this choice of constants,

$$\eta(t) = e^{-i\|\vec{\beta}\|t}\left\{\frac{ae^{-i\omega_0 t} + \overline{d}e^{i\omega_0 t}e^{-i\phi}}{\sqrt{2}}\right\}\beta_1 + e^{i\|\vec{\beta}\|t}\left\{\frac{be^{-i\omega_0 t} - \overline{c}e^{i\omega_0 t}e^{-i\phi}}{\sqrt{2}}\right\}\beta_2$$

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[1] In the following equations, we will be identifying i with its quaternionic version $\vec{i}$.

By making use of the relations,

$$\begin{aligned}
\overline{\beta_1}\beta_1 &= 1 \\
\overline{\beta_2}\beta_2 &= 1 \\
\vec{j}\beta_1 &= -e^{-i\phi}\beta_2 \\
\vec{j}\beta_2 &= e^{-i\phi}\beta_1 \\
\overline{\beta_1}\beta_2 &= \beta_2\overline{\beta_1} \\
&= -\vec{j}e^{-i\phi} \\
\beta_1\overline{\beta_2} &= \overline{\beta_2}\beta_1 \\
&= \vec{j}e^{-i\phi}
\end{aligned}$$

We can deduce that,

$$\eta(t) \quad = \quad \frac{e^{-i\omega_0 t}ae^{-i\|\vec{\beta}\|t}}{\sqrt{2}}\beta_1 + \frac{e^{-i\omega_0 t}be^{i\|\vec{\beta}\|t}}{\sqrt{2}}\beta_2 + \frac{e^{i\omega_0 t}\overline{c}e^{i\|\vec{\beta}\|t}}{\sqrt{2}}\vec{j}\beta_1 + \frac{e^{i\omega_0 t}\overline{d}e^{-i\|\vec{\beta}\|t}}{\sqrt{2}}\vec{j}\beta_2$$

We note that the above equation is really a Spinor solution. That is, by viewing $\beta_i = \beta_{i,0} + \beta_{i,1}\vec{j} \rightarrow (\beta_{i,0}, \beta_{i,1}) \in \mathbb{C} \oplus \mathbb{C}$ where $\beta_{i,0}, \beta_{i,1}$ are complex numbers, we can view $\beta_i \in \mathbb{C} \oplus \mathbb{C}$ thus giving η as a Spinor. However, we can provide for a different representation for $\eta(t)$ that makes use of the fact that each summand $\beta_i, \vec{j}\beta_i$ is a real 4 vector multiplied by complex factor component wise. This representation will allow for the introduction of hidden variables in the representation of $\eta(t)$ that will give physical significance to the correspondence between different solutions of the QFP that map to the same solution of the SPE. We will do this by making use of the following identity. Let $\vec{x} = (x_0, x_1, x_2, x_3)^T$ be a 4 vector. Clearly, this 4 vector can also be viewed as a Quaternion x under the natural representation $x = x_0 + x_1\vec{i} + x_2\vec{j} + x_3\vec{i}\vec{j}$. Note that, as a 4-vector calculation, for any complex number $c_0 + c_1 i$,

$$\begin{aligned}
Re\left((c_0+c_1 i)\left(\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \pm i \begin{bmatrix} -x_1 \\ x_0 \\ -x_3 \\ x_2 \end{bmatrix}\right)\right) &= c_0 \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \mp c_1 \begin{bmatrix} -x_1 \\ x_0 \\ -x_3 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} \begin{pmatrix} x_0 & -x_1 \\ x_1 & x_0 \end{pmatrix} \begin{pmatrix} c_0 \\ \mp c_1 \end{pmatrix} \\ \begin{pmatrix} x_2 & -x_3 \\ x_3 & x_2 \end{pmatrix} \begin{pmatrix} c_0 \\ \mp c_1 \end{pmatrix} \end{bmatrix} \\
&= (c_0 \mp c_1 i) \begin{bmatrix} x_0 + i x_1 \\ x_2 + i x_3 \end{bmatrix} \\
&= (c_0 \mp c_1 i)(x_0 + x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{ij})
\end{aligned}$$

We have identified in the above last two equations, a real 4-vector with its Quaternion counterpart. We note that,

$$\begin{bmatrix} -x_1 \\ x_0 \\ -x_3 \\ x_2 \end{bmatrix} = \rho_L(i) \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Hence, we have shown the following identities:

$$\begin{aligned}
Re(c(\vec{x} - \overrightarrow{(ix)})) &= cx \\
Re(c(\vec{x} + \overrightarrow{(ix)})) &= \bar{c}x
\end{aligned}$$

Where $\vec{x}, \overrightarrow{(ix)}$ are real 4 vectors and c is complex number. The left hand side of the above equations involve component wise multiplication of the complex number c with the components of the vectors $\vec{x}, \overrightarrow{(ix)}$. The right hand side of the above equations involve Quaternion multiplication of the complex numbers $c, \bar{c}$ with the Quaternion x.

Consider,

$$\begin{aligned}
\begin{bmatrix} \cos(\frac{\theta}{2}) \mid y_-\rangle \\ \sin(\frac{\theta}{2})e^{i\phi} \mid y_-\rangle \end{bmatrix} &= \frac{1}{\sqrt{2}} \left\{ \begin{bmatrix} \cos(\frac{\theta}{2}) \\ 0 \\ \sin(\frac{\theta}{2})\cos(\phi) \\ \sin(\frac{\theta}{2})\sin(\phi) \end{bmatrix} + i \begin{bmatrix} 0 \\ -\cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})\sin(\phi) \\ -\sin(\frac{\theta}{2})\cos(\phi) \end{bmatrix} \right\} \\
&= \frac{1}{\sqrt{2}} \left\{ \vec{\beta_1} - i\overrightarrow{(i\beta_1)} \right\}
\end{aligned}$$

Where, we have identified β_1 with its corresponding 4 vector $\vec{\beta_1}$ and $i\beta_1$ with its corresponding 4 vector $\overrightarrow{(i\beta_1)}$. Hence,

$$\begin{aligned}
Re\left(ae^{-i\omega_0 t}e^{-i\|\vec{\beta}\|t} \begin{bmatrix} \cos(\frac{\theta}{2}) \mid y_-\rangle \\ \sin(\frac{\theta}{2})e^{i\phi} \mid y_-\rangle \end{bmatrix} \right) &= Re\left(ae^{-i\omega_0 t}e^{-i\|\vec{\beta}\|t} \frac{\vec{\beta_1} - i\overrightarrow{(i\beta_1)}}{\sqrt{2}} \right) \\
&= \frac{e^{-i\omega_0 t}ae^{-i\|\vec{\beta}\|t}}{\sqrt{2}}\beta_1
\end{aligned}$$

Using the identities,

$$\begin{aligned}
\begin{bmatrix} \sin(\frac{\theta}{2}) \mid y_-\rangle \\ -\cos(\frac{\theta}{2})e^{i\phi} \mid y_-\rangle \end{bmatrix} &= \frac{\vec{\beta}_2 - i\overrightarrow{(i\beta_2)}}{\sqrt{2}} \\
\begin{bmatrix} -\sin(\frac{\theta}{2})e^{i\phi} \mid y_+\rangle \\ \cos(\frac{\theta}{2}) \mid y_+\rangle \end{bmatrix} &= \frac{\overrightarrow{(j\beta_1)} + i\overrightarrow{(ij\beta_1)}}{\sqrt{2}} \\
\begin{bmatrix} \cos(\frac{\theta}{2})e^{i\phi} \mid y_+\rangle \\ \sin(\frac{\theta}{2}) \mid y_+\rangle \end{bmatrix} &= \frac{\overrightarrow{(j\beta_2)} + i\overrightarrow{(ij\beta_2)}}{\sqrt{2}}
\end{aligned}$$

Similarly, we can show that,

$$\begin{aligned}
Re\left(e^{-i\omega_0 t}be^{i\|\vec{\beta}\|t}\begin{bmatrix} \sin(\frac{\theta}{2}) \mid y_-\rangle \\ -\cos(\frac{\theta}{2})e^{i\phi} \mid y_-\rangle \end{bmatrix}\right) &= \frac{e^{-i\omega_0 t}be^{i\|\vec{\beta}\|t}}{\sqrt{2}}\beta_2 \\
Re\left(e^{-i\omega_0 t}ce^{-i\|\vec{\beta}\|t}\begin{bmatrix} -\sin(\frac{\theta}{2})e^{i\phi} \mid y_+\rangle \\ \cos(\frac{\theta}{2}) \mid y_+\rangle \end{bmatrix}\right) &= \frac{e^{i\omega_0 t}\overline{c}e^{i\|\vec{\beta}\|t}}{\sqrt{2}}\vec{j}\beta_1 \\
Re\left(e^{-i\omega_0 t}de^{i\|\vec{\beta}\|t}\begin{bmatrix} \cos(\frac{\theta}{2})e^{i\phi} \mid y_+\rangle \\ \sin(\frac{\theta}{2}) \mid y_+\rangle \end{bmatrix}\right) &= \frac{e^{i\omega_0 t}\overline{d}e^{-i\|\vec{\beta}\|t}}{\sqrt{2}}\vec{j}\beta_2
\end{aligned}$$

The result follows. □

The next proposition will give the analogous conditions on the constants a, b, c, d of Proposition (2.5.6) that corresponds to the conditions given by Proposition (2.5.3).

Proposition 2.5.7. Let $\eta(t)$ be a solution of the E-L equations and choose the constants a, b, c, d as they are given by Proposition (2.5.6). Then, $L(t, \eta(t), \dot{\eta(t)}) = 0$ if and only if $ad = bc$.

Proof. Recall that, from the proof of Proposition (2.5.6) and the result of Proposition (2.5.4) that,

$$\begin{aligned}
C_+ &= e^{\frac{\vec{i}\phi}{2}}\left(\frac{b+\overline{d}e^{-\vec{i}\phi}\vec{j}}{\sqrt{2}}\right)\left(\sin(\frac{\theta}{2})-\cos(\frac{\theta}{2})\vec{j}\right)e^{\frac{-\vec{i}\phi}{2}}\\
C_- &= e^{\frac{\vec{i}\phi}{2}}\left(\frac{-\overline{c}e^{-\vec{i}\phi}+a\vec{j}}{\sqrt{2}}\right)\left(\sin(\frac{\theta}{2})-\cos(\frac{\theta}{2})\vec{j}\right)e^{\frac{-\vec{i}\phi}{2}}\\
\overline{C_-} &= e^{\frac{\vec{i}\phi}{2}}\left(\sin(\frac{\theta}{2})+\cos(\frac{\theta}{2})\vec{j}\right)\left(\frac{-ce^{\vec{i}\phi}-a\vec{j}}{\sqrt{2}}\right)e^{\frac{-\vec{i}\phi}{2}}\\
\vec{\beta_0} &= \vec{i}\left(\cos(\theta)+\sin(\theta)e^{\vec{i}\phi}\vec{j}\right)
\end{aligned}$$

Clearly, by direct calculation,

$$\begin{aligned}
C_+\overline{C_-} &= e^{\frac{\vec{i}\phi}{2}}\left(\frac{b+\overline{d}e^{-\vec{i}\phi}\vec{j}}{\sqrt{2}}\right)\left(\frac{-ce^{\vec{i}\phi}-a\vec{j}}{\sqrt{2}}\right)e^{\frac{-\vec{i}\phi}{2}}\\
-\vec{i} &= \left(\sin(\frac{\theta}{2})-\cos(\frac{\theta}{2})\vec{j}\right)e^{\frac{-\vec{i}\phi}{2}}\vec{\beta_0}e^{\frac{\vec{i}\phi}{2}}\left(\sin(\frac{\theta}{2})+\cos(\frac{\theta}{2})\vec{j}\right)\\
C_+\vec{\beta_0}\overline{C_-} &= e^{\frac{\vec{i}\phi}{2}}\left(\frac{b+\overline{d}e^{-\vec{i}\phi}\vec{j}}{\sqrt{2}}\right)(-\vec{i})\left(\frac{-ce^{\vec{i}\phi}-a\vec{j}}{\sqrt{2}}\right)e^{\frac{-\vec{i}\phi}{2}}
\end{aligned}$$

Note that, for any Quaternion $\alpha = \alpha_0 + \vec{\alpha}$, we have that,

$$\begin{aligned}
e^{\theta\vec{x}}\alpha e^{-\theta\vec{x}} &= e^{\theta\vec{x}}\alpha_0 e^{-\theta\vec{x}} + e^{\theta\vec{x}}\vec{\alpha}e^{-\theta\vec{x}}\\
&= \alpha_0 + e^{\theta\vec{x}}\vec{\alpha}e^{-\theta\vec{x}}
\end{aligned}$$

Clearly, $e^{\theta\vec{x}}\vec{\alpha}e^{-\theta\vec{x}}$ is purely imaginary. Hence,

$$Re\left(e^{\theta\vec{x}}\alpha e^{-\theta\vec{x}}\right) = Re(\alpha)$$

Hence,

$$\begin{aligned}
Re\left(C_+\overline{C_-}\right) &= Re\left(\left(\frac{b+\overline{d}e^{-\vec{i}\phi}\vec{j}}{\sqrt{2}}\right)\left(\frac{-ce^{\vec{i}\phi}-a\vec{j}}{\sqrt{2}}\right)\right)\\
&= \frac{1}{2}Re\left(-bce^{\vec{i}\phi}-ba\vec{j}-\overline{d}ce^{-2\vec{i}\phi}\vec{j}+\overline{da}e^{-\vec{i}\phi}\right)\\
&= \frac{1}{2}Re\left(-bce^{i\phi}+\overline{da}e^{-i\phi}\right)\\
Re\left(C_+\vec{\beta}_0\overline{C_-}\right) &= Re\left(\left(\frac{b+\overline{d}e^{-\vec{i}\phi}\vec{j}}{\sqrt{2}}\right)(-\vec{i})\left(\frac{-ce^{\vec{i}\phi}-a\vec{j}}{\sqrt{2}}\right)\right)\\
&= \frac{1}{2}Re\left((b+\overline{d}e^{-\vec{i}\phi}\vec{j})(\vec{i}ce^{\vec{i}\phi}+\vec{i}a\vec{j})\right)\\
&= \frac{1}{2}Re\left(\vec{i}bce^{\vec{i}\phi}+\vec{i}ba\vec{j}-\overline{\vec{i}dc}e^{-2\vec{i}\phi}\vec{j}+\overline{\vec{i}da}e^{-\vec{i}\phi}\right)\\
&= \frac{1}{2}Re\left(i(bce^{i\phi}+\overline{da}e^{-i\phi})\right)\\
&= \frac{1}{2}Im\left(bce^{i\phi}+\overline{da}e^{-i\phi}\right)
\end{aligned}$$

Thus, by Proposition (2.5.3), $L=0$ if and only if there are real numbers γ, ϵ such that,

$$\begin{aligned}
-bce^{i\phi}+\overline{da}e^{-i\phi} &= i\epsilon\\
bce^{i\phi}+\overline{da}e^{-i\phi} &= \gamma
\end{aligned}$$

Therefore,

$$\begin{aligned}
2\overline{da}e^{-i\phi} &= \gamma+i\epsilon\\
2bce^{i\phi} &= \gamma-i\epsilon
\end{aligned}$$

In particular,

$$\begin{aligned}
\overline{2bce^{i\phi}} &= \overline{\gamma-i\epsilon}\\
&= \gamma+i\epsilon\\
&= 2\overline{da}e^{-i\phi}
\end{aligned}$$

Hence, $bc = da$. $\square$

2.5.2 SPE And QFP Correspondence

We will provide for a map between solutions to the quaternionic Foucault pendulum and solutions to the SPE analogous to the correspondence given by Proposition (2.4.1). We will see that the $L = 0$ condition will couple the 4 independent oscillators in the QFP further to reduce the number of free parameters in the solutions of the E-L equations of the QFP to the number free parameters of the spin $\frac{1}{2}$ system. Thus making the map between the solution space of the QFP and the solution space of the SPE possible. We start with a Lemma that will provide for the map used in the correspondence.

Lemma 2.5.8. *Let a, b, c, d be complex numbers satisfying $ad = bc$. Then, there are complex numbers A, B, f, g such that,*

$$\begin{aligned} a &= \sqrt{2}Af \\ b &= \sqrt{2}Ag \\ c &= \sqrt{2}Bf \\ d &= \sqrt{2}Bg \end{aligned}$$

We note that if a, b, c, d can be written in the above form for some constants A, B, f, g then $ad = bc$ by direct computation.

Proof. Let the following be the complex polar representation of the complex numbers a, b, c, d.

$$\begin{aligned} a &= r_1 e^{i\theta_1} \\ b &= r_2 e^{i\theta_2} \\ c &= r_3 e^{i\theta_3} \\ d &= r_4 e^{i\theta_4} \end{aligned}$$

Note that the $ad = bc$ conditions forces,

$$\begin{aligned} r_1 r_4 &= r_2 r_3 \\ \theta_1 + \theta_4 &= \theta_2 + \theta_3 \bmod 2\pi \end{aligned}$$

Let us assume first that both a, d are not equal to zero. Then, clearly $r_1, r_2, r_3, r_4 \neq 0$. Thus, the following choice of A, B, f, g will suffice,

$$\begin{aligned} f &= \frac{e^{i\theta_1}}{r_4} \\ g &= \frac{e^{i\theta_2}}{r_3} \\ A &= \frac{r_4 r_1}{\sqrt{2}} \\ &= \frac{r_3 r_2}{\sqrt{2}} \\ B &= \frac{e^{i(\theta_4 - \theta_2)} r_4 r_3}{\sqrt{2}} \\ &= \frac{e^{i(\theta_3 - \theta_1)} r_4 r_3}{\sqrt{2}} \end{aligned}$$

Let us assume that $a = 0, d \neq 0, c = 0$. Then, $f = 0, g = 1, A = \frac{b}{\sqrt{2}}, B = \frac{d}{\sqrt{2}}$ suffices.

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$\square$

The complex constants f, g, A, B of Lemma(2.5.8) will be given interpretations in the next results. The complex constants $f' = f\sqrt{\mid A \mid^2 + \mid B \mid^2}, g' = g\sqrt{\mid A \mid^2 + \mid B \mid^2}$ will correspond to different solutions of the SPE, and the complex constants A, B to hidden variables that are independent of the SPE solutions.

Proposition 2.5.9. Let $\eta(t)$ be a solution to the E-L equations of the QFP. Consider the SPE with constant magnetic field $\vec{B}$ and a negatively charged particle (that is γ is negative). Assume further that $\|\vec{\beta}\| = -\frac{\gamma\|\vec{B}\|}{2}$. Assume further that $L(t, \eta(t), \dot{\eta}(t)) = 0$. Let a, b, c, d be constants of Proposition (2.5.6) in the representation of $\eta(t)$. Clearly, by Proposition (2.5.7), $ad = bc$ and by Lemma(2.5.8) there are complex constants A, B, f, g such that:

$$\begin{aligned} a &= \sqrt{2}Af \\ b &= \sqrt{2}Ag \\ c &= \sqrt{2}Bf \\ d &= \sqrt{2}Bg \end{aligned}$$

Then,

$$\frac{(\overline{A} - \overline{B}\vec{j})}{\sqrt{\mid A \mid^2 + \mid B \mid^2}}\eta(t) = e^{-i\omega_0 t}\left\{e^{i\|\vec{\beta}\|t}f'\beta_1 + e^{-i\|\vec{\beta}\|t}g'\beta_2\right\}$$

Is a solution of the SPE where $f' = f\sqrt{\mid A \mid^2 + \mid B \mid^2}, g' = g\sqrt{\mid A \mid^2 + \mid B \mid^2}$ and $\|\vec{\beta}\| = -\frac{\gamma\|\vec{B}\|}{2} = -\beta$.

Proof. Recall, from the proof of Proposition (2.5.6) that:

$$\begin{aligned}\eta(t) &= \frac{e^{-i\omega_0 t} a e^{-i\|\vec{\beta}\|t}}{\sqrt{2}}\beta_1 + \frac{e^{-i\omega_0 t} b e^{i\|\vec{\beta}\|t}}{\sqrt{2}}\beta_2 + \frac{e^{i\omega_0 t}\overline{c} e^{i\|\vec{\beta}\|t}}{\sqrt{2}}\vec{j}\beta_1 + \frac{e^{i\omega_0 t}\overline{d} e^{-i\|\vec{\beta}\|t}}{\sqrt{2}}\vec{j}\beta_2 \\ &= e^{-i\omega_0 t} A\left\{f e^{-i\|\vec{\beta}\|t}\beta_1 + g e^{i\|\vec{\beta}\|t}\beta_2\right\} + e^{i\omega_0 t}\overline{B}\left\{\overline{f} e^{i\|\vec{\beta}\|t}\vec{j}\beta_1 + \overline{g} e^{-i\|\vec{\beta}\|t}\vec{j}\beta_2\right\}\end{aligned}$$

A direct calculation yields,

$$(\overline{A} - \overline{B}\vec{j})\eta(t) = (\mid A\mid^2 + \mid B\mid^2)e^{-i\omega_0 t}\left\{e^{i\|\vec{\beta}\|t} f\beta_1 + e^{-i\|\vec{\beta}\|t} g\beta_2\right\}$$

Clearly, from this it follows that,

$$\begin{aligned}\frac{(\overline{A} - \overline{B}\vec{j})}{\sqrt{\mid A\mid^2 + \mid B\mid^2}}\eta(t) &= e^{-i\omega_0 t}\left\{e^{i\|\vec{\beta}\|t} f'\beta_1 + e^{-i\|\vec{\beta}\|t} g'\beta_2\right\} \\ &= e^{-i\omega_0 t}\left\{e^{-i\frac{\gamma\|\vec{B}\|}{2}t} f'\beta_1 + e^{i\frac{\gamma\|\vec{B}\|}{2}t} g'\beta_2\right\}\end{aligned}$$

Which is clearly a solution to the SPE with $\beta = \frac{\gamma\|\vec{B}\|}{2}$.

□

We note that if we parametrize $\eta(t) = (\eta_0(t), \eta_1(t), \eta_2(t), \eta_3(t))^T$ as a 4-d function with real valued coordinate functions $\eta_i(t)$. Then, using the complex constants A, B, we can define a map between the Quaternion representation to a Spinor representation by letting,

$$\begin{aligned}\chi_+(t) &= \frac{(\eta_0 + i\eta_1)\overline{A} + (\eta_2 - i\eta_3)\overline{B}}{\sqrt{\mid A\mid^2 + \mid B\mid^2}} \\ \chi_-(t) &= \frac{(-\eta_0 + i\eta_1)\overline{B} + (\eta_2 + i\eta_3)\overline{A}}{\sqrt{\mid A\mid^2 + \mid B\mid^2}}\end{aligned}$$

We will call this map $\theta_{A,B}(\eta(t)) = (\chi_+(t), \chi_-(t))$. We will denote by θ the map $\theta_{1,0}$. We note that, by direct calculation,

$$\theta_{A,B}(\eta(t)) = \theta\left(\frac{(\overline{A} - \overline{B}\vec{j})}{\sqrt{\mid A\mid^2 + \mid B\mid^2}}\eta(t)\right)$$

In particular, the last proposition can be paraphrased as the following Corollary.

Corollary 2.5.10. *Let* $\eta(t)$ *be a solution to the E-L equations of the QFP with constant* $\beta(t) = \vec{\beta}$ *parameter. Assume further that* $L(t, \eta(t), \dot{\eta}(t)) = 0$. *Then, there is a unit Quaternion* u *such that,*

$$\theta\left(u\eta(t)\right)$$

Is a solution of the SPE with $-\frac{\gamma\|\vec{B}\|}{2} = \|\vec{\beta}\|$ *for a negatively charged particle subjected to a uniform magnetic field* $\vec{B}$.

Corollary(2.5.10) is the analogous of Proposition (2.4.1) for the QFP .

2.6 Results for the Time Varying Magnetic Field Case

We will show that the converse of Corollary(2.5.10) is a partial correspondence of the QFP solution set and the solution set of the SPE for an arbitrary time-varying magnetic field.

Proposition 2.6.1. Let $\eta(t)$ be the a solution to the SPE of a negatively charged particle under a time-varying magnetic field $\vec{B} = \|\vec{B}\|\vec{\beta_0}$ and rest energy ω_0 where $\vec{\beta_0}$ is a time-varying unit vector. Consider the QFP with time-varying $\vec{\beta}$ parameter equal to $\frac{-\gamma\|\vec{B}\|}{2}\vec{\beta_0}$. Then, for any unit Quaternion γ, the function $\gamma\eta(t)$ is a solution to the E-L equations of the QFP. Further, because any solution of the SPE has constant norm, by Proposition (2.5.3), the solution $\gamma\eta(t)$ satisfies the $L(t, \gamma\eta(t), \gamma\dot{\eta}(t)) = 0$ condition as well.

Proof. We are assuming that $\vec{\beta}(t) = \frac{-\gamma\|\vec{B(t)}\|}{2}\vec{\beta_0}(t)$ and that $\|\vec{\beta}\| = \frac{-\gamma\|\vec{B(t)}\|}{2}$ which is consistent as γ is negative because we are studying the state of a negatively charged particle. Clearly, the SPE in quaternionic notation is,

$$\begin{aligned}\dot{\eta}(t) &= \eta(t)\left(\frac{\gamma\|\vec{B(t)}\|}{2}\vec{\beta}_0\right) - i\omega_0\eta(t)\\ &= -\eta(t)\vec{\beta} - i\omega_0\eta(t)\end{aligned}$$

Also, the E-L equations for the QFP is,

$$\ddot{\eta}(t) + 2\dot{\eta}(t)\vec{\beta} + \eta(t)\left(-\|\vec{\beta}\|^2 + \dot{\vec{\beta}} + \omega_0^2\right) = 0 \tag{2.8}$$

It suffices to show that if $\eta(t)$ satisfies the SPE, then it must also satisfy the E-L equations of the QFP. The γ constant on the left $\eta(t)$ can be shown to respect the algebraic operations that are to follow. Let us assume that $\eta(t)$ satisfies the SPE. Note that,

$$\begin{aligned}\ddot{\eta}(t) &= -\dot{\eta}(t)\vec{\beta}(t) - \eta(t)\dot{\vec{\beta}}(t) - i\omega_0\dot{\eta}(t)\\ &= -\dot{\eta}(t)\vec{\beta}(t) - \eta(t)\dot{\vec{\beta}}(t) - i\omega_0\left(-\eta(t)\vec{\beta}(t) - i\omega_0\eta(t)\right)\\ &= -\dot{\eta}(t)\vec{\beta}(t) - \eta(t)\dot{\vec{\beta}}(t) + i\omega_0\eta(t)\vec{\beta}(t) - \omega_0^2\eta(t)\end{aligned}$$

Hence,

$$\begin{aligned}\ddot{\eta}(t) + 2\dot{\eta}(t)\vec{\beta} + \eta(t)\left(-\|\vec{\beta}\|^2 + \dot{\vec{\beta}} + \omega_0^2\right) &= \left(i\omega_0\eta(t) + \dot{\eta}(t)\right)\vec{\beta} - \eta(t)\|\vec{\beta}\|^2\\ &= -\eta(t)\vec{\beta}\vec{\beta} - \eta(t)\|\vec{\beta}\|^2\\ &= \eta(t)\|\vec{\beta}\|^2 - \eta(t)\|\vec{\beta}\|^2\\ &= 0\end{aligned}$$

□

CHAPTER 3

SUMMARY AND CONCLUSION

In the above sections, we discussed the properties of the Foucault pendulum as classical analogs of the spin $\frac{1}{2}$ system. These properties include the Berry or geometric phase, the presence of the Zeeman energy splitting phenomenon, and the superposition of the normal modes. These similarities motivated the formulation of an equivalence among solutions of the Schrodinger-Pauli-Equation and the modified Focault pendulum which was given by Proposition(2.4.1). Proposition(2.4.1) had the shortcoming of being applicable to magnetic fields in the y direction only. This motivated the definition of the quaternionic Foucault pendulum by first generalizing the complex Lagrangian of the modified Foucault pendulum to a quaternionic Lagrangian, second generalizing the real valued parameter β to a purely imaginary Quaternion, and third defining the QFP as the solution to the E-L equations of the generalized quaternionic Lagrangian.

Using the quaternionic structure of the Lagrangian of the QFP, two groups were found to be symmetry groups of the QFP Lagrangian. These groups were defined using left multiplication by a unit or right multiplication by a unit. The constants of motion associated with these groups were found using Noether's theorem and the infinitesimal generators of the Lie algebras of both of these groups. These constants were compared to their counterparts in the SPE by postulating a Lagrangian for the SPE in quaternionic notation. It was also shown that any QFP with constant β parameter was equivalent to a QFP in canonical form with β parameter equal to $\alpha\vec{k}$. That is, any QFP with constant β parameter is equivalent to the dynamics of two independent modified Foucault pendulums at the same latitude and of the same length. We called the equivalent QFP with β parameter equal to $\alpha\vec{k}$ the canonical form of the QFP.

We then closed the discussion with an extensive derivation of the equivalence between solutions of the SPE with time independent magnetic field and solutions of the QFP with time independent β parameter. The main achievement of this extensive derivation was the determination that the $L = 0$ condition is necessary and sufficient in the SPE equivalence with the QFP. This result is summarized by Corollary(2.5.10), which gives the existence of a unit quaternion u that makes the equivalence between the QFP and SPE possible as a many-to-one map. This is a many-to-one map, in the sense that there are additional parameters in the solution to Equation (2.8) that can be altered without affecting the corresponding quantum solution, including an overall phase. From a quantum perspective, these additional parameters would be called ”hidden variables”.

The similarites between the dynamics of the Foucault pendulum and the dynamics of the spin $\frac{1}{2}$ system has been explored before in the work of Klyshko [D.N93] but only in the context of the Berry phase. Section (2.3) shows that the analogy goes beyond the Berry phase analog. Prior efforts to find a classical analog of the spin $\frac{1}{2}$ system have made use of the physical angular momentum vector in real space as the analog for spin. Under such working assumption, a physical rotation of the angular momentum vector by 2π does not yield a π geometric phase without making additional reference to elements outside of the state itself. This is illustrated by Feynman's coffee cup demonstration in Feynman and Weinberg [FR87].

We close the discussion by posing the question of whether or not it is possible to construct a working mechanical or electrical version of the classical oscillators described in Section (2.4) for the QFP. Such construction would make a remarkable demonstration of the dynamics of an unmeasured electron spin state.

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APPENDIX A

APPENDIX

We will derive a few geometrical results about Quaternions, solve two quaternionic Ordinary Differential Equations (ODEs), and give a formulation of Noether's theorem on constants of motion in the context of Lie Groups.

A.1 A Few Geometrical Results about Quaternions

In this section, we will cover a few geometrical results about Quaternions that will prove useful in the derivation of the results that are covered in the manuscript.

The non-zero Quaternions $\mathbb{H}^*$ equipped with the multiplicative product makes them a 4 dimensional real Lie group. A Lie group that is isomorphic to $SU(2) \times \mathbb{R}^+$, where $SU(2)$ is the Lie group of 2×2 complex matrices that are unitary and of determinant 1, and $\mathbb{R}^+$ is the multiplicative group of positive real numbers. The $SU(2)$ component is isomorphic to the group of Quaternions that have norm 1. The $\mathbb{R}^+$ component corresponds to the image of the Norm map.

One can define the Exponential map for Quaternions using the standard definition of the Exponential function.

Definition A.1.1. Given a Quaternion w. The, Exponential of w is defined as:

$$\begin{aligned} exp(w) &= 1 + \frac{w^1}{1!} + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{w^k}{k!} \end{aligned}$$

Proposition A.1.2. Let $w = w_0 + \vec{w}$ be a Quaternion. Then,

$$exp(w) \quad = \quad exp(w_0)(\cos\|\vec{w}\| + \frac{\vec{w}}{\|\vec{w}\|}\sin\|\vec{w}\|)$$

The Exponential map can be viewed as a map from the Lie Algebra to the Lie Group. Based on the definition given, one can show that the Lie Algebra of the Lie subgroup of Quaternions of Norm 1 is given by the purely imaginary Quaternions. We will use the Exponential map to determine the infinitesimal generator of a Lie group element.

We close this section with a Group theoretic result about Quaternions.

Proposition A.1.3. Let $\vec{\beta}$ and $\vec{\eta}$ be purely imaginary Quaternions. Then, there exist a (not necessarily unique) unit Quaternion γ and a real number a such that,

$$\overline{\gamma}\vec{\beta}\gamma \quad = \quad a\vec{\eta}$$

Proof. Without loss of generality, we can assume $\vec{\eta} = \vec{k}$. That is, it suffices to show that for arbitrary $\vec{\beta}$, there are γ and a such that $\overline{\gamma}\vec{\beta}\gamma = a\vec{k}$. Once this is shown, we can find γ_1, a_1 and γ_2, a_2 such that,

$$\begin{aligned} \overline{\gamma_1}\vec{\beta}\gamma_1 &= a_1\vec{k} \\ \overline{\gamma_2}\vec{\eta}\gamma_2 &= a_2\vec{k} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\overline{\gamma_1}\vec{\beta}\gamma_1}{a_1} &= \vec{k} \\ &= \frac{\overline{\gamma_2}\vec{\eta}\gamma_2}{a_2} \end{aligned}$$

Thus,

$$\begin{aligned} \overline{\gamma_1\gamma_2}\vec{\beta}\gamma_1\gamma_2 &= \overline{\gamma_2}\overline{\gamma_1}\vec{\beta}\gamma_1\gamma_2 \\ &= \frac{a_1}{a_2}\vec{\eta} \end{aligned}$$

Therefore, $\gamma = \gamma_1\gamma_2$ and $a = \frac{a_1}{a_2}$ will give the desired result.

We proceed to show that for given β, there exist a unit Quaternion γ and a non-zero real number a such that $\overline{\gamma}\vec{\beta}\gamma = a\vec{k}$.

Let $\vec{u}$ be an arbitrary unit vector, and θ be the angle between $\vec{u}$ and $\vec{\beta_0}$ where $\vec{\beta_0} = \frac{\vec{\beta}}{||\vec{\beta}||}$. Note that for,

$$\begin{aligned} \gamma &= \cos\theta + \sin\theta\frac{\vec{\beta_0}\times\vec{u}}{||\vec{\beta_0}\times\vec{u}||} \\ &= \exp(\theta\frac{\vec{\beta_0}\times\vec{u}}{||\vec{\beta_0}\times\vec{u}||}) \end{aligned}$$

We have the relation,

$$\overline{\gamma}\vec{\beta_0}\gamma = (4\cos^2\theta - 1)\vec{\beta_0} - 2\cos\theta\vec{u}$$

In particular, if we choose $\vec{u}$ so that $\theta = \frac{\pi}{3}$,

$$\begin{aligned} \gamma &= \frac{1}{2} + \vec{\beta_0}\times\vec{u} \qquad \text{(A.1)} \\ \overline{\gamma}\vec{\beta_0}\gamma &= -\vec{u} \end{aligned}$$

The last equation satisfies the conclusion of the proposition if $\vec{u} = \vec{k}$ and $\vec{\beta}$ can be joined to $\vec{k}$ by a geodesic arc of length $\frac{\pi}{3}$. Note that, if $\vec{\beta}$ and $\vec{k}$ could be joined by a piecewise path of geodesic arcs each of length $\frac{\pi}{3}$, the geodesic path will yield a series of γ_i's and the ordered product of all the γ_i's will give,

$$\overline{\gamma}\vec{\beta_0}\gamma = (-1)^l\vec{k}$$

Where l is the number of geodesic arcs in the path joining $\vec{\beta_0}$ and $\vec{k}$, $\gamma = \Pi_{i=1}^{l}\gamma_i$, and γ_i is the γ constructed by Equation (A.1).

Hence, the result follows if we are able to show that any two unit vectors in the unit sphere in $\mathbb{R}^3$ can be joined by piecewise path of geodesic arcs each of length $\frac{\pi}{3}$. We leave it as an exercise to the reader to show that one can always find such a path and the length of this path is at most 4. $\square$

A.2 Special Quaternionic ODEs

We will consider the solution space of the following first order differential equation in quaternionic space.

$$\dot{\eta}(t) \quad = \quad \alpha\eta(t) + \eta(t)\gamma \tag{A.2}$$

And of the following second order differential equation in quaternionic space.

$$0 \quad = \quad \ddot{\eta}(t) + 2\dot{\eta}(t)\vec{\beta} + (\omega_0^2 - \|\vec{\beta}\|^2)\eta(t) \tag{A.3}$$

Where α and γ are two fixed Quaternions, $\vec{\beta}$ is a purely imaginary Quaternion, and ω_0 is a real number.

The Schrodinger Pauli Equation (SPE) for the spin $\frac{1}{2}$ particle will be shown to be special case of ODE (A.2), and the Euler-Lagrange Equations for the quaternionic Foucault pendulum will be shown to be given by ODE (A.3).

Proposition A.2.1. Let α and γ be two fixed Quaternions. Then, the following ODE,

$$\dot{\eta}(t) \quad = \quad \gamma\eta(t) + \eta(t)\alpha$$

Has solution,

$$\eta(t) \quad = \quad exp(\gamma t)Cexp(\alpha t)$$

Where C is a quaternionic constant.

Proof. Recall that using the definition of the exponential function, we get that for $\exp(\gamma t)$,

$$\begin{aligned}
\frac{d\exp(\gamma t)}{dt} &= \frac{d}{dt}\left\{\sum_{l=0}^{\infty}\frac{(\gamma t)^l}{l!}\right\} \\
&= \frac{d}{dt}\left\{\sum_{l=0}^{\infty}\frac{t^l(\gamma)^l}{l!}\right\} \\
&= \sum_{l=0}^{\infty}\frac{t^l(\gamma)^{l+1}}{l!} \\
&= \gamma\left(\sum_{l=0}^{\infty}\frac{t^l(\gamma)^l}{l!}\right) \\
&= \left(\sum_{l=0}^{\infty}\frac{t^l(\gamma)^l}{l!}\right)\gamma
\end{aligned}$$

Where we have used the fact that t commutes with any Quaternion because it is real. Thus, we get that,

$$\begin{aligned}
\frac{d\exp(\gamma t)}{dt} &= \gamma\exp(\gamma t) \\
&= \exp(\gamma t)\gamma
\end{aligned}$$

Now, applying the product rule of differentiation for functions of one real variable to $\eta(t) = exp(\gamma t) * C * exp(\alpha t)$, we deduce:

$$\begin{aligned}
\dot{\eta}(t) &= \gamma exp(\gamma t)Cexp(\alpha t) + exp(\gamma t)Cexp(\alpha t)\alpha \\
&= \gamma\eta(t) + \eta(t)\alpha
\end{aligned}$$

Hence, $\eta(t) = exp(\gamma t)Cexp(\alpha t)$ is a four dimensional solution set to the ODE over $\mathbb{R}$. Clearly, the solution set of the ODE is four dimensional over $\mathbb{R}$. Hence, the result follows.

□

Proposition A.2.2. Let $\vec{\beta}$ be a purely imaginary Quaternion and ω_0 a real number. Then, the second order quaternionic ODE:

$$0 = \ddot{\eta}(t) + 2\dot{\eta}(t)\vec{\beta} + (\omega_0^2 - \|\vec{\beta}\|^2)\eta(t)$$

Has solution:

$$\eta(t) = C_+ \exp\left(\frac{(-\|\vec{\beta}\| + \omega_0)\vec{\beta}}{\|\vec{\beta}\|}t\right) + C_- \exp\left(\frac{(-\|\vec{\beta}\| - \omega_0)\vec{\beta}}{\|\vec{\beta}\|}t\right)$$

Where C_+, C_- are quaternionic constants.

Proof. Let us assume a solution of the form $\eta(t) = C\exp(\alpha t)$ where C is a quaternionic constant. Note that, for this $\eta(t)$, we must have for arbitrary t:

$$\begin{aligned} 0 &= \ddot{\eta}(t) + 2\dot{\eta}(t)\vec{\beta} + (\omega_0^2 - \|\vec{\beta}\|^2)\eta(t) \\ &= C\exp(\alpha t)\left(\alpha^2 + 2\alpha\vec{\beta} + (\omega_0^2 - \|\vec{\beta}\|)\right) \end{aligned}$$

Hence, the ODE is satisfied if and only if α is a root of the following quadratic equation over $\mathbb{H}$,

$$\alpha^2 + 2\alpha\vec{\beta} + (\omega_0^2 - \|\vec{\beta}\|) = 0$$

We will show that this quadratic equation has exactly two quaternionic roots α_+, α_-. Hence, the general solution to the ODE will be given as:

$$\eta(t) = C_+\exp(\alpha_+ t) + C_-\exp(\alpha_- t)$$

The result will follow by giving the exact formula for α_+, α_-. Let $\alpha = \alpha_0 + \vec{\alpha}$. Then,

$$\begin{aligned} \alpha^2 &= \alpha_0^2 - \|\vec{\alpha}\|^2 + 2\alpha_0\vec{\alpha} \\ \alpha\vec{\beta} &= -\langle\vec{\alpha}, \vec{\beta}\rangle + \alpha_0\vec{\beta} + \vec{\alpha} \times \vec{\beta} \end{aligned}$$

Hence,

$$0 = (\alpha_0^2 - \|\vec{\alpha}\|^2 - 2\langle\vec{\alpha}, \vec{\beta}\rangle + \omega_0^2 - \|\vec{\beta}\|^2) + 2 * (\alpha_0\vec{\alpha} + \alpha_0\vec{\beta} + \vec{\alpha} \times \vec{\beta})$$

Thus, the following conditions must be satisfied,

$$0 = \alpha_0^2 - \|\vec{\alpha}\|^2 - 2\langle\vec{\alpha}, \vec{\beta}\rangle + \omega_0^2 - \|\vec{\beta}\|^2 \tag{A.4}$$

$$0 = \alpha_0\vec{\alpha} + \alpha_0\vec{\beta} + \vec{\alpha} \times \vec{\beta} \tag{A.5}$$

Clearly, from Equation (A.5) we deduce $\vec{\alpha} \times \vec{\beta} = -\alpha_0(\vec{\alpha} + \vec{\beta})$. In particular, this means that $\vec{\alpha} \times \vec{\beta}$ lies in the plane spanned by $\vec{\alpha}$ and $\vec{\beta}$. Note that $\vec{\alpha} \times \vec{\beta}$ is always perpendicular to the plane spanned by $\vec{\alpha}$ and $\vec{\beta}$ unless $\vec{\alpha}$ and $\vec{\beta}$ are linearly dependent. Hence, in order to satisfy Equation (A.5), we must have $\alpha_0 = 0$, $\vec{\alpha} \times \vec{\beta} = \vec{0}$, and $\vec{\alpha}$ and $\vec{\beta}$ be linearly dependent. Thus $\vec{\alpha} = k\vec{\beta}$. Clearly, with this condition Equation (A.5) is satisfied trivially. Note that Equation (A.4) is equivalent to,

$$\begin{aligned} \omega_0^2 &= \|\vec{\alpha}\|^2 + 2\langle\vec{\alpha}, \vec{\beta}\rangle + \|\vec{\beta}\|^2 \\ &= \|\vec{\alpha} + \vec{\beta}\|^2 \\ &= (k+1)^2\|\vec{\beta}\|^2 \end{aligned}$$

This gives a solution for $k = -1 \pm \frac{\omega_0}{\|\vec{\beta}\|}$. Hence, the roots to the quadratic equation in $\mathbb{H}$ is given as:

$$\begin{aligned}
\alpha_+ &= \frac{-\|\vec{\beta}\| + \omega_0}{\|\vec{\beta}\|}\vec{\beta} \\
\alpha_- &= \frac{-\|\vec{\beta}\| - \omega_0}{\|\vec{\beta}\|}\vec{\beta}
\end{aligned}$$

The result follows. □

A.3 Noether's Theorem for Lie Groups

In the following, we shall be interested in finding the constants of motion associated with a symmetry of a given dynamical system that is defined by a Lagrangian L. Specifically, the Foucault pendulum and the quaternionic Foucault pendulum. Both of these systems make use of a real valued Lagrangian L defined over a division ring ($\mathbb{C}$ or $\mathbb{H}$). Also, for both of these systems the division ring at hand can be viewed as a Lie group under the right regular product of Equations (1.1) and (1.3). We will see that subgroups of these Lie groups induce symmetries of L. Hence, it is natural to talk about the symmetry group of the Lagrangian L as a Lie group as well.

Definition A.3.1. Let $L(t, x, \dot{x})$ be a Lagrangian of a system that is real valued, where $x, \dot{x} \in \mathbb{R}^n$. That is, L is defined on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. Let a Lie group G act on $\mathbb{R}^n$ and $\mathbb{R}$, and be of dimension m. We will call G a symmetry group of L whenever for all $g \in G$,

$$L(g \cdot t, g \cdot x, g \cdot \dot{x}) \quad = \quad L(t, x, \dot{x})$$

Where $\cdot$ is the action of G on $\mathbb{R}^n$ and $\mathbb{R}$

It is well known, by Noether's theorem, that a symmetry of the Lagrangian corresponds to a constant of motion of the solutions to the Euler-Lagrange equations. The following result summarizes this result in the context of Lie groups and gives an explicit formula for these constants.

Proposition A.3.2. Let $L(t, \vec{x}, \dot{\vec{x}})$ be a Lagrangian of a system that is real valued, where $\vec{x}, \dot{\vec{x}} \in \mathbb{R}^n$. That is, L is defined on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. Let G be a Lie group of dimension m that is a symmetry group of L. Define $\vec{p}$ as having components $p_i = \frac{\partial L}{\partial \dot{x_i}}$, where $x = (x_1, \cdots, x_n)^T$. Thus, $\vec{p}(\vec{x}, \dot{\vec{x}}) = \frac{\partial L}{\partial \dot{\vec{x}}}$. Let $R_g(\vec{x}) = g \cdot \vec{x}$ be the diffeomorphism induced by g under the action of G on $\mathbb{R}^n$. Let ξ_g be the vector field generated by the infinitesimal generator of R_g. Then, for given $g \in G$, the following quantity,

$$S(R_g) \quad = \quad \langle \vec{p}(\vec{x}, \dot{\vec{x}}), \xi_g(\vec{x}) \rangle$$

Is the constant of motion that corresponds to the symmetry given by g.

A consequence of Proposition (A.3.2) is that one only needs to calculate the constants of motion given by the vector fields of the generators of the Lie algebra of G to determine all the constants of motions of G. This is because any constant of motion induced by G is a linear combination of the constants of motions induced by the generators of the Lie algebra of G. Hence, by Proposition (A.3.2) there are only m linearly independent constants of motion induced by G.

In Chapter 2, we will apply Proposition (A.3.2) to get all the constants of motion of the Foucault pendulum and the quaternionic Foucault pendulum that are associated with the induced symmetry group in the corresponding background Lie group $\mathbb{C}$ or $\mathbb{H}$.