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Geometric Algebra: An Introduction with Applications in Euclidean and Conformal Geometry

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GEOMETRIC ALGEBRA:
AN INTRODUCTION WITH APPLICATIONS IN EUCLIDEAN AND CONFORMAL GEOMETRY

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ABSTRACT

GEOMETRIC ALGEBRA:
AN INTRODUCTION WITH APPLICATIONS IN EUCLIDEAN
AND CONFORMAL GEOMETRY

by Richard A. Miller

This thesis presents an introduction to geometric algebra for the uninitiated. It contains examples of how some of the more traditional topics of mathematics can be reexpressed in terms of geometric algebra along with proofs of several important theorems from geometry. We introduce the conformal model. This is a current topic among researchers in geometric algebra as it is finding wide applications in computer graphics and robotics. The appendices provide a list of some of the notational conventions used in the literature, a reference list of formulas and identities used in geometric algebra along with some of their derivations, and a glossary of terms.

DEDICATION

This thesis is affectionately dedicated to my wife, Denise.

ACKNOWLEDGEMENTS

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CHAPTER 1

INTRODUCTION

In 1966, David Hestenes, a theoretical physicist at Arizona State University, published the book, *Space-Time Algebra*, a rewrite of his Ph.D. thesis [DL03, p. 122]. Hestenes had realized that Dirac algebras and Pauli matrices could be unified in a matrix-free form, which he presented in his book. This was the first major step in developing a unified, coordinate-free, geometric algebra and calculus for all of physics. His book applied the long-forgotten geometric product of William Clifford to the study of classical physics.

Hestenes' success with applying Clifford's geometric product and his continued publications have inspired a new generation of physicists and mathematicians to re-examine geometric algebra. Acceptance of geometric algebra is growing and, while not everyone is in full agreement, it is now hard to find any area of physics to which geometric algebra cannot or has not been applied without some degree of success [DL03, p. 124]. Some of the better known successful applications include fields such as classical physics, space-time, relativity, quantum theory, differential geometry, computer graphics, and robotics.

What makes this geometric algebra so flexible that it can be applied to so many areas? It turns out that geometric algebra provides a generalized theory that encompasses many of the mathematical topics that have been around for years, such as complex numbers, quaternions, matrix algebra, vectors, tensor and spinor algebras, and the algebra of differential forms. Geometric algebra provides a method of expressing geometrical relationships through algebraic equations.

What is geometric algebra? Geometric algebra extends the concept of a vector as a one-dimensional segment of a line with direction, orientation, and magnitude. Geometric algebra extends this concept to multiple dimensions. A *bivector*, for example, is like a two-dimensional vector. It is a two-dimensional segment of a plane with direction, orientation, and magnitude. A *trivector* is like a three-dimensional vector. It is a three-dimensional segment of a space with direction, orientation, and magnitude.

Geometric algebra traces its history back to the nineteenth century. During this century mathematicians like Hamilton, Grassmann, Gibbs, and Clifford struggled with and developed the mathematics of complex numbers, quaternions, and vectors [DL03]. A common thread to their research was to find the answer to the question, "Just what does it mean to take the product of two vectors?"

William Rowan Hamilton (1805–1865) was an Irish mathematician and the inventor of quaternions. He worked for years trying to extend his theory of complex numbers to three dimensions [DL03, p. 8-10]. On October 16^{th}, 1843, while out walking with his wife along the Royal Canal in Dublin, he had an epiphany and carved the equation $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ into the Broome Bridge. Although quaternions found favor with physicists like James Clerk Maxwell, they were weighed down by the increasingly dogmatic arguments over their interpretation and were eventually displaced by the hybrid system of vector algebra promoted by Gibbs.

Hermann Gunther Grassmann (1809–1877) was a German high school mathematics teacher [DL03, p. 12,13,17]. He introduced the *outer* or *exterior product* which, in modern notation, we write as $\mathbf{a} \wedge \mathbf{b}$, or "$\mathbf{a}$ wedge $\mathbf{b}$." The outer product provided a means of encoding a plane, without relying on the notion of a vector perpendicular to it. It was a new mathematical entity for encoding an oriented plane and today we call it a *bivector*. Grassmann published the first edition of his geometric calculus, *Lineale Ausdehnungslehre*, in 1844, the same year Hamilton announced his discovery of the quaternions. Grassmann's work did not achieve the same impact as the quaternions; however, and it was many years before his ideas were understood and appreciated by other mathematicians. Perhaps he was not taken seriously by his contemporaries because he was a high school teacher.

Josiah Willard Gibbs (1839–1903) was an American scientist/mathematician who is credited with the invention of modern vector calculus [Kle08], [DL03, p. 10,11,17]. He spent the majority of his career at Yale, where he was professor of mathematical physics from 1871 until his death. Gibbs worked on developing Grassmann's exterior algebra from 1880 to 1884. He produced a vector calculus well-suited to the needs of physicists. He distinguished the difference between the dot and cross products of two vectors. Gibbs's lecture notes on vector calculus were privately printed in 1881 and 1884 for the use of his students and were later adapted by Edwin Bidwell Wilson into a textbook, *Vector Analysis*, published in 1901. The success of Gibbs' vector calculus is evident, as today it is a major part of every science, technology, engineering, and mathematics (STEM) curriculum.

William Kingdon Clifford (1845–1879) was a British mathematician and professor of Applied Mathematics at University College, London [DL03, p.20,21]. Clifford appears to have been one of the small number of mathematicians at the time to be significantly influenced by Grassmann's work. Clifford introduced his *geometric algebra* by uniting the inner and outer product into a single *geometric product*. This product is associative like Grassmann's product, but has the crucial extra feature of being invertible. Although his work is often referred to as *Clifford Algebra*, we use Clifford's original choice—*geometric algebra*.

David Hestenes (1933–Present) is the chief architect of geometric algebra as we know it today [DL03, p. 122-124]. Throughout much of his career, he has worked to develop geometric algebra as a unified language for mathematics and physics. Hestenes, now retired, is the prime mover behind the contemporary resurgence of interest in geometric algebra and other offshoots of Clifford algebras as ways of formalizing theoretical physics. In 1984, he published *Clifford Algebra to Geometric Calculus*. This book, coauthored by Garret Sobczyk and subtitled *A Unified Language for Mathematics and Physics*, develops a complete system of Geometric Calculus. In 1986 Hestenes published *New Foundations for Classical Mechanics*. This book presents many of the topics covered in a classical undergraduate physics class using the language of geometric calculus.

Chapter 2 of this thesis presents an introduction to geometric algebra for the uninitiated. Chapter 3 contains examples of how some of the more traditional topics of mathematics can be re-expressed in terms of geometric algebra and gives proofs of several important theorems from geometry. Chapter 4 introduces the conformal model. This is a current topic among researchers in geometric algebra as it is finding wide applications in computer graphics and robotics. Because geometric algebra is a relatively new topic, authors of the literature have not yet standardized their notation. Appendix A provides a list of some of the notational conventions used in the literature. Appendix B provides a reference list of formulas and identities used in geometric algebra, along with some of their derivations. Appendix C provides a glossary of terms.

Geometric algebra is really all about how we multiply vectors, i.e., about the geometric product and how we interpret that product. Hestenes once wrote that,

> "Physicists had not learned properly how to multiply vectors and, as a result of attempts to overcome this, have evolved a variety of mathematical systems and notations that has come to resemble Babel." [GLD93]

It is our belief that geometric algebra is here to stay and may someday be a part of every mainstream mathematics and physics program.

CHAPTER 2

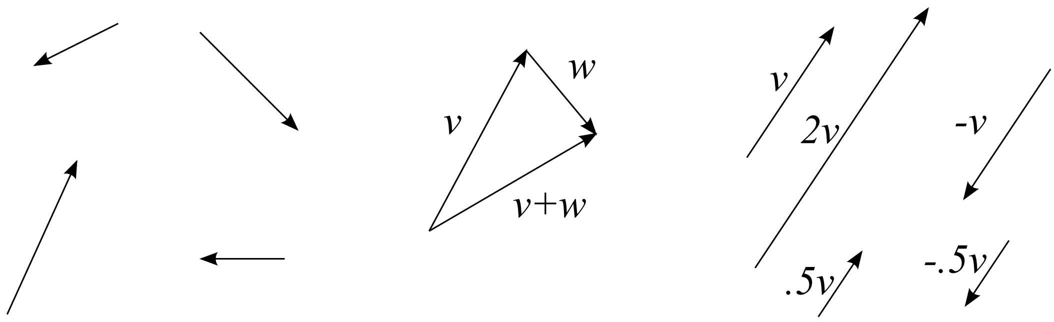
A BRIEF INTRODUCTION TO GEOMETRIC ALGEBRA

This chapter is intended to provide a brief introduction to geometric algebra for the uninitiated. No proofs are provided. This material is intentionally presented with a lack of rigor in hopes of bringing the reader up-to-speed as quickly as possible on the objects, operations, and properties of geometric algebra. We limit our discussion of geometric algebra to two and three dimensions for simplicity, but part of the power and beauty of geometric algebra is that it extends to any dimension. A primary motivation for geometric algebra is to unify all of modern physics under one umbrella. For this reason, the theory has been well-developed for higher dimensions. Much of the material and terminology used in this chapter follows Macdonald [Mac10].

2.1 Scalars, Vectors, and Vector Spaces

When students begin their study of physics, they learn that there are two types of physical quantities: *scalar* and *vector*. Those quantities that have only magnitude and no direction, like temperature and volume, are called *scalar* quantities. Quantities that have both magnitude and direction, like force and velocity, are called *vector* quantities. This thesis uses lower case letters a, b, c to represent scalars and lower case bold letters $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to represent vectors.

Introductory physics classes emphasize the visualization of a vector quantity, such as force or velocity, as a line segment with an arrowhead at the end. The length of the line segment shows the magnitude of the vector and the arrowhead indicates the direction (see Figure 2.1a). This visualization of vectors as *oriented lengths* is used to introduce the concepts of vector addition and scalar multiplication (see Figures 2.1b and 2.1c).



(a) Examples (b) Vector Addition (c) Scalar Multiplication

Figure 2.1: Vectors

In upper-division math courses we are introduced to the more abstract concept of a *vector space*. We learn that any set on which vector addition and scalar multiplication operations are defined is a vector space, provided that certain properties hold. This concept of a vector space is formally defined in Definition 2.1.1.

Definition 2.1.1 (Vector space).

A **vector space** V *is a set of objects called* **vectors**. *There are two operations defined on* V *called* **vector addition** *and* **scalar multiplication**. *There also exists a zero vector* $\mathbf{0}$. *Axioms V0–V7 must be satisfied for all vectors* $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *and all scalars* a *and* $b \in \mathbb{R}$ *(in this thesis, we limit scalars to the field of real numbers):*

V0:	$\mathbf{v} + \mathbf{w} \in V$ *and* $a\mathbf{v} \in V$,	*closure*
V1:	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$,	*commutativity*
V2:	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$,	*associativity*
V3:	$\mathbf{v} + \mathbf{0} = \mathbf{v}$,	*additive identity*
V4:	$\forall\ \mathbf{v} \in V, \exists\ \mathbf{w} \in\ V$ *s.t.* $\mathbf{v} + \mathbf{w} = \mathbf{0}$,	*additive inverse*
V5:	$1\mathbf{v} = \mathbf{v}$,	*multiplicative identity*
V6:	$(ab)\mathbf{v} = a(b\mathbf{v})$,	*associativity*
V7:	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$	*distributive properties*
	$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.	

Along with this definition of a vector space comes the realization that there are many different mathematical objects, other than oriented lengths, that qualify as vectors. Take, for example, the set of all polynomials of order less than or equal to n, where n is some positive integer. This set is a vector space. The abstraction of the concept of a vector makes visualization much more difficult. But at the same time, it vastly increases the range of applications to which the theory can be applied.

Geometric algebra introduces several new vector spaces that have important significance to geometry. In addition, the direct sum of these vector spaces is also very important in geometric algebra. The following sections describe these vector spaces and introduce two new operations, the *outer product* and the *geometric product.*

2.2 Oriented Lengths

For the next few sections, we refer to the common vectors of Euclidean 3-space as *oriented lengths.*

Definition 2.2.1 (Oriented length).

An **oriented length v** *is an oriented segment of a line. The line segment has an* **initial point** *and an* **end point** *with an arrowhead (see Figure 2.2). The length of* **v** *is called its* **norm***, denoted by* $|\mathbf{v}|$.

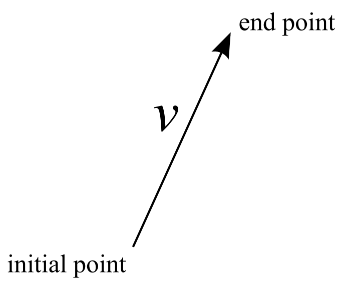


Figure 2.2: Oriented length

Two oriented lengths are said to be *equivalent* if they have the same *attitude*, *orientation*, and *norm*. These three properties of an oriented length are important for future discussion, so we formally define them here.

Definition 2.2.2 (Properties of oriented lengths[1]).

attitude - *Two oriented lengths have the same attitude if they lie on the same line or on parallel lines.*

orientation - *Two oriented lengths have the same orientation if they have the same attitude and their arrows point in the same direction. Thus,* **v** *and* $-\mathbf{v}$ *have opposite orientations.*

[1] The terms *attitude*, *orientation*, and *norm* for oriented lengths are defined in some detail in Dorst, Fontijne, and Mann [DFM07, p. 26].

norm - *Two oriented lengths have the same norm if they have the same length.*

For example, if $\mathbf{v}$ is an oriented length, then $-\mathbf{v}$ is an oriented length with the same attitude and norm, but with opposite orientation.

We now define scalar multiplication, vector addition, and zero for oriented lengths.

Definition 2.2.3.

Scalar multiplication: *The scalar multiple* $a\mathbf{u}$ *is an oriented length parallel to* $\mathbf{u}$ *with norm* $|a|\,|\mathbf{u}|$. *If* $a > 0$ *then* $a\mathbf{u}$ *has the same orientation as* $\mathbf{u}$. *If* $a < 0$ *then* $a\mathbf{u}$ *has the opposite orientation as* $\mathbf{u}$. *If* $a = 0$ *then* $a\mathbf{u} = \mathbf{0}$*, the zero oriented length (see below).*

Vector addition: *To determine the sum of two oriented lengths* $\mathbf{v}$ *and* $\mathbf{w}$*, we move the oriented length* $\mathbf{w}$ *parallel to itself so that its initial point coincides with the end point of the oriented length* $\mathbf{v}$ *(see Figure 2.3). The sum* $\mathbf{v} + \mathbf{w}$ *then equals the oriented length whose initial point equals the initial point of* $\mathbf{v}$ *and whose end point equals the end point of the moved oriented length* $\mathbf{w}$.

Zero: *It is useful to define the* zero oriented length*, denoted by* $\mathbf{0}$*, with the following properties:*

$$\begin{aligned} a\mathbf{0} &= \mathbf{0}, \quad \textit{for all scalars } a, \\ 0\mathbf{u} &= \mathbf{0}, \quad \textit{for all oriented lengths } \mathbf{u}, \\ \mathbf{u} + \mathbf{0} &= \mathbf{u}, \quad \textit{for all oriented lengths } \mathbf{u}. \end{aligned} \tag{2.1}$$

Note that attitude and orientation are not defined for $\mathbf{0}$*; the norm of* $\mathbf{0}$ *is 0.*

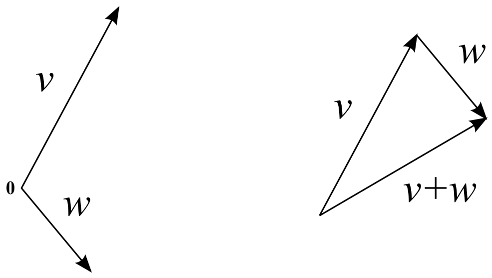


Figure 2.3: The sum of two oriented lengths.

2.2.1 The Inner Product

An important operation defined on two oriented lengths is the *inner product.*

Definition 2.2.4 (Inner product).

The **inner product** *of the nonzero oriented lengths* $\mathbf{u}$ *and* $\mathbf{v}$ *is the scalar defined by*

$$\boxed{\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta, \qquad 0 \le \theta \le \pi}, \tag{2.2}$$

where θ *is the angle between the oriented lengths when one is moved parallel to itself so that the initial points of the two oriented lengths coincide (see Figure 2.4). We also define* $\mathbf{v} \cdot \mathbf{0} = \mathbf{0} \cdot \mathbf{v} = 0$ *for all* $\mathbf{v} \in V$*. Note that* $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.

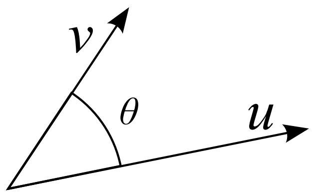


Figure 2.4: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$.

An inner product of oriented lengths has the following properties:

Theorem 2.2.5 (Inner product properties).

Let $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *be oriented lengths and let* a *be a scalar. Then*

I1:	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,	*commutativity*
I2:	$(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v})$,	*homogeneity*
	$\mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v})$,	
I3:	$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$,	*distributive*
	$\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}$,	
I4:	*If* $\mathbf{v} \neq \mathbf{0}$, *then* $\mathbf{v} \cdot \mathbf{v} > 0$.	*positivity*

A vector space, along with an inner product, is called an *inner-product space.* Now that we have the inner product, we can use it to test if two oriented lengths are *orthogonal.*

Theorem 2.2.6 (Orthogonal vectors test).

Vectors $\mathbf{u}$ *and* $\mathbf{v}$ *are* **orthogonal** *if and only if* $\mathbf{u} \cdot \mathbf{v} = 0$. *Note that the* **zero oriented length 0** *is orthogonal to every vector.*

Let $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ be any three unit oriented lengths in Euclidean 3-space such that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j. \end{cases}$$

then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ forms an orthonormal basis for the vector space of oriented lengths in Euclidean 3-space (see Figure 2.5). The *dimension* of any finite-dimensional vector space is defined to be the the number of elements in any basis for that vector space. Thus, Euclidean 3-space has dimension three.

With this basis, any oriented length $\mathbf{v}$ in Euclidean 3-space can be expressed

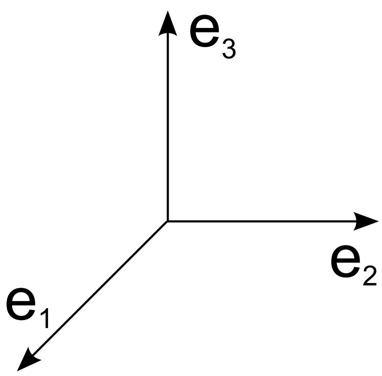


Figure 2.5: An orthonormal basis.

as a unique linear combination of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$.

$$\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$$

Oriented lengths in $\mathbb{R}^3$ satisfy axioms V0–V7 of Definition 2.1.1 and thus form a vector space of dimension three.

2.3 Oriented Areas

Now that we have a clear understanding of what an oriented length is, we would like to turn our attention to the vector space of *oriented areas.*

Definition 2.3.1 (Oriented area).

An **oriented area B** *is an oriented segment or region of a plane. These "segments" or "regions" of a plane must have well-defined, finite areas. The area of* **B** *is called its* **norm**, *denoted by* $|\mathbf{B}|$. *These "segments" or "regions" of a plane must also have a well-defined orientation (usually denoted as clockwise or counterclockwise).*

We use the uppercase bold letter **B** to represent oriented areas (see Figure 2.6). Just like oriented lengths, two oriented areas $\mathbf{B}_1$ and $\mathbf{B}_2$ are considered

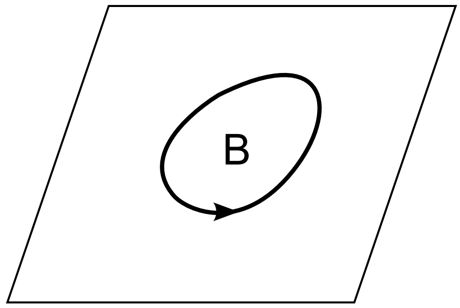


Figure 2.6: An oriented area **B** in a plane.

equivalent ($\mathbf{B}_1 \sim \mathbf{B}_2$) if they have the same *attitude*, *orientation*, and *norm*. The following definition captures these ideas.

Definition 2.3.2 (**Properties of oriented areas**[2]).

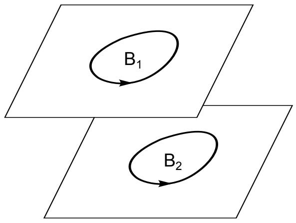
attitude - *Two oriented areas have the same attitude if they lie on the same plane or on parallel planes.*

orientation - *Two oriented areas have the same orientation if they have the same attitude and the same rotational direction, clockwise or counterclockwise, as we traverse the perimeter of the region.*

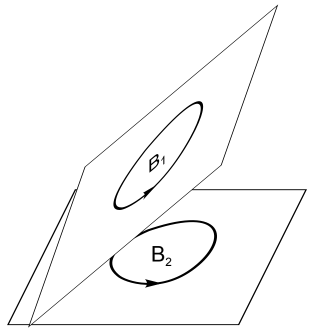
norm - *Two oriented areas have the same norm if they have the same area.*

See Figure 2.7.

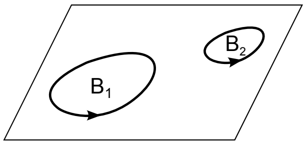
[2] The terms *attitude*, *orientation*, and *norm* for oriented areas are defined in some detail in Dorst, Fontijne, and Mann [DFM07, p. 27–28].



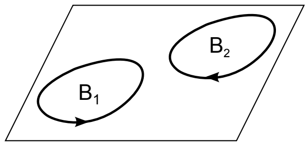
(a) $\mathbf{B}_1 \sim \mathbf{B}_2$
Equivalent.



(b) $\mathbf{B}_1 \nsim \mathbf{B}_2$
Different Attitudes.



(c) $\mathbf{B}_1 \nsim \mathbf{B}_2$
Different norms.



(d) $\mathbf{B}_1 \sim -\mathbf{B}_2$
Different Orientations.

Figure 2.7: Oriented area equivalency.

An oriented area $\mathbf{B}_1$ of any shape can be shown to be equivalent to an oriented rectangle in the same plane. Suppose we have two perpendicular oriented lengths $\mathbf{u}$ and $\mathbf{v}$ in the same plane as $\mathbf{B}_1$. Suppose also that the rectangle, $\mathbf{B}_2$, formed by $\mathbf{u}$ and $\mathbf{v}$, has the same area as $\mathbf{B}_1$, i.e.,

$$|\mathbf{u}|\,|\mathbf{v}| = |\mathbf{B}_1|.$$

We assign an orientation to $\mathbf{B}_2$ to be the same as that of $\mathbf{B}_1$, thus the two oriented areas are equivalent (see Figure 2.8).

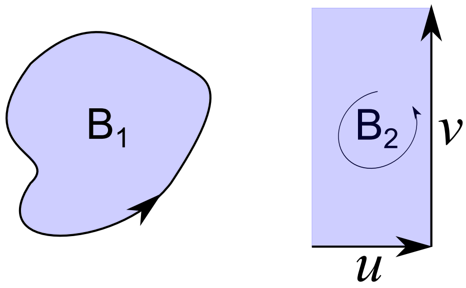


Figure 2.8: Equivalent oriented areas.

We now define scalar multiplication, vector addition, and zero for oriented areas.

Definition 2.3.3.

Scalar multiplication: *The scalar multiple* $a\mathbf{B}$ *is an oriented area in the same plane as* $\mathbf{B}$ *or in a plane parallel to* $\mathbf{B}$ *with norm* $|a|\,|\mathbf{B}|$*. If* $a > 0$ *then* $a\mathbf{B}$ *has the same orientation as* $\mathbf{B}$*. If* $a < 0$ *then* $a\mathbf{B}$ *has the opposite orientation. If* $a = 0$ *then* $a\mathbf{B} = \mathbf{0}$*, the zero oriented area (see Figure 2.9).*

Vector addition: *There are two cases.*

(1) Suppose that the oriented areas $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are in the same or parallel planes. Let* $\mathbf{B}$ *be a* unit oriented area*—i.e., an oriented area with norm of one—parallel to* $\mathbf{B}_1$ *and* $\mathbf{B}_2$*. Then* $\mathbf{B}_1 \sim b_1\mathbf{B}$ *and* $\mathbf{B}_2 \sim b_2\mathbf{B}$*, where* b_1 *and* b_2 *are scalars with* $|b_i| = |B_i|$*. Define* $\mathbf{B}_1 + \mathbf{B}_2 = (b_1 + b_2)\mathbf{B}$*.*

(2) Suppose that the oriented areas $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *lie in nonparallel planes. In Euclidean 3-space, two nonparallel planes must intersect in a line* l *(see Figure 2.10).*

(a) *Choose an oriented length* $\mathbf{w}$ *on the line of intersection of the two planes containing* $\mathbf{B}_1$ *and* $\mathbf{B}_2$.

(b) *Construct an oriented length* $\mathbf{u}$ *in the* $\mathbf{B}_1$ *plane such that when the initial point of* $\mathbf{u}$ *is placed on the initial point of* $\mathbf{w}$, $\mathbf{u}$ *indicates the orientation of* $\mathbf{B}_1$, $\mathbf{u} \perp \mathbf{w}$, *and* $|\mathbf{u}|\,|\mathbf{w}| = |\mathbf{B}_1|$.

(c) *Likewise, construct an oriented length* $\mathbf{v}$ *in the* $\mathbf{B}_2$ *plane such that when the initial point of* $\mathbf{v}$ *is placed on the initial point of* $\mathbf{w}$, $\mathbf{v}$ *indicates the orientation of* $\mathbf{B}_2$, $\mathbf{v} \perp \mathbf{w}$, *and* $|\mathbf{v}|\,|\mathbf{w}| = |\mathbf{B}_2|$.

(d) *Define* $\mathbf{B}_1 + \mathbf{B}_2$ *to be the oriented area formed from a rectangle with sides* $\mathbf{u} + \mathbf{v}$ *and* $\mathbf{w}$ *and with orientation given by* $\mathbf{u} + \mathbf{v}$. [3]

Zero: *The zero oriented area, denoted by* $\mathbf{0}$, *has the following properties:*

$a\mathbf{0} = \mathbf{0}$, *for all scalars* a,

$0\mathbf{B} = \mathbf{0}$, *for all oriented areas* $\mathbf{B}$,

$\mathbf{B} + \mathbf{0} = \mathbf{B}$, *for all oriented areas* $\mathbf{B}$.

As with the zero oriented length, attitude and orientation are not defined for the zero oriented area; the norm of $\mathbf{0}$ *is 0.*

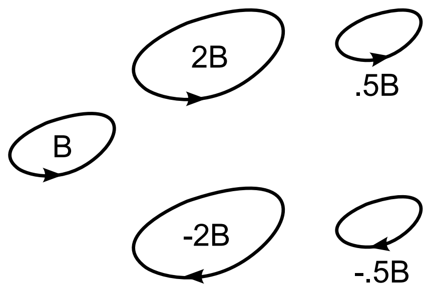


Figure 2.9: Scalar multiplication of oriented areas.

[3] Note that oriented area addition is similar to finding the diagonal of a parallelogram, but in this case we are finding the diagonal plane of a parallelepiped.

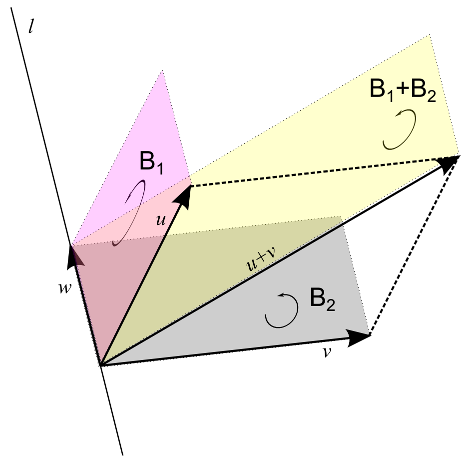


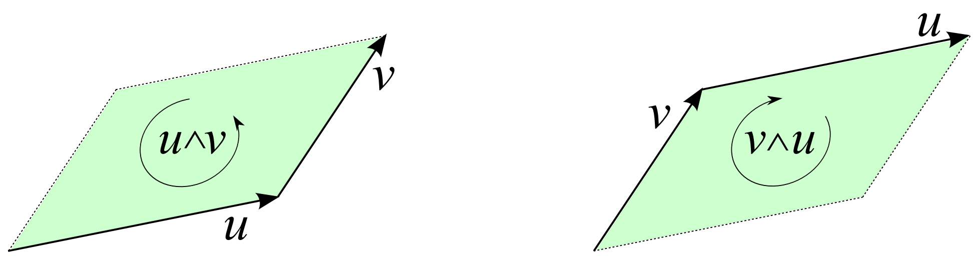
Figure 2.10: Addition of oriented areas.

2.3.1 The Outer Product

As previously stated, two oriented lengths can be used to construct an oriented area. We now formalize this concept by defining the *outer product* of two oriented lengths. This new operation is a key concept of geometric algebra.

If a point is moved a distance and direction specified by a non-zero oriented length **u**, it sweeps out a directed line segment. If this line segment is then moved a distance and direction specified by a second non-zero, non-parallel oriented length **v**, it sweeps out a parallelogram (see Figure 2.11a). Because the oriented area corresponding to this parallelogram is uniquely determined by this geometric construction, it may be regarded as a kind of "product" of the oriented lengths **u** and **v**.[4] If either of the oriented lengths happens to be zero, or if they are parallel, then no parallelogram is constructed. In this case we define the outer product to be the zero oriented area.

[4] This description of the outer product comes from Hestenes [Hes03, p. 22–23].



(a) $\mathbf{u} \wedge \mathbf{v}$. (b) $\mathbf{v} \wedge \mathbf{u}$.

Figure 2.11: Outer product.

Definition 2.3.4 (Outer product).

The **outer product** *of two non-zero, non-parallel oriented lengths* $\mathbf{u}$ *and* $\mathbf{v}$*, denoted by* $\mathbf{u} \wedge \mathbf{v}$*, is an oriented area in the plane of* $\mathbf{u}$ *and* $\mathbf{v}$ *formed by sweeping the oriented length* $\mathbf{u}$ *along the length of the oriented length* $\mathbf{v}$*. Its* norm *is the area of the parallelogram with edges* $\mathbf{u}$ *and* $\mathbf{v}$*. Its* orientation *can be determined by sliding the second oriented length* $\mathbf{v}$ *parallel to itself until its initial point touches the end point of the first oriented length* $\mathbf{u}$*. The orientation is then visualized as traversing the perimeter of the parallelogram in the direction indicated by arrows on* $\mathbf{u}$ *and* $\mathbf{v}$*. If either of the oriented lengths happens to be zero, or if they are parallel, then we define the outer product* $\mathbf{u} \wedge \mathbf{v}$ *to be the zero oriented area. A "wedge" is used to denote this new kind of multiplication to distinguish it from the "dot" denoting the inner product.*

Note that the orientation of $\mathbf{v} \wedge \mathbf{u}$ is opposite that of $\mathbf{u} \wedge \mathbf{v}$. Thus, $\mathbf{v} \wedge \mathbf{u} = -(\mathbf{u} \wedge \mathbf{v})$ (see Figure 2.11b). As long as neither oriented length is zero and they are not parallel, the oriented area generated by an outer product has the shape of a parallelogram. It is important to remember that oriented areas, in general, have no shape. It is the area that is important, not the shape.

Theorem 2.3.5 (Properties of the outer product).

O1:	$\mathbf{u} \wedge \mathbf{u} = \mathbf{0}$,	*parallel*
O2:	$\mathbf{u} \wedge \mathbf{v} = -(\mathbf{v} \wedge \mathbf{u})$,	*antisymmetry*
O3:	$(a\mathbf{u}) \wedge \mathbf{v} = a(\mathbf{u} \wedge \mathbf{v})$,	*homogeneity*
	$\mathbf{u} \wedge (a\mathbf{v}) = a(\mathbf{u} \wedge \mathbf{v})$,	
O4:	$(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w}$,	*distributive*
	$\mathbf{w} \wedge (\mathbf{u} + \mathbf{v}) = \mathbf{w} \wedge \mathbf{u} + \mathbf{w} \wedge \mathbf{v}$.	

Definition 2.3.6 (Unit oriented area).

Let $\{\mathbf{b}_1, \mathbf{b}_2\}$ *be an orthonormal basis for a plane that contains the origin, i.e., of a two-dimensional homogeneous subspace of* $\mathbb{R}^n$. *Because both* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *are unit oriented lengths,* $\mathbf{b}_1 \wedge \mathbf{b}_2$ *forms a 1-by-1 square with area (norm) of one. This is referred to as a* **unit oriented area**.

Theorem 2.3.7.

Let $\{\mathbf{b}_1, \mathbf{b}_2\}$ *be an orthonormal basis for a homogeneous plane. Orient the plane with* $\mathbf{b}_1 \wedge \mathbf{b}_2$. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in the plane. Let* θ *be the oriented angle from* $\mathbf{u}$ *to* $\mathbf{v}$. *Then*

$$\boxed{\mathbf{u} \wedge \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \sin\theta\,(\mathbf{b}_1 \wedge \mathbf{b}_2), \qquad -\pi < \theta \leq \pi}. \tag{2.3}$$

We can express any oriented length in $\mathbb{R}^3$ as a linear combination of the standard basis $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$. Let $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ be two oriented lengths in $\mathbb{R}^3$, where u_i and v_i are scalars. The outer product $\mathbf{u} \wedge \mathbf{v}$

can be calculated as:

$$\begin{aligned}\mathbf{u}\wedge\mathbf{v} = \ & u_1v_1(\mathbf{e}_1\wedge\mathbf{e}_1) + u_2v_1(\mathbf{e}_2\wedge\mathbf{e}_1) + u_3v_1(\mathbf{e}_3\wedge\mathbf{e}_1) \\ & + u_1v_2(\mathbf{e}_1\wedge\mathbf{e}_2) + u_2v_2(\mathbf{e}_2\wedge\mathbf{e}_2) + u_3v_2(\mathbf{e}_3\wedge\mathbf{e}_2) \\ & + u_1v_3(\mathbf{e}_1\wedge\mathbf{e}_3) + u_2v_3(\mathbf{e}_2\wedge\mathbf{e}_3) + u_3v_3(\mathbf{e}_3\wedge\mathbf{e}_3) \\ = \ & (u_2v_3 - u_3v_2)\,\mathbf{e}_2\wedge\mathbf{e}_3 + (u_3v_1 - u_1v_3)\,\mathbf{e}_3\wedge\mathbf{e}_1 + (u_1v_2 - u_2v_1)\,\mathbf{e}_1\wedge\mathbf{e}_2.\end{aligned} \tag{2.4}$$

The following equation is useful for remembering the calculation of Eq. (2.4).

$$\mathbf{u}\wedge\mathbf{v} = \begin{vmatrix} \mathbf{e}_2\wedge\mathbf{e}_3 & \mathbf{e}_3\wedge\mathbf{e}_1 & \mathbf{e}_1\wedge\mathbf{e}_2 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{2.5}$$

This calculation is very similar to that of the cross product $\mathbf{u}\times\mathbf{v}$, only the cross product results in a vector perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$, whereas the outer product results in an oriented area in the plane of $\mathbf{u}$ and $\mathbf{v}$.

Theorem 2.3.8.

The oriented areas $\{\mathbf{e}_2\wedge\mathbf{e}_3,\ \mathbf{e}_3\wedge\mathbf{e}_1,\ \mathbf{e}_1\wedge\mathbf{e}_2\}$ *form a basis for the oriented areas in Euclidean 3-space (see Figure 2.12).*

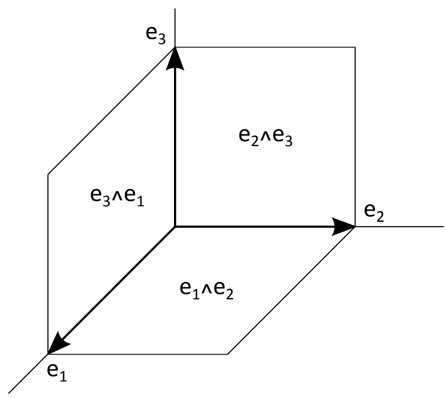


Figure 2.12: Oriented area basis.

Proof. Figure 2.8 shows that every oriented area is an outer product of two vectors. Then Eq. (2.4) shows that the three oriented areas span the space of bivectors. Now consider

$$a_1\, \mathbf{e}_2 \wedge \mathbf{e}_3 + a_2\, \mathbf{e}_3 \wedge \mathbf{e}_1 + a_3\, \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_2 \wedge (a_1\mathbf{e}_3 - a_3\mathbf{e}_1) + a_2\, \mathbf{e}_3 \wedge \mathbf{e}_1. \tag{2.6}$$

The term on the right of the equal sign in Eq. (2.6) is the sum of two oriented areas. Figure 2.13 can be helpful in visualizing this when $a_i = 1$. According to Figure 2.10, the sum of two nonzero oriented areas in different planes is nonzero. Therefore, the sum of the terms in Eq. (2.6) is zero only when $a_1 = a_2 = a_3 = 0$. Thus, the three oriented areas are linearly independent. □

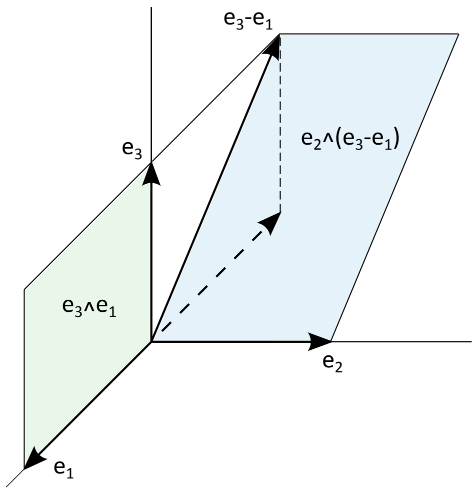


Figure 2.13: Visual aid for Eq. (2.6).

Oriented areas satisfy axioms V0–V7 of Definition 2.1.1 and thus form a vector space of dimension three.

2.4 Oriented Volumes

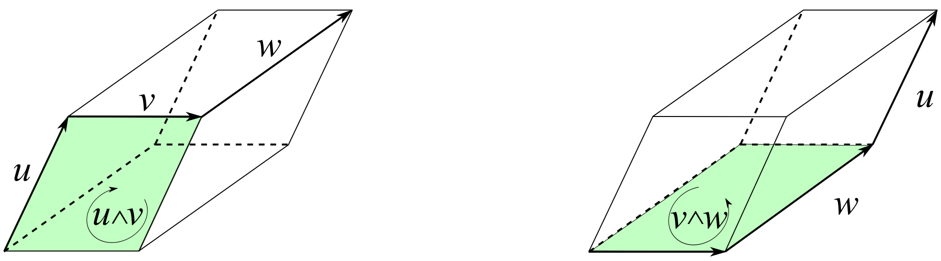
In addition to oriented lengths and oriented areas, we now add a third vector space, that of *oriented volumes.*

Definition 2.4.1 (Oriented volume).

An **oriented volume** $\mathbf{T}$ *is an oriented segment of (i.e., solid in) three dimensional space. The volume of* $\mathbf{T}$ *is called its* **norm**, *denoted by* $|\mathbf{T}|$. *We use the uppercase bold letter* $\mathbf{T}$ *to represent an oriented volume.*

The theory of the outer product, as described so far, calls for a generalization[5]. Just as a plane segment is swept out by a moving line segment, a "space segment" is swept out by a moving plane segment. Thus, the points on an oriented parallelogram, specified by the oriented area $\mathbf{u} \wedge \mathbf{v}$ moving a distance and direction specified by an oriented length $\mathbf{w}$ (not in the $\mathbf{u} \wedge \mathbf{v}$ plane,) sweep out an oriented parallelepiped (see Figure 2.14a), which defines a new kind of directed volume $\mathbf{T}$ called an oriented volume. In this way we define the outer product of the oriented area $\mathbf{u} \wedge \mathbf{v}$ with the oriented length $\mathbf{w}$. Thus, we write

$$\mathbf{T} = (\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}.$$



(a) $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$. (b) $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$.

Figure 2.14: Oriented volumes.

[5] This description comes from Hestenes [Hes03, p. 26–28].

Oriented volumes contain no information about the shape of the solid. Two oriented volumes are equivalent if they have the same *attitude*, *orientation*, and *norm*.

Definition 2.4.2 (Properties of oriented volumes[6]).

attitude - *In Euclidean 3-space, there is only one attitude for all oriented volumes. In dimensions greater than three, the attitude represents the three-dimensional subspace that contains the oriented volume.*

orientation - *The orientation of an oriented volume is usually referred to as handedness. For an oriented volume formed by the outer product* $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$*, we use the right-hand rule: we curl the fingers of our right hand around* $\mathbf{u} \wedge \mathbf{v}$ *consistent with its orientation. If the third oriented length* $\mathbf{w}$ *is in the direction of our thumb, then volume is right-handed, otherwise it is left-handed. Two oriented volumes have the same orientation if they have the same handedness. The volume spanned by* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ *(in that order), has the opposite orientation from that spanned by* $\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3$ *(in that order), i.e., moving the plane segment* $\mathbf{e}_1 \wedge \mathbf{e}_2$ *along the line segment* $\mathbf{e}_3$ *has the opposite orientation from that of moving it along the line segment* $-\mathbf{e}_3$*.*

norm - *Two oriented volumes have the same norm if they have the same volume.*

We noted several properties of the outer product in Theorem 2.3.5. We now need to add one new result, namely, the conclusion that the outer product should obey the associative rule.

[6] The terms *attitude*, *orientation*, and *norm* for oriented volumes are defined in some detail in Dorst, Fontijne, and Mann [DFM07, p. 33–35].

Theorem 2.4.3 (Properties of the outer product).

O5: $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$. *associativity*

In figure 2.14, it is easy to see that both $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$ and $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$ are right-handed systems. The orientation of an oriented area depends on the order of its factors. The anticommutative property together with the associative property imply that exchanging any pair of factors in an outer product reverses the orientation of the result. From this we can derive the following equalities:

$$\mathbf{T} = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u} = \mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v}, \tag{2.7}$$

$$-\mathbf{T} = \mathbf{u} \wedge \mathbf{w} \wedge \mathbf{v} = \mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} = \mathbf{w} \wedge \mathbf{v} \wedge \mathbf{u}. \tag{2.8}$$

The outer product $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ of three oriented lengths $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$ represents the oriented volume of the parallelepiped with edges $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3). \tag{2.9}$$

The oriented volume $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ has norm one. Any oriented volume in Euclidean 3-space can be expressed in terms of $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$, thus the set $\{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\}$ forms an "orthonormal" basis for oriented volumes in Euclidean 3-space, although orthogonality is a mute point here.

$$\mathbf{T} = t\,(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3), \qquad \text{where } t \text{ is a scalar.}$$

We now define scalar multiplication, vector addition, and zero for oriented volumes.

Definition 2.4.4.

Scalar multiplication: *The scalar multiple* $a\mathbf{T}$ *is an oriented volume with norm* $|a|\,|\mathbf{T}|$. *If* $a > 0$ *then* $a\mathbf{T}$ *has the same orientation as* $\mathbf{T}$. *If* $a < 0$ *then* $a\mathbf{T}$ *has the opposite orientation as* $\mathbf{T}$. *If* $a = 0$ *then* $a\mathbf{T} = \mathbf{0}$, *the zero oriented volume (see below). That is, if* $t = |\mathbf{T}|$, *then* $a\mathbf{T} = a\,(t\,(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)) = a\,t\,(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$.

Vector addition: $\mathbf{T}_1 + \mathbf{T}_2$ *is an oriented volume such that*

$$\mathbf{T}_1 + \mathbf{T}_2 = t_1\,(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) + t_2\,(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = (t_1 + t_2)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3).$$

Zero: *The zero oriented volume, denoted by* $\mathbf{0}$, *has the following properties:*

$$a\mathbf{0} = \mathbf{0}, \quad \text{for all scalars } a,$$

$$0\mathbf{T} = \mathbf{0}, \quad \text{for all oriented volumes } \mathbf{T},$$

$$\mathbf{T} + \mathbf{0} = \mathbf{T}, \quad \text{for all oriented volumes } \mathbf{T}.$$

Once again, attitude and orientation are not defined for the zero oriented volume; the norm of $\mathbf{0}$ *is 0.*

Oriented volumes satisfy axioms V0–V7 of Definition 2.1.1 and thus form a vector space of dimension one.

2.5 Scalars

Although we do not normally think of scalars as forming a vector space, a closer examination reveals that scalars have much in common with the oriented volumes of Euclidean 3-space. Two scalars are equivalent if they have the same *attitude*, *orientation*, and *norm*.

Definition 2.5.1 (Properties of scalars).

attitude - *There is only one attitude for all scalars. We can visualize scalars as points on the origin. All points on the origin have the same attitude.*

orientation - *Two scalars have the same orientation if they have the same sign (i.e., positive or negative).*

norm - *Two scalars have the same norm if they have the same magnitude. The magnitude or* norm *of a scalar is its absolute value denoted by* $|a|$.

The set $\{1\}$ forms an "orthonormal" basis for scalars. Any scalar s can be expressed in terms of of this basis.

$$s = s\,1, \qquad \text{where } s \text{ is a scalar.}$$

Scalars satisfy axioms V0–V7 of Definition 2.1.1 and thus form a vector space of dimension one.

2.6 The $\mathbb{G}^3$ Vector Space

We have used the terms *oriented length*, *oriented area*, and *oriented volume* because they are geometrically descriptive, but they are not standard. From now on, we use the standard terms *vector* for an oriented length $\mathbf{v}$, *bivector* for an oriented area $\mathbf{B}$, and *trivector* for an oriented volume $\mathbf{T}$.

The geometric algebra $\mathbb{G}^3$ is a vector space formed from the direct sum of the scalar, vector, bivector, and trivector vector spaces, along with an additional operation called the geometric product.

Definition 2.6.1 (**The $\mathbb{G}^3$ vector space**).

The $\mathbb{G}^3$ vector space consists of objects of the form

$$\mathbf{M} = s + \mathbf{v} + \mathbf{B} + \mathbf{T}$$

where s is a **scalar**, $\mathbf{v}$ *is a* **vector**, $\mathbf{B}$ *is a* **bivector**, *and* $\mathbf{T}$ *is a* **trivector**. $\mathbf{M}$ *is referred to as a* **multivector**.

Definition 2.6.2 (**Grade**).

Each component of a multivector has an associated **grade**. *The grade indicates the number of vector factors in a non-zero component, e.g.,*

Component	***Grade***	***Subspace***
scalar	*0*	$\mathbb{R}$
vector	*1*	$\mathbb{R}^3$
bivector	*2*	$\bigwedge^2 \mathbb{R}^3$
trivector	*3*	$\bigwedge^3 \mathbb{R}^3$

Where $\bigwedge^k \mathbb{R}^3$ represents the grade k subspace of $\mathbb{G}^3$ and $\mathbb{G}^3$ is the direct sum of the four subspaces, i.e.,

$$\mathbb{G}^3 = \mathbb{R} \oplus \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3 \oplus \bigwedge^3 \mathbb{R}^3$$

Two multivectors $\mathbf{M}_1$ and $\mathbf{M}_2$ are equivalent if and only if their components are equal.

Let $\mathbf{M}_1 = s_1 + \mathbf{v}_1 + \mathbf{B}_1 + \mathbf{T}_1$ and $\mathbf{M}_2 = s_2 + \mathbf{v}_2 + \mathbf{B}_2 + \mathbf{T}_2$.

Then $\mathbf{M}_1 = \mathbf{M}_2$ if and only if $s_1 = s_2$, $\mathbf{v}_1 = \mathbf{v}_2$, $\mathbf{B}_1 = \mathbf{B}_2$, and $\mathbf{T}_1 = \mathbf{T}_2$.

We now define scalar multiplication, vector addition, and zero for $\mathbb{G}^3$.

Definition 2.6.3.

Scalar multiplication: *If* $\mathbf{M} = s + \mathbf{v} + \mathbf{B} + \mathbf{T}$, *then* $a\mathbf{M} = as + a\mathbf{v} + a\mathbf{B} + a\mathbf{T}$.

Vector addition: *If* $\mathbf{M}_1 = s_1 + \mathbf{v}_1 + \mathbf{B}_1 + \mathbf{T}_1$ *and* $\mathbf{M}_2 = s_2 + \mathbf{v}_2 + \mathbf{B}_2 + \mathbf{T}_2$ *then*

$$\mathbf{M}_1 + \mathbf{M}_2 = (s_1 + s_2) + (\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{B}_1 + \mathbf{B}_2) + (\mathbf{T}_1 + \mathbf{T}_2).$$

Zero: *The zero multivector has zero for all its components (scalar, vector, bivector, and trivector). We denote the zero multivector by* $\mathbf{0} = (0, \mathbf{0}, \mathbf{0}, \mathbf{0})$.

The set $\{\underbrace{1,}_{\text{1 scalar}} \underbrace{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3,}_{\text{3 vectors}} \underbrace{\mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_1 \wedge \mathbf{e}_2,}_{\text{3 bivectors}} \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\text{1 trivector}}\}$

forms a basis for $\mathbb{G}^3$. Any multivector $\mathbf{M} \in \mathbb{G}^3$ can be expressed in terms of this basis.

The space $\mathbb{G}^3$ satisfies axioms V0–V7 of Definition 2.1.1 and thus forms a vector space of dimension eight.

2.6.1 The Geometric Product

The geometric product is the key innovation of geometric algebra. It is what gives geometric algebra its power as a mathematical tool.

Definition 2.6.4 (Geometric product).

The **geometric product uv** *of vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *is a scalar plus a bivector:*

$$\boxed{\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}}. \tag{2.10}$$

Theorem 2.6.5 (Properties of the geometric product).

G1:	$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v}$, *if* $\mathbf{u} \parallel \mathbf{v}$,	*inner product if parallel vectors*
G2:	$\mathbf{uv} = \mathbf{u} \wedge \mathbf{v}$, *if* $\mathbf{u} \perp \mathbf{v}$,	*outer product if perpendicular vectors*
G3:	$(a\mathbf{u})\,\mathbf{v} = a\,(\mathbf{uv})$,	*homogeneity*
	$\mathbf{u}\,(a\mathbf{v}) = a\,(\mathbf{uv})$,	
G4:	$(\mathbf{u} + \mathbf{v})\,\mathbf{w} = \mathbf{uw} + \mathbf{vw}$,	*distributive*
	$\mathbf{w}\,(\mathbf{u} + \mathbf{v}) = \mathbf{wu} + \mathbf{wv}$,	
G5:	$\mathbf{u}\,(\mathbf{vw}) = (\mathbf{uv})\,\mathbf{w}$.	*associativity*

If we reverse the order of the terms in the geometric product, we get

$$\mathbf{vu} = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \wedge \mathbf{u} = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \wedge \mathbf{v}. \tag{2.11}$$

We now combine Eq. (2.10) with Eq. (2.11), through addition and subtraction, to get the following important identities:

$$\boxed{\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu})}, \tag{2.12}$$

$$\boxed{\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu})}. \tag{2.13}$$

2.6.2 Multiplicative Inverse of a Vector

In general, the geometric product of two vectors produces a scalar and a bivector. Thus, the geometric product is not closed over the set of vectors but is closed over the larger graded ring of multivectors. Although so far we have focused on the inner, outer, and geometric products of two vectors, it can be shown that these products are defined for all multivectors and form a closed algebra of multivectors.

To say that there exists a multiplicative identity element $\mathbf{1}$ for a vector $\mathbf{v}$ under the geometric product implies that $\mathbf{1}\,\mathbf{v} = \mathbf{v}\,\mathbf{1} = \mathbf{v}$. Although there is no vector $\mathbf{1}$ with this property, if we consider the broader set of multivectors, the scalar 1 works nicely. We now extend our product definitions to include scalar-vector products.

Definition 2.6.6 (**Scalar-vector products**).

Let α be a scalar and $\mathbf{v}$ be a vector, then

$$\textit{Inner:} \qquad \alpha \cdot \mathbf{v} = \mathbf{v} \cdot \alpha = 0 \tag{2.14}$$

$$\textit{Outer:} \qquad \alpha \wedge \mathbf{v} = \mathbf{v} \wedge \alpha = \alpha\,\mathbf{v} \tag{2.15}$$

$$\textit{Geometric:} \qquad \alpha\,\mathbf{v} = \alpha \cdot \mathbf{v} + \alpha \wedge \mathbf{v} = 0 + \alpha\,\mathbf{v} = \alpha\,\mathbf{v} \tag{2.16}$$

Now that we have defined scalar-vector products, it is possible to define the multiplicative inverse of a nonzero vector $\mathbf{v}$ with respect to the geometric product. If $\mathbf{v} \neq 0$, then $\mathbf{v}\,\mathbf{v} = \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \wedge \mathbf{v} = |\mathbf{v}|^2 + 0 = |\mathbf{v}|^2$. Thus, we have,

$$1 = \frac{\mathbf{v}\,\mathbf{v}}{\mathbf{v}\,\mathbf{v}} = \frac{1}{|\mathbf{v}|^2}\,\mathbf{v}\,\mathbf{v} = \left(\frac{1}{|\mathbf{v}|^2}\,\mathbf{v}\right)\mathbf{v} = \mathbf{v}\left(\frac{1}{|\mathbf{v}|^2}\,\mathbf{v}\right),$$

where the scalar 1 plays the role of the multiplicative identity element. This leads us to the following definition:

Definition 2.6.7 (**Multiplicative inverse of a vector**).

Let $\mathbf{v}$ be a nonzero vector. Then the **multiplicative inverse** *of $\mathbf{v}$ is:*

$$\boxed{\mathbf{v}^{-1} = \frac{1}{|\mathbf{v}|^2}\,\mathbf{v}}\,. \tag{2.17}$$

2.6.3 Applying the Geometric Product to Orthonormal Vectors

When we apply the geometric product operation to the orthonormal basis vectors of Euclidean 3-space $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we get several useful identities:

$$\mathbf{e}_1^2 = \mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_1 \cdot \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_1 = 1 + 0 = 1,$$

$$\mathbf{e}_2^2 = \mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_2 + \mathbf{e}_2 \wedge \mathbf{e}_2 = 1 + 0 = 1,$$

$$\mathbf{e}_3^2 = \mathbf{e}_3\mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_3 + \mathbf{e}_3 \wedge \mathbf{e}_3 = 1 + 0 = 1,$$

$$\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_3 = 0 + \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_2 \wedge \mathbf{e}_3,$$

$$\mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_3 \cdot \mathbf{e}_1 + \mathbf{e}_3 \wedge \mathbf{e}_1 = 0 + \mathbf{e}_3 \wedge \mathbf{e}_1 = \mathbf{e}_3 \wedge \mathbf{e}_1,$$

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2 = 0 + \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2.$$

These last three orthonormal bivectors show up so often in equations that going forward we shall abbreviate them as:

$$\mathbf{e}_2\mathbf{e}_3 \equiv \mathbf{e}_{23}, \qquad \mathbf{e}_3\mathbf{e}_1 \equiv \mathbf{e}_{31}, \qquad \mathbf{e}_1\mathbf{e}_2 \equiv \mathbf{e}_{12}.$$

The highest grade component in a geometric algebra is called a *pseudoscalar*. For example, a trivector—grade 3—is the the highest grade component in $\mathbb{G}^3$, thus a trivector is a pseudoscalar in $\mathbb{G}^3$. In the next chapter we also discuss the geometric algebra $\mathbb{G}^2$ where a bivector—grade 2—is a pseudoscalar. The name pseudoscalar comes from the study of dual spaces where the pseudoscalar plays the role that the scalar does in normal space. When discussing $\mathbb{G}^3$, we sometimes abbreviate the unit trivector $(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$—also called the *unit pseudoscalar*—as

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \equiv \mathbf{e}_{123} \equiv \mathbf{I}.$$

Using these abbreviations it is possible to express the basis for $\mathbb{G}^3$, $\{1,\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_2 \wedge \mathbf{e}_3,\ \mathbf{e}_3 \wedge \mathbf{e}_1,\ \mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\}$, in the more compact form $\{\underbrace{1,}_{\text{1 scalar}}\ \underbrace{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,}_{\text{3 vectors}}\ \underbrace{\mathbf{e}_{23},\ \mathbf{e}_{31},\ \mathbf{e}_{12},}_{\text{3 bivectors}}\ \underbrace{\mathbf{e}_{123}}_{\text{1 trivector}}\}$.

So far, we have defined the geometric product for two vectors and for scalar-vector combinations. If we define the geometric product for all of the basis elements of $\mathbb{G}^3$, we can then use that definition to extend the geometric product to all multivectors in $\mathbb{G}^3$. For example, consider the product of two unit bivectors:

$$\mathbf{e}_{12}\mathbf{e}_{23} = (\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_2\mathbf{e}_3) = \mathbf{e}_1(\mathbf{e}_2\mathbf{e}_2)\mathbf{e}_3 = \mathbf{e}_1 1\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_{13}$$

$$\mathbf{e}_{12}\mathbf{e}_{31} = (\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_3\mathbf{e}_1) = (-\mathbf{e}_2\mathbf{e}_1)(-\mathbf{e}_1\mathbf{e}_3) = \mathbf{e}_2(\mathbf{e}_1\mathbf{e}_1)\mathbf{e}_3 = \mathbf{e}_2 1\mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_{23}$$

$$\mathbf{e}_{23}\mathbf{e}_{31} = (\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_3\mathbf{e}_1) = \mathbf{e}_2(\mathbf{e}_3\mathbf{e}_3)\mathbf{e}_1 = \mathbf{e}_2 1\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_{21} = -\mathbf{e}_{12}$$

And lastly, consider the square of the unit pseudoscalar:

$$\mathbf{I}^2 = \mathbf{e}_{123}^2 = \mathbf{e}_{123}\mathbf{e}_{123} = (\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = -1. \tag{2.18}$$

Definition 2.6.8 (Geometric product of $\mathbb{G}^3$ basis elements).

The geometric product of any two basis elements of $\mathbb{G}^3$ is defined in Table 2.1.

Table 2.1: Multiplication table for the $\mathbb{G}^3$ geometric product.

GP	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{123}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{123}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{123}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\mathbf{e}_3$	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	$\mathbf{e}_{12}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	-1	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_1$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$\mathbf{e}_{12}$	-1	$-\mathbf{e}_{23}$	$-\mathbf{e}_2$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	-1	$-\mathbf{e}_3$
$\mathbf{e}_{123}$	$\mathbf{e}_{123}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	-1

An element of a geometric algebra is called a *blade* if it can be expressed as the outer product of k linearly independent vectors. A k-blade is a blade of grade k whose constituent vectors span a k-dimensional subspace.

For example, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are all 1-blades because they each span a 1-dimensional subspace of Euclidean 3-space. In addition, any nontrivial linear combination of these 1-blades is also a 1-blade. Likewise, $\mathbf{e}_{12}$, $\mathbf{e}_{23}$, and $\mathbf{e}_{31}$ are 2-blades because their constituent vectors each span a 2-dimensional subspace. Any nontrivial linear combination of these 2-blades is also a 2-blade. The scalar 1 is called a 0-blade and the unit pseudoscalar $\mathbf{I}$ is called a 3-blade.

In the literature, the term k-vector is often used. A *k-vector* is any linear combination of k-blades. In $\mathbb{G}^3$, all k-vectors are k-blades.

When all the vectors in a geometric product are orthogonal, the orientation is changed if any two vectors are swapped.

$$\mathbf{e}_{21} = \mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_{12},$$

$$\mathbf{e}_{32} = \mathbf{e}_3\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_{23},$$

$$\mathbf{e}_{13} = \mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_1 = -\mathbf{e}_{31}.$$

Likewise, for the unit trivector, all even (or odd) permutations are equivalent:

$$\mathbf{e}_{123} = \mathbf{e}_{231} = \mathbf{e}_{312} = \mathbf{I},$$

$$\mathbf{e}_{132} = \mathbf{e}_{213} = \mathbf{e}_{321} = -\mathbf{I}.$$

We have now introduced all the fundamental objects, operations, and properties of geometric algebra. The inner product, outer product, and geometric product were all defined as operations on vectors. We extend the definitions of the basic operations as needed to include bivectors, trivectors, and scalars. We started with scalars and vectors. We were then able to define the bivector as the outer product of two vectors. The trivector was shown to be the outer product of a vector and a bivector or, via the associative property, the outer product of three vectors.

We have obtained an 8-dimensional vector space, $\mathbb{G}^3$, that is equipped with a multiplication—the geometric product. We then have a ring over the field $\mathbb{R}$, where the underlying set, $\mathbb{G}^3$, and the addition and $\mathbf{0}$ are the same in the ring and vector space and

$$a(\mathbf{M}_1\mathbf{M}_2) = (a\mathbf{M}_1)\mathbf{M}_2 = \mathbf{M}_1(a\mathbf{M}_2)$$

holds for all $a \in \mathbb{R}$ and $\mathbf{M}_1, \mathbf{M}_2 \in \mathbb{G}^3$. A vector space with such a ring structure is an *associative algebra* over the field $\mathbb{R}$. An associative algebra that arises from a vector space with a scalar product in the same manner as this example does from Euclidean 3-space, is called a *Clifford algebra*.

CHAPTER 3

APPLICATIONS OF THE VECTOR SPACE MODEL

3.1 The Vector Space Model

In the previous chapter we introduced the basic concepts of geometric algebra, where $\mathbb{G}^3$ (a vector space of dimension eight) was used to model geometric objects (scalars, vectors, bivectors, and trivectors) in Euclidean 3-space.

In the following chapters of this thesis, we learn that by adding one or two additional dimensions to the space we are studying, we gain certain advantages. For example, if we use $\mathbb{G}^5$ to represent Euclidean 3-space, then we can

- remove the origin as being a special point
- support points and lines at infinity
- provide a single geometric mechanism for representing lines, circles, planes, and spheres
- enjoy all the normal features of Euclidean space.

The different approaches to representing Euclidean spaces are referred to as *models*. When we represent $\mathbb{R}^2$ with $\mathbb{G}^2$ or $\mathbb{R}^3$ with $\mathbb{G}^3$, we are using the *vector space model*. When we represent $\mathbb{R}^2$ with $\mathbb{G}^3$ or $\mathbb{R}^3$ with $\mathbb{G}^4$, we are using the *homogeneous model*. When we represent $\mathbb{R}^2$ with $\mathbb{G}^4$ or $\mathbb{R}^3$ with $\mathbb{G}^5$, we are using the *conformal model*.

In all of the previous chapters we have been using the vector space model. Using this model, we treat vectors, bivectors, and trivectors as having no fixed position, i.e., they can be moved anywhere parallel to themselves without changing their orientation. With the vector space model we can perform many operations on subspaces. We can span them, project them, and rotate them, but we do not move them off the origin.

This chapter focuses on a few applications of the vector space model. We present examples to help relate geometric algebra to other more familiar topics in mathematics. Geometric algebra should be viewed as an abstract structure that subsumes many of the more familiar concepts of mathematics, like vector calculus, complex numbers, quaternions, matrix theory, tensors, and so on. Geometric algebra facilitates a simple expression of a full range of geometric ideas.

3.2 Plane Trigonometry

Using what we currently know about geometric algebra, we now derive some of the identities of plane trigonometry.[1] In this section we are dealing exclusively with Euclidean 2-space, $\mathbb{R}^2$. We define $\mathbb{G}^2$ as a subalgebra of $\mathbb{G}^3$ generated by $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_2\}$. In $\mathbb{G}^2$ the unit pseudoscalar $\mathbf{I}_{(2)}$ is a bivector[2], i.e., $\mathbf{I}_{(2)} = \mathbf{e}_1 \wedge \mathbf{e}_2$.

3.2.1 Law of Cosines

Consider a triangle with sides of length $a, b,$ and c and interior angles $\alpha,\ \beta,$ and γ (see Figure 3.1a). If we redraw the triangle using vectors $\mathbf{a},\ \mathbf{b},$ and $\mathbf{c}$

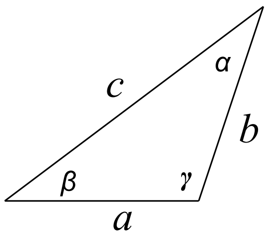
[1] This material is adapted from Hestenes [Hes03].

[2] Note that the symbol $\mathbf{I}$ for the unit pseudoscalar can have different meanings depending on the dimension of the vector space you are dealing with. We use the subscript $_{(2)}$ to distinguish the unit pseudoscalar for $\mathbb{G}^2$, $\mathbf{I}_{(2)} = \mathbf{e}_1 \wedge \mathbf{e}_2$, from the unit pseudoscalar for $\mathbb{G}^3$, $\mathbf{I} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$.

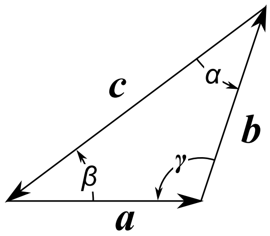
for the sides so that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0, \tag{3.1}$$

and we assign directions to the angles so that they are all counterclockwise, we get Figure 3.1b.



(a) Lengths and Angles



(b) Vectors and Angles

Figure 3.1: Plane trigonometry

The lengths of the vectors are $|\mathbf{a}| = a$, $|\mathbf{b}| = b$, and $|\mathbf{c}| = c$. The angle α starts at $\mathbf{c}$ and ends on $-\mathbf{b}$, thus the geometric product of $\mathbf{c}$ and $-\mathbf{b}$ is $-\mathbf{cb} = -\mathbf{c}\cdot\mathbf{b} - \mathbf{c}\wedge\mathbf{b} = cb\cos\alpha + cb\mathbf{I}_{(2)}\sin\alpha$, where $\mathbf{I}_{(2)}$ is the unit pseudoscalar that defines the orientation of the of the plane containing the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. We can compute the geometric product of the vectors that define the angles β and γ in a similar manner. This leads us to the following three equations:

$$-\mathbf{cb} = -\mathbf{c}\cdot\mathbf{b} - \mathbf{c}\wedge\mathbf{b} = cb\cos\alpha + cb\mathbf{I}_{(2)}\sin\alpha, \tag{3.2a}$$

$$-\mathbf{ac} = -\mathbf{a}\cdot\mathbf{c} - \mathbf{a}\wedge\mathbf{c} = ac\cos\beta + ac\mathbf{I}_{(2)}\sin\beta, \tag{3.2b}$$

$$-\mathbf{ba} = -\mathbf{b}\cdot\mathbf{a} - \mathbf{b}\wedge\mathbf{a} = ba\cos\gamma + ba\mathbf{I}_{(2)}\sin\gamma. \tag{3.2c}$$

For each of these three equations we can separate the geometric product into its scalar and bivector parts.

$$\mathbf{a}\cdot\mathbf{b} = -ab\cos\gamma, \qquad \mathbf{a}\wedge\mathbf{b} = ab\,\mathbf{I}_{(2)}\sin\gamma, \tag{3.3a}$$

$$\mathbf{b}\cdot\mathbf{c} = -bc\cos\alpha, \qquad \mathbf{b}\wedge\mathbf{c} = bc\,\mathbf{I}_{(2)}\sin\alpha, \tag{3.3b}$$

$$\mathbf{c}\cdot\mathbf{a} = -ca\cos\beta, \qquad \mathbf{c}\wedge\mathbf{a} = ca\,\mathbf{I}_{(2)}\sin\beta. \tag{3.3c}$$

Now if we solve Eq. (3.1) for $\mathbf{c}$ and then square both sides, we get

$$\begin{aligned}
\mathbf{c} &= -\mathbf{a} - \mathbf{b}, \\
\mathbf{c}^2 &= (-\mathbf{a} - \mathbf{b})^2 \\
&= (-\mathbf{a} - \mathbf{b})(-\mathbf{a} - \mathbf{b}) \\
&= \mathbf{a}^2 + \mathbf{ab} + \mathbf{ba} + \mathbf{b}^2 \\
&= \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{ab} + \mathbf{ba} \\
&= \mathbf{a}^2 + \mathbf{b}^2 + 2\left[\tfrac{1}{2}(\mathbf{ab} + \mathbf{ba})\right] \\
\mathbf{c}^2 &= \mathbf{a}^2 + \mathbf{b}^2 + 2\left[\mathbf{a}\cdot\mathbf{b}\right] \\
c^2 &= a^2 + b^2 + 2\,\mathbf{a}\cdot\mathbf{b} \\
c^2 &= a^2 + b^2 - 2ab\cos\gamma.
\end{aligned} \tag{3.4}$$

Where the last step is obtained by substitution of Eq. (3.3a) for $\mathbf{a}\cdot\mathbf{b}$. This equation (3.4) is the familiar *Law of Cosines* from trigonometry.

3.2.2 Law of Sines

Next, we start with Eq. (3.1) and take the outer product first by $\mathbf{a}$ and then by $\mathbf{b}$ or $\mathbf{c}$. We get

$$\begin{aligned} \mathbf{a}+\mathbf{b}+\mathbf{c} &= 0, \\ (\mathbf{a}+\mathbf{b}+\mathbf{c})\wedge\mathbf{a} &= 0\wedge\mathbf{a} \\ \mathbf{a}\wedge\mathbf{a}+\mathbf{b}\wedge\mathbf{a}+\mathbf{c}\wedge\mathbf{a} &= 0 \\ 0+\mathbf{b}\wedge\mathbf{a}+\mathbf{c}\wedge\mathbf{a} &= 0 \\ -\mathbf{a}\wedge\mathbf{b}+\mathbf{c}\wedge\mathbf{a} &= 0 \\ \mathbf{a}\wedge\mathbf{b} &= \mathbf{c}\wedge\mathbf{a}. \end{aligned} \qquad \begin{aligned} \mathbf{a}+\mathbf{b}+\mathbf{c} &= 0, \\ (\mathbf{a}+\mathbf{b}+\mathbf{c})\wedge\mathbf{b} &= 0\wedge\mathbf{b} \\ \mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{b}+\mathbf{c}\wedge\mathbf{b} &= 0 \\ \mathbf{a}\wedge\mathbf{b}+0+\mathbf{c}\wedge\mathbf{b} &= 0 \\ \mathbf{a}\wedge\mathbf{b}-\mathbf{b}\wedge\mathbf{c} &= 0 \\ \mathbf{a}\wedge\mathbf{b} &= \mathbf{b}\wedge\mathbf{c}. \end{aligned}$$

Combining these two equations we get

$$\mathbf{a}\wedge\mathbf{b} = \mathbf{b}\wedge\mathbf{c} = \mathbf{c}\wedge\mathbf{a}. \tag{3.5}$$

Now substituting from Eq. (3.3a), (3.3b), and (3.3c) into Eq. (3.5) and multiplying each term on the right by the inverse of $abc\,\mathbf{I}_{(2)}$ and noting that the sine terms are all scalars—we get

$$\mathbf{a}\wedge\mathbf{b} = \mathbf{b}\wedge\mathbf{c} = \mathbf{c}\wedge\mathbf{a}$$

$$ab\,\mathbf{I}_{(2)}\sin\gamma = bc\,\mathbf{I}_{(2)}\sin\alpha = ca\,\mathbf{I}_{(2)}\sin\beta$$

$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} = \frac{\sin\beta}{b}. \tag{3.6}$$

Thus, we have derived the *Law of Sines*, Eq. (3.6), using geometric algebra.

3.2.3 Area of a Triangle

Eq. (3.5) can be regarded as giving three equivalent ways of determining the area of a triangle. We have previously noted that $\mathbf{a}\wedge\mathbf{b}$ is the oriented area of a parallelogram; our triangle has only half that area. Hence, the oriented area **A** of the triangle is given by

$$\mathbf{A} = \tfrac{1}{2}\,\mathbf{a}\wedge\mathbf{b} = \tfrac{1}{2}\,\mathbf{b}\wedge\mathbf{c} = \tfrac{1}{2}\,\mathbf{c}\wedge\mathbf{a}.$$

Using Eq. (3.3a), (3.3b), and (3.3c) we can also show that

$$|\mathbf{A}| = \tfrac{1}{2}(\mathbf{a}\wedge\mathbf{b})\mathbf{I}_{(2)}^{-1} = \tfrac{1}{2}(\mathbf{b}\wedge\mathbf{c})\mathbf{I}_{(2)}^{-1} = \tfrac{1}{2}(\mathbf{c}\wedge\mathbf{a})\mathbf{I}_{(2)}^{-1},$$

$$|\mathbf{A}| = \tfrac{1}{2}\,ab\sin\gamma \quad = \tfrac{1}{2}\,bc\sin\alpha \quad = \tfrac{1}{2}\,ca\sin\beta. \tag{3.7}$$

Thus, the area is one-half the base times the height.

3.2.4 Sum of the Interior Angles of a Triangle

If we multiply Eq. (3.2a), (3.2b), and (3.2c) together and then divide by $a^2b^2c^2$, we get

$$(-\mathbf{ac})(-\mathbf{cb})(-\mathbf{ba}) = ac(\cos\beta + \mathbf{I}_{(2)}\sin\beta)\,cb(\cos\alpha + \mathbf{I}_{(2)}\sin\alpha)\,ba(\cos\gamma + \mathbf{I}_{(2)}\sin\gamma),$$

$$\frac{-\mathbf{accbba}}{a^2b^2c^2} = \frac{accbba\,(\cos\beta + \mathbf{I}_{(2)}\sin\beta)\,(\cos\alpha + \mathbf{I}_{(2)}\sin\alpha)\,(\cos\gamma + \mathbf{I}_{(2)}\sin\gamma)}{a^2b^2c^2},$$

$$-\,\mathbf{a}^{-1}\mathbf{c}^{-1}\mathbf{cb}^{-1}\mathbf{ba} = [(\cos\beta + \mathbf{I}_{(2)}\sin\beta)\,(\cos\alpha + \mathbf{I}_{(2)}\sin\alpha)]\,(\cos\gamma + \mathbf{I}_{(2)}\sin\gamma),$$

$$-\,1 = [(\cos\beta\cos\alpha - \sin\beta\sin\alpha)\; + \mathbf{I}_{(2)}(\cos\beta\sin\alpha + \sin\beta\cos\alpha)](\cos\gamma + \mathbf{I}_{(2)}\sin\gamma)$$

$$-\,1 = [\cos(\alpha+\beta) + \mathbf{I}_{(2)}\sin(\alpha+\beta)]\,(\cos\gamma + \mathbf{I}_{(2)}\sin\gamma)$$

$$-\,1 = (\cos(\alpha+\beta)\cos\gamma - \sin(\alpha+\beta)\sin\gamma)$$

$$\quad + \mathbf{I}_{(2)}(\cos(\alpha+\beta)\sin\gamma + \sin(\alpha+\beta)\cos\gamma)$$

$$-\,1 = \cos(\alpha+\beta+\gamma) + \mathbf{I}_{(2)}\sin(\alpha+\beta+\gamma). \tag{3.8}$$

This shows that successive rotations through the three interior angles of a triangle are equivalent to a straight angle. We can conclude that $\alpha+\beta+\gamma=\pi$.

3.3 Matrix Representation

It turns out that the normal basis elements of $\mathbb{G}^3$, $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{123}\}$, can be represented by square matrices. [3]

We represent the unit scalar 1 as the 4×4 identity matrix:

$$1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.9}$$

The unit orthogonal vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are represented as

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{3.10}$$

The unit orthogonal bivectors $\mathbf{e}_{12}$, $\mathbf{e}_{23}$, and $\mathbf{e}_{31}$ are represented as

$$\mathbf{e}_{12} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{e}_{23} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_{31} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.11}$$

And, the unit trivector $\mathbf{e}_{123}$ is represented as

$$\mathbf{e}_{123} = \mathbf{I} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{3.12}$$

[3] Much of the material in this section follows that of Snygg [Sny10].

Consider the vector subspace of $M_4(\mathbb{R}) = \{4 \times 4 \text{ real matrices}\}$ spanned by these eight matrices. Representing scalars, vectors, bivectors, and trivectors in terms of these square matrices enables us to visualize the geometric product as matrix multiplication.

It is easy to verify that the set of matrices given above satisfies the definition of a geometric product for $\mathbb{G}^3$ as given in Table 2.1. In particular,

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1, \tag{3.13}$$

$$\mathbf{e}_{12} + \mathbf{e}_{21} = \mathbf{e}_{23} + \mathbf{e}_{32} = \mathbf{e}_{31} + \mathbf{e}_{13} = 0, \tag{3.14}$$

$$\mathbf{I}^2 = \mathbf{e}_{123}^2 = -1. \tag{3.15}$$

Now let us consider the geometric product of two vectors. Suppose $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, then

$$\begin{aligned}\mathbf{uv} = &(u_1v_1 + u_2v_2 + u_3v_3) + \\ &u_2v_3\mathbf{e}_2\mathbf{e}_3 + u_3v_2\mathbf{e}_3\mathbf{e}_2 + u_3v_1\mathbf{e}_3\mathbf{e}_1 + u_1v_3\mathbf{e}_1\mathbf{e}_3 + u_1v_2\mathbf{e}_1\mathbf{e}_2 + u_2v_1\mathbf{e}_2\mathbf{e}_1.\end{aligned}$$

Using the relations of Eq. (3.14), we have

$$\begin{aligned}\mathbf{uv} =&(u_1v_1 + u_2v_2 + u_3v_3) + \\ &(u_2v_3 - u_3v_2)\mathbf{e}_2\mathbf{e}_3 + (u_3v_1 - u_1v_3)\mathbf{e}_3\mathbf{e}_1 + (u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2.\end{aligned} \tag{3.16}$$

Notice that $\mathbf{uv}$ consists of a scalar part

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3, \tag{3.17}$$

and a bivector part

$$\mathbf{u} \wedge \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{e}_2\mathbf{e}_3 + (u_3v_1 - u_1v_3)\mathbf{e}_3\mathbf{e}_1 + (u_1v_2 - u_2v_1)\mathbf{e}_1\mathbf{e}_2. \tag{3.18}$$

We note that the coefficients of $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, and $\mathbf{e}_1\mathbf{e}_2$ that appear in the outer product $\mathbf{u} \wedge \mathbf{v}$ are the three components of the cross product $\mathbf{u} \times \mathbf{v}$. Instead of being the components of a vector perpendicular to $\mathbf{u}$ and $\mathbf{v}$, they are components of the bivector $\mathbf{u} \wedge \mathbf{v}$. It is this distinction which enables the outer product to be defined in any dimension. There is a strong connection between the cross product and the outer product, but whereas the cross product requires the introduction of a third dimension perpendicular to the two vector factors, the outer product does not.

3.4 Generalized Complex Numbers

One of the achievements of Clifford's geometric algebra is that it generalizes complex arithmetic to spaces of arbitrary dimension.[4]

We initially limit our discussion to the $\mathbb{G}^2$ subalgebra of $\mathbb{G}^3$ consisting of the $\mathbf{e}_{12}$-plane. This subalgebra is a 4-dimensional real algebra with a basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}$. The basis elements obey the multiplication table shown in Table 3.1.

Table 3.1: Multiplication table for the $\mathbb{G}^2$ geometric product.

GP	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

[4] Much of the material in this section follows Lounesto [Lou03, p. 26], Doran and Lasenby [DL03, p. 26–28] and Dorst, Fontijne, and Mann [DFM07, p. 177–178].

The basis elements span the subspaces consisting of

basis	*subspace*	*grade*	*description*
1	$\mathbb{R}$	0	*scalars*
$\mathbf{e}_1, \mathbf{e}_2$	$\mathbb{R}^2$	1	*vectors*
$\mathbf{e}_{12}$	$\bigwedge^2 \mathbb{R}^2$	2	*bivectors*,

where $\bigwedge^2 \mathbb{R}^2$ represents the grade 2 subspace of $\mathbb{G}^2$, i.e., the subspace of bivectors.

Thus, the geometric algebra $\mathbb{G}^2$ contains copies of $\mathbb{R}$, $\mathbb{R}^2$, and $\bigwedge^2 \mathbb{R}^2$, and it is the direct sum of its subspaces, elements of grades 0, 1, and 2:

$$\mathbb{G}^2 = \mathbb{R} \oplus \mathbb{R}^2 \oplus \bigwedge^2 \mathbb{R}^2.$$

The geometric algebra $\mathbb{G}^2$ is also a direct sum of its even part $\mathbb{G}^{2+} = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2$ and its odd part $\mathbb{G}^{2-} = \mathbb{R}^2$,

$$\mathbb{G}^2 = \mathbb{G}^{2+} \oplus \mathbb{G}^{2-}$$

The even part is not only a subspace but also a subalgebra. It consists of elements of the form $x + y\mathbf{e}_{12}$, where $x, y \in \mathbb{R}$ and $\mathbf{e}_{12}^2 = -1$. This even subalgebra $\mathbb{G}^{2+}$ is isomorphic to the complex numbers $\mathbb{C}$.

Definition 3.4.1 (Unit pseudoscalar).

The bivector $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_{12} = \mathbf{I}_{(2)}$ *is called the* **unit pseudoscalar** *of the oriented plane containing the orthogonal unit vectors* $\mathbf{e}_1$ *and* $\mathbf{e}_2$*.*

This unit pseudoscalar uniquely determines an oriented plane in Euclidean 3-space. An important property of unit pseudoscalar $\mathbf{I}_{(2)}$ is that $\mathbf{I}_{(2)}^2 = -1$.

In $\mathbb{G}^{2+}$ we represent a complex number $\mathbf{Z}$ as $\mathbf{Z} = a + b\mathbf{I}_{(2)}$, where $a, b \in \mathbb{R}$. Consider what happens when we take a complex number $\mathbf{Z}$ and multiply it on the right by the unit pseudoscalar $\mathbf{I}_{(2)}$.

$$\mathbf{Z}\mathbf{I}_{(2)} = (a + b\mathbf{I}_{(2)})\mathbf{I}_{(2)} = a\mathbf{I}_{(2)} + b\mathbf{I}_{(2)}^2 = a\mathbf{I}_{(2)} - b = -b + a\mathbf{I}_{(2)}.$$

The result of right multiplying by the unit pseudoscalar is to rotate the complex number $\mathbf{Z}$ by 90° counterclockwise (see Figure 3.2).

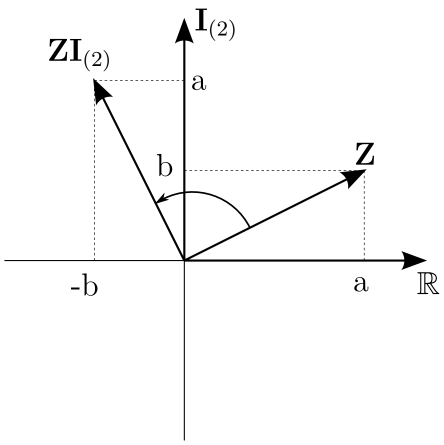


Figure 3.2: Right multiplying $\mathbf{Z}$ by the unit pseudoscalar.

Likewise, consider what happens when we multiply the same complex number $\mathbf{Z}$ by the unit pseudoscalar $\mathbf{I}_{(2)}$ on the left.

$$\mathbf{I}_{(2)}\mathbf{Z} = \mathbf{I}_{(2)}(a + b\mathbf{I}_{(2)}) = a\mathbf{I}_{(2)} + b\mathbf{I}_{(2)}^2 = a\mathbf{I}_{(2)} - b = -b + a\mathbf{I}_{(2)}.$$

Once again, the result of left multiplying by the unit pseudoscalar is to rotate the vector by 90° counterclockwise. Just as with multiplication by i in the ordinary complex numbers, multiplication of a complex number in $\mathbb{G}^{2+}$ by $\mathbf{I}_{(2)}$ is commutative and it rotates a complex number by 90° counterclockwise in the plane.

Suppose we would like to rotate a complex number $Z = a + b\mathbf{I}_{(2)}$ in the complex plane by some angle other than 90°. For this we need to define the exponential function.

Definition 3.4.2 (Exponential).

Define the **exponential** *as* $e^{\mathbf{I}_{(2)}\theta} = \cos\theta + \mathbf{I}_{(2)}\sin\theta$.

The exponential is the sum of a scalar and a bivector. For any given angle θ, the norm of the exponential is always 1.

The exponential can also be visualized as a geometric product of two unit vectors. It captures the angle between them and the orientation of the rotation with respect to the unit pseudoscalar. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors in the $\mathbf{I}_{(2)}$-plane and θ is the angle measured from $\mathbf{u}_1$ to $\mathbf{u}_2$, then

$$\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_1 \wedge \mathbf{u}_2 = u_1 u_2 \cos\theta + u_1 u_2 \mathbf{I}_{(2)} \sin\theta,$$

but $u_1 = u_2 = 1$, thus

$$\mathbf{u}_1\mathbf{u}_2 = \cos\theta + \mathbf{I}_{(2)}\sin\theta = e^{\mathbf{I}_{(2)}\theta}.$$

By extension from a basis, the geometric product of any two vectors is a complex number. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in the $\mathbf{I}_{(2)}$-plane. Then

$$\mathbf{Z} = \mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = uv\cos\theta + uv\sin\theta\mathbf{I}_{(2)} = uve^{\mathbf{I}_{(2)}\theta},$$

$$\mathbf{Z} = a + b\mathbf{I}_{(2)} = re^{\mathbf{I}_{(2)}\theta},$$

$$\text{where} \quad a = uv\cos\theta, \quad b = uv\sin\theta,$$

$$r = uv = \sqrt{a^2 + b^2}, \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{b}{a}\right).$$

If we wish to rotate a complex number $\mathbf{Z}$ that exists in the $\mathbf{I}_{(2)}$-plane by an angle θ, we can right- or left-multiply $\mathbf{Z}$ by $e^{\mathbf{I}_{(2)}\theta}$.

$$\begin{aligned}
\mathbf{Z}e^{\mathbf{I}_{(2)}\theta} &= (a + b\mathbf{I}_{(2)})(\cos\theta + \mathbf{I}_{(2)}\sin\theta) \\
&= a\cos\theta + a\sin\theta\,\mathbf{I}_{(2)} + b\cos\theta\,\mathbf{I}_{(2)} + b\sin\theta\,\mathbf{I}_{(2)}^2 \\
&= (a\cos\theta - b\sin\theta) + (a\sin\theta + b\cos\theta)\,\mathbf{I}_{(2)}.
\end{aligned}$$

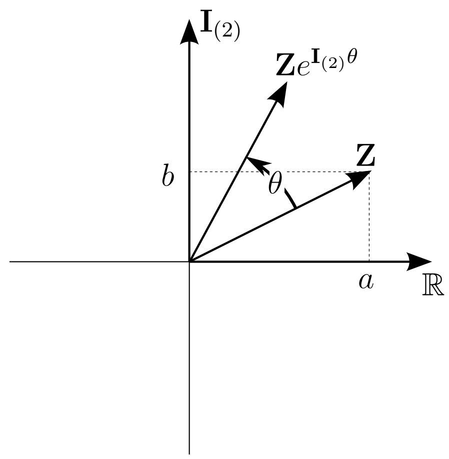


Figure 3.3: Multiplying $\mathbf{Z}$ by an exponential.

See Figure 3.3.

Note that if $\theta = 90°$, then

$$e^{\mathbf{I}_{(2)}\theta} = \cos 90° + \mathbf{I}_{(2)} \sin 90° = 0 + \mathbf{I}_{(2)} = \mathbf{I}_{(2)},$$

which confirms our previous example that multiplication of a complex number by $\mathbf{I}_{(2)}$ causes a 90° counterclockwise rotation.

In the preceding discussion we viewed the pseudoscalar $\mathbf{I}_{(2)}$ as representing the $\mathbf{e}_{12}$-plane in Euclidean 3-space formed by the two standard orthonormal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. If we consider $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as the standard basis for $\mathbb{R}^3$, then any plane in Euclidean 3-space can be expressed as a linear combination of the basis bivectors $\{\mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}\}$, thus we should more properly think of $b\mathbf{I}_{(2)}$ as any plane through the origin in $\mathbb{R}^3$ where

$$b\mathbf{I}_{(2)} = b_1\mathbf{e}_{23} + b_2\mathbf{e}_{31} + b_3\mathbf{e}_{12},$$

such that $b = \sqrt{b_1^2 + b_2^2 + b_3^2}$ and $|\mathbf{I}_{(2)}| = 1$.

Because we have defined a complex number in $\mathbb{G}^{2+}$ to be of the form

$\mathbf{Z} = a + b\mathbf{I}_{(2)}$, a *generalized complex number* has the form

$$\mathbf{Z} = a + b\mathbf{I}_{(2)} = a + b_1\mathbf{e}_{23} + b_2\mathbf{e}_{31} + b_3\mathbf{e}_{12}. \tag{3.19}$$

Generalized complex numbers are just multivectors with a scalar part and a bivector part. Table 3.2 does a side-by-side comparison of the algebra of complex numbers with that of the geometric subalgebra of a plane, $\mathbb{G}^{2+}$.

Table 3.2: Complex Numbers vs. $\mathbb{G}^{2+}$ Geometric Algebra

Property	**Complex Numbers**	$\mathbb{G}^{2+}$ **Geometric Algebra**
rectangular	$z = a + bi$	$\mathbf{Z} = a + b\mathbf{I}_{(2)}$
polar[a]	$z = re^{i\theta}$	$\mathbf{Z} = re^{\mathbf{I}_{(2)}\theta}$
geometric product	N/A	$\mathbf{Z} = \mathbf{uv}$
rectangular conjugate	$\overline{z} = a - bi$	$\overline{\mathbf{Z}} = \mathbf{vu} = a - b\mathbf{I}_{(2)}$
polar conjugate	$\overline{z} = re^{-i\theta}$	$\overline{\mathbf{Z}} = \mathbf{vu} = re^{-\mathbf{I}_{(2)}\theta}$
i and $\mathbf{I}_{(2)}$ squared	$i^2 = -1$	$\mathbf{I}_{(2)}^2 = -1$
rotate 90° CCW	zi	$\mathbf{Z}\mathbf{I}_{(2)}$
rotate 90° CW	$-zi$	$-\mathbf{Z}\mathbf{I}_{(2)}$
commutativity	$zi = iz$	$\mathbf{Z}\mathbf{I}_{(2)} = \mathbf{I}_{(2)}\mathbf{Z}$
norm squared	$\lvert z\rvert^2 = z\overline{z} = a^2 + b^2$	$\lvert\mathbf{Z}\rvert^2 = \mathbf{Z}\overline{\mathbf{Z}} = a^2 + b^2$
rotate by θ	$ze^{i\theta}$	$\mathbf{Z}e^{\mathbf{I}_{(2)}\theta}$

[a] Where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

We have used the symbol $\mathbf{I}_{(2)}$ for the unit pseudoscalar, rather than the tempting alternative i. It is important to make this distinction. Even though $\mathbb{G}^{2+}$ is isomorphic with the field of complex numbers, we see that i and $\mathbf{I}_{(2)}$ behave differently when we multiply them with vectors.

Consider what happens when we multiply a real vector $\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2$ by i.

$$\mathbf{v}i = (a\mathbf{e}_1 + b\mathbf{e}_2)i = ai\mathbf{e}_1 + bi\mathbf{e}_2,$$

we get a complex vector. Also note that

$$\mathbf{v}i = i\mathbf{v},$$

thus multiplication of a vector by i is commutative.

Next, consider what happens when we take a vector $\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2$ in the $\mathbf{e}_{12}$-plane and multiply it on the right by the unit pseudoscalar $\mathbf{I}_{(2)}$.

$$\mathbf{v}\mathbf{I}_{(2)} = (a\mathbf{e}_1 + b\mathbf{e}_2)\mathbf{e}_{12} = a\mathbf{e}_1\mathbf{e}_{12} + b\mathbf{e}_2\mathbf{e}_{12} = a\mathbf{e}_2 - b\mathbf{e}_1 = -b\mathbf{e}_1 + a\mathbf{e}_2,$$

The result of right-multiplying by the unit pseudoscalar is to rotate the vector by 90° counterclockwise (see Figure 3.4).

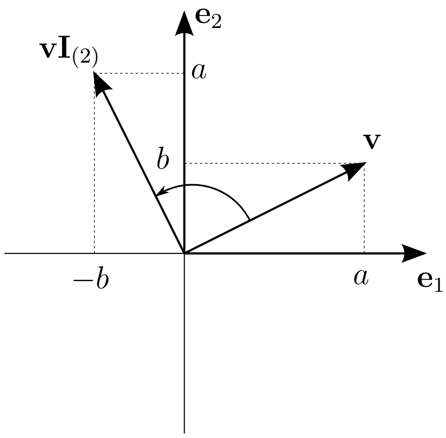


Figure 3.4: Right multiplying $\mathbf{v}$ by the unit pseudoscalar.

Likewise, consider what happens when we multiply the same vector $\mathbf{v}$ by the unit pseudoscalar $\mathbf{I}_{(2)}$ on the left.

$$\mathbf{I}_{(2)}\mathbf{v} = \mathbf{e}_{12}(a\mathbf{e}_1 + b\mathbf{e}_2) = a\mathbf{e}_{12}\mathbf{e}_1 + b\mathbf{e}_{12}\mathbf{e}_2 = -a\mathbf{e}_2 + b\mathbf{e}_1 = b\mathbf{e}_1 - a\mathbf{e}_2.$$

The result of left-multiplying by the unit pseudoscalar is to rotate the vector by 90° clockwise (see Figure 3.5).

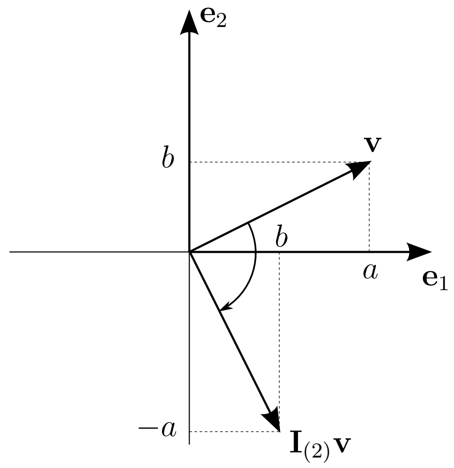


Figure 3.5: Left multiplying $\mathbf{v}$ by the unit pseudoscalar.

It should be noted that our pseudoscalar $\mathbf{I}_{(2)}$ anticommutes with $\mathbf{e}_1$ and $\mathbf{e}_2$ and thus $\mathbf{I}_{(2)}$ anticommutes with every vector in the $\mathbf{e}_{12}$-plane. From the above discussion we see that if we limit ourselves to the algebra of $\mathbb{G}^{2+}$, $\mathbf{I}_{(2)}$ behaves exactly the same as i, but if we broaden our algebra to include all of $\mathbb{G}^2$, then $\mathbf{I}_{(2)}$ behaves quite differently from i.

Another name for a generalized complex number is *spinor*. The name spinor is suggestive of its action on a vector under geometric multiplication. Each spinor can be regarded as an algebraic representation of a rotation-dilation. Generalized complex numbers, or spinors, form a subalgebra of $\mathbb{G}^3$.

3.5 The Quaternions

It was shown in the previous section that complex numbers can be subsumed in the broader frame of geometric algebra. The same is true of the quaternions. William Rowan Hamilton (1805–1865) developed an algebra generated by four fundamental units $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with which many aspects of mechanics could be handled [DL03, p. 8-10]. Hamilton's quaternion took the form

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

$$\text{where} \qquad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

$$\text{and} \qquad \mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}.$$

The quaternions form a skew field, a noncommutative division ring, under addition and multiplication.

If we take our orthonormal basis for the bivectors $\{\mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}\}$ and reverse their signs, we get the 2-blades

$$\mathbf{i} = \mathbf{e}_{32}, \quad \mathbf{j} = \mathbf{e}_{13}, \quad \mathbf{k} = \mathbf{e}_{21}.$$

With this definition, all of the identities listed above hold. Hamilton identified pure quaternions (null scalar part) with vectors, but we now see that they are actually bivectors. To set up an isomorphism between quaternions and bivectors, we must flip the signs of the bivectors. This shows that the quaternions are a *left-handed* set of bivectors, whereas Hamilton and others attempted to view $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as a right-handed set of vectors. Not surprisingly, this was a potential source of great confusion and meant one had to be extremely careful when applying quaternions in vector algebra [DL03, p. 34].

Quaternions are just multivectors with a scalar part and a bivector part. Like generalized complex numbers, quaternions are spinors that form a subalgebra of $\mathbb{G}^3$.

3.6 Reflections

We now take a closer look at the geometric product $\mathbf{uv}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$. If we right multiply this product by the inverse of $\mathbf{v}$ ($\mathbf{v}^{-1}$), we should retrieve the vector $\mathbf{u}$.

$$\begin{aligned}\mathbf{u} &= \mathbf{u}\left(\mathbf{v}\mathbf{v}^{-1}\right) = \left(\mathbf{u}\mathbf{v}\right)\mathbf{v}^{-1} \\ &= \left(\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}\right)\mathbf{v}^{-1} \\ &= \left(\mathbf{u}\cdot\mathbf{v}\right)\mathbf{v}^{-1} + \left(\mathbf{u}\wedge\mathbf{v}\right)\mathbf{v}^{-1}.\end{aligned} \tag{3.20}$$

Because $\mathbf{u}$ is a vector and $\left(\mathbf{u}\cdot\mathbf{v}\right)\mathbf{v}^{-1}$ is a vector, we see that $\left(\mathbf{u}\wedge\mathbf{v}\right)\mathbf{v}^{-1}$ is also a vector. The first term is the component of $\mathbf{u}$ in the $\mathbf{v}$ direction, i.e., the orthogonal *projection* of $\mathbf{u}$ onto $\mathbf{v}$ ($\mathbf{u}_{\|\mathbf{v}}$). The second term represents the component of $\mathbf{u}$ that contains no $\mathbf{v}$ component at all, i.e., the *rejection* of $\mathbf{u}$ by $\mathbf{v}$ ($\mathbf{u}_{\perp\mathbf{v}}$). Thus, we have:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_{\|\mathbf{v}} + \mathbf{u}_{\perp\mathbf{v}}, \\ \text{where} \quad \mathbf{u}_{\|\mathbf{v}} &= \left(\mathbf{u}\cdot\mathbf{v}\right)\mathbf{v}^{-1}, \\ \text{and} \quad \mathbf{u}_{\perp\mathbf{v}} &= \left(\mathbf{u}\wedge\mathbf{v}\right)\mathbf{v}^{-1}. \end{aligned}$$

We have decomposed the vector $\mathbf{u}$ relative to $\mathbf{v}$, providing its $\mathbf{v}$-component and non-$\mathbf{v}$-component (see Figure 3.6).

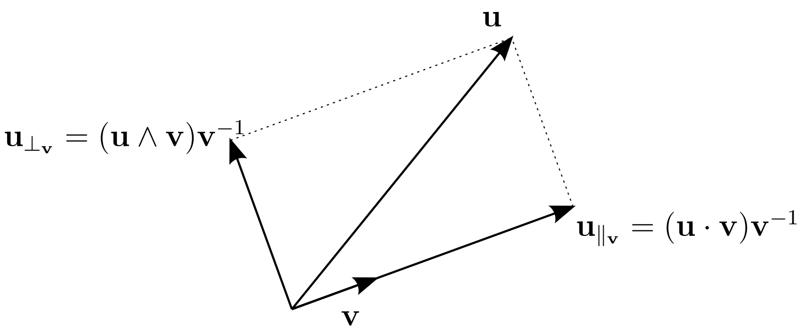


Figure 3.6: Projection and rejection of $\mathbf{u}$ relative to $\mathbf{v}$.

We now consider what happens when we left multiply the geometric product

$\mathbf{uv}$ by $\mathbf{v}^{-1}$.

$$
\begin{aligned}
\mathbf{v}^{-1}\mathbf{uv} &= \mathbf{v}^{-1}(\mathbf{uv}) \\
&= \frac{\mathbf{v}}{|\mathbf{v}|^2}(\mathbf{uv}) && \text{(def. of vector inverse)} \\
&= \mathbf{vu}\frac{\mathbf{v}}{|\mathbf{v}|^2} && \text{(because scalars commute)} \\
&= \mathbf{vuv}^{-1} = (\mathbf{vu})\,\mathbf{v}^{-1} \\
&= (\mathbf{v}\cdot\mathbf{u} + \mathbf{v}\wedge\mathbf{u})\,\mathbf{v}^{-1} \\
&= (\mathbf{v}\cdot\mathbf{u})\,\mathbf{v}^{-1} + (\mathbf{v}\wedge\mathbf{u})\,\mathbf{v}^{-1} \\
&= (\mathbf{u}\cdot\mathbf{v})\,\mathbf{v}^{-1} - (\mathbf{u}\wedge\mathbf{v})\,\mathbf{v}^{-1}. && (3.21)
\end{aligned}
$$

If we now compare Eq. (3.21) to Eq. (3.20), we see that the first term (the projection) is the same, but the second term—the rejection—has the opposite sign. The resulting of sandwiching the vector $\mathbf{u}$ between the vectors $\mathbf{v}$ and $\mathbf{v}^{-1}$ ($\mathbf{v}^{-1}\mathbf{uv}$) has, in effect, *reflected* the $\mathbf{u}$ vector across the line of the $\mathbf{v}$ vector in the plane of the two vectors (see Figure 3.7).

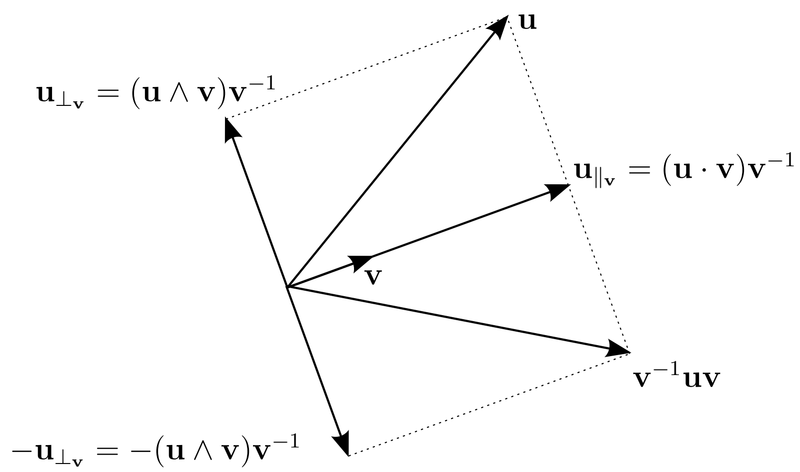


Figure 3.7: Reflection of $\mathbf{u}$ in $\mathbf{v}$.

The magnitude and orientation of the vector $\mathbf{v}$ are irrelevant to the outcome, because the inversion removes any scalar factor. Only the attitude of the line matters. We have proved the following theorem:

Theorem 3.6.1.

The reflection of a vector $\mathbf{u}$ *in a vector* $\mathbf{v}$ *is:*

$$\textit{Reflection of } \mathbf{u} \textit{ in } \mathbf{v}: \qquad \boxed{\mathbf{u} \;\mapsto\; \mathbf{v}^{-1}\mathbf{u}\mathbf{v}}. \tag{3.22}$$

Sandwiching operations are quite common in geometric algebra. It is important to note that the objects of geometric algebra, vectors in this case, are also the operators of a geometric transform, reflection in this case. One of the powers of geometric algebra is that geometric objects and operators are all treated the same.

3.7 Rotations

The usual way to specify a rotation in Euclidean 3-space is by its axis $\mathbf{n}$ and a scalar rotational angle θ. Instead we specify the rotation by the "plane $\mathbf{I}_{(2)}$", i.e., the plane with pseudoscalar $\mathbf{I}_{(2)}$ orthogonal to the axis (the "plane of rotation") and θ (the "size of the rotation").

Suppose that the rotation carries the vector $\mathbf{u}$ to a vector $\mathbf{u}'$.

Consider the first case where $\mathbf{u}$ is in the plane of rotation $\mathbf{I}_{(2)}$. Then so is $\mathbf{u}'$ (see Figure 3.8).

From previous sections, we have:

$$\begin{aligned}
\mathbf{u}\mathbf{u}' &= \mathbf{u}\cdot\mathbf{u}' + \mathbf{u}\wedge\mathbf{u}' \\
&= |\mathbf{u}|\,|\mathbf{u}'|\,\cos\theta + |\mathbf{u}|\,|\mathbf{u}'|\,\mathbf{I}_{(2)}\sin\theta \\
&= |\mathbf{u}|\,|\mathbf{u}'|\,(\cos\theta + \mathbf{I}_{(2)}\sin\theta) \\
&= |\mathbf{u}|\,|\mathbf{u}'|\,e^{\mathbf{I}_{(2)}\theta}.
\end{aligned} \tag{3.23}$$

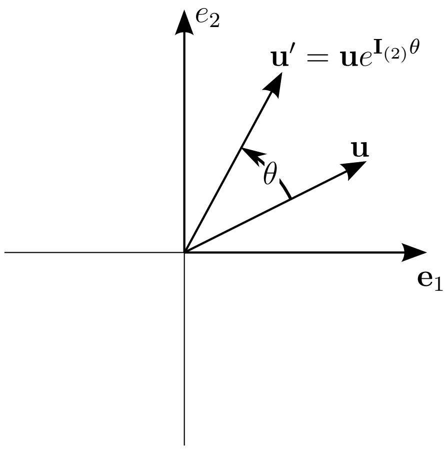


Figure 3.8: Rotation in the $\mathbf{I}_{(2)}$ plane by angle θ.

If we multiply both sides of Eq. (3.23) by $\mathbf{u}$ on the left, and use the fact that $|\mathbf{u}| = |\mathbf{u}'|$ and $\mathbf{u}^2 = |\mathbf{u}|\,|\mathbf{u}| = |\mathbf{u}|\,|\mathbf{u}'|$, we obtain:

$$\mathbf{u}\mathbf{u}' = |\mathbf{u}|\,|\mathbf{u}'|\,e^{\mathbf{I}_{(2)}\theta}$$

$$|\mathbf{u}|\,|\mathbf{u}|\,\mathbf{u}' = \mathbf{u}\,|\mathbf{u}|\,|\mathbf{u}|\,e^{\mathbf{I}_{(2)}\theta}$$

$$\mathbf{u}' = \mathbf{u}\,e^{\mathbf{I}_{(2)}\theta}. \tag{3.24}$$

Figure 3.8 depicts Eq. (3.24) in action: $e^{\mathbf{I}_{(2)}\theta}$ rotates $\mathbf{u}$ to $\mathbf{u}'$ in the plane of $\mathbf{I}_{(2)}$.

We now consider the more general case where the vector $\mathbf{u}$ is not necessarily in the plane of rotation $\mathbf{I}_{(2)}$. We decompose $\mathbf{u}$ into its projection onto the plane $\mathbf{I}_{(2)}$, $\mathbf{u}_{\|}$ and its rejection, the component perpendicular to the plane $\mathbf{I}_{(2)}$, $\mathbf{u}_{\perp}$, thus $\mathbf{u} = \mathbf{u}_{\|} + \mathbf{u}_{\perp}$ (see Figure 3.9).

Here is the key: as $\mathbf{u}$ rotates to $\mathbf{u}'$, $\mathbf{u}_{\|}$ rotates to $\mathbf{u}_{\|}e^{\mathbf{I}_{(2)}\theta}$ and $\mathbf{u}_{\perp}$ is carried

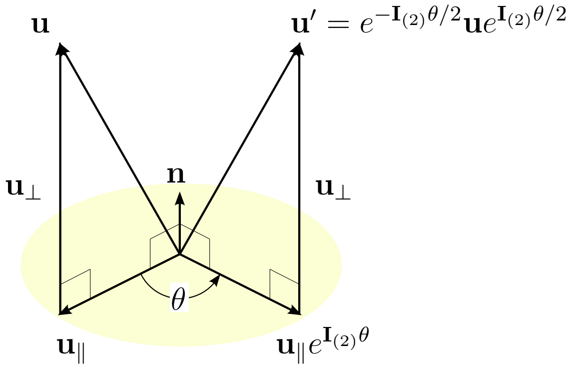


Figure 3.9: Rotation in any plane by angle θ.

along unchanged. Thus,

$$\begin{aligned}\mathbf{u}' &= \mathbf{u}_\parallel e^{\mathbf{I}_{(2)}\theta} + \mathbf{u}_\perp \\ &= \mathbf{u}_\parallel e^{\mathbf{I}_{(2)}\theta/2} e^{\mathbf{I}_{(2)}\theta/2} + \mathbf{u}_\perp e^{-\mathbf{I}_{(2)}\theta/2} e^{\mathbf{I}_{(2)}\theta/2} \\ &= e^{-\mathbf{I}_{(2)}\theta/2}\mathbf{u}_\parallel e^{\mathbf{I}_{(2)}\theta/2} + e^{-\mathbf{I}_{(2)}\theta/2}\mathbf{u}_\perp e^{\mathbf{I}_{(2)}\theta/2} \\ &= e^{-\mathbf{I}_{(2)}\theta/2}\mathbf{u}e^{\mathbf{I}_{(2)}\theta/2}. \end{aligned} \tag{3.25}$$

We have proved the following theorem which gives an elegant algebraic representation of rotations in three dimensions.

Theorem 3.7.1.

Let $R_{\mathbf{I}_{(2)}\theta}(\mathbf{u})$ denote the rotation of a vector $\mathbf{u}$ by the angle θ around an axis perpendicular to the plane $\mathbf{I}_{(2)}$. Then

$$\textit{Rotation:} \qquad \boxed{R_{\mathbf{I}_{(2)}\theta}(\mathbf{u}) = e^{-\mathbf{I}_{(2)}\theta/2}\mathbf{u}e^{\mathbf{I}_{(2)}\theta/2}}. \tag{3.26}$$

We say the quaternion $e^{-\mathbf{I}_{(2)}\theta/2}$ represents the rotation $R_{\mathbf{I}_{(2)}\theta}$.

We know from geometry that two reflections make a rotation, so we can also visualize this rotation as the result of two successive reflections. Suppose that $\mathbf{v}$ and

$\mathbf{w}$ are two unit vectors in the plane of rotation and that the angle from $\mathbf{v}$ to $\mathbf{w}$ is $\theta/2$. We first reflect $\mathbf{u}$ in $\mathbf{v}$, giving $\mathbf{v}^{-1}\mathbf{u}\mathbf{v}$. Then we reflect this result in $\mathbf{w}$ giving

$$\mathbf{w}^{-1}\left(\mathbf{v}^{-1}\mathbf{u}\mathbf{v}\right)\mathbf{w} = (\mathbf{w}^{-1}\mathbf{v}^{-1})\mathbf{u}(\mathbf{v}\mathbf{w}) = (\mathbf{v}\mathbf{w})^{-1}\mathbf{u}(\mathbf{v}\mathbf{w}).$$

See Figure 3.10.

The net result of the two reflections is the same as the rotation obtained with $e^{-\mathbf{I}_{(2)}\theta/2}\mathbf{u}e^{\mathbf{I}_{(2)}\theta/2}$, where $\mathbf{v}\mathbf{w} = e^{\mathbf{I}_{(2)}\theta/2}$ and $(\mathbf{v}\mathbf{w})^{-1} = e^{-\mathbf{I}_{(2)}\theta/2}$. Notice that to get a rotation of angle θ, the angle between $\mathbf{v}$ and $\mathbf{w}$ must be $\theta/2$ and both $\mathbf{v}$ and $\mathbf{w}$ must be in the plane of rotation $\mathbf{I}_{(2)}$. The magnitude and orientation of the vectors $\mathbf{v}$ and $\mathbf{w}$ are irrelevant to the outcome, because the inversion removes any scalar factor. Only the attitude of the lines matters. In summary, to rotate an object by an angle θ with respect to a plane, we need only choose any two vectors $\mathbf{v}$ and $\mathbf{w}$ in that plane such that the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\theta/2$.

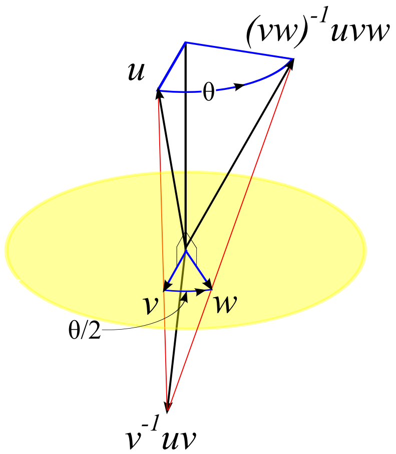


Figure 3.10: Rotation viewed as two successive reflections.

Once again, notice the sandwiching operation where the vector to be transformed ($\mathbf{u}$) is sandwiched between two operators. One of the most powerful aspects of geometric algebra is the ability to realize isometries as sandwich operations of the form

$$\mathbf{X} \mapsto \mathbf{g}^{-1}\mathbf{X}\mathbf{g}$$

where $\mathbf{X}$ is any geometric element of the algebra and $\mathbf{g}$ is a specific geometric element, unique to the isometry. The operator $\mathbf{g}$ is in general a *versor*, i.e., it can be expressed as the product of 1-vectors.

3.8 Geometry

In this section we use geometric algebra to prove several theorems of geometry. First we present several lemmas that are helpful in proving other lemmas and theorems that follow.

3.8.1 The Inverse of a Bivector

Many of the lemmas and theorems in this section deal with inverse bivectors, so we first discuss the meaning of the inverse of a multivector [5].

The inverse of a multivector $\mathbf{A}$, if it exists, is denoted by $\mathbf{A}^{-1}$ or $\frac{1}{\mathbf{A}}$. We can divide any multivector $\mathbf{B}$ by $\mathbf{A}$ in two ways, by multiplying by $\mathbf{A}^{-1}$ on the left,

$$\mathbf{A}^{-1}\mathbf{B} = \frac{1}{\mathbf{A}}\mathbf{B},$$

or on the right,

$$\mathbf{B}\mathbf{A}^{-1} = \mathbf{B}\frac{1}{\mathbf{A}} = \mathbf{B}/\mathbf{A}.$$

The *left division* is not equivalent to the *right division* unless $\mathbf{B}$ commutes with $\mathbf{A}^{-1}$, in which case the division can be denoted unambiguously by

$$\frac{\mathbf{B}}{\mathbf{A}}.$$

[5] This description of multivector inverse comes from Hestenes [Hes03, p. 37].

This is always the case when both $\mathbf{A}$ and $\mathbf{B}$ are bivectors in the same plane, as shown in Lemma 3.8.3 and Lemma 3.8.4 below.

Definition 3.8.1 (Reversion).

Reversion $\sim$ *is an operation that takes a k-blade* $\mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k$ *and produces its* **reverse***:*

$$\textbf{reversion:} \qquad \widetilde{\mathbf{B}} = \mathbf{b}_k \wedge \mathbf{b}_{k-1} \wedge \cdots \wedge \mathbf{b}_1,$$

which just has all vectors of $\mathbf{B}$ *in reverse order. The notation of the tilde is chosen to be reminiscent of an editor's notation for an interchange of terms.*[6]

Definition 3.8.2 (Inverse of a blade).

The **inverse of a blade B** *is*

$$\mathbf{B}^{-1} = \frac{\widetilde{\mathbf{B}}}{\mathbf{B}\widetilde{\mathbf{B}}}. \tag{3.27}$$

3.8.2 Lemmas

Lemma 3.8.3.

The inverse of a bivector, if it exists, is a bivector in the same plane as the original bivector.

Proof.

Let $\mathbf{A}$ be an invertible bivector. Without loss of generality, we assume that $\mathbf{A}$ lies in the $\mathbf{e}_1 \wedge \mathbf{e}_2$ plane, viz.,

$$\mathbf{A} = \alpha \mathbf{e}_{12}, \qquad \text{where } \alpha \text{ is a scalar.}$$

[6] Some literature denotes reversion by $\mathbf{B}^\dagger$ instead of $\widetilde{\mathbf{B}}$, because it is related to complex conjugation in a certain context.

Then,

$$\mathbf{A}^{-1} = \frac{\widetilde{\mathbf{A}}}{\mathbf{A}\widetilde{\mathbf{A}}} = \frac{\alpha \mathbf{e}_{21}}{\alpha^2 \mathbf{e}_{12}\mathbf{e}_{21}} = \frac{1}{\alpha}\mathbf{e}_{21} = -\frac{1}{\alpha}\mathbf{e}_{12}.$$

Therefore, $\mathbf{A}^{-1}$ is a bivector in the same plane as $\mathbf{A}$. □

Lemma 3.8.4.

If $\mathbf{A}$ *and* $\mathbf{B}$ *are are bivectors in the same plane, then their geometric product commutes, i.e.,*

$$\mathbf{AB} = \mathbf{BA}.$$

Proof.

Let $\mathbf{A}$ and $\mathbf{B}$ be a bivectors that lie in the same plane. Without loss of generality, we assume that $\mathbf{A}$ and $\mathbf{B}$ lie in the $\mathbf{e}_1 \wedge \mathbf{e}_2$ plane, i.e.,

$$\mathbf{A} = \alpha \mathbf{e}_{12}, \quad \mathbf{B} = \beta \mathbf{e}_{12}, \quad \text{where } \alpha \text{ and } \beta \text{ are scalars.}$$

Then,

$$\mathbf{AB} = \alpha \mathbf{e}_{12} \beta \mathbf{e}_{12} = -\alpha\beta \quad \text{and} \quad \mathbf{BA} = \beta \mathbf{e}_{12} \alpha \mathbf{e}_{12} = -\alpha\beta.$$

Therefore,

$$\mathbf{AB} = \mathbf{BA}.$$

□

Lemma 3.8.5.

Let $\mathbf{A}, \mathbf{B}, \mathbf{A}', \mathbf{B}'$ *all represent bivectors in the same plane, where* $\mathbf{A} \neq \mathbf{A}'$, $\mathbf{B} \neq \pm\mathbf{B}'$, $\mathbf{B} \neq \mathbf{0}$, $\mathbf{B}' \neq \mathbf{0}$.

$$\text{If} \quad \frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{A}'}{\mathbf{B}'}, \quad \text{then} \quad \frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{A} \pm \mathbf{A}'}{\mathbf{B} \pm \mathbf{B}'}. \tag{3.28}$$

Proof.

Because $\mathbf{A}$ and $\mathbf{B}$ are in the same plane, $\mathbf{A}$ commutes with $\mathbf{B}^{-1}$ by Lemma 3.8.3 and Lemma 3.8.4. That is,

$$\mathbf{AB}^{-1} = \mathbf{B}^{-1}\mathbf{A} = \frac{\mathbf{A}}{\mathbf{B}}.$$

Likewise, $\mathbf{A}'$ commutes with $(\mathbf{B}')^{-1}$. But $\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{A}'}{\mathbf{B}'}$ implies that $\mathbf{A}' = m\mathbf{A}$ and $\mathbf{B}' = m\mathbf{B}$ for some $m \in \mathbb{R}$ such that $m \neq 0, \pm 1$. Then we have

$$\frac{\mathbf{A}}{\mathbf{B}} = 1\frac{\mathbf{A}}{\mathbf{B}} = \frac{(1 \pm m)}{(1 \pm m)}\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{A} \pm m\mathbf{A}}{\mathbf{B} \pm m\mathbf{B}} = \frac{\mathbf{A} \pm \mathbf{A}'}{\mathbf{B} \pm \mathbf{B}'}.$$

□

Lemma 3.8.5 also works when $\mathbf{A}$ and $\mathbf{B}$ are scalars.

Definition 3.8.6 (Altitude of a triangle).

An **altitude of a triangle** *is a segment from a vertex perpendicular to the line in which the opposite side—the* **base***—lies (see Figure 3.11 where the bases* $\overline{AB}, \overline{BC}, \overline{CA}$ *have respective altitudes of* a_1, a_2, a_3*). When referring the the length of an altitude line segment, we use the term* **height***.*

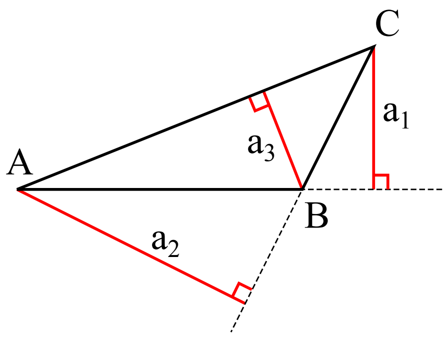
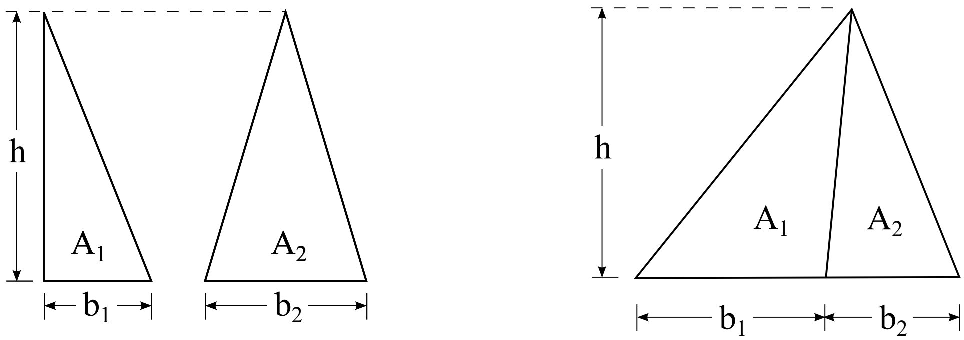


Figure 3.11: Altitude of a triangle

Lemma 3.8.7.

When two triangles have the same height, the ratio of their areas equals the ratio of their bases.



(a) Configuration 1 (b) Configuration 2

Figure 3.12: Lemma 3.8.7

Proof.

Let A_1 and A_2 be the areas of two triangles with respective bases of b_1 and b_2 and sharing a common height of h (see Figure 3.12). Then $A_1 = \frac{1}{2}b_1h, \quad A_2 = \frac{1}{2}b_2h$. Therefore

$$\frac{A_1}{A_2} = \frac{\frac{1}{2}b_1h}{\frac{1}{2}b_2h} = \frac{b_1}{b_2}. \tag{3.29}$$

□

We use the configuration shown in Figure 3.12b for proving the theorems that follow.

We now present an extension of Lemma 3.8.7 without proof.

Lemma 3.8.8.

When two triangles have the same height, the ratio of the codirectional bivectors representing their areas equals the ratio of the codirectional vectors representing their bases, e.g.,

$$\frac{\mathbf{A}_1}{\mathbf{A}_2} = \frac{\mathbf{a}-\mathbf{x}}{\mathbf{x}-\mathbf{b}}. \tag{3.30}$$

See Figure 3.13.

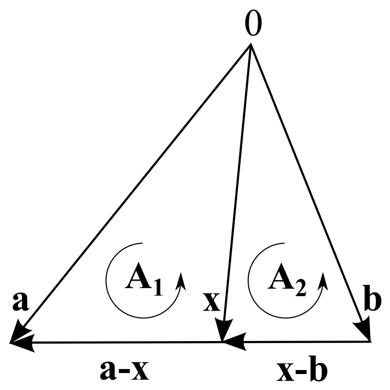


Figure 3.13: Lemma 3.8.8, using vectors and bivectors

Lemma 3.8.9.

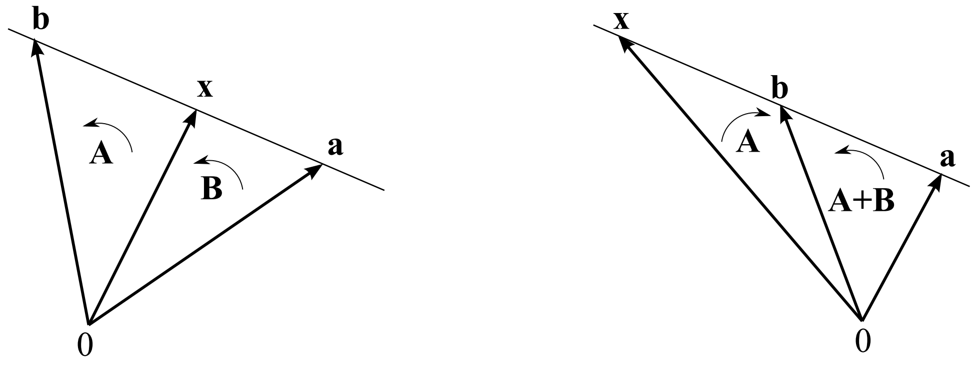
Three vectors $\mathbf{x}$, $\mathbf{a}$, *and* $\mathbf{b}$, *with initial points on the origin, have end points that are collinear if and only if* $\mathbf{x}$ *can be expressed as a linear combination of* $\mathbf{a}$ *and* $\mathbf{b}$, *viz.,*

$$\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b}, \tag{3.31a}$$

$$\textit{where} \quad \alpha + \beta = 1. \tag{3.31b}$$

See Figure 3.14 (a) and (b).

Proof.



(a) Configuration a (b) Configuration b

Figure 3.14: Lemma 3.8.9

(⇒) Suppose that three vectors $\mathbf{x}$, $\mathbf{a}$, and $\mathbf{b}$ with initial points on the origin have end points that are collinear. Let $\mathbf{A}$ be the oriented area of triangle $\triangle 0\mathbf{xb}$, $\mathbf{A} = \frac{1}{2}(\mathbf{x} \wedge \mathbf{b})$ and let $\mathbf{B}$ be the oriented area of triangle $\triangle 0\mathbf{ax}$, $\mathbf{B} = \frac{1}{2}(\mathbf{a} \wedge \mathbf{x})$. From Eq. (3.30) we have

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{b} - \mathbf{x}}{\mathbf{x} - \mathbf{a}}$$

$$\mathbf{x} - \mathbf{a} = \frac{\mathbf{B}}{\mathbf{A}}(\mathbf{b} - \mathbf{x})$$

$$\mathbf{x} - \mathbf{a} = \frac{\mathbf{B}}{\mathbf{A}}\mathbf{b} - \frac{\mathbf{B}}{\mathbf{A}}\mathbf{x}$$

$$\mathbf{x} + \frac{\mathbf{B}}{\mathbf{A}}\mathbf{x} = \mathbf{a} + \frac{\mathbf{B}}{\mathbf{A}}\mathbf{b}$$

$$\mathbf{x} = \frac{\mathbf{a} + \frac{\mathbf{B}}{\mathbf{A}}\mathbf{b}}{1 + \frac{\mathbf{B}}{\mathbf{A}}}$$

$$\mathbf{x} = \left(\frac{\mathbf{A}}{\mathbf{A} + \mathbf{B}}\right)\mathbf{a} + \left(\frac{\mathbf{B}}{\mathbf{A} + \mathbf{B}}\right)\mathbf{b}$$

$$\mathbf{x} = \left(\frac{A}{A+B}\right)\mathbf{a} + \left(\frac{B}{A+B}\right)\mathbf{b}.$$

Where the last step is justified because **A** and **B** are in the same plane and are codirectional.

$$\text{Let} \qquad \alpha = \frac{A}{A+B} \qquad \text{and} \qquad \beta = \frac{B}{A+B},$$

$$\text{then} \qquad x = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{and} \qquad \alpha + \beta = 1.$$

($\Leftarrow$) Suppose that three vectors $\mathbf{x}$, $\mathbf{a}$, and $\mathbf{b}$ with initial points on the origin satisfy the equation $\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b}$ where $\alpha + \beta = 1$. Then $\alpha = 1 - \beta$, thus

$$\mathbf{x} = (1-\beta)\mathbf{a} + \beta\mathbf{b}$$

$$\mathbf{x} = \mathbf{a} - \beta\mathbf{a} + \beta\mathbf{b}$$

$$\mathbf{x} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}). \tag{3.32}$$

Eq. (3.32) indicates that $\mathbf{x}$ is the sum of $\mathbf{a}$ and a scalar multiple of $\mathbf{b} - \mathbf{a}$, thus the end point of $\mathbf{x}$ is collinear with the end points of $\mathbf{a}$ and $\mathbf{b}$ (see Figure 3.15). □

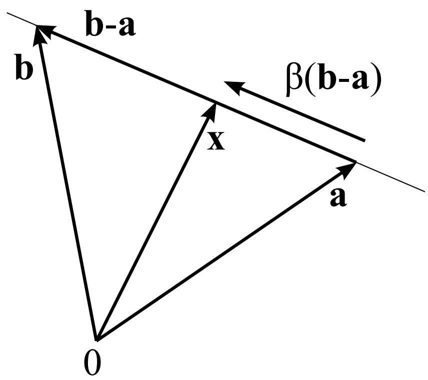


Figure 3.15: Lemma 3.8.9 proof

Lemma 3.8.10.

Three distinct points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *lie on a line if and only if there exist scalars* α, β, γ, *not all zero, such that* $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0$ *and* $\alpha + \beta + \gamma = 0$.

Proof.

($\Rightarrow$) Suppose that the three distinct points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie on a line. Then by Eqs. (3.31a and d), we have

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{and} \quad \alpha + \beta = 1,$$

$$0 = \alpha\mathbf{a} + \beta\mathbf{b} - \mathbf{c},$$

$$0 = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}, \quad \text{where} \quad \gamma = -1,$$

$$0 = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}, \quad \text{where} \quad \alpha + \beta + \gamma = 0.$$

($\Leftarrow$) Suppose that three distinct points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are related by the equations $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0$, where $\alpha + \beta + \gamma = 0$ and at least one of α, β, γ is nonzero. Without loss of generality we may assume that $\gamma \neq 0$. Then we have

$$-\gamma\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \quad \text{and} \quad -\gamma = \alpha + \beta,$$

$$\mathbf{c} = \left(\frac{\alpha}{-\gamma}\right)\mathbf{a} + \left(\frac{\beta}{-\gamma}\right)\mathbf{b},$$

$$\mathbf{c} = \left(\frac{\alpha}{\alpha + \beta}\right)\mathbf{a} + \left(\frac{\beta}{\alpha + \beta}\right)\mathbf{b},$$

$$\mathbf{c} = \alpha'\mathbf{a} + \beta'\mathbf{b}, \quad \text{where} \quad \alpha' = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad \beta' = \frac{\beta}{\alpha + \beta},$$

$$\mathbf{c} = \alpha'\mathbf{a} + \beta'\mathbf{b}, \quad \text{where} \quad \alpha' + \beta' = 1.$$

Therefore, the three points are collinear by Lemma 3.8.9. □

3.8.3 Ceva's Theorem

Ceva's theorem is a historically important theorem from geometry that was first published in 1678 by the Italian Giovanni Ceva [Eve72, p. 63]. A *cevian line* is a line passing through the vertex of a triangle that does not coincide with a side of the triangle. A cevian line cuts the opposite side of a triangle at a point. Ceva's theorem examines the directed lengths of the sides of a triangle that have been cut by cevian lines. It uses the product of ratios to determine if three cevian lines are concurrent at a point (see Figure 3.16).

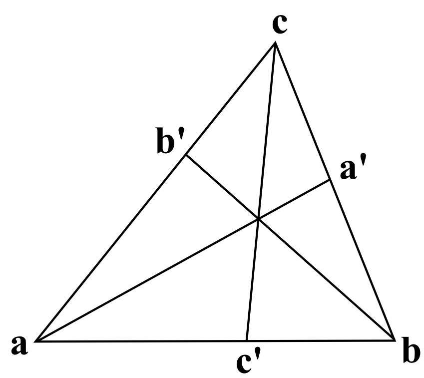


Figure 3.16: Ceva's Theorem.

Theorem 3.8.11 (Ceva's Theorem).

Three cevian lines aa', bb', cc' of an ordinary triangle $\triangle abc$ are concurrent if and only if

$$\left(\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}}\right)\left(\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}\right)=+1.$$

Proof of Ceva's Theorem.

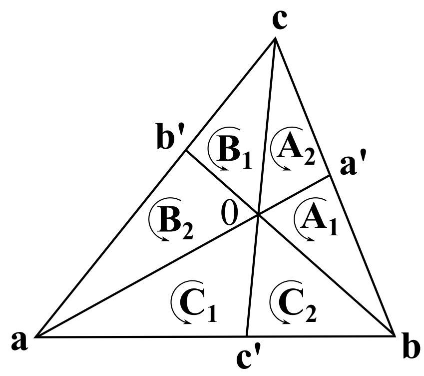


Figure 3.17: Proof of Ceva's Theorem.

(⇒) Suppose that three cevian lines aa', bb', cc' of an ordinary triangle $\triangle abc$ are concurrent at the point 0 as shown in Figure 3.17. Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C}_1, \mathbf{C}_2$ be bivectors whose norms are equal to the areas of the smaller triangles.

Closer examination of Figure 3.17 reveals that each pair of smaller triangles shares a common height, viz.,

$$\begin{aligned}
&\text{triangles } \mathbf{A}_1 \text{ and } \mathbf{A}_2, \text{ with respect to bases } \mathbf{b}-\mathbf{a}' \text{ and } \mathbf{a}'-\mathbf{c},\\
&\text{triangles } \mathbf{B}_1 \text{ and } \mathbf{B}_2, \text{ with respect to bases } \mathbf{c}-\mathbf{b}' \text{ and } \mathbf{b}'-\mathbf{a},\\
&\text{triangles } \mathbf{C}_1 \text{ and } \mathbf{C}_2, \text{ with respect to bases } \mathbf{a}-\mathbf{c}' \text{ and } \mathbf{c}'-\mathbf{b}.
\end{aligned}$$

By Eq. (3.30) we have

$$\begin{aligned}
\frac{\mathbf{A}_1}{\mathbf{A}_2} &= \frac{\mathbf{b}\wedge\mathbf{a}'}{\mathbf{a}'\wedge\mathbf{c}} = \frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\\
\frac{\mathbf{B}_1}{\mathbf{B}_2} &= \frac{\mathbf{c}\wedge\mathbf{b}'}{\mathbf{b}'\wedge\mathbf{a}} = \frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}} \qquad (3.33)\\
\frac{\mathbf{C}_1}{\mathbf{C}_2} &= \frac{\mathbf{a}\wedge\mathbf{c}'}{\mathbf{c}'\wedge\mathbf{b}} = \frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}.
\end{aligned}$$

Next we consider the same ratios of sides, but this time we equate these ratios to areas of the larger triangle $\triangle abc$. Once again, these larger pairs of triangles share a common height. For example,

$$\begin{aligned}
&\text{triangles } \triangle\, baa' \text{ and } \triangle\, a'ac, \text{ with respect to bases } \mathbf{b} - \mathbf{a}' \text{ and } \mathbf{a}' - \mathbf{c},\\
&\text{triangles } \triangle\, cbb' \text{ and } \triangle\, b'ba, \text{ with respect to bases } \mathbf{c} - \mathbf{b}' \text{ and } \mathbf{b}' - \mathbf{a},\\
&\text{triangles } \triangle\, acc' \text{ and } \triangle\, c'cb, \text{ with respect to bases } \mathbf{a} - \mathbf{c}' \text{ and } \mathbf{c}' - \mathbf{b}.
\end{aligned}$$

Then by Eq. (3.30) we have

$$\begin{aligned}
\frac{\mathbf{b} - \mathbf{a}'}{\mathbf{a}' - \mathbf{c}} &= \frac{(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{a}' - \mathbf{a})}{(\mathbf{a}' - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})} = \frac{\mathbf{A}_1 + \mathbf{C}_1 + \mathbf{C}_2}{\mathbf{A}_2 + \mathbf{B}_1 + \mathbf{B}_2}\\
\frac{\mathbf{c} - \mathbf{b}'}{\mathbf{b}' - \mathbf{a}} &= \frac{(\mathbf{c} - \mathbf{b}) \wedge (\mathbf{b}' - \mathbf{b})}{(\mathbf{b}' - \mathbf{b}) \wedge (\mathbf{a} - \mathbf{b})} = \frac{\mathbf{B}_1 + \mathbf{A}_1 + \mathbf{A}_2}{\mathbf{B}_2 + \mathbf{C}_1 + \mathbf{C}_2} \qquad (3.34)\\
\frac{\mathbf{a} - \mathbf{c}'}{\mathbf{c}' - \mathbf{b}} &= \frac{(\mathbf{a} - \mathbf{c}) \wedge (\mathbf{c}' - \mathbf{c})}{(\mathbf{c}' - \mathbf{c}) \wedge (\mathbf{b} - \mathbf{c})} = \frac{\mathbf{C}_1 + \mathbf{B}_1 + \mathbf{B}_2}{\mathbf{C}_2 + \mathbf{A}_1 + \mathbf{A}_2}.
\end{aligned}$$

Equating the formulas in Eq. (3.33) with those in Eq. (3.34) and applying Lemma 3.8.5, we get

$$\begin{aligned}
\frac{\mathbf{A}_1}{\mathbf{A}_2} &= \frac{\mathbf{A}_1 + \mathbf{C}_1 + \mathbf{C}_2}{\mathbf{A}_2 + \mathbf{B}_1 + \mathbf{B}_2} = \frac{\mathbf{C}_1 + \mathbf{C}_2}{\mathbf{B}_1 + \mathbf{B}_2}\\
\frac{\mathbf{B}_1}{\mathbf{B}_2} &= \frac{\mathbf{B}_1 + \mathbf{A}_1 + \mathbf{A}_2}{\mathbf{B}_2 + \mathbf{C}_1 + \mathbf{C}_2} = \frac{\mathbf{A}_1 + \mathbf{A}_2}{\mathbf{C}_1 + \mathbf{C}_2} \qquad (3.35)\\
\frac{\mathbf{C}_1}{\mathbf{C}_2} &= \frac{\mathbf{C}_1 + \mathbf{B}_1 + \mathbf{B}_2}{\mathbf{C}_2 + \mathbf{A}_1 + \mathbf{A}_2} = \frac{\mathbf{B}_1 + \mathbf{B}_2}{\mathbf{A}_1 + \mathbf{A}_2}.
\end{aligned}$$

If we take the product of the factors in Eqs. (3.35), we get

$$\left(\frac{\mathbf{A}_1}{\mathbf{A}_2}\right)\left(\frac{\mathbf{B}_1}{\mathbf{B}_2}\right)\left(\frac{\mathbf{C}_1}{\mathbf{C}_2}\right) = \left(\frac{\mathbf{C}_1 + \mathbf{C}_2}{\mathbf{B}_1 + \mathbf{B}_2}\right)\left(\frac{\mathbf{A}_1 + \mathbf{A}_2}{\mathbf{C}_1 + \mathbf{C}_2}\right)\left(\frac{\mathbf{B}_1 + \mathbf{B}_2}{\mathbf{A}_1 + \mathbf{A}_2}\right) = +1. \qquad (3.36)$$

And by Eqs. (3.33) this is equivalent to

$$\left(\frac{\mathbf{b} - \mathbf{a}'}{\mathbf{a}' - \mathbf{c}}\right)\left(\frac{\mathbf{c} - \mathbf{b}'}{\mathbf{b}' - \mathbf{a}}\right)\left(\frac{\mathbf{a} - \mathbf{c}'}{\mathbf{c}' - \mathbf{b}}\right) = +1. \qquad (3.37)$$

(⇐) Suppose that three cevian lines aa', bb', $c\hat{c}$ of an ordinary triangle $\triangle abc$ divide their respective opposite sides such that the product

$$\left(\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}}\right)\left(\frac{\mathbf{a}-\hat{\mathbf{c}}}{\hat{\mathbf{c}}-\mathbf{b}}\right) = +1. \tag{3.38}$$

Let cc' be the cevian line concurrent with cevian lines aa' and bb' (see Figure 3.18). In the first part of this proof, we showed that for concurrent cevian lines aa', bb', cc'

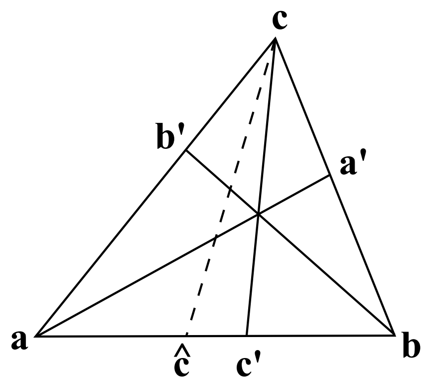


Figure 3.18: Converse of Ceva's Theorem.

we have Eq. (3.37). If we equate the left hand sides of Eq. (3.37) and (3.38) we get

$$\left(\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}}\right)\left(\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}\right) = \left(\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}}\right)\left(\frac{\mathbf{a}-\hat{\mathbf{c}}}{\hat{\mathbf{c}}-\mathbf{b}}\right).$$

After cancelation of like terms, we have

$$\left(\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}\right) = \left(\frac{\mathbf{a}-\hat{\mathbf{c}}}{\hat{\mathbf{c}}-\mathbf{b}}\right) = r, \qquad \text{where } r \text{ is a scalar ratio.}$$

Solving these equations for $\mathbf{c}'$ and $\hat{\mathbf{c}}$ in terms of $\mathbf{a}, \mathbf{b}$, and r we get

$$\mathbf{c}' = \frac{\mathbf{a}+r\mathbf{b}}{r+1} \qquad \text{and} \qquad \hat{\mathbf{c}} = \frac{\mathbf{a}+r\mathbf{b}}{r+1}.$$

Therefore, $\hat{\mathbf{c}}$ must equal $\mathbf{c}'$, and the three cevian lines are concurrent. □

3.8.4 Menelaus' Theorem

Another close companion to Ceva's theorem is Menelaus' theorem. This theorem dates back to ancient Greece, where Menelaus of Alexandria was an astronomer who lived in the first century A.D. [Eve72, p. 63]. A *menelaus point* is a point lying on a side of a triangle but not coinciding with a vertex of the triangle. Menelaus' Theorem examines the directed lengths of the sides of a triangle that has been cut by a transversal through three menelaus points. It uses the product of ratios to determine if three menelaus points are collinear (see Figure 3.19). Note that it is common in the literature for the terms *point* and *vector* to be interchanged. A vector with its initial point at the origin determines a unique *point* at its end point, e.g., points a, b, c can be uniquely represented by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

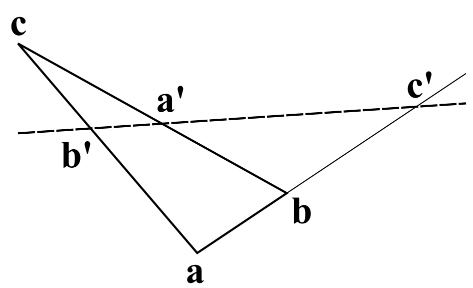


Figure 3.19: Menelaus' Theorem.

Theorem 3.8.12 (**Menelaus' Theorem**).

Three menelaus points $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ *on the sides* $\mathbf{bc}, \mathbf{ca}, \mathbf{ab}$ *of an ordinary triangle* $\triangle\mathbf{abc}$ *are collinear if and only if*

$$\left(\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}}\right)\left(\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}\right) = -1.$$

Proof of Menelaus' Theorem.

($\Rightarrow$) Suppose that three menelaus points $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ on sides $\mathbf{bc}, \mathbf{ca}, \mathbf{ab}$ of an ordinary triangle $\triangle\mathbf{abc}$ are collinear. Let $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4, \mathbf{C}$ be bivectors whose norms are equal to the areas of the smaller triangles (see Figure 3.20). Consider the

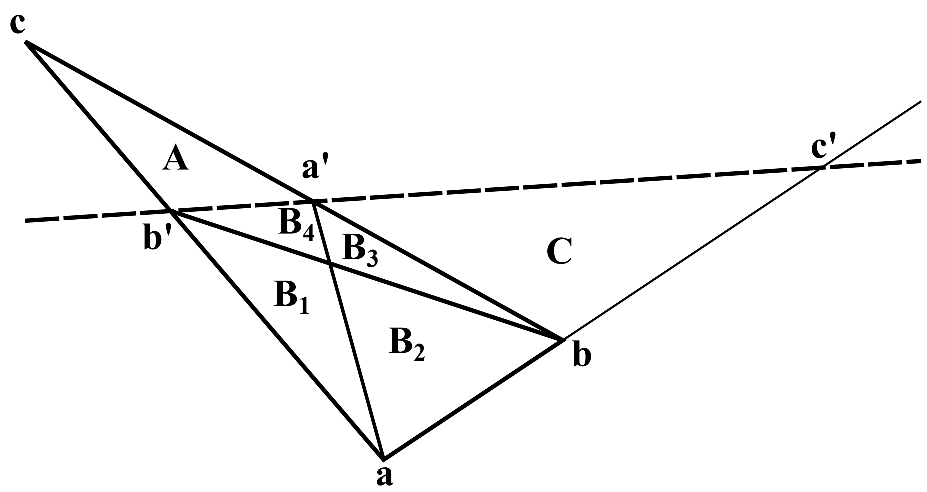


Figure 3.20: Proof of Menelaus' Theorem.

ratio of the side starting at $\mathbf{b}$, through $\mathbf{a}'$, to $\mathbf{c}$:

$$\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}} \overset{1}{=} \frac{\mathbf{B}_2+\mathbf{B}_3}{\mathbf{A}+\mathbf{B}_1+\mathbf{B}_4} \overset{2}{=} \frac{\mathbf{B}_3+\mathbf{B}_4}{\mathbf{A}} \overset{3}{=} \frac{\mathbf{B}_2-\mathbf{B}_4}{\mathbf{B}_1+\mathbf{B}_4}. \tag{3.39}$$

In Eq. (3.39), step (1) follows from the extended version of Lemma 3.8.7 where $\mathbf{a}$ is the common vertex. Step (2) follows from Lemma 3.8.7 where $\mathbf{b}'$ is the common vertex. Step (3) uses Lemma 3.8.5 to take the difference of the first two steps.

Consider the ratio of the side starting at $\mathbf{c}$, through $\mathbf{b'}$, to $\mathbf{a}$:

$$\frac{\mathbf{c}-\mathbf{b'}}{\mathbf{b'}-\mathbf{a}} \stackrel{1}{=} \frac{\mathbf{A}+\mathbf{B}_3+\mathbf{B}_4}{\mathbf{B}_1+\mathbf{B}_2} \stackrel{2}{=} \frac{\mathbf{A}}{\mathbf{B}_1+\mathbf{B}_4} \stackrel{3}{=} \frac{\mathbf{B}_3+\mathbf{B}_4}{\mathbf{B}_2-\mathbf{B}_4}. \tag{3.40}$$

In Eq. (3.40), step (1) follows from Lemma 3.8.7 where $\mathbf{b}$ is the common vertex. Step (2) follows from Lemma 3.8.7 where $\mathbf{a'}$ is the common vertex. Step (3) uses Lemma 3.8.5 to take the difference of the first two steps.

Consider the ratio of the side starting at $\mathbf{a}$, to $\mathbf{c'}$, and back to $\mathbf{b}$:

$$\frac{\mathbf{a}-\mathbf{c'}}{\mathbf{c'}-\mathbf{b}} \stackrel{1}{=} -\frac{\mathbf{B}_1+\mathbf{B}_2+\mathbf{B}_3+\mathbf{B}_4+\mathbf{C}}{\mathbf{B}_3+\mathbf{B}_4+\mathbf{C}} \stackrel{2}{=} -\frac{\mathbf{B}_2+\mathbf{B}_3+\mathbf{C}}{\mathbf{C}} \stackrel{3}{=} -\frac{\mathbf{B}_1+\mathbf{B}_4}{\mathbf{B}_3+\mathbf{B}_4}. \tag{3.41}$$

In Eq. (3.41), step (1) follows from Lemma 3.8.7 where $\mathbf{b'}$ is the common vertex. Step (2) follows from Lemma 3.8.7 where $\mathbf{a'}$ is the common vertex. Step (3) uses Lemma 3.8.5 to take the difference of the first two steps. Notice that a negative sign was introduced because the menelaus point $\mathbf{c'}$ cuts the side ab exterior to the triangle.

We now take the product of the right-most terms of Eq. (3.39), (3.40), and (3.41) to get

$$\left(\frac{\mathbf{B}_2-\mathbf{B}_4}{\mathbf{B}_1+\mathbf{B}_4}\right)\left(\frac{\mathbf{B}_3+\mathbf{B}_4}{\mathbf{B}_2-\mathbf{B}_4}\right)\left(-\frac{\mathbf{B}_1+\mathbf{B}_4}{\mathbf{B}_3+\mathbf{B}_4}\right) = -1.$$

Therefore,

$$\left(\frac{\mathbf{b}-\mathbf{a'}}{\mathbf{a'}-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b'}}{\mathbf{b'}-\mathbf{a}}\right)\left(\frac{\mathbf{a}-\mathbf{c'}}{\mathbf{c'}-\mathbf{b}}\right) = -1. \tag{3.42}$$

($\Leftarrow$) Suppose that three menelaus points $a', b', \hat{c}$ on sides bc, ca, ab of triangle abc satisfy the equation

$$\left(\frac{\mathbf{b}-\mathbf{a'}}{\mathbf{a'}-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b'}}{\mathbf{b'}-\mathbf{a}}\right)\left(\frac{\mathbf{a}-\hat{\mathbf{c}}}{\hat{\mathbf{c}}-\mathbf{b}}\right) = -1. \tag{3.43}$$

Let c' be a menelaus point on side ab that is collinear with a' and b' (see Figure 3.21). In the first part of this proof, we showed that for collinear menelaus

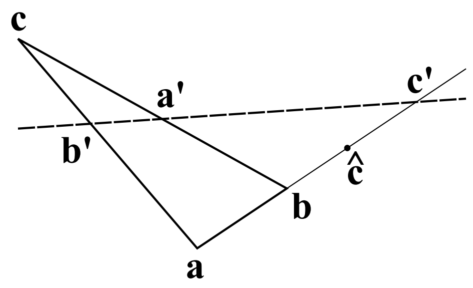


Figure 3.21: Converse of Menelaus' Theorem.

points a', b', c' we have Eq. (3.42). If we equate the left-hand sides of Eq. (3.42) and (3.43) we get

$$\left(\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}}\right)\left(\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}\right)=\left(\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}}\right)\left(\frac{\mathbf{a}-\hat{\mathbf{c}}}{\hat{\mathbf{c}}-\mathbf{b}}\right).$$

After cancelation of like terms, we have

$$\left(\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}\right)=\left(\frac{\mathbf{a}-\hat{\mathbf{c}}}{\hat{\mathbf{c}}-\mathbf{b}}\right)=r, \qquad \text{where } r \text{ is a scalar ratio.}$$

Solving these equations for $\mathbf{c}'$ and $\hat{\mathbf{c}}$ in terms of $\mathbf{a}, \mathbf{b}$, and r we get

$$\mathbf{c}'=\frac{\mathbf{a}+r\mathbf{b}}{r+1} \qquad \text{and} \qquad \hat{\mathbf{c}}=\frac{\mathbf{a}+r\mathbf{b}}{r+1}.$$

Therefore, $\hat{\mathbf{c}}$ must equal $\mathbf{c}'$ and the three menelaus points are collinear. □

3.8.5 Desargues' Two-Triangle Theorem

Gérard Desargues was a French mathematician and engineer who is considered one of the founders of projective geometry. In 1636 Desargues published a work on perspective that included his now famous two-triangle theorem [Eve72, p. 70]. This theorem has become basic in the present-day theory of projective geometry. In 1639 Desargues published a book, *Brouillon Projet d'une atteinte aux événemens des rencontres du Cône avec un Plan*, that marked the first advance in synthetic geometry since the time of the ancient Greeks [Eve72, p. 59].

Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *copolar* if AA', BB', CC' are concurrent at the point S; they are said to be *coaxial* if the points of intersection P (of BC and $B'C'$), Q (of CA and $C'A'$), R (of AB and $A'B'$) are collinear (see Figure 3.22).

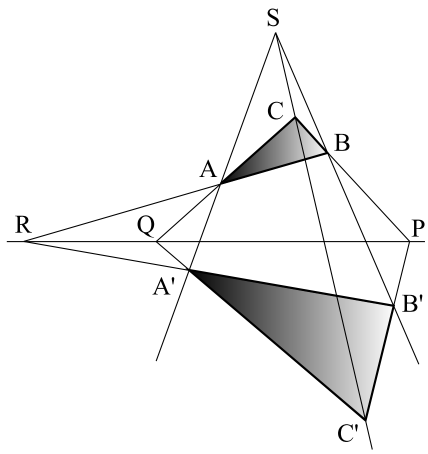


Figure 3.22: Desargues' Two-Triangle Theorem.

Theorem 3.8.13 (Desargues' Two-Triangle Theorem).

Two ordinary triangles are copolar if and only if they are coaxial.

Proof of Desargues' Two-Triangle Theorem.

Let the vertices of triangles $\triangle ABC$ and $\triangle A'B'C'$ be represented by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ respectively. Let the points P, Q, R, S be represented by the vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ respectively (see Figure 3.23).

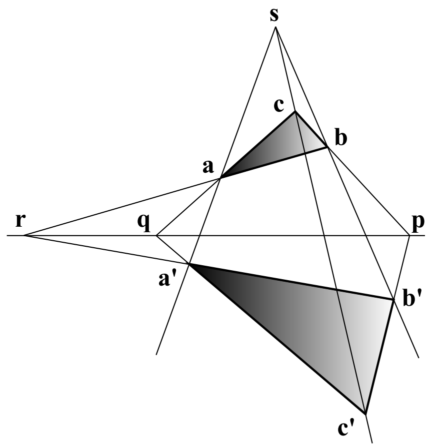


Figure 3.23: Proof of Desargues' Two-Triangle Theorem.

($\Rightarrow$) Suppose that triangles $\triangle\mathbf{abc}$ and $\triangle\mathbf{a}'\mathbf{b}'\mathbf{c}'$ are copolar at point $\mathbf{s}$. Then by Eq. (3.31a) and (3.31d) of Lemma 3.8.9, there exist scalars $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ such that

$$\mathbf{s} = \alpha\mathbf{a} + \alpha'\mathbf{a}' = \beta\mathbf{b} + \beta'\mathbf{b}' = \gamma\mathbf{c} + \gamma'\mathbf{c}', \tag{3.44a}$$

$$\alpha + \alpha' = \beta + \beta' = \gamma + \gamma' = 1. \tag{3.44b}$$

Then from Eqs. (3.44a and b) we get

$$\beta\mathbf{b} - \gamma\mathbf{c} = -(\beta'\mathbf{b}' - \gamma'\mathbf{c}'),$$

$$\text{and} \quad \beta - \gamma = -(\beta' - \gamma').$$

Thus, if $\beta - \gamma \neq 0$, we have

$$\frac{\beta\mathbf{b} - \gamma\mathbf{c}}{\beta - \gamma} = \frac{\beta'\mathbf{b}' - \gamma'\mathbf{c}'}{\beta' - \gamma'} = \mathbf{p}. \tag{3.45a}$$

Where $\mathbf{p}$ is the intersection of the line from $\mathbf{c}$ to $\mathbf{b}$ and the line from $\mathbf{c}'$ to $\mathbf{b}'$. The left-hand ratio is a point on the line through the points $\mathbf{b}$ and $\mathbf{c}$ because it can be expressed as a linear combination of $\mathbf{b}$ and $\mathbf{c}$ where the coefficients sum to 1 (Lemma 3.8.9). Likewise, the right-hand ratio is a point on the line through the points $\mathbf{b}'$ and $\mathbf{c}'$ because it can be expressed as a linear combination of $\mathbf{b}'$ and $\mathbf{c}'$ where the coefficients sum to 1 (Lemma 3.8.9). Thus, these ratios equal $\mathbf{p}$, the point of intersection of these two lines.

Following similar derivations, we find expressions for $\mathbf{q}$, the intersection of the line from $\mathbf{c}$ to $\mathbf{a}$ and the line from $\mathbf{c}'$ to $\mathbf{a}'$, and $\mathbf{r}$, the intersection of the line from $\mathbf{b}$ to $\mathbf{a}$ and the line from $\mathbf{b}'$ to $\mathbf{a}'$,

$$\frac{\gamma\mathbf{c} - \alpha\mathbf{a}}{\gamma - \alpha} = \frac{\gamma'\mathbf{c}' - \alpha'\mathbf{a}'}{\gamma' - \alpha'} = \mathbf{q}, \tag{3.45b}$$

$$\frac{\alpha\mathbf{a} - \beta\mathbf{b}}{\alpha - \beta} = \frac{\alpha'\mathbf{a}' - \beta'\mathbf{b}'}{\alpha' - \beta'} = \mathbf{r}. \tag{3.45c}$$

Rearranging Eqs. (3.45a, b, and c), we get

$$(\beta - \gamma)\mathbf{p} = \beta\mathbf{b} - \gamma\mathbf{c}, \tag{3.46a}$$

$$(\gamma - \alpha)\mathbf{q} = \gamma\mathbf{c} - \alpha\mathbf{a}, \tag{3.46b}$$

$$(\alpha - \beta)\mathbf{r} = \alpha\mathbf{a} - \beta\mathbf{b}. \tag{3.46c}$$

Adding the left and right sides of Eqs. (3.46a, b, and c), we get

$$\begin{aligned}
&(\beta - \gamma)\mathbf{p} + (\gamma - \alpha)\mathbf{q} + (\alpha - \beta)\mathbf{r} = \gamma\mathbf{c} - \alpha\mathbf{a} + \beta\mathbf{b} - \gamma\mathbf{c} + \alpha\mathbf{a} - \beta\mathbf{b},\\
&(\beta - \gamma)\mathbf{p} + (\gamma - \alpha)\mathbf{q} + (\alpha - \beta)\mathbf{r} = 0,\\
&\text{where} \quad (\beta - \gamma) + (\gamma - \alpha) + (\alpha - \beta) = 0.
\end{aligned}$$

Thus, by Lemma 3.8.10, the points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are collinear, provided not all of $\beta - \gamma$, $\gamma - \alpha$, and $\alpha - \beta$ are zero.

If $\beta - \gamma = 0$, then the lines from $\mathbf{b}$ to $\mathbf{c}$ and $\mathbf{b}'$ to $\mathbf{c}'$ are parallel and $\mathbf{p}$ would be a point at infinity. If $\beta - \gamma \neq 0$, $\gamma - \alpha \neq 0$, and $\alpha - \beta \neq 0$, then we are assured that each pair of lines intersects in an ordinary point and that we are dealing with an ordinary triangle. This theorem can be extended to include points at infinity.

($\Leftarrow$) Suppose that triangles $\triangle\mathbf{abc}$ and $\triangle\mathbf{a'b'c'}$ are coaxial through the points $\mathbf{p}, \mathbf{q}, \mathbf{r}$, and let $\mathbf{s}$ be the point of intersection of the line through $\mathbf{a'}$ and $\mathbf{a}$ and the line through $\mathbf{b'}$ and $\mathbf{b}$. Then the triangles $\triangle\mathbf{aa'q}$ and $\triangle\mathbf{bb'p}$ are copolar through $\mathbf{r}$. Therefore, by the first part of this proof, they are coaxial through $\mathbf{c'}, \mathbf{c}, \mathbf{s}$. Thus, the coaxial triangles $\triangle\mathbf{abc}$ and $\triangle\mathbf{a'b'c'}$ are copolar. $\square$

CHAPTER 4

THE CONFORMAL MODEL

The *conformal model* is an alternative model to the *Euclidean model* for solving three-dimensional geometric problems. The creation of the conformal model involves embedding Euclidean space in a higher dimensional space. Two extra dimensions are added to the Euclidean space being modeled. These extra dimensions have particular properties such that linear subspaces in conformal space represent geometric entities of interest in Euclidean space.

In this chapter we develop the conformal model and show how it can be used to represent points, lines, circles, planes, and spheres.[1] The conformal model can support points and lines at infinity. It also supports the conformal transformations of translation, rotation, reflection, inversion, and scaling using rotors.

4.1 Development of the Conformal Model

When first introduced to the conformal model, one often wonders "why go to all this trouble?" It turns out that this model has some real advantages in its ability to model geometric objects and express transformations. The reader can be assured that any effort to understand this model will be rewarded in the end.

Suppose we have a line in Euclidean 1-space with a unit basis vector $\mathbf{e}_1$ (see Figure 4.1). Any point on this line can be expressed as a vector, a scalar

[1] The organization of this chapter follows that of Vince [Vin08, Ch. 11] and Doran & Lasenby [DL03, Ch. 10].

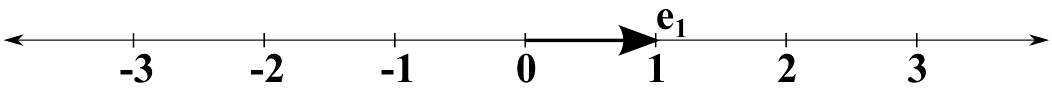


Figure 4.1: Line in Euclidean 1-space

multiple of $\mathbf{e}_1$, viz.,

$$\mathbf{x} = x\mathbf{e}_1.$$

4.1.1 Add a Dimension for Stereographic Projection

We can use *stereographic projection* to map points from a one-dimensional line onto a unit circle in two-dimensional space. To do this, we introduce a new dimension perpendicular to $\mathbf{e}_1$ with unit basis vector $\mathbf{e}$. Next, we construct a unit circle centered at the origin 0 of $\mathbf{e}_1$ and $\mathbf{e}$. To map any point $\mathbf{x}$ on the $\mathbf{e}_1$ line to a point $\mathbf{x}'$ on the unit circle, we construct a line from the point $\mathbf{x}$ to the end of the vector $\mathbf{e}$. Wherever this line cuts the unit circle is the point $\mathbf{x}'$ (see Figure 4.2).

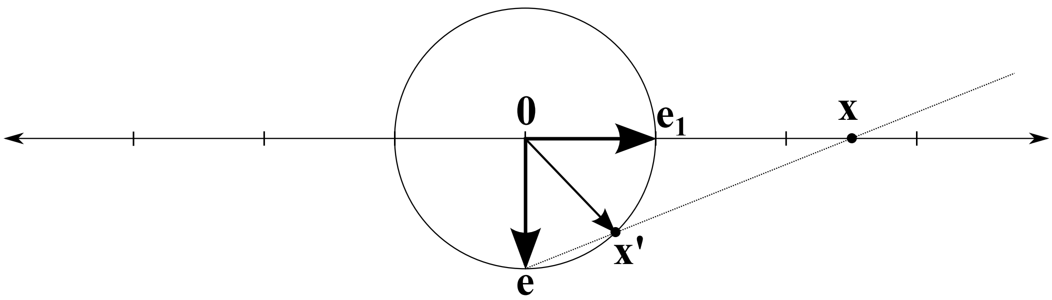


Figure 4.2: Stereographic projection

Stereographic projection allows us to work with points at infinity. The point at infinity on the $\mathbf{e}_1$ line is mapped to the point $\mathbf{e}$ on the unit circle. The stereographic projection $\mathbf{x}'$ can be expressed as a vector in terms of its components $\mathbf{e}_1$ and $\mathbf{e}$.

$$\mathbf{x}' = \frac{2x}{1+x^2}\,\mathbf{e}_1 - \frac{1-x^2}{1+x^2}\,\mathbf{e}, \qquad \text{where} \quad \mathbf{x}'\cdot\mathbf{x}' = \mathbf{x}'^{\,2} = 1. \tag{4.1}$$

Equation (4.1) can be derived as follows:

Let $\mathbf{x} = x\mathbf{e}_1$ and $\mathbf{x}' = \mathbf{e} + \lambda(\mathbf{x}-\mathbf{e})$, where $\lambda \in \mathbb{R}$. Note that

$$\mathbf{e}\cdot\mathbf{e} = 1, \quad \mathbf{x}'\cdot\mathbf{x}' = 1, \quad \mathbf{e}\cdot\mathbf{x} = 0, \quad \mathbf{x}\cdot\mathbf{x} = \mathbf{x}^2.$$

When we take the inner product of $\mathbf{x}'$ with itself, we get

$$\begin{aligned}
\mathbf{x}'\cdot\mathbf{x}' &= (\mathbf{e}+\lambda(\mathbf{x}-\mathbf{e}))\cdot(\mathbf{e}+\lambda(\mathbf{x}-\mathbf{e}))\\
\mathbf{x}'\cdot\mathbf{x}' &= \mathbf{e}\cdot\mathbf{e} + 2\lambda\mathbf{e}\cdot(\mathbf{x}-\mathbf{e}) + \lambda^2(\mathbf{x}-\mathbf{e})\cdot(\mathbf{x}-\mathbf{e})\\
\mathbf{x}'\cdot\mathbf{x}' &= \mathbf{e}\cdot\mathbf{e} + 2\lambda(\mathbf{e}\cdot\mathbf{x} - \mathbf{e}\cdot\mathbf{e}) + \lambda^2(\mathbf{x}\cdot\mathbf{x} - 2\mathbf{e}\cdot\mathbf{x} + \mathbf{e}\cdot\mathbf{e})\\
1 &= 1 + 2\lambda(0-1) + \lambda^2(\mathbf{x}^2 - 0 + 1)\\
1 &= 1 - 2\lambda + \lambda^2(1+\mathbf{x}^2)\\
0 &= \lambda^2(1+\mathbf{x}^2) - 2\lambda.
\end{aligned} \tag{4.2}$$

Solving Eq. (4.2) for λ, we get

$$\lambda = 0 \quad \text{or} \quad \lambda = \frac{2}{1+\mathbf{x}^2}.$$

If $\lambda = 0$, then $\mathbf{x}' = \mathbf{e}$ and $\mathbf{x}$ would be the point at infinity. For all other points $\mathbf{x}$ on the $\mathbf{e}_1$ axis, the equation for $\mathbf{x}'$ would be

$$\begin{aligned}
\mathbf{x}' &= \mathbf{e} + \lambda(\mathbf{x}-\mathbf{e})\\
\mathbf{x}' &= \mathbf{e} + \frac{2}{1+\mathbf{x}^2}(\mathbf{x}-\mathbf{e})\\
\mathbf{x}' &= \frac{\mathbf{e}(1+\mathbf{x}^2) + 2(\mathbf{x}-\mathbf{e})}{1+\mathbf{x}^2}\\
\mathbf{x}' &= \frac{\mathbf{e} + \mathbf{x}^2\mathbf{e} + 2\mathbf{x} - 2\mathbf{e}}{1+\mathbf{x}^2}
\end{aligned}$$

$$\mathbf{x}' = \frac{2\mathbf{x} + \mathbf{x}^2\mathbf{e} - \mathbf{e}}{1 + \mathbf{x}^2}$$

$$\mathbf{x}' = \frac{2x}{1 + x^2}\mathbf{e}_1 - \frac{1 - x^2}{1 + x^2}\mathbf{e}.$$

Thus, we have derived Eq. (4.1).

Using Eq. (4.1), any point $\mathbf{x}$ on the $\mathbf{e}_1$ line can be mapped to a point $\mathbf{x}'$ on the unit circle in the $\mathbf{e}_1, \mathbf{e}$ plane, e.g.,

$$\begin{aligned} \mathbf{x} = 0\mathbf{e}_1 &\quad \mapsto \quad \mathbf{x}' = -\mathbf{e}, \\ \mathbf{x} = 1\mathbf{e}_1 &\quad \mapsto \quad \mathbf{x}' = \mathbf{e}_1, \\ \mathbf{x} = \infty\mathbf{e}_1 &\quad \mapsto \quad \mathbf{x}' = \mathbf{e}. \end{aligned}$$

4.1.2 Add a Dimension for Homogeneous Coordinates

Before we go any further, we need to introduce the concept of *signature*. So far we have been working with *Inner-Product Spaces*, i.e., vector spaces equipped with positive definite, symmetric bilinear forms over the real numbers. At this point, we find it useful to relax our assumptions a little bit by dispensing with the assumption that the form under consideration is positive definite. Although we still use the term "inner product," along with the "dot product" notation to indicate the value of our symmetric bilinear form, we now have the possibility that a vector $\mathbf{v}$ satisfies $\mathbf{v} \cdot \mathbf{v} < 0$ or $\mathbf{v} \cdot \mathbf{v} = 0$.

Definition 4.1.1 (Signature).

If an n-dimensional vector space has p independent unit vectors satisfying $\mathbf{e}_i \cdot \mathbf{e}_i = 1$, and q independent unit vectors satisfying $\mathbf{e}_i \cdot \mathbf{e}_i = -1$, then it is customary to to say that the space has p "positive dimensions" and q "negative *dimensions" (with $n = p + q$). Note that we assume that our symmetric bilinear form is non-degenerate, thus avoiding "zero dimensions." Instead of $\mathbb{R}^n$, we then write $\mathbb{R}^{p,q}$ and call (p, q) the* **signature** *of the space. An n-dimensional Euclidean space is then written as $\mathbb{R}^{n,0}$, the representational space for the conformal model as $\mathbb{R}^{n+1,1}$, and an n-dimensional Minkowski space as $\mathbb{R}^{n-1,1}$ [DFM07, p. 586].*

Next, we add another dimension to the stereographic projection described in the previous section. This new dimension is identified by the unit vector $\bar{\mathbf{e}}$, where $\bar{\mathbf{e}}$ is perpendicular to both $\mathbf{e}_1$ and $\mathbf{e}$. We give this new vector a negative signature, meaning $\bar{\mathbf{e}} \cdot \bar{\mathbf{e}} = \bar{\mathbf{e}}^2 = -1$ ($\bar{\mathbf{e}}^2 = \bar{\mathbf{e}}\bar{\mathbf{e}} = \bar{\mathbf{e}} \cdot \bar{\mathbf{e}} + \bar{\mathbf{e}} \wedge \bar{\mathbf{e}} = \bar{\mathbf{e}} \cdot \bar{\mathbf{e}} + 0 = \bar{\mathbf{e}} \cdot \bar{\mathbf{e}} = |\bar{\mathbf{e}}|^2 = -1$). Finally, we lift the unit circle off the $\mathbf{e}_1\mathbf{e}$-plane so that its center lies one unit above the origin at the end of the $\bar{\mathbf{e}}$ vector (see Figure 4.3).

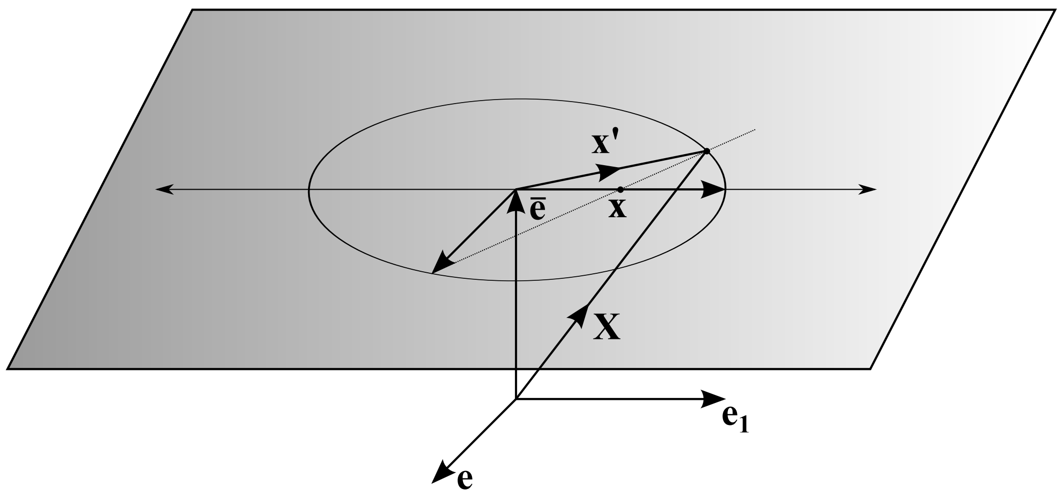


Figure 4.3: Addition of the $\bar{\mathbf{e}}$ dimension

The addition of this new dimension provides us with another degree of freedom that we use in our expression for mapping a real point into the conformal model. Thus, we have

$$\mathbf{X} = \mathbf{x}' + \bar{\mathbf{e}},$$
$$\mathbf{X} = \frac{2x}{1+x^2}\,\mathbf{e}_1 - \frac{1-x^2}{1+x^2}\,\mathbf{e} + \bar{\mathbf{e}}, \tag{4.3}$$

where we have substituted Eq. (4.1) for $\mathbf{x}'$.

Note that

$$\mathbf{X}^2 = \left(\frac{2x}{1+x^2}\,\mathbf{e}_1 - \frac{1-x^2}{1+x^2}\,\mathbf{e} + \bar{\mathbf{e}}\right)^2$$

$$= \left(\frac{2x}{1+x^2}\,\mathbf{e}_1 - \frac{1-x^2}{1+x^2}\,\mathbf{e} + \bar{\mathbf{e}}\right)\left(\frac{2x}{1+x^2}\,\mathbf{e}_1 - \frac{1-x^2}{1+x^2}\,\mathbf{e} + \bar{\mathbf{e}}\right)$$

$$= \left\{\begin{array}{rcrcrc} & \frac{4x^2}{(1+x^2)^2}\,\mathbf{e}_1^2 & - & \frac{2x\,(1-x^2)}{(1+x^2)^2}\,\mathbf{e}_1\mathbf{e} & + & \frac{2x}{1+x^2}\mathbf{e}_1\bar{\mathbf{e}} \\ - & \frac{(1-x^2)\,2x}{(1+x^2)^2}\,\mathbf{e}\mathbf{e}_1 & + & \frac{(1-x^2)^2}{(1+x^2)^2}\,\mathbf{e}^2 & - & \frac{1-x^2}{1+x^2}\,\mathbf{e}\bar{\mathbf{e}} \\ + & \frac{2x}{1+x^2}\,\bar{\mathbf{e}}\mathbf{e}_1 & - & \frac{1-x^2}{1+x^2}\,\bar{\mathbf{e}}\mathbf{e} & + & \bar{\mathbf{e}}^2 \end{array}\right.$$

$$= \left\{\begin{array}{rcrcrc} & \frac{4x^2}{(1+x^2)^2}\,1 & - & \frac{2x\,(1-x^2)}{(1+x^2)^2}\,\mathbf{e}_1\mathbf{e} & + & \frac{2x}{1+x^2}\mathbf{e}_1\bar{\mathbf{e}} \\ + & \frac{2x\,(1-x^2)}{(1+x^2)^2}\,\mathbf{e}_1\mathbf{e} & + & \frac{(1-x^2)^2}{(1+x^2)^2}\,1 & - & \frac{1-x^2}{1+x^2}\,\mathbf{e}\bar{\mathbf{e}} \\ - & \frac{2x}{1+x^2}\,\mathbf{e}_1\bar{\mathbf{e}} & + & \frac{1-x^2}{1+x^2}\,\mathbf{e}\bar{\mathbf{e}} & - & 1 \end{array}\right.$$

$$= \frac{4x^2}{(1+x^2)^2} + \frac{(1-x^2)^2}{(1+x^2)^2} - 1$$

$$= \frac{4x^2 + (1-x^2)(1-x^2) - (1+x^2)(1+x^2)}{(1+x^2)^2}$$

$$= \frac{4x^2 + 1 - 2x^2 + x^4 - 1 - 2x^2 - x^4}{(1+x^2)^2}$$

$$= 0.$$

We see that the negative signature of $\bar{\mathbf{e}}$ ensures that $\mathbf{X}$ is always a *null vector*, i.e., $\mathbf{X}^2 = \mathbf{X}\mathbf{X} = \mathbf{X} \cdot \mathbf{X} + \mathbf{X} \wedge \mathbf{X} = |\mathbf{X}|^2 + 0 = |\mathbf{X}|^2 = 0$.

Definition 4.1.2 (**Null vector**).

A **null vector** *is a vector* $\mathbf{v}$ *whose inner product with itself is zero, i.e.,* $\mathbf{v} \cdot \mathbf{v} = 0$. *Null vectors show up when we work with spaces of mixed signature (both positive and negative), as with the conformal model, for example. Null vectors may be nonzero. However, the zero vector is a null vector. Null vectors are not invertible.*

The equation $\mathbf{X}^2 = 0$ is homogeneous: if it is satisfied for $\mathbf{X}$, it is satisfied for any nonzero scalar multiple of $\mathbf{X}$ (e.g., $\sigma\mathbf{X}$, where $\sigma \in \mathbb{R}$). Any nonzero scalar multiple of the null vector $\mathbf{X}$ is also a null vector that lies on the line that passes through the same point on the unit circle as $\mathbf{X}$ does. The extra degree of freedom we get by adding the $\bar{\mathbf{e}}$ dimension has created what is called a *homogeneous coordinate system* where we consider all such nonzero scalar multiples of $\mathbf{X}$ to represent the same point on the unit circle. Thus, all nonzero scalar multiples of $\mathbf{X}$ form an equivalence class.

The locus of all lines that pass from the origin through the unit circle is called the *null cone*.

We can simplify Eq. (4.3) by multiplying each term by the scalar $\sigma(1 + x^2)$, giving,

$$\mathbf{X} = \sigma(2x\mathbf{e}_1 - (1 - x^2)\mathbf{e} + (1 + x^2)\bar{\mathbf{e}}), \tag{4.4}$$

where σ is an arbitrary nonzero scalar. Thus, Eq. (4.4) defines an equivalence class of null vectors $\mathbf{X} \in \mathbb{R}^{2,1}$ used to represent a single vector $\mathbf{x} = x\mathbf{e}_1 \in \mathbb{R}^1$.

The vectors $\mathbf{e}$ and $\bar{\mathbf{e}}$ have extended the space $\mathbb{R}^1$ to $\mathbb{R}^{2,1}$. The basis elements for this new conformal space are $\{\mathbf{e}_1, \mathbf{e}, \bar{\mathbf{e}}\}$, where $\mathbf{e}_1^2 = 1$, $\mathbf{e}^2 = 1$, and $\bar{\mathbf{e}}^2 = -1$.

The inner, outer, and geometric products of $\{\mathbf{e}_1, \mathbf{e}, \bar{\mathbf{e}}\}$ are summarized in Table 4.1.

Table 4.1: Multiplication tables for $\mathbf{e}_1$, $\mathbf{e}$, $\bar{\mathbf{e}}$.

$\cdot$	$\mathbf{e}_1$	$\mathbf{e}$	$\bar{\mathbf{e}}$
$\mathbf{e}_1$	1	0	0
$\mathbf{e}$	0	1	0
$\bar{\mathbf{e}}$	0	0	-1

$\wedge$	$\mathbf{e}_1$	$\mathbf{e}$	$\bar{\mathbf{e}}$
$\mathbf{e}_1$	0	$\mathbf{e}_1 \wedge \mathbf{e}$	$\mathbf{e}_1 \wedge \bar{\mathbf{e}}$
$\mathbf{e}$	$-\mathbf{e}_1 \wedge \mathbf{e}$	0	$\mathbf{e} \wedge \bar{\mathbf{e}}$
$\bar{\mathbf{e}}$	$-\mathbf{e}_1 \wedge \bar{\mathbf{e}}$	$-\mathbf{e} \wedge \bar{\mathbf{e}}$	0

GP	$\mathbf{e}_1$	$\mathbf{e}$	$\bar{\mathbf{e}}$
$\mathbf{e}_1$	1	$\mathbf{e}_1\mathbf{e}$	$\mathbf{e}_1\bar{\mathbf{e}}$
$\mathbf{e}$	$-\mathbf{e}_1\mathbf{e}$	1	$\mathbf{e}\bar{\mathbf{e}}$
$\bar{\mathbf{e}}$	$-\mathbf{e}_1\bar{\mathbf{e}}$	$-\mathbf{e}\bar{\mathbf{e}}$	-1

4.1.3 Change Variables from $\mathbf{e}$, $\bar{\mathbf{e}}$ to $\mathbf{n}$, $\bar{\mathbf{n}}$

Here is a recap of what we have done so far:

(1) Stereographic Projection - We added the dimension $\mathbf{e}$ perpendicular to the Euclidean dimension $\mathbf{e}_1$. This allows us to map all points on the Euclidean line $\mathbf{e}_1$—including the point at infinity—onto the unit circle.

(2) Homogeneous Projection - We added the dimension $\bar{\mathbf{e}}$ perpendicular to the Euclidean dimension $\mathbf{e}_1$ and perpendicular to $\mathbf{e}$. We then raised the unit circle off of the $\mathbf{e}_1\mathbf{e}$-plane, placing its center on the end point of the $\bar{\mathbf{e}}$ vector. Any point on the unit circle can now be identified by an equivalence class of nonzero vectors that lie on a line that passes from the origin through that point on the unit circle. Thus, an equivalence class of vectors all refer to the same point on the Euclidean line $\mathbf{e}_1$ vis-à-vis the point on the unit circle. With the choice of the negative signature for $\bar{\mathbf{e}}$, we insure that all nonzero vectors $\mathbf{X} \in \mathbb{R}^{2,1}$ lying on a line that passes from the origin through the unit circle, and, therefore, representing points on the Euclidean line $\mathbf{e}_1$, are null vectors, i.e., $\mathbf{X}^2 = 0$.

Unfortunately, the basis vectors $\mathbf{e}$ and $\bar{\mathbf{e}}$ that we used to get additional dimensions have no geometric interpretation. If we use a change of variables to replace these two basis vectors with two null vectors, then it is possible to assign a geometric interpretation to these null vectors. We can achieve this by replacing our basis vectors $\mathbf{e}$ and $\bar{\mathbf{e}}$ with the null vectors $\mathbf{n}$ and $\overline{\mathbf{n}}$ defined as:

$$\boxed{\mathbf{n} = \mathbf{e} + \bar{\mathbf{e}}}, \qquad \boxed{\overline{\mathbf{n}} = \mathbf{e} - \bar{\mathbf{e}}}. \tag{4.5}$$

Note that just like $\mathbf{e}$ and $\bar{\mathbf{e}}$, both $\mathbf{n}$ and $\overline{\mathbf{n}}$ are perpendicular to the Euclidean space spanned by $\mathbf{e}_1$, i.e., $\mathbf{n} \cdot \mathbf{e}_1 = 0$ and $\overline{\mathbf{n}} \cdot \mathbf{e}_1 = 0$. These vectors satisfy the following relations:

$$\boxed{\mathbf{n}^2 = \overline{\mathbf{n}}^2 = 0}, \qquad \boxed{\mathbf{n} \cdot \overline{\mathbf{n}} = 2}, \qquad \boxed{\mathbf{n} \wedge \overline{\mathbf{n}} = -2\mathbf{e}\bar{\mathbf{e}}}. \tag{4.6}$$

The inner, outer, and geometric products of $\{\mathbf{e}_1, \mathbf{n}, \overline{\mathbf{n}}\}$ are summarized in Table 4.2.

Table 4.2: Multiplication tables for $\mathbf{e}_1$, $\mathbf{n}$, $\overline{\mathbf{n}}$.

$\cdot$	$\mathbf{e}_1$	$\mathbf{n}$	$\overline{\mathbf{n}}$
$\mathbf{e}_1$	1	0	0
$\mathbf{n}$	0	0	2
$\overline{\mathbf{n}}$	0	2	0

$\wedge$	$\mathbf{e}_1$	$\mathbf{n}$	$\overline{\mathbf{n}}$
$\mathbf{e}_1$	0	$\mathbf{e}_1 \wedge \mathbf{n}$	$\mathbf{e}_1 \wedge \overline{\mathbf{n}}$
$\mathbf{n}$	$-\mathbf{e}_1 \wedge \mathbf{n}$	0	$-2\mathbf{e}\bar{\mathbf{e}}$
$\overline{\mathbf{n}}$	$-\mathbf{e}_1 \wedge \overline{\mathbf{n}}$	$2\mathbf{e}\bar{\mathbf{e}}$	0

GP	$\mathbf{e}_1$	$\mathbf{n}$	$\overline{\mathbf{n}}$
$\mathbf{e}_1$	1	$\mathbf{e}_1\mathbf{n}$	$\mathbf{e}_1\overline{\mathbf{n}}$
$\mathbf{n}$	$-\mathbf{e}_1\mathbf{n}$	0	$2 - 2\mathbf{e}\bar{\mathbf{e}}$
$\overline{\mathbf{n}}$	$-\mathbf{e}_1\overline{\mathbf{n}}$	$2 + 2\mathbf{e}\bar{\mathbf{e}}$	0

Substituting Eqs. (4.5) into Eq. (4.4), we get

$$\mathbf{X} = \sigma(2x\mathbf{e}_1 + x^2\mathbf{n} - \overline{\mathbf{n}}), \tag{4.7}$$

where σ is an arbitrary nonzero scalar.

This configuration can be visualized as shown in Figure 4.4. Points on the line determined by $\mathbf{e}_1$ are mapped to null vectors on the *null cone* using Eq. (4.7).

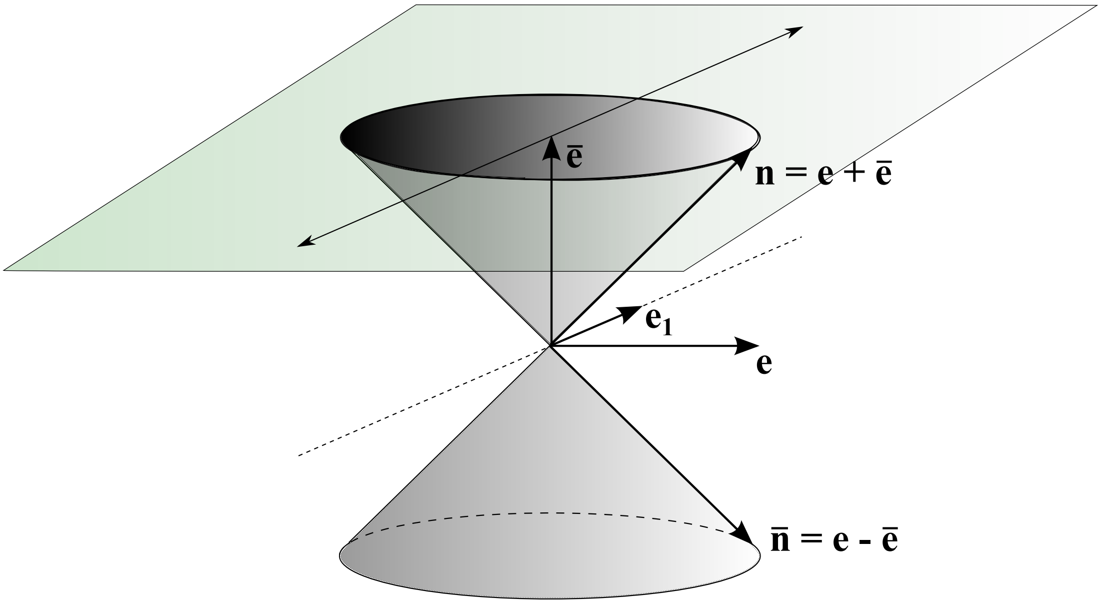


Figure 4.4: The conformal model of a line.

In the sections to follow, we have need of an identity related to $\mathbf{n}$ and $\bar{\mathbf{n}}$. This identity is now derived.

Identity 4.1.3 ($\mathbf{n}\bar{\mathbf{n}}\mathbf{n} = 4\mathbf{n}$)**.**

$$
\begin{aligned}
\mathbf{n}\bar{\mathbf{n}}\mathbf{n} &= (\mathbf{e}+\bar{\mathbf{e}})(\mathbf{e}-\bar{\mathbf{e}})(\mathbf{e}+\bar{\mathbf{e}}) \\
&= (\mathbf{e}^2 - \mathbf{e}\bar{\mathbf{e}} + \bar{\mathbf{e}}\mathbf{e} - \bar{\mathbf{e}}^2)(\mathbf{e}+\bar{\mathbf{e}}) \\
&= (1 - \mathbf{e}\bar{\mathbf{e}} - \mathbf{e}\bar{\mathbf{e}} + 1)(\mathbf{e}+\bar{\mathbf{e}}) \\
&= (2 - 2\mathbf{e}\bar{\mathbf{e}})(\mathbf{e}+\bar{\mathbf{e}}) \\
&= 2\mathbf{e} + 2\bar{\mathbf{e}} - 2\mathbf{e}\bar{\mathbf{e}}\mathbf{e} - 2\mathbf{e}\bar{\mathbf{e}}^2 \\
&= 2\mathbf{e} + 2\bar{\mathbf{e}} + 2\mathbf{e}^2\bar{\mathbf{e}} + 2\mathbf{e} \\
&= 2\mathbf{e} + 2\bar{\mathbf{e}} + 2\bar{\mathbf{e}} + 2\mathbf{e} \\
&= 4\mathbf{e} + 4\bar{\mathbf{e}} = 4(\mathbf{e}+\bar{\mathbf{e}}) = 4\mathbf{n}.
\end{aligned}
$$

$$\therefore \quad \boxed{\mathbf{n}\bar{\mathbf{n}}\mathbf{n} = 4\mathbf{n}}\,. \tag{4.8}$$

4.1.4 Generalization to Euclidean 3-Space

The preceding discussion, including Eq. (4.7), shows how to map a one-dimensional Euclidean space into a three-dimensional conformal model. It is easy to extend these findings to three-dimensional Euclidean space. First, for the one-dimensional case, we let the vector $\mathbf{x}$ be defined as $\mathbf{x} = x\mathbf{e}_1$ where $x^2 = \mathbf{x} \cdot \mathbf{x} = \mathbf{x}^2$. Then Eq. (4.7) becomes

$$\mathbf{X} = \sigma(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}), \tag{4.9}$$

where σ is an arbitrary nonzero scalar.

Next, we replace $\mathbf{x} = x\mathbf{e}_1$ with $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, and use Eq. (4.9) to define $\mathbf{X}$, the conformal mapping of $\mathbf{x} \in \mathbb{R}^3$ into $\mathbb{R}^{4,1}$. Note that $\mathbf{e}$ and $\bar{\mathbf{e}}$ are perpendicular to all three Euclidean basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and that $\mathbf{X}$ lies on a line that intersects a unit hypersphere.

Because σ can be any arbitrary nonzero scalar, we choose the value of one-half as being convenient for expressing the *standard form of the conformal image of* $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$,

$$\text{Standard Form:} \qquad \boxed{\mathbf{X} = \tfrac{1}{2}(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}})}\,.^2 \tag{4.10}$$

If we are given an arbitrary null vector $\mathbf{X}$ in conformal space, it is frequently convenient to convert it into the standard form of Eq. (4.10) so that the Euclidean coordinates can be easily extracted. If $\mathbf{X}$ is in standard form, then

$$-\mathbf{X} \cdot \mathbf{n} = -\tfrac{1}{2}(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}) \cdot \mathbf{n} = 1. \tag{4.11}$$

[2] Different authors use different standard forms of the conformal equation. See Appendix A.

An arbitrary null vector $\mathbf{X}$ can be *normalized* by dividing it by $-\mathbf{X}\cdot\mathbf{n}$.

$$\text{Normalization:} \qquad \boxed{\widehat{\mathbf{X}} = \frac{\mathbf{X}}{-\mathbf{X}\cdot\mathbf{n}}}. \tag{4.12}$$

For example, given the vector

$$\mathbf{X} = 48\mathbf{e}_1 + 20\mathbf{e}_2 + 16\mathbf{e}_3 + 17\mathbf{e} + 57\bar{\mathbf{e}}, \quad \text{where } \mathbf{X}^2 = \mathbf{X}\cdot\mathbf{X} = 0,$$

we have

$$\begin{aligned} -\mathbf{X}\cdot\mathbf{n} &= -(48\mathbf{e}_1 + 20\mathbf{e}_2 + 16\mathbf{e}_3 + 17\mathbf{e} + 57\bar{\mathbf{e}})\cdot\mathbf{n} \\ &= -(48\mathbf{e}_1 + 20\mathbf{e}_2 + 16\mathbf{e}_3 + 17\mathbf{e} + 57\bar{\mathbf{e}})\cdot(\mathbf{e} + \bar{\mathbf{e}}) \\ &= -(17 - 57) \\ &= 40. \end{aligned}$$

Therefore the normalized form of $\mathbf{X}$ is

$$\begin{aligned} \widehat{\mathbf{X}} &= \frac{\mathbf{X}}{-\mathbf{X}\cdot\mathbf{n}} \\ &= \frac{48\mathbf{e}_1 + 20\mathbf{e}_2 + 16\mathbf{e}_3 + 17\mathbf{e} + 57\bar{\mathbf{e}}}{40} \\ &= \tfrac{48}{40}\mathbf{e}_1 + \tfrac{20}{40}\mathbf{e}_2 + \tfrac{16}{40}\mathbf{e}_3 + \tfrac{17}{40}\mathbf{e} + \tfrac{57}{40}\bar{\mathbf{e}} \\ &= \tfrac{6}{5}\mathbf{e}_1 + \tfrac{1}{2}\mathbf{e}_2 + \tfrac{2}{5}\mathbf{e}_3 + \tfrac{17}{40}\mathbf{e} + \tfrac{57}{40}\bar{\mathbf{e}} \\ &= \tfrac{6}{5}\mathbf{e}_1 + \tfrac{1}{2}\mathbf{e}_2 + \tfrac{2}{5}\mathbf{e}_3 + \tfrac{37}{40}\mathbf{n} - \tfrac{1}{2}\bar{\mathbf{n}}. \end{aligned}$$

And from this formula, the Euclidean coordinates of the point x can be read as $\mathbf{x} = (\frac{6}{5}, \frac{1}{2}, \frac{2}{5})$.

4.1.5 Geometric Interpretation of n and $\bar{\mathbf{n}}$

We would like to be able to give a geometric interpretation to $\mathbf{n}$ and $\bar{\mathbf{n}}$ in Eq. (4.10). If $\mathbf{x}$ is at the origin of the Euclidean space, i.e., $\mathbf{x} = \mathbf{0}$, then by

Eq. (4.10), $\mathbf{X} = -\frac{1}{2}\overline{\mathbf{n}}$, which implies that

$$\boxed{\overline{\mathbf{n}} \text{ represents the origin}}.$$

To interpret $\mathbf{n}$, we first need to take the inner product $\mathbf{X} \cdot \overline{\mathbf{n}}$:

$$\begin{aligned}
\mathbf{X} \cdot \overline{\mathbf{n}} &= \tfrac{1}{2}(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}) \cdot \overline{\mathbf{n}} \\
&= \tfrac{1}{2}(2\mathbf{x} \cdot \overline{\mathbf{n}} + \mathbf{x}^2(\mathbf{n} \cdot \overline{\mathbf{n}}) - \overline{\mathbf{n}} \cdot \overline{\mathbf{n}}) \\
&= \tfrac{1}{2}(0 + 2\mathbf{x}^2 + 0) \\
&= \mathbf{x}^2.
\end{aligned}$$

Multiplying the homogeneous vector $\mathbf{X}$ by a scalar doesn't change the Euclidean point it represents. We multiply $\mathbf{X}$ by the scalar $\sigma = 2/(\mathbf{X} \cdot \overline{\mathbf{n}}) = 2/\mathbf{x}^2$ and take the limit as $\mathbf{x}$ approaches infinity.

$$\begin{aligned}
\lim_{\mathbf{x}\to\infty} \sigma\mathbf{X} &= \lim_{\mathbf{x}\to\infty} \left(\frac{2}{\mathbf{X} \cdot \overline{\mathbf{n}}}\right) \mathbf{X} \\
&= \lim_{\mathbf{x}\to\infty} \left(\frac{2}{\mathbf{x}^2}\right) \tfrac{1}{2}(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}) \\
&= \lim_{\mathbf{x}\to\infty} \left(\mathbf{n} + \frac{2\mathbf{x} - \overline{\mathbf{n}}}{\mathbf{x}^2}\right) \\
&= \mathbf{n}. \qquad (4.13)
\end{aligned}$$

From Eq. (4.13) we see that

$$\boxed{\mathbf{n} \text{ represents the point at infinity}}.$$

4.2 Conformal Transformations

We now examine conformal transformations using the conformal model. *Conformal transformations* are defined as transformations that leave angles invariant. Conformal transformations of $\mathbb{R}^3$ form a group under composition—the *conformal group*—the main elements of which are translations, rotations, dilations, and inversions. Translations and rotations form a subgroup of the larger conformal group called the *Euclidean group.* The Euclidean group consists of transformations that leave the distance between points invariant.

Definition 4.2.1 (**Versor**).

A **versor** *is a geometric product of invertible vectors. Examples of versors are* $\mathbf{u}, \mathbf{uv}, \mathbf{uvw}$*, etc. A versor is called* even *or* odd *depending upon whether it has an even or odd number of vector terms.*

In the conformal model, conformal transformations are implemented with a sandwiching operation using a versor, i.e.,

$$\text{even versors:} \qquad \mathbf{X} \mapsto \mathbf{V}\mathbf{X}\mathbf{V}^{-1},$$
$$\text{odd versors:} \qquad \mathbf{X} \mapsto -\mathbf{V}\mathbf{X}\mathbf{V}^{-1},$$

where $\mathbf{V}$ is a versor and $\mathbf{X} = \frac{1}{2}(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}})$ is a conformal space representation of a vector $\mathbf{x} \in \mathbb{R}^3$. In this section we drop the $\frac{1}{2}$ from the standard form and use $\mathbf{X} = 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}$ when deriving the transformation versors. This makes the formulas a little easier to follow and does not impact the final results.

We use the following approach in our study of conformal transformations:

- Present the versor $\mathbf{V}$ used to generate the transformation.
- Derive how the versor transforms $\mathbf{x}$, $\mathbf{n}$, and $\overline{\mathbf{n}}$, i.e., $\mathbf{x} \mapsto \pm\mathbf{V}\mathbf{x}\mathbf{V}^{-1}$, $\mathbf{n} \mapsto \pm\mathbf{V}\mathbf{n}\mathbf{V}^{-1}$, and $\overline{\mathbf{n}} \mapsto \pm\mathbf{V}\overline{\mathbf{n}}\mathbf{V}^{-1}$.
- Use the above information and the distributive property to derive how the versor $\mathbf{V}$ transforms $\mathbf{X} = 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}$, the conformal model representation

of $\mathbf{x}$, i.e.,

$$\begin{aligned}\pm\mathbf{V}\mathbf{X}\mathbf{V}^{-1} &= \pm(\mathbf{V}\,(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}})\,\mathbf{V}^{-1}) \\ &= \pm(2\mathbf{V}\mathbf{x}\mathbf{V}^{-1} + \mathbf{x}^2\mathbf{V}\mathbf{n}\mathbf{V}^{-1} - \mathbf{V}\overline{\mathbf{n}}\mathbf{V}^{-1}).\end{aligned}$$

4.2.1 Translation

Consider the operation of translating an object in $\mathbb{R}^3$ in a direction and a distance indicated by the vector $\mathbf{a}$ (see Figure 4.5).

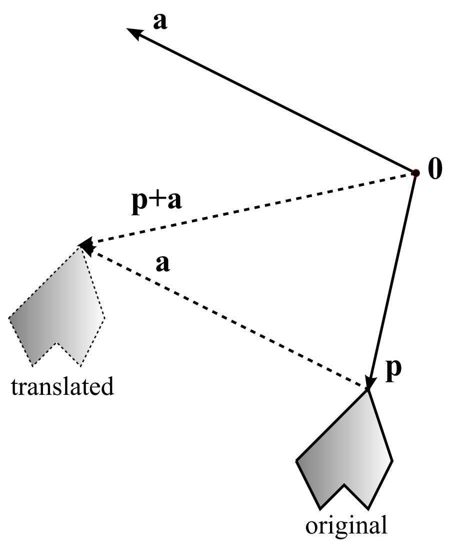


Figure 4.5: Translation using vector $\mathbf{a}$

Each point on the object is translated by the vector $\mathbf{a}$. If the location of a point on the object is indicated by the vector $\mathbf{p}$, then $\mathbf{p} + \mathbf{a}$ locates the same point on the translated object.

If $f_{\mathbf{a}}$ is a function that translates objects using the vector $\mathbf{a}$ then, $f_{\mathbf{a}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by $f_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} + \mathbf{a}$.

Definition 4.2.2 (Rotor).

A **rotor** *is a geometric product of an even number of invertible unit vectors. Rotors have the useful property that* $R^{-1} = \widetilde{R}$ *(where* $\widetilde{R}$ *is the reversion of* R*). Examples of rotors are* $\mathbf{uv}, \mathbf{vw}, \mathbf{uw}$*, if* $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$*. Rotors can always be expressed as exponentials, e.g.,* $\mathbf{uv} = e^{\mathbf{I}_{(2)}\theta}$*, where* $\mathbf{I}_{(2)}$ *is the unit pseudoscalar in the* $\mathbf{uv}$*-plane and* θ *is the angle from* $\mathbf{u}$ *to* $\mathbf{v}$*. Rotors are even unit versors.*

To implement translation in the conformal model, we use the rotor

$$\mathbf{R} = \mathbf{T_a} = e^{\mathbf{na}/2},$$

where $\mathbf{a} \in \mathbb{R}^3$ is the translation vector and $\mathbf{n} \cdot \mathbf{a} = 0$.

The term $\mathbf{na} = \mathbf{n} \cdot \mathbf{a} + \mathbf{n} \wedge \mathbf{a} = 0 + \mathbf{n} \wedge \mathbf{a} = \mathbf{n} \wedge \mathbf{a}$ is a bivector, and $(\mathbf{na})^2 = \mathbf{nana} = -\mathbf{nnaa} = -0\mathbf{a}^2 = 0$, thus $\mathbf{na}$ is a *null bivector.*

Consider the series expansion for the exponential,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Replacing x with $\mathbf{na}/2$, we get

$$e^{\mathbf{na}/2} = 1 + \frac{\mathbf{na}}{2} + \frac{(\frac{\mathbf{na}}{2})^2}{2!} + \frac{(\frac{\mathbf{na}}{2})^3}{3!} + \cdots .$$

Because $\mathbf{na}$ is a null bivector, all terms with power greater than one become zero and we have

Translation rotors: $$\boxed{\mathbf{T_a} = e^{\mathbf{na}/2} = 1 + \frac{\mathbf{na}}{2}} \quad \text{and} \quad \boxed{\widetilde{\mathbf{T}}_\mathbf{a} = e^{-\mathbf{na}/2} = 1 - \frac{\mathbf{na}}{2}}. \tag{4.14}$$

For clarity, we apply these translation rotors separately to the vectors $\mathbf{x}$, $\mathbf{n}$, and $\overline{\mathbf{n}}$. We then use the distributive property,

$$\mathbf{RX\widetilde{R}} = \mathbf{R}\left(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}\right)\widetilde{\mathbf{R}} = \mathbf{R}2\mathbf{x}\widetilde{\mathbf{R}} + \mathbf{R}\mathbf{x}^2\mathbf{n}\widetilde{\mathbf{R}} - \mathbf{R}\overline{\mathbf{n}}\widetilde{\mathbf{R}}. \tag{4.15}$$

to show that the sandwich operation $\mathbf{T_a X \widetilde{T}_a}$ corresponds to a translation of the vector $\mathbf{x}$ in $\mathbb{R}^3$.

For $\mathbf{x} \in \mathbb{R}^3$ we have

$$\begin{aligned}
\mathbf{T_a x \widetilde{T}_a} &= (1 + \frac{\mathbf{na}}{2})\,\mathbf{x}\,(1 - \frac{\mathbf{na}}{2}) \\
&= (\mathbf{x} + \frac{\mathbf{nax}}{2})(1 - \frac{\mathbf{na}}{2}) \\
&= \mathbf{x} - \frac{\mathbf{xna}}{2} + \frac{\mathbf{nax}}{2} - \frac{\mathbf{naxna}}{4} \\
&= \mathbf{x} + \frac{\mathbf{nxa}}{2} + \frac{\mathbf{nax}}{2} - \frac{\mathbf{n^2axa}}{4} \\
&= \mathbf{x} + \mathbf{n}\left(\frac{\mathbf{xa} + \mathbf{ax}}{2}\right) - 0 = \mathbf{x} + \mathbf{n}(\mathbf{a} \cdot \mathbf{x})
\end{aligned}$$

$$\boxed{\mathbf{T_a x \widetilde{T}_a} = \mathbf{x} + (\mathbf{a} \cdot \mathbf{x})\mathbf{n}}. \tag{4.16}$$

For $\mathbf{n} \in \mathbb{R}^{4,1}$, the point at infinity, we have

$$\begin{aligned}
\mathbf{T_a n \widetilde{T}_a} &= (1 + \frac{\mathbf{na}}{2})\,\mathbf{n}\,(1 - \frac{\mathbf{na}}{2}) \\
&= (\mathbf{n} + \frac{\mathbf{nan}}{2})(1 - \frac{\mathbf{na}}{2}) \\
&= (\mathbf{n} - \frac{\mathbf{n^2a}}{2})(1 - \frac{\mathbf{na}}{2}) \\
&= (\mathbf{n} - 0)(1 - \frac{\mathbf{na}}{2}) \\
&= \mathbf{n}\,(1 - \frac{\mathbf{na}}{2}) \\
&= \mathbf{n} - \frac{\mathbf{n^2a}}{2} = \mathbf{n} - 0 = \mathbf{n}
\end{aligned}$$

$$\boxed{\mathbf{T_a n \widetilde{T}_a} = \mathbf{n}}. \tag{4.17}$$

For $\overline{\mathbf{n}} \in \mathbb{R}^{4,1}$, the point at the origin, we have

$$\begin{aligned}
\mathbf{T_a}\overline{\mathbf{n}}\widetilde{\mathbf{T}}_\mathbf{a} &= (1+\frac{\mathbf{na}}{2})\,\overline{\mathbf{n}}\,(1-\frac{\mathbf{na}}{2}) \\
&= (\overline{\mathbf{n}}+\frac{\mathbf{na}\overline{\mathbf{n}}}{2})(1-\frac{\mathbf{na}}{2}) \\
&= \overline{\mathbf{n}} - \frac{\overline{\mathbf{n}}\mathbf{na}}{2} + \frac{\mathbf{na}\overline{\mathbf{n}}}{2} - \frac{\mathbf{na}\overline{\mathbf{n}}\mathbf{na}}{4} \\
&= \overline{\mathbf{n}} - \frac{\overline{\mathbf{n}}\mathbf{na}}{2} - \frac{\mathbf{n}\overline{\mathbf{n}}\mathbf{a}}{2} - \frac{\mathbf{n}\overline{\mathbf{n}}\mathbf{n}\mathbf{a}^2}{4} \\
&\overset{5}{=} \overline{\mathbf{n}} - \left(\frac{\overline{\mathbf{n}}\mathbf{n}+\mathbf{n}\overline{\mathbf{n}}}{2}\right)\mathbf{a} - \frac{4\mathbf{n}\mathbf{a}^2}{4} \\
&= \overline{\mathbf{n}} - (\mathbf{n}\cdot\overline{\mathbf{n}})\,\mathbf{a} - \mathbf{n}\mathbf{a}^2 = \overline{\mathbf{n}} - 2\mathbf{a} - \mathbf{a}^2\mathbf{n}
\end{aligned}$$

$$\boxed{\mathbf{T_a}\overline{\mathbf{n}}\widetilde{\mathbf{T}}_\mathbf{a} = \overline{\mathbf{n}} - 2\mathbf{a} - \mathbf{a}^2\mathbf{n}}\,. \tag{4.18}$$

Where step (5) uses Identity 4.1.3, viz., $\mathbf{n}\overline{\mathbf{n}}\mathbf{n} = 4\mathbf{n}$.

We now return to our equation for conformal mapping $\mathbf{X} = 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}$. Let $F : \mathbb{R}^3 \to \mathbb{R}^{4,1}$, where $F(x) = \mathbf{X} = 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}$. Then,

$$\begin{aligned}
\mathbf{T_a}F(\mathbf{x})\widetilde{\mathbf{T}}_\mathbf{a} &= \mathbf{T_a}(2\mathbf{x}+\mathbf{x}^2\mathbf{n}-\overline{\mathbf{n}})\widetilde{\mathbf{T}}_\mathbf{a} \\
&= \mathbf{T_a}2\mathbf{x}\widetilde{\mathbf{T}}_\mathbf{a} + \mathbf{T_a}\mathbf{x}^2\mathbf{n}\widetilde{\mathbf{T}}_\mathbf{a} - \mathbf{T_a}\overline{\mathbf{n}}\widetilde{\mathbf{T}}_\mathbf{a} \\
&\overset{3}{=} (2\mathbf{x}+2(\mathbf{a}\cdot\mathbf{x})\mathbf{n}) + (\mathbf{x}^2\mathbf{n}) - (\overline{\mathbf{n}} - 2\mathbf{a} - \mathbf{a}^2\mathbf{n}) \\
&= 2(\mathbf{x}+\mathbf{a}) + (\mathbf{ax}+\mathbf{xa})\mathbf{n} + \mathbf{x}^2\mathbf{n} + \mathbf{a}^2\mathbf{n} - \overline{\mathbf{n}} \\
&= 2(\mathbf{x}+\mathbf{a}) + (\mathbf{x}^2+\mathbf{ax}+\mathbf{xa}+\mathbf{a}^2)\,\mathbf{n} - \overline{\mathbf{n}} \\
&= 2(\mathbf{x}+\mathbf{a}) + (\mathbf{x}+\mathbf{a})^2\,\mathbf{n} - \overline{\mathbf{n}}.
\end{aligned}$$

$$\text{Conformal Translation:} \qquad \boxed{\mathbf{T_a}F(\mathbf{x})\widetilde{\mathbf{T}}_\mathbf{a} = 2(\mathbf{x}+\mathbf{a}) + (\mathbf{x}+\mathbf{a})^2\,\mathbf{n} - \overline{\mathbf{n}}}\,. \tag{4.19}$$

Where step (3) comes from Eqs. (4.16), (4.17), and (4.18). Note that

$$\mathbf{T_a}F(\mathbf{x})\widetilde{\mathbf{T}}_\mathbf{a} = F(\mathbf{x}+\mathbf{a}),$$

which performs the conformal transformation corresponding to the translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$. In conformal space, translations are handled as rotations in the $\mathbf{na}$ plane.

4.2.2 Rotation

In Section 3.7 we learned that rotating a vector $\mathbf{x} \in \mathbb{R}^3$ about the origin, in the $\mathbf{I}_{(2)}$-plane, can be accomplished by a sandwiching operation

$$\mathbf{R}\mathbf{x}\widetilde{\mathbf{R}}.$$

The rotor $\mathbf{R}$ was defined as

$$\text{Rotation operators:} \qquad \boxed{\mathbf{R} = e^{\mathbf{I}_{(2)}\theta/2}} \quad \text{and} \quad \boxed{\widetilde{\mathbf{R}} = e^{-\mathbf{I}_{(2)}\theta/2}}. \tag{4.20}$$

where θ is the angle of rotation and $\mathbf{I}_{(2)}$ is a unit pseudoscalar for the plane of rotation. The formula for rotating the vector $\mathbf{x}$ is

$$\boxed{\mathbf{R}\mathbf{x}\widetilde{\mathbf{R}} = e^{\mathbf{I}_{(2)}\theta/2}\,\mathbf{x}\,e^{-\mathbf{I}_{(2)}\theta/2}}. \tag{4.21}$$

We now see what happens when we apply this rotor to the vector $\mathbf{n} \in \mathbb{R}^{4,1}$, the point at infinity. Without loss of generality we can assume that $\mathbf{I}_{(2)} = \mathbf{b}_1\mathbf{b}_2$ where $\mathbf{b}_1$ and $\mathbf{b}_2$ are orthogonal unit vectors in the $\mathbf{I}_{(2)}$-plane. Then we have

$$\begin{aligned}
\mathbf{n}\mathbf{I}_{(2)} &= \mathbf{n}\mathbf{b}_1\mathbf{b}_2 \\
\mathbf{n}\mathbf{I}_{(2)} &= \mathbf{b}_1\mathbf{b}_2\mathbf{n} \\
\mathbf{n}\mathbf{I}_{(2)} &= \mathbf{I}_{(2)}\mathbf{n}.
\end{aligned} \tag{4.22}$$

and

$$\mathbf{I}_{(2)}\mathbf{n}\mathbf{I}_{(2)} = \mathbf{I}_{(2)}\mathbf{I}_{(2)}\mathbf{n}$$
$$\mathbf{I}_{(2)}\mathbf{n}\mathbf{I}_{(2)} = -\mathbf{n}. \tag{4.23}$$

Then expanding Eq. (4.21) and using the results of Eqs. (4.22) and (4.23) we have,

$$\begin{aligned}
\mathbf{R}\mathbf{n}\widetilde{\mathbf{R}} &= e^{\mathbf{I}_{(2)}\theta/2}\,\mathbf{n}\,e^{-\mathbf{I}_{(2)}\theta/2} \\
&= (\cos\tfrac{\theta}{2} + \mathbf{I}_{(2)}\sin\tfrac{\theta}{2})\,\mathbf{n}\,(\cos\tfrac{\theta}{2} - \mathbf{I}_{(2)}\sin\tfrac{\theta}{2}) \\
&= \mathbf{n}\cos^2\tfrac{\theta}{2} - \mathbf{n}\mathbf{I}_{(2)}\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2} + \mathbf{I}_{(2)}\mathbf{n}\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} - \mathbf{I}_{(2)}\mathbf{n}\mathbf{I}_{(2)}\sin^2\tfrac{\theta}{2} \\
&= \mathbf{n}\cos^2\tfrac{\theta}{2} - \mathbf{I}_{(2)}\mathbf{n}\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2} + \mathbf{I}_{(2)}\mathbf{n}\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2} + \mathbf{n}\sin^2\tfrac{\theta}{2} \\
&= \mathbf{n}\,(\cos^2\tfrac{\theta}{2} + \sin^2\tfrac{\theta}{2}) \\
&= \mathbf{n}.
\end{aligned}$$

$$\boxed{\mathbf{R}\mathbf{n}\widetilde{\mathbf{R}} = \mathbf{n}}. \tag{4.24}$$

If we replace $\mathbf{n}$ with $\overline{\mathbf{n}}$, in the above calculations, all results turn out the same. Thus, we get

$$\boxed{\mathbf{R}\overline{\mathbf{n}}\widetilde{\mathbf{R}} = \overline{\mathbf{n}}}. \tag{4.25}$$

The rotor $\mathbf{R}$ commutes with both $\mathbf{n}$ and $\overline{\mathbf{n}}$ because it is an even element of $\mathbb{R}^{4,1}$.

Now we apply the rotor $\mathbf{R}$ to the conformal space representation $\mathbf{X}$ of the Euclidean vector $\mathbf{x}$,

$$\begin{aligned}
\mathbf{X} = F(\mathbf{x}) &= 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}} \\
\mathbf{R}F(\mathbf{x})\widetilde{\mathbf{R}} &= \mathbf{R}(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}})\widetilde{\mathbf{R}} \\
&= \mathbf{R}2\mathbf{x}\widetilde{\mathbf{R}} + \mathbf{R}\mathbf{x}^2\mathbf{n}\widetilde{\mathbf{R}} - \mathbf{R}\overline{\mathbf{n}}\widetilde{\mathbf{R}} \\
&= 2\mathbf{R}\mathbf{x}\widetilde{\mathbf{R}} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}.
\end{aligned}$$

$$\text{Conformal Rotation:} \qquad \boxed{\mathbf{R}F(\mathbf{x})\widetilde{\mathbf{R}} = 2\mathbf{R}\mathbf{x}\widetilde{\mathbf{R}} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}}\,, \tag{4.26}$$

where $\mathbf{R}\mathbf{x}\widetilde{\mathbf{R}}$ is as shown in Eq. (4.21).

Suppose that we wish to rotate about a point other than the origin. This can be achieved by translating that point to the origin, doing the rotation and then translating back to the original point. In terms of $\mathbf{X} = F(\mathbf{x})$, the result is

$$\mathbf{X} \mapsto \mathbf{T_a}(\mathbf{R}(\mathbf{T_{-a}}\mathbf{X}\widetilde{\mathbf{T}}_{\mathbf{-a}})\widetilde{\mathbf{R}})\widetilde{\mathbf{T}}_{\mathbf{a}}.$$

If we combine operators, we get

$$\text{Generator for a rotation about a point } \mathbf{a}: \qquad \boxed{\mathbf{R}' = \mathbf{T_a}\mathbf{R}\mathbf{T_{-a}} = \left(1 + \frac{\mathbf{na}}{2}\right) e^{\mathbf{I}_{(2)}\theta/2} \left(1 + \frac{\mathbf{an}}{2}\right)}\,.$$

The combined transformation becomes

$$\mathbf{X} \mapsto \mathbf{R}'\mathbf{X}\widetilde{\mathbf{R}}'.$$

The conformal model has freed us from treating the origin as a special point. Rotations about any point are handled in the same manner, and are still generated by a bivector.

4.2.3 Inversion

An inversion in the origin in $\mathbb{R}^3$ consists of the map

$$\mathbf{x} \mapsto \frac{\mathbf{x}}{\mathbf{x}^2}.$$

The inverse of an invertible vector $\mathbf{x}$, as shown in Section 2.6.2, is a vector with the same attitude and orientation as $\mathbf{x}$, but with norm $1/|\mathbf{x}|$. We use the

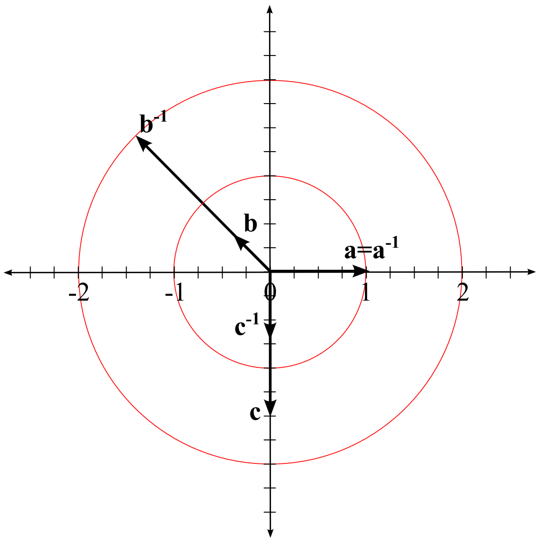


Figure 4.6: Inversion in the origin.

notation $\mathbf{x}^{-1}$ to denote the inverse of $\mathbf{x}$. If $|\mathbf{x}|$ is less than 1, then $|\mathbf{x}^{-1}|$ is greater than one and vice versa. If $|\mathbf{x}| = 1$, then $|\mathbf{x}^{-1}| = 1$ (see Figure 4.6).

If we take the equation $F(\mathbf{x}) = 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}$ and replace $\mathbf{x}$ with $\mathbf{x}^{-1}$, we get

$$\begin{aligned} F(\mathbf{x}^{-1}) &= 2\mathbf{x}^{-1} + \mathbf{x}^{-2}\mathbf{n} - \overline{\mathbf{n}}, \\ &= \frac{1}{\mathbf{x}^2}(2\mathbf{x} + \mathbf{n} - \mathbf{x}^2\overline{\mathbf{n}}). \end{aligned}$$

But we are using the homogeneous representation for conformal space where nonzero scalar multiples are arbitrary, so we can ignore the $\frac{1}{\mathbf{x}^2}$ factor and obtain

$$F(\mathbf{x}^{-1}) = 2\mathbf{x} - \mathbf{x}^2\overline{\mathbf{n}} + \mathbf{n}.$$

We see that an inversion in the origin in $\mathbb{R}^3$ consists of the following maps in conformal space,

$$\mathbf{n} \mapsto -\overline{\mathbf{n}}, \quad \text{and} \quad \overline{\mathbf{n}} \mapsto -\mathbf{n}.$$

This map can be achieved by a reflection in $\mathbf{e}$, where the versor $\mathbf{V} = \mathbf{V}^{-1} = \mathbf{e}$. For

$\mathbf{n}$ we have

$$\begin{aligned} -\mathbf{V}\mathbf{n}\mathbf{V}^{-1} &= -\mathbf{e}\mathbf{n}\mathbf{e} \\ &= -\mathbf{e}\left(\mathbf{e}+\bar{\mathbf{e}}\right)\mathbf{e} \\ &= -\mathbf{e}\mathbf{e}\mathbf{e} - \mathbf{e}\bar{\mathbf{e}}\mathbf{e} \\ &= -\mathbf{e}+\bar{\mathbf{e}} = -(\mathbf{e}-\bar{\mathbf{e}}) = -\overline{\mathbf{n}}. \end{aligned}$$

$$\boxed{-\mathbf{V}\mathbf{n}\mathbf{V}^{-1} = -\overline{\mathbf{n}}}. \tag{4.27}$$

And for $\overline{\mathbf{n}}$ we have

$$\begin{aligned} -\mathbf{V}\overline{\mathbf{n}}\mathbf{V}^{-1} &= -\mathbf{e}\overline{\mathbf{n}}\mathbf{e} \\ &= -\mathbf{e}\left(\mathbf{e}-\bar{\mathbf{e}}\right)\mathbf{e} \\ &= -\mathbf{e}\mathbf{e}\mathbf{e} + \mathbf{e}\bar{\mathbf{e}}\mathbf{e} \\ &= -\mathbf{e}-\bar{\mathbf{e}}, = -(\mathbf{e}+\bar{\mathbf{e}}) = -\mathbf{n}. \end{aligned}$$

$$\boxed{-\mathbf{V}\overline{\mathbf{n}}\mathbf{V}^{-1} = -\mathbf{n}}. \tag{4.28}$$

If we apply this reflection to the vector $\mathbf{x} \in \mathbf{R}^3$, we get

$$-\mathbf{V}\mathbf{x}\mathbf{V}^{-1} = -\mathbf{e}\mathbf{x}\mathbf{e} = \mathbf{e}\mathbf{e}\mathbf{x} = 1\mathbf{x} = \mathbf{x}.$$

$$\boxed{-\mathbf{V}\mathbf{x}\mathbf{V}^{-1} = \mathbf{x}}. \tag{4.29}$$

We apply this reflection to the conformal space representation $\mathbf{X}$ of the Euclidean vector $\mathbf{x}$.

$$\begin{aligned} \mathbf{X} = F(\mathbf{x}) &= 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}, \\ -\mathbf{V}F(\mathbf{x})\mathbf{V}^{-1} &= -\mathbf{e}\left(2\mathbf{x}+\mathbf{x}^2\mathbf{n}-\overline{\mathbf{n}}\right)\mathbf{e} \\ &= -\mathbf{e}2\mathbf{x}\mathbf{e} - \mathbf{e}\mathbf{x}^2\mathbf{n}\mathbf{e} - (-\mathbf{e}\overline{\mathbf{n}}\mathbf{e}) \\ &= 2\mathbf{x} - \mathbf{x}^2\overline{\mathbf{n}} + \mathbf{n}. \end{aligned}$$

$$\text{Conformal Inversion:} \qquad \boxed{-\mathbf{V}F(\mathbf{x})\mathbf{V}^{-1} = 2\mathbf{x} - \mathbf{x}^2\overline{\mathbf{n}} + \mathbf{n}}. \tag{4.30}$$

Note that

$$-\mathbf{V}F(\mathbf{x})\mathbf{V}^{-1} = \mathbf{x}^2 F(\mathbf{x}^{-1}).$$

Thus, a reflection in $\mathbf{e}$ corresponds to an inversion in the original Euclidean space.

To find the generator of an inversion in an arbitrary point $\mathbf{a} \in \mathbb{R}^3$, we translate to the origin, invert, and translate back to the original point. In terms of $\mathbf{X} = F(\mathbf{x})$, the result is

$$\mathbf{X} \mapsto \mathbf{T}_{\mathbf{a}}(\mathbf{V}(\mathbf{T}_{-\mathbf{a}}\mathbf{X}\widetilde{\mathbf{T}}_{-\mathbf{a}})\mathbf{V}^{-1})\widetilde{\mathbf{T}}_{\mathbf{a}}.$$

If we combine operators, the resulting generator is

$$\begin{aligned}
\mathbf{V}' = \mathbf{T}_{\mathbf{a}}\mathbf{V}\mathbf{T}_{-\mathbf{a}} &= \left(1 + \frac{\mathbf{na}}{2}\right) \mathbf{e} \left(1 + \frac{\mathbf{an}}{2}\right) \\
&= \mathbf{e} + \frac{\mathbf{ean}}{2} + \frac{\mathbf{nae}}{2} + \frac{\mathbf{naean}}{4} \\
&= \mathbf{e} - \left(\frac{\mathbf{en} + \mathbf{ne}}{2}\right) \mathbf{a} - \frac{\mathbf{nea}^2\mathbf{n}}{4} \\
&= \mathbf{e} - (\mathbf{e} \cdot \mathbf{n})\, \mathbf{a} - \frac{\mathbf{a}^2\mathbf{nen}}{4} \\
&= \mathbf{e} - [\mathbf{e} \cdot (\mathbf{e} + \bar{\mathbf{e}})]\, \mathbf{a} - \frac{\mathbf{a}^2}{4} \left[\mathbf{n} \left(\frac{\mathbf{n} + \overline{\mathbf{n}}}{2}\right) \mathbf{n}\right] \\
&= \mathbf{e} - [\mathbf{e} \cdot \mathbf{e} + \mathbf{e} \cdot \bar{\mathbf{e}}]\, \mathbf{a} - \frac{\mathbf{a}^2}{4} \left[\frac{\mathbf{nnn} + \mathbf{n}\overline{\mathbf{n}}\mathbf{n}}{2}\right] \\
&\overset{7}{=} \mathbf{e} - [1 + 0]\, \mathbf{a} - \frac{\mathbf{a}^2}{4} \left[\frac{0 + 4\mathbf{n}}{2}\right] \\
&= \mathbf{e} - \mathbf{a} - \frac{\mathbf{a}^2}{2}\mathbf{n}.
\end{aligned}$$

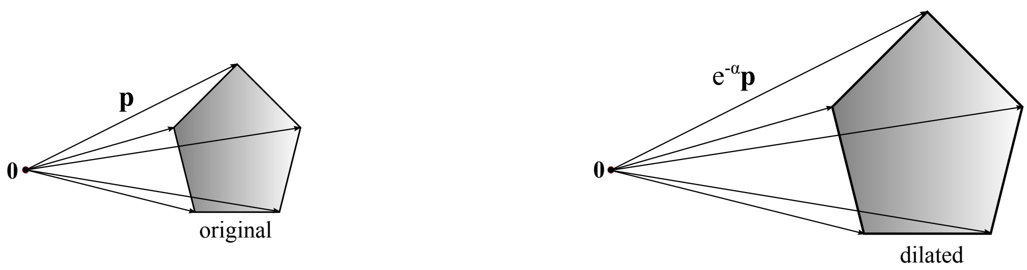
Generator for an inversion in a point $\mathbf{a}$:
$$\boxed{\mathbf{V}' = \mathbf{T_a}\mathbf{V}\mathbf{T_{-a}} = \mathbf{e} - \mathbf{a} - \frac{\mathbf{a}^2}{2}\mathbf{n}}. \tag{4.31}$$

Where step (7) uses Identity 4.1.3, viz., $\mathbf{n}\bar{\mathbf{n}}\mathbf{n} = 4\mathbf{n}$. The combined transform becomes

$$\mathbf{X} \mapsto \mathbf{V}'\mathbf{X}\mathbf{V}'^{-1}.$$

4.2.4 Dilation

We use the term *dilation* to refer to either the *contraction* or *expansion* of geometric objects. When we perform a dilation with respect to the origin, all vectors from the origin are multiplied by the same scalar constant (see Figure 4.7).



(a) Original image. (b) Dilated image.

Figure 4.7: Dilation with respect to the origin.

Although the size of the object has changed, all the angles remain the same. Thus, dilation is a conformal transformation. If the absolute value of the scalar multiple is less than one, then we are shrinking the object. If it is greater than one, then we are stretching the object.

We use the exponential $e^{-\alpha}$ where $\alpha \in \mathbb{R}$ to represent the scalar multiple. In this way the scalar multiple is always a positive number. When α is positive, we have a contraction. When α is negative we have an expansion. A dilation of the vector $\mathbf{x}$ relative to the origin in $\mathbb{R}^3$ is expressed as

$$\mathbf{x} \mapsto e^{-\alpha}\mathbf{x}.$$

If we take the equation $F(\mathbf{x}) = 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}$ and replace $\mathbf{x}$ with $e^{-\alpha}\mathbf{x}$, we get

$$\begin{aligned} F(e^{-\alpha}\mathbf{x}) &= 2e^{-\alpha}\mathbf{x} + e^{-2\alpha}\mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}} \\ &= e^{-\alpha}\left(2\mathbf{x} + \mathbf{e}^{-\alpha}\mathbf{x}^2\mathbf{n} - e^{\alpha}\overline{\mathbf{n}}\right). \end{aligned} \tag{4.32}$$

Once again we can ignore the scalar multiple $e^{-\alpha}$ due to the homogenous coordinates used in $\mathbb{R}^{4,1}$. If we compare Eq. (4.32) to the standard form Eq. (4.10), we see that we are looking for a conformal space dilation operator that maps

$$\mathbf{x} \mapsto \mathbf{x}, \quad \mathbf{n} \mapsto e^{-\alpha}\mathbf{n}, \quad \text{and} \quad \overline{\mathbf{n}} \mapsto e^{\alpha}\overline{\mathbf{n}}.$$

Such an operator does exist and it takes the form of a rotor $\mathbf{D}_\alpha$, where

$$\mathbf{D}_\alpha F(\mathbf{x})\widetilde{\mathbf{D}}_\alpha = 2\mathbf{x} + \mathbf{e}^{-\alpha}\mathbf{x}^2\mathbf{n} - e^{\alpha}\overline{\mathbf{n}}.$$

The rotor $\mathbf{D}_\alpha$ is defined as

Dilation operators:

$$\boxed{\mathbf{D}_\alpha = \cosh(\alpha/2) + \sinh(\alpha/2)\,\mathbf{N}} \quad \text{and} \quad \boxed{\widetilde{\mathbf{D}}_\alpha = \cosh(\alpha/2) - \sinh(\alpha/2)\,\mathbf{N}}. \tag{4.33}$$

where $\mathbf{N} = \mathbf{e}\bar{\mathbf{e}} = \frac{1}{2}(\overline{\mathbf{n}} \wedge \mathbf{n})$.

We now list several identities involving $\mathbf{N}$ that are used in the discussion that follows.

$$\mathbf{n}\mathbf{N} = (\mathbf{e} + \bar{\mathbf{e}})\,\mathbf{e}\bar{\mathbf{e}} = \mathbf{n}. \tag{4.34a}$$

$$\mathbf{N}\mathbf{n} = \mathbf{e}\bar{\mathbf{e}}\,(\mathbf{e} + \bar{\mathbf{e}}) = -\mathbf{n}. \tag{4.34b}$$

$$\mathbf{n}\mathbf{N} = -\mathbf{N}\mathbf{n}. \tag{4.34c}$$

$$\overline{\mathbf{n}}\mathbf{N} = (\mathbf{e} - \bar{\mathbf{e}})\,\mathbf{e}\bar{\mathbf{e}} = -\overline{\mathbf{n}}. \tag{4.34d}$$

$$\mathbf{N}\overline{\mathbf{n}} = \mathbf{e}\bar{\mathbf{e}}\,(\mathbf{e} - \bar{\mathbf{e}}) = \overline{\mathbf{n}}. \tag{4.34e}$$

$$\overline{\mathbf{n}}\mathbf{N} = -\mathbf{N}\overline{\mathbf{n}}. \tag{4.34f}$$

$$\mathbf{N}^2 = (\mathbf{e}\bar{\mathbf{e}})^2 = 1. \tag{4.34g}$$

Next we apply the dilation operator $\mathbf{D}_\alpha$ to the vectors $\mathbf{x}$, $\mathbf{n}$, and $\overline{\mathbf{n}}$. For $\mathbf{x} \in \mathbb{R}^3$, we have

$$\begin{aligned}
\mathbf{x}\widetilde{\mathbf{D}}_\alpha &= \mathbf{x}\,(\cosh(\alpha/2) - \sinh(\alpha/2)\,\mathbf{N}) \\
&= \cosh(\alpha/2)\mathbf{x} - \sinh(\alpha/2)\,\mathbf{x}\mathbf{N} \\
&= \cosh(\alpha/2)\mathbf{x} - \sinh(\alpha/2)\,\mathbf{x}\mathbf{e}\bar{\mathbf{e}} \\
&= \cosh(\alpha/2)\mathbf{x} - \sinh(\alpha/2)\,\mathbf{e}\bar{\mathbf{e}}\mathbf{x} \\
&= (\cosh(\alpha/2) - \sinh(\alpha/2)\,\mathbf{N})\,\mathbf{x} \\
&= \widetilde{\mathbf{D}}_\alpha\mathbf{x}.
\end{aligned}$$

This allows us to rewrite the transformation equations as follows:

$$\begin{aligned}
\mathbf{D}_\alpha \mathbf{x} \widetilde{\mathbf{D}}_\alpha &= \mathbf{D}_\alpha \widetilde{\mathbf{D}}_\alpha \mathbf{x} \\
&= (\cosh(\alpha/2) + \sinh(\alpha/2)\,\mathbf{N})\,(\cosh(\alpha/2) - \sinh(\alpha/2)\,\mathbf{N})\,\mathbf{x} \\
&= (\cosh^2(\alpha/2) - \sinh^2(\alpha/2)\,\mathbf{N}^2)\,\mathbf{x} \\
&\overset{4}{=} (\cosh^2(\alpha/2) - \sinh^2(\alpha/2))\,\mathbf{x} \\
&\overset{5}{=} 1\mathbf{x} = \mathbf{x}.
\end{aligned}$$

$$\boxed{\mathbf{D}_\alpha \mathbf{x} \widetilde{\mathbf{D}}_\alpha = \mathbf{x}}. \tag{4.35}$$

Where step (4) uses the identity in Eq. (4.34g) and step (5) uses the hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$.

For $\mathbf{n} \in \mathbb{R}^{4,1}$, we have

$$\begin{aligned}
\mathbf{n} \widetilde{\mathbf{D}}_\alpha &= \mathbf{n}\,(\cosh(\alpha/2) - \sinh(\alpha/2)\,\mathbf{N}) \\
&= \cosh(\alpha/2)\mathbf{n} - \sinh(\alpha/2)\,\mathbf{n} \\
&\overset{3}{=} \cosh(\alpha/2)\mathbf{n} + \sinh(\alpha/2)\,\mathbf{N}\mathbf{n} \\
&= (\cosh(\alpha/2) + \sinh(\alpha/2)\,\mathbf{N})\,\mathbf{n} \\
&= \mathbf{D}_\alpha \mathbf{n}.
\end{aligned}$$

where step (3) uses the identity in Eq. (4.34c). This allows us to rewrite the transformation equations as follows:

$$\begin{aligned}
\mathbf{D}_\alpha \mathbf{n} \widetilde{\mathbf{D}}_\alpha &= \mathbf{D}_\alpha^2 \mathbf{n} \\
&= (\cosh(\alpha/2) + \sinh(\alpha/2)\,\mathbf{N})^2\,\mathbf{n} \\
&= (\cosh^2(\alpha/2) + 2\cosh(\alpha/2)\sinh(\alpha/2)\,\mathbf{N} + \sinh^2(\alpha/2)\,\mathbf{N}^2)\,\mathbf{n} \\
&= \cosh^2(\alpha/2)\,\mathbf{n} + 2\cosh(\alpha/2)\sinh(\alpha/2)\,\mathbf{N}\mathbf{n} + \sinh^2(\alpha/2)\,\mathbf{n}
\end{aligned}$$

$$\overset{5}{=} \cosh^2(\alpha/2)\,\mathbf{n} - 2\cosh(\alpha/2)\sinh(\alpha/2)\,\mathbf{n} + \sinh^2(\alpha/2)\,\mathbf{n}$$
$$= (\cosh^2(\alpha/2) - 2\cosh(\alpha/2)\sinh(\alpha/2) + \sinh^2(\alpha/2))\,\mathbf{n}$$
$$\overset{7}{=} (2\cosh^2(\alpha/2) - (\cosh^2(\alpha/2) - \sinh^2(\alpha/2)) - 2\cosh(\alpha/2)\sinh(\alpha/2))\,\mathbf{n}$$
$$\overset{8}{=} (2\cosh^2(\alpha/2) - 1 - 2\cosh(\alpha/2)\sinh(\alpha/2))\,\mathbf{n}$$
$$= ([2\cosh^2(\alpha/2) - 1] - [2\cosh(\alpha/2)\sinh(\alpha/2)])\,\mathbf{n}$$
$$\overset{10}{=} (\cosh(\alpha) - \sinh(\alpha))\,\mathbf{n} = e^{-\alpha}\mathbf{n}.$$

$$\boxed{\mathbf{D}_\alpha \mathbf{n} \widetilde{\mathbf{D}}_\alpha = e^{-\alpha}\mathbf{n}}. \tag{4.36}$$

Where step (5) uses Eq. (4.34b), step (7) adds and subtracts $\cosh^2(\alpha/2)$, step (8) uses the hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$, and step (10) uses the hyperbolic double-angle formulas.

A similar calculation for $\overline{\mathbf{n}}$ produces the result

$$\boxed{\mathbf{D}_\alpha \overline{\mathbf{n}} \widetilde{\mathbf{D}}_\alpha = e^{\alpha}\overline{\mathbf{n}}}. \tag{4.37}$$

We now apply this dilation operator to the conformal space representation $\mathbf{X}$ of the Euclidean vector $\mathbf{x}$.

$$\begin{aligned}
\mathbf{X} = F(\mathbf{x}) &= 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}, \\
\mathbf{D}_\alpha F(\mathbf{x})\widetilde{\mathbf{D}}_\alpha &= \mathbf{D}_\alpha \left(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}\right) \widetilde{\mathbf{D}}_\alpha \\
&= \mathbf{D}_\alpha 2\mathbf{x}\widetilde{\mathbf{D}}_\alpha + \mathbf{D}_\alpha \mathbf{x}^2\mathbf{n}\widetilde{\mathbf{D}}_\alpha - \mathbf{D}_\alpha\overline{\mathbf{n}}\widetilde{\mathbf{D}}_\alpha \\
&= 2\mathbf{x} + \mathbf{x}^2 e^{-\alpha}\mathbf{n} - e^{\alpha}\overline{\mathbf{n}}.
\end{aligned}$$

Conformal Dilation: $$\boxed{\mathbf{D}_\alpha F(\mathbf{x})\widetilde{\mathbf{D}}_\alpha = 2\mathbf{x} + \mathbf{x}^2 e^{-\alpha}\mathbf{n} - e^{\alpha}\overline{\mathbf{n}}}. \tag{4.38}$$

Note that

$$\mathbf{D}_\alpha F(\mathbf{x})\widetilde{\mathbf{D}}_\alpha = e^{\alpha}F(e^{-\alpha}\mathbf{x}).$$

4.3 Some Background Needed for Representing Geometric Objects with the Conformal Model

Before we continue our discussion of how we represent geometric objects using the conformal model, it is necessary that we provide some background information on duality, null spaces, linear transformations, outermorphisms, and determinants.

4.3.1 Duality

In projective geometry we are introduced to the *principle of duality.* This principle highlights the symmetry between geometric objects such as points and lines.

With duality, if we have a true proposition about "points" and "lines," we can obtain another true proposition by interchanging the two words. For example:

> Any two distinct *points* determine one, and only one, *line* on which they both lie.
>
> Any two distinct *lines* determine one, and only one, *point* through which they both pass.

In geometric algebra there is also a symmetry between geometric objects of different grades which allows us to use the principle of duality to extend propositions about a set of objects to their duals. Scalars, vectors, bivectors, and trivectors all have duals that depend on the dimension of the space we are working with.

Definition 4.3.1 (Dual). [3]

For a given geometric algebra $\mathbb{G}^n$ *with a unit pseudoscalar* $\mathbf{I}$*, the* **dual** *of a multivector* $\mathbf{A}$*, denoted by* $\mathbf{A}^*$*, is*

$$\mathbf{A}^* = \mathbf{A}\mathbf{I}^{-1}. \tag{4.39}$$

As an example, suppose we are working with $\mathbb{R}^3$, where the unit pseudoscalar is $\mathbf{I} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. Then the duals of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are bivectors:

$$\mathbf{e}_1^* = \mathbf{e}_1\mathbf{I}^{-1} = \mathbf{e}_1\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_2\mathbf{e}_3,$$

$$\mathbf{e}_2^* = \mathbf{e}_2\mathbf{I}^{-1} = \mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_3\mathbf{e}_1,$$

$$\mathbf{e}_3^* = \mathbf{e}_3\mathbf{I}^{-1} = \mathbf{e}_3\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2.$$

The duals of the bivectors $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, $\mathbf{e}_1\mathbf{e}_2$ are vectors:

$$(\mathbf{e}_2\mathbf{e}_3)^* = \mathbf{e}_2\mathbf{e}_3\mathbf{I}^{-1} = \mathbf{e}_2\mathbf{e}_3\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_1,$$

$$(\mathbf{e}_3\mathbf{e}_1)^* = \mathbf{e}_3\mathbf{e}_1\mathbf{I}^{-1} = \mathbf{e}_3\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_2,$$

$$(\mathbf{e}_1\mathbf{e}_2)^* = \mathbf{e}_1\mathbf{e}_2\mathbf{I}^{-1} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3.$$

The dual of the scalar 1 is the trivector $-\mathbf{I}$.

$$1^* = 1\mathbf{I}^{-1} = 1\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -\mathbf{I}.$$

The dual of the trivector $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{I}$ is the scalar 1.

$$(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^* = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{I}^{-1} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = 1.$$

If $\mathbf{B}_k$ is a blade of grade k, then the dual of $\mathbf{B}_k$ returns the *orthogonal complement* in $\mathbb{R}^n$ of the subspace represented by $\mathbf{B}_k$. For example, if $\mathbf{B} \in \mathbb{G}^3$ and $\mathbf{B} = \mathbf{e}_1 \wedge \mathbf{e}_2$, then $\mathbf{B}^* = \mathbf{e}_3$. The dual, $\mathbf{B}_k^*$, has a grade of $n - k$, where n is the grade of the pseudoscalar for the space. If $\mathbf{B}_k$ represents a subspace U, then $\mathbf{B}_k^*$ represents $U^{\perp}$.

[3] Different authors use different definitions for dual. See Appendix A.

4.3.2 Inner Product and Outer Product Null Spaces

Before discussing the representations of lines, planes, circles, and spheres in the conformal model, we need to look at how blades are used to represent geometric objects.

Typically, in $\mathbb{R}^3$, a non-zero vector $\mathbf{a}$ is used to "represent" a plane through the origin with normal vector $\mathbf{a}$. The plane is the set $\{\mathbf{x} \,|\, \mathbf{x} \cdot \mathbf{a} = 0\}$. Note that any non-zero multiple of the vector $\mathbf{a}$, $\lambda\mathbf{a}$, also "represents" the same plane, so the representative, $\lambda\mathbf{a}$, is a homogeneous coordinate representative of the plane.

In geometric algebra, the vector $\mathbf{a}$ is a 1-blade and the plane $\{\mathbf{x} \,|\, \mathbf{x} \cdot \mathbf{a} = 0\}$ is called the *Inner Product Null Space* (IPNS) of the blade $\mathbf{a}$. Let the linearly independent vectors $\mathbf{b}$ and $\mathbf{c}$ be in the IPNS of the 1-blade $\mathbf{a}$. In geometric algebra, $\mathbf{b} \wedge \mathbf{c}$ is a 2-blade (or bivector) and $\mathbf{x} \wedge \mathbf{b} \wedge \mathbf{c} = 0$ if and only if $\mathbf{x} = \beta\mathbf{b} + \gamma\mathbf{c}$. Thus, the set $\{\mathbf{x} \,|\, \mathbf{x} \wedge \mathbf{b} \wedge \mathbf{c} = 0\}$, called the *Outer Product Null Space* (OPNS) of the bivector $\mathbf{b} \wedge \mathbf{c}$, is the same plane as the IPNS of the 1-blade $\mathbf{a}$. We may take any non-zero multiple of $\mathbf{b} \wedge \mathbf{c}$, $\lambda(\mathbf{b} \wedge \mathbf{c})$ as a "representative" of the plane.

Because $(\mathbf{b} \wedge \mathbf{c})^* = (\mathbf{b} \wedge \mathbf{c})\mathbf{I}^{-1} = \lambda\mathbf{a}$, we see that the OPNS of the bivector $\mathbf{b} \wedge \mathbf{c}$, and the IPNS of the of the 1-blade $\mathbf{a}$ are related by duality in geometric algebra. This occurs in all dimensions, and, in particular, in $\mathbb{R}^{4,1}$.

In the conformal model of Euclidean space we have each point $\mathbf{x}$ in $\mathbb{R}^3$ mapped to an image point $\mathbf{X} = 2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}$, in $\mathbb{R}^{4,1}$ on the null cone. We show that certain blades can be taken as representatives of their corresponding (inner or outer product) null spaces, and how their intersection with the null cone in $\mathbb{R}^{4,1}$ then represents geometric objects in $\mathbb{R}^3$.

4.3.3 Linear Transformations

Linear transformations form a major part of linear algebra. All of the transformations we discussed in Section 4.2 are linear transformations on $\mathbb{R}^{4,1}$.

Vector rotation as described in Section 3.7 is an example of a linear transformation, e.g.,

$$\begin{aligned} R(\lambda\mathbf{u} + \mu\mathbf{v}) &= R\,(\lambda\mathbf{u} + \mu\mathbf{v})\widetilde{R}, \\ &= R\lambda\mathbf{u}\widetilde{R} + R\mu\mathbf{v}\widetilde{R}, \\ &= \lambda R\mathbf{u}\widetilde{R} + \mu R\mathbf{v}\widetilde{R}, \\ &= \lambda R(\mathbf{u}) + \mu R(\mathbf{v}). \end{aligned}$$

4.3.4 Outermorphisms

Definition 4.3.2 (Outermorphism).

An **outermorphism** *is a linear transformation $F : \mathbb{G}^m \to \mathbb{G}^n$ which preserves outer products.*

$$F(\mathbf{M}_1 \wedge \mathbf{M}_2) = F(\mathbf{M}_1) \wedge F(\mathbf{M}_2), \tag{4.40}$$

for all multivectors $\mathbf{M}_1$, $\mathbf{M}_2 \in \mathbb{G}^m$.

Outermorphisms have nice algebraic properties that are essential to their geometrical usage [DFM07, p. 101]:

- **Blades remain blades.** Oriented subspaces are transformed to oriented subspaces.
- **Grades are preserved.** A linear transformation F maps vectors to vectors. It follows that $\text{grade}(F(\mathbf{B})) = \text{grade}(\mathbf{B})$ for all blades $\mathbf{B}$.

- **Factorization is preserved.** If blades $\mathbf{A}$ and $\mathbf{B}$ have a blade $\mathbf{C}$ in common (so that they may be written as $\mathbf{A} = \mathbf{A}' \wedge \mathbf{C}$ and $\mathbf{B} = \mathbf{B}' \wedge \mathbf{C}$, for an appropriately chosen $\mathbf{A}'$ and $\mathbf{B}'$.), then $F(\mathbf{A})$ and $F(\mathbf{B})$ have $F(\mathbf{C})$ in common. Geometrically this means that the intersection of subspaces is preserved.

Reflections and rotations, as described in Sections 3.6 and 3.7, are examples of outermorphisms. Macdonald [Mac10, p. 126] proves that reflection is an outermorphism, i.e.,

$$R(\mathbf{u} \wedge \mathbf{v}) = R(\mathbf{u}) \wedge R(\mathbf{v}).$$

Thus, we can rotate a bivector by rotating each vector and then taking the outer product of the result.

4.3.5 Determinants

Definition 4.3.3 (**Determinant of a linear transformation** [Mac10, p. 151])**.**

Let F be a one-to-one linear transformation on $\mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathbf{I}$ be the unit pseudoscalar of $\mathbb{G}^n$, an n-vector. Because F preserves grades, $F(\mathbf{I})$ is also an n-vector. Thus, it is a scalar multiple of $\mathbf{I}$,

$$F(\mathbf{I}) = \sigma \, \mathbf{I}.$$

The scalar σ is exactly the same as the **determinant** *of an $n \times n$ matrix representation of F. Thus, we have,*

$$F(\mathbf{I}) = \det(F) \, \mathbf{I}. \tag{4.41}$$

The determinant has a simple geometric meaning:

> The determinant of a linear transformation F on $\mathbb{R}^n$ is the factor by which F multiplies oriented volumes of n-dimensional parallelograms.

The following theorem is presented without proof:

Theorem 4.3.4 ([Mac12, p. 16])**.**

Let $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ *be vectors in* $\mathbb{R}^n$ *and let* $\mathfrak{B} = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ *be an orthonormal basis for* $\mathbb{R}^n$*. Then*

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n = \det \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{e}_1 & \mathbf{v}_2 \cdot \mathbf{e}_1 & \cdots & \mathbf{v}_n \cdot \mathbf{e}_1 \\ \mathbf{v}_1 \cdot \mathbf{e}_2 & \mathbf{v}_2 \cdot \mathbf{e}_2 & \cdots & \mathbf{v}_n \cdot \mathbf{e}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_1 \cdot \mathbf{e}_n & \mathbf{v}_2 \cdot \mathbf{e}_n & \cdots & \mathbf{v}_n \cdot \mathbf{e}_n \end{bmatrix} \mathbf{I}. \tag{4.42}$$

This theorem says that we can find the determinant of a matrix composed of n column vectors from $\mathbb{R}^n$, by taking the outer product of those vectors and multiplying the result by $\mathbf{I}^{-1}$.

4.4 Representing Geometric Objects with the Conformal Model

4.4.1 Points in the Conformal Model

As we have already seen in the conformal model, a point in Euclidean space is represented by a null vector in $\mathbb{R}^{4,1}$ using the equation

$$\text{Point:} \qquad \boxed{\mathbf{P} = \tfrac{1}{2}(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}})}. \tag{4.43}$$

Because this is a null vector in the conformal model, the inner product of any point with itself is always zero.

$$\mathbf{P} \cdot \mathbf{P} = 0.$$

4.4.2 Inner Product of Two Points in the Conformal Model

Suppose we have two Euclidean points, $\mathbf{x}_1$ and $\mathbf{x}_2$, that define a line in Euclidean space. These two points map to vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ respectively in conformal space.

We now examine the scalar product $\mathbf{X}_1 \cdot \mathbf{X}_2$.

$$
\begin{aligned}
\mathbf{X}_1 \cdot \mathbf{X}_2 &= \tfrac{1}{2}(2\mathbf{x}_1 + \mathbf{x}_1^2\mathbf{n} - \overline{\mathbf{n}}) \cdot \tfrac{1}{2}(2\mathbf{x}_2 + \mathbf{x}_2^2\mathbf{n} - \overline{\mathbf{n}}) \\
&= (\mathbf{x}_1 + \tfrac{1}{2}\mathbf{x}_1^2\mathbf{n} - \tfrac{1}{2}\overline{\mathbf{n}}) \cdot (\mathbf{x}_2 + \tfrac{1}{2}\mathbf{x}_2^2\mathbf{n} - \tfrac{1}{2}\overline{\mathbf{n}}) \\
&= \left\{ \begin{array}{rrrrrr} & \mathbf{x}_1 \cdot \mathbf{x}_2 & + & \frac{1}{2}\mathbf{x}_2^2(\mathbf{x}_1 \cdot \mathbf{n}) & - & \frac{1}{2}(\mathbf{x}_1 \cdot \overline{\mathbf{n}}) \\ + & \frac{1}{2}\mathbf{x}_1^2(\mathbf{n} \cdot \mathbf{x}_2) & + & \frac{1}{4}\mathbf{x}_1^2\mathbf{x}_2^2(\mathbf{n} \cdot \mathbf{n}) & - & \frac{1}{4}\mathbf{x}_1^2(\mathbf{n} \cdot \overline{\mathbf{n}}) \\ - & \frac{1}{2}(\overline{\mathbf{n}} \cdot \mathbf{x}_2) & - & \frac{1}{4}\mathbf{x}_2^2(\overline{\mathbf{n}} \cdot \mathbf{n}) & + & \frac{1}{4}(\overline{\mathbf{n}} \cdot \overline{\mathbf{n}}) \end{array} \right. \\
&= \left\{ \begin{array}{rrrrrr} & \mathbf{x}_1 \cdot \mathbf{x}_2 & + & 0 & - & 0 \\ + & 0 & + & 0 & - & \frac{1}{2}\mathbf{x}_1^2 \\ - & 0 & - & \frac{1}{2}\mathbf{x}_2^2 & + & 0 \end{array} \right. \\
&= -\tfrac{1}{2}\left(\mathbf{x}_2^2 - 2\mathbf{x}_2 \cdot \mathbf{x}_1 + \mathbf{x}_1^2\right) \\
&= -\tfrac{1}{2}\left(\mathbf{x}_2 - \mathbf{x}_1\right) \cdot \left(\mathbf{x}_2 - \mathbf{x}_1\right) \\
&= -\tfrac{1}{2}\left|\mathbf{x}_2 - \mathbf{x}_1\right|^2.
\end{aligned}
$$

Because $|x_2 - x_1|^2$ is the square of the distance between the two points, we have shown that the inner product of two points in the conformal model encodes the distance between the two points on a Euclidean line.

This result is of fundamental importance because it follows that any transformation of null vectors in conformal space which leaves the inner product invariant can correspond to a transformation in Euclidean space which leaves distance and angles invariant.

For two points, $\mathbf{P}_1$ and $\mathbf{P}_2$, in the conformal model, we have

$$\boxed{\mathbf{P}_1 \cdot \mathbf{P}_2 \text{ encodes the distance between the two points}}. \tag{4.44}$$

4.4.3 Lines in the Conformal Model

In the conformal model, lines are represented by the outer product of two ordinary points and the point at infinity, i.e., two points define a line and every line includes a point at infinity.

$$\text{Line:} \quad \boxed{\mathbf{L}_l = \mathbf{P}_1 \wedge \mathbf{P}_2 \wedge \mathbf{n}}, \tag{4.45}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are the conformal mappings of two Euclidean points on the line, i.e., $\mathbf{P}_1 = F(\mathbf{p}_1)$ and $\mathbf{P}_2 = F(\mathbf{p}_2)$.

The equation for a line is represented as an outer product null space (OPNS). That is to say

$$\text{OPNS}_{\text{line}} = \{\mathbf{X} \,|\, \mathbf{X} \wedge \mathbf{P}_1 \wedge \mathbf{P}_2 \wedge \mathbf{n} = 0\}.$$

To show that this null set really does represent a line in $\mathbb{R}^3$, we have the following

$$\begin{aligned}
0 &= \mathbf{X} \wedge \mathbf{P}_1 \wedge \mathbf{P}_2 \wedge \mathbf{n} \\
&= \tfrac{1}{2}(2\mathbf{x} + \mathbf{x}^2\mathbf{n} - \overline{\mathbf{n}}) \wedge \tfrac{1}{2}(2\mathbf{p}_1 + \mathbf{p}_1^2\mathbf{n} - \overline{\mathbf{n}}) \wedge \tfrac{1}{2}(2\mathbf{p}_2 + \mathbf{p}_2^2\mathbf{n} - \overline{\mathbf{n}}) \wedge \mathbf{n} \\
&\overset{4}{=} \tfrac{1}{8}\left[(2\mathbf{x} - \overline{\mathbf{n}}) \wedge (2\mathbf{p}_1 - \overline{\mathbf{n}}) \wedge (2\mathbf{p}_2 - \overline{\mathbf{n}}) \wedge \mathbf{n}\right] \\
&= \tfrac{1}{8}\left[(4(\mathbf{x} \wedge \mathbf{p}_1) - 2(\mathbf{x} \wedge \overline{\mathbf{n}}) - 2(\overline{\mathbf{n}} \wedge \mathbf{p}_1) + \overline{\mathbf{n}} \wedge \overline{\mathbf{n}}) \wedge (2\mathbf{p}_2 - \overline{\mathbf{n}}) \wedge \mathbf{n}\right] \\
&\overset{6}{=} \tfrac{1}{8}\left[(4(\mathbf{x} \wedge \mathbf{p}_1) - 2(\mathbf{x} \wedge \overline{\mathbf{n}}) + 2(\mathbf{p}_1 \wedge \overline{\mathbf{n}})) \wedge (2\mathbf{p}_2 - \overline{\mathbf{n}}) \wedge \mathbf{n}\right] \\
&= \begin{cases} \tfrac{1}{8}[(8(\mathbf{x} \wedge \mathbf{p}_1 \wedge \mathbf{p}_2) - 4(\mathbf{x} \wedge \mathbf{p}_1 \wedge \overline{\mathbf{n}}) - 4(\mathbf{x} \wedge \overline{\mathbf{n}} \wedge \mathbf{p}_2) \\ + 2(\mathbf{x} \wedge \overline{\mathbf{n}} \wedge \overline{\mathbf{n}}) + 4(\mathbf{p}_1 \wedge \overline{\mathbf{n}} \wedge \mathbf{p}_2) - 2(\mathbf{p}_1 \wedge \overline{\mathbf{n}} \wedge \overline{\mathbf{n}})) \wedge \mathbf{n}] \end{cases}
\end{aligned}$$

$$\overset{8}{=} \frac{1}{8}\left[(8(\mathbf{x} \wedge \mathbf{p}_1 \wedge \mathbf{p}_2) - 4(\mathbf{x} \wedge \mathbf{p}_1 \wedge \overline{\mathbf{n}}) + 4(\mathbf{x} \wedge \mathbf{p}_2 \wedge \overline{\mathbf{n}}) - 4(\mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \overline{\mathbf{n}})) \wedge \mathbf{n}\right]$$

$$= \begin{cases} \frac{1}{8}[8(\mathbf{x} \wedge \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{n}) - 4(\mathbf{x} \wedge \mathbf{p}_1 \wedge \overline{\mathbf{n}} \wedge \mathbf{n}) \\ +4(\mathbf{x} \wedge \mathbf{p}_2 \wedge \overline{\mathbf{n}} \wedge \mathbf{n}) - 4(\mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \overline{\mathbf{n}} \wedge \mathbf{n})] \end{cases}$$

$$= \begin{cases} \frac{1}{8}[8(\mathbf{x} \wedge \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{n}) + 4(\mathbf{x} \wedge \mathbf{p}_1 \wedge \mathbf{n} \wedge \overline{\mathbf{n}}) \\ -4(\mathbf{x} \wedge \mathbf{p}_2 \wedge \mathbf{n} \wedge \overline{\mathbf{n}}) + 4(\mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{n} \wedge \overline{\mathbf{n}})] \end{cases}$$

$$= \frac{1}{8}\left[8(\mathbf{x} \wedge \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{n}) + 4(\mathbf{x} \wedge \mathbf{p}_1 - \mathbf{x} \wedge \mathbf{p}_2 + \mathbf{p}_1 \wedge \mathbf{p}_2) \wedge \mathbf{n} \wedge \overline{\mathbf{n}}\right]$$

$$= \frac{1}{8}\left[8(\mathbf{x} \wedge \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{n}) + 4(\mathbf{x} \wedge (\mathbf{p}_1 - \mathbf{p}_2) + \mathbf{p}_1 \wedge \mathbf{p}_2) \wedge \mathbf{n} \wedge \overline{\mathbf{n}}\right]$$

$$0 = \mathbf{x} \wedge \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{n} + \frac{1}{2}(\mathbf{x} \wedge (\mathbf{p}_1 - \mathbf{p}_2) + \mathbf{p}_1 \wedge \mathbf{p}_2) \wedge \mathbf{n} \wedge \overline{\mathbf{n}}. \tag{4.46}$$

Step (4) takes advantage of the fact that all terms with $\mathbf{n}$ become zero when we take the outer product with $\mathbf{n}$ from the left. Steps (6) and (8) use the identities $\overline{\mathbf{n}} \wedge \overline{\mathbf{n}} = 0$ and $\overline{\mathbf{n}} \wedge \mathbf{p}_i = -\mathbf{p}_i \wedge \overline{\mathbf{n}}$.

Because the two terms in Eq. (4.46) are linearly independent, they must both be zero. The first term, $\mathbf{x} \wedge \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{n} = 0$, implies that

$$\mathbf{x} = \alpha\mathbf{p}_1 + \beta\mathbf{p}_2, \qquad \text{where} \quad \alpha,\, \beta \in \mathbb{R}. \tag{4.47}$$

The second term is zero if and only if

$$\begin{aligned} \mathbf{x} \wedge (\mathbf{p}_2 - \mathbf{p}_1) &= \mathbf{p}_1 \wedge \mathbf{p}_2 \\ (\alpha\mathbf{p}_1 + \beta\mathbf{p}_2) \wedge (\mathbf{p}_2 - \mathbf{p}_1) &= \mathbf{p}_1 \wedge \mathbf{p}_2 \\ \alpha\mathbf{p}_1 \wedge \mathbf{p}_2 - \alpha\mathbf{p}_1 \wedge \mathbf{p}_1 + \beta\mathbf{p}_2 \wedge \mathbf{p}_2 - \beta\mathbf{p}_2 \wedge \mathbf{p}_1 &= \mathbf{p}_1 \wedge \mathbf{p}_2 \\ \alpha\mathbf{p}_1 \wedge \mathbf{p}_2 + \beta\mathbf{p}_1 \wedge \mathbf{p}_2 &= \mathbf{p}_1 \wedge \mathbf{p}_2 \\ (\alpha + \beta)\mathbf{p}_1 \wedge \mathbf{p}_2 &= \mathbf{p}_1 \wedge \mathbf{p}_2. \end{aligned}$$

Therefore,

$$\alpha + \beta = 1. \tag{4.48}$$

From Eqs. (4.47) and (4.48) and Lemma 3.8.9 we see that $\mathbf{x}$ is a point on the line through $\mathbf{p}_1$ and $\mathbf{p}_2$.

A line can also be viewed as a circle with infinite radius.

4.4.4 Circles in the Conformal Model

In the conformal model, circles are represented by the outer product of three non-collinear ordinary points, i.e., three non-collinear points define a circle.

$$\text{Circle:} \quad \boxed{\mathbf{L}_c = \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}}, \tag{4.49}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are the conformal mappings of three non-collinear Euclidean points on the circle, i.e., $\mathbf{A} = F(\mathbf{a})$, $\mathbf{B} = F(\mathbf{b})$, and $\mathbf{C} = F(\mathbf{c})$.

The equation for a circle is represented as an outer product null space (OPNS). That is to say

$$\text{OPNS}_{\text{circle}} = \{\mathbf{X} \,|\, \mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} = 0\}.$$

A circle is a two dimensional object, so for simplicity we consider a circle in the xy-plane. The conformal model for a 2-dimensional space is $\mathbb{R}^{3,1}$. We first note that $\mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}$ is a 4-blade in a 4-dimensional space $\mathbb{R}^{3,1}$. From Theorem 4.3.4 we have

$$0 = \mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} = \det \begin{bmatrix} \mathbf{X} \cdot \mathbf{e}_1 & \mathbf{A} \cdot \mathbf{e}_1 & \mathbf{B} \cdot \mathbf{e}_1 & \mathbf{C} \cdot \mathbf{e}_1 \\ \mathbf{X} \cdot \mathbf{e}_2 & \mathbf{A} \cdot \mathbf{e}_2 & \mathbf{B} \cdot \mathbf{e}_2 & \mathbf{C} \cdot \mathbf{e}_2 \\ \mathbf{X} \cdot \mathbf{n} & \mathbf{A} \cdot \mathbf{n} & \mathbf{B} \cdot \mathbf{n} & \mathbf{C} \cdot \mathbf{n} \\ \mathbf{X} \cdot \overline{\mathbf{n}} & \mathbf{A} \cdot \overline{\mathbf{n}} & \mathbf{B} \cdot \overline{\mathbf{n}} & \mathbf{C} \cdot \overline{\mathbf{n}} \end{bmatrix} \mathbf{I},$$

where $\mathbf{I} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{n} \wedge \overline{\mathbf{n}}$. This implies that

$$\det \begin{bmatrix} \mathbf{X}\cdot\mathbf{e}_1 & \mathbf{A}\cdot\mathbf{e}_1 & \mathbf{B}\cdot\mathbf{e}_1 & \mathbf{C}\cdot\mathbf{e}_1 \\ \mathbf{X}\cdot\mathbf{e}_2 & \mathbf{A}\cdot\mathbf{e}_2 & \mathbf{B}\cdot\mathbf{e}_2 & \mathbf{C}\cdot\mathbf{e}_2 \\ \mathbf{X}\cdot\mathbf{n} & \mathbf{A}\cdot\mathbf{n} & \mathbf{B}\cdot\mathbf{n} & \mathbf{C}\cdot\mathbf{n} \\ \mathbf{X}\cdot\overline{\mathbf{n}} & \mathbf{A}\cdot\overline{\mathbf{n}} & \mathbf{B}\cdot\overline{\mathbf{n}} & \mathbf{C}\cdot\overline{\mathbf{n}} \end{bmatrix} = 0.$$

Because the determinant of a matrix equals the determinant of its transpose, we have

$$\det \begin{bmatrix} \mathbf{X}\cdot\mathbf{e}_1 & \mathbf{X}\cdot\mathbf{e}_2 & \mathbf{X}\cdot\mathbf{n} & \mathbf{X}\cdot\overline{\mathbf{n}} \\ \mathbf{A}\cdot\mathbf{e}_1 & \mathbf{A}\cdot\mathbf{e}_2 & \mathbf{A}\cdot\mathbf{n} & \mathbf{A}\cdot\overline{\mathbf{n}} \\ \mathbf{B}\cdot\mathbf{e}_1 & \mathbf{B}\cdot\mathbf{e}_2 & \mathbf{B}\cdot\mathbf{n} & \mathbf{B}\cdot\overline{\mathbf{n}} \\ \mathbf{C}\cdot\mathbf{e}_1 & \mathbf{C}\cdot\mathbf{e}_2 & \mathbf{C}\cdot\mathbf{n} & \mathbf{C}\cdot\overline{\mathbf{n}} \end{bmatrix} = 0.$$

By substituting all the inner products from the standard form Eq. (4.10) into the matrix, we get

$$\det \begin{bmatrix} 2x_1 & 2x_2 & -2 & 2\mathbf{x}^2 \\ 2a_1 & 2a_2 & -2 & 2\mathbf{a}^2 \\ 2b_1 & 2b_2 & -2 & 2\mathbf{b}^2 \\ 2b_1 & 2c_2 & -2 & 2\mathbf{c}^2 \end{bmatrix} = 0,$$

where subscripts, as in x_1, x_2, represent the $\mathbf{e}_1$, $\mathbf{e}_2$ components of the vector $\mathbf{x} \in \mathbb{R}^2$.

We can factor out a 2 from each row and rearrange the columns of the matrix without impacting the results to get

$$\det \begin{bmatrix} \mathbf{x}^2 & x_1 & x_2 & -1 \\ \mathbf{a}^2 & a_1 & a_2 & -1 \\ \mathbf{b}^2 & b_2 & b_2 & -1 \\ \mathbf{c}^2 & c_3 & c_2 & -1 \end{bmatrix} = 0, \tag{4.50}$$

Equation (4.50) is in the standard form for representing a circle in $\mathbb{R}^2$ with a determinant [Che08, p. 3-14]. The variable $\mathbf{x}$ represents the locus of all points on a circle passing through points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

We note that both circles and lines are represented by trivectors in the conformal model. In fact, lines can be viewed as circles with infinite radius. If the three points that define a circle are collinear, then the circle is a straight line, i.e.,

$$\mathbf{L}_c \wedge \mathbf{n} = 0 \quad \Rightarrow \quad \text{straight line.}$$

This explains why the earlier derivation of a line through $\mathbf{p}_1$ and $\mathbf{p}_2$ led to the trivector $\mathbf{P}_1 \wedge \mathbf{P}_2 \wedge \mathbf{n}$, which explicitly includes the point at infinity.

The center of the circle is the point

$$\text{Center:} \quad \boxed{\mathbf{P}_0 = \mathbf{L}_c \mathbf{n} \mathbf{L}_c}\,. \tag{4.51}$$

The operation $\mathbf{L}_c \mathbf{M} \mathbf{L}_c$ reflects any object $\mathbf{M}$ in the circle $\mathbf{L}_c$. Eq. (4.51) says that the center of a circle is the image of the point at infinity under a reflection in a circle.

4.4.5 Planes in the Conformal Model

In the conformal model, planes are represented by the outer product of three non-collinear ordinary points and the point at infinity, i.e., three non-collinear points define a plane and every plane includes points at infinity.

$$\text{Plane:} \quad \boxed{\mathbf{L}_p = \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{n}}\,, \tag{4.52}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are the conformal mappings of three Euclidean points on the plane, i.e., $\mathbf{A} = F(\mathbf{a})$, $\mathbf{B} = F(\mathbf{b})$, and $\mathbf{C} = F(\mathbf{c})$.

The equation for a plane is represented as an outer product null space (OPNS). That is to say

$$\text{OPNS}_{\text{plane}} = \{\mathbf{X} \,|\, \mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{n} = 0\}.$$

To show that this null set really does represent a plane in $\mathbb{R}^3$, we would first note that $\mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{n}$ is a 5-blade in a 5-dimensional space $\mathbb{R}^{4,1}$. From Theorem 4.3.4 we have

$$0 = \mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{n} = \det \begin{bmatrix} \mathbf{X}\cdot\mathbf{e}_1 & \mathbf{A}\cdot\mathbf{e}_1 & \mathbf{B}\cdot\mathbf{e}_1 & \mathbf{C}\cdot\mathbf{e}_1 & \mathbf{n}\cdot\mathbf{e}_1 \\ \mathbf{X}\cdot\mathbf{e}_2 & \mathbf{A}\cdot\mathbf{e}_2 & \mathbf{B}\cdot\mathbf{e}_2 & \mathbf{C}\cdot\mathbf{e}_2 & \mathbf{n}\cdot\mathbf{e}_2 \\ \mathbf{X}\cdot\mathbf{e}_3 & \mathbf{A}\cdot\mathbf{e}_3 & \mathbf{B}\cdot\mathbf{e}_3 & \mathbf{C}\cdot\mathbf{e}_3 & \mathbf{n}\cdot\mathbf{e}_3 \\ \mathbf{X}\cdot\mathbf{n} & \mathbf{A}\cdot\mathbf{n} & \mathbf{B}\cdot\mathbf{n} & \mathbf{C}\cdot\mathbf{n} & \mathbf{n}\cdot\mathbf{n} \\ \mathbf{X}\cdot\overline{\mathbf{n}} & \mathbf{A}\cdot\overline{\mathbf{n}} & \mathbf{B}\cdot\overline{\mathbf{n}} & \mathbf{C}\cdot\overline{\mathbf{n}} & \mathbf{n}\cdot\overline{\mathbf{n}} \end{bmatrix} \mathbf{I},$$

where $\mathbf{I} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{n} \wedge \overline{\mathbf{n}}$. This implies that

$$\det \begin{bmatrix} \mathbf{X}\cdot\mathbf{e}_1 & \mathbf{A}\cdot\mathbf{e}_1 & \mathbf{B}\cdot\mathbf{e}_1 & \mathbf{C}\cdot\mathbf{e}_1 & \mathbf{n}\cdot\mathbf{e}_1 \\ \mathbf{X}\cdot\mathbf{e}_2 & \mathbf{A}\cdot\mathbf{e}_2 & \mathbf{B}\cdot\mathbf{e}_2 & \mathbf{C}\cdot\mathbf{e}_2 & \mathbf{n}\cdot\mathbf{e}_2 \\ \mathbf{X}\cdot\mathbf{e}_3 & \mathbf{A}\cdot\mathbf{e}_3 & \mathbf{B}\cdot\mathbf{e}_3 & \mathbf{C}\cdot\mathbf{e}_3 & \mathbf{n}\cdot\mathbf{e}_3 \\ \mathbf{X}\cdot\mathbf{n} & \mathbf{A}\cdot\mathbf{n} & \mathbf{B}\cdot\mathbf{n} & \mathbf{C}\cdot\mathbf{n} & \mathbf{n}\cdot\mathbf{n} \\ \mathbf{X}\cdot\overline{\mathbf{n}} & \mathbf{A}\cdot\overline{\mathbf{n}} & \mathbf{B}\cdot\overline{\mathbf{n}} & \mathbf{C}\cdot\overline{\mathbf{n}} & \mathbf{n}\cdot\overline{\mathbf{n}} \end{bmatrix} = 0.$$

Because the determinant of a matrix equals the determinant of its transpose, we have

$$\det \begin{bmatrix} \mathbf{X}\cdot\mathbf{e}_1 & \mathbf{X}\cdot\mathbf{e}_2 & \mathbf{X}\cdot\mathbf{e}_3 & \mathbf{X}\cdot\mathbf{n} & \mathbf{X}\cdot\overline{\mathbf{n}} \\ \mathbf{A}\cdot\mathbf{e}_1 & \mathbf{A}\cdot\mathbf{e}_2 & \mathbf{A}\cdot\mathbf{e}_3 & \mathbf{A}\cdot\mathbf{n} & \mathbf{A}\cdot\overline{\mathbf{n}} \\ \mathbf{B}\cdot\mathbf{e}_1 & \mathbf{B}\cdot\mathbf{e}_2 & \mathbf{B}\cdot\mathbf{e}_3 & \mathbf{B}\cdot\mathbf{n} & \mathbf{B}\cdot\overline{\mathbf{n}} \\ \mathbf{C}\cdot\mathbf{e}_1 & \mathbf{C}\cdot\mathbf{e}_2 & \mathbf{C}\cdot\mathbf{e}_3 & \mathbf{C}\cdot\mathbf{n} & \mathbf{C}\cdot\overline{\mathbf{n}} \\ \mathbf{n}\cdot\mathbf{e}_1 & \mathbf{n}\cdot\mathbf{e}_2 & \mathbf{n}\cdot\mathbf{e}_3 & \mathbf{n}\cdot\mathbf{n} & \mathbf{n}\cdot\overline{\mathbf{n}} \end{bmatrix} = 0.$$

By substituting all the inner products from the standard form Eq. (4.10) into the matrix, we get

$$\det \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 & -2 & 2\mathbf{x}^2 \\ 2a_1 & 2a_2 & 2a_3 & -2 & 2\mathbf{a}^2 \\ 2b_1 & 2b_2 & 2b_3 & -2 & 2\mathbf{b}^2 \\ 2c_1 & 2c_2 & 2c_3 & -2 & 2\mathbf{c}^2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = 0,$$

where subscripts, as in x_1, x_2, x_3, represent the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ components of the vector $\mathbf{x} \in \mathbb{R}^3$.

Taking advantage of the zeros in row 5, the determinant of this matrix is 2 times the determinant of the submatrix formed by crossing out row 5, column 5. This leads to

$$\det \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 & -2 \\ 2a_1 & 2a_2 & 2a_3 & -2 \\ 2b_1 & 2b_2 & 2b_3 & -2 \\ 2c_1 & 2c_2 & 2c_3 & -2 \end{bmatrix} = 0,$$

We can factor out a 2 from each row without impacting the results to get

$$\det \begin{bmatrix} x_1 & x_2 & x_3 & -1 \\ a_1 & a_2 & a_3 & -1 \\ b_1 & b_2 & b_3 & -1 \\ c_1 & c_2 & c_3 & -1 \end{bmatrix} = 0, \tag{4.53}$$

Equation (4.53) is in the standard form for representing a plane in $\mathbb{R}^3$ with a determinant [Che08, p. 3-14]. The variable $\mathbf{x}$ represents the locus of all points on a plane passing through points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

A plane can be viewed as a sphere with infinite radius.

4.4.6 Spheres in the Conformal Model

In the conformal model, spheres are represented by the outer product of four ordinary points, no three of which are collinear, i.e., four points, no three of which are collinear, define a sphere.

$$\text{Sphere:} \quad \boxed{\mathbf{L}_s = \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}}, \tag{4.54}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are the conformal mappings of four Euclidean points on the sphere, i.e., $\mathbf{A} = F(\mathbf{a})$, $\mathbf{B} = F(\mathbf{b})$, $\mathbf{C} = F(\mathbf{c})$, and $\mathbf{D} = F(\mathbf{d})$.

The equation for a sphere is represented as an outer product null space (OPNS). That is to say

$$\text{OPNS}_{\text{sphere}} = \{\mathbf{X} \,|\, \mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D} = 0\}.$$

To show that this null set really does represent a sphere in $\mathbb{R}^3$, we would first note that $\mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}$ is a 5-blade in a 5-dimensional space $\mathbb{R}^{4,1}$. From Theorem 4.3.4 we have

$$0 = \mathbf{X} \wedge \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D} = \det \begin{bmatrix} \mathbf{X}\cdot\mathbf{e}_1 & \mathbf{A}\cdot\mathbf{e}_1 & \mathbf{B}\cdot\mathbf{e}_1 & \mathbf{C}\cdot\mathbf{e}_1 & \mathbf{D}\cdot\mathbf{e}_1 \\ \mathbf{X}\cdot\mathbf{e}_2 & \mathbf{A}\cdot\mathbf{e}_2 & \mathbf{B}\cdot\mathbf{e}_2 & \mathbf{C}\cdot\mathbf{e}_2 & \mathbf{D}\cdot\mathbf{e}_2 \\ \mathbf{X}\cdot\mathbf{e}_3 & \mathbf{A}\cdot\mathbf{e}_3 & \mathbf{B}\cdot\mathbf{e}_3 & \mathbf{C}\cdot\mathbf{e}_3 & \mathbf{D}\cdot\mathbf{e}_3 \\ \mathbf{X}\cdot\mathbf{n} & \mathbf{A}\cdot\mathbf{n} & \mathbf{B}\cdot\mathbf{n} & \mathbf{C}\cdot\mathbf{n} & \mathbf{D}\cdot\mathbf{n} \\ \mathbf{X}\cdot\overline{\mathbf{n}} & \mathbf{A}\cdot\overline{\mathbf{n}} & \mathbf{B}\cdot\overline{\mathbf{n}} & \mathbf{C}\cdot\overline{\mathbf{n}} & \mathbf{D}\cdot\overline{\mathbf{n}} \end{bmatrix} \mathbf{I},$$

where $\mathbf{I} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{n} \wedge \overline{\mathbf{n}}$. This implies that

$$\det \begin{bmatrix} \mathbf{X}\cdot\mathbf{e}_1 & \mathbf{A}\cdot\mathbf{e}_1 & \mathbf{B}\cdot\mathbf{e}_1 & \mathbf{C}\cdot\mathbf{e}_1 & \mathbf{D}\cdot\mathbf{e}_1 \\ \mathbf{X}\cdot\mathbf{e}_2 & \mathbf{A}\cdot\mathbf{e}_2 & \mathbf{B}\cdot\mathbf{e}_2 & \mathbf{C}\cdot\mathbf{e}_2 & \mathbf{D}\cdot\mathbf{e}_2 \\ \mathbf{X}\cdot\mathbf{e}_3 & \mathbf{A}\cdot\mathbf{e}_3 & \mathbf{B}\cdot\mathbf{e}_3 & \mathbf{C}\cdot\mathbf{e}_3 & \mathbf{D}\cdot\mathbf{e}_3 \\ \mathbf{X}\cdot\mathbf{n} & \mathbf{A}\cdot\mathbf{n} & \mathbf{B}\cdot\mathbf{n} & \mathbf{C}\cdot\mathbf{n} & \mathbf{D}\cdot\mathbf{n} \\ \mathbf{X}\cdot\overline{\mathbf{n}} & \mathbf{A}\cdot\overline{\mathbf{n}} & \mathbf{B}\cdot\overline{\mathbf{n}} & \mathbf{C}\cdot\overline{\mathbf{n}} & \mathbf{D}\cdot\overline{\mathbf{n}} \end{bmatrix} = 0.$$

Because the determinant of a matrix equals the determinant of its transpose, we have

$$\det \begin{bmatrix} \mathbf{X}\cdot\mathbf{e}_1 & \mathbf{X}\cdot\mathbf{e}_2 & \mathbf{X}\cdot\mathbf{e}_3 & \mathbf{X}\cdot\mathbf{n} & \mathbf{X}\cdot\overline{\mathbf{n}} \\ \mathbf{A}\cdot\mathbf{e}_1 & \mathbf{A}\cdot\mathbf{e}_2 & \mathbf{A}\cdot\mathbf{e}_3 & \mathbf{A}\cdot\mathbf{n} & \mathbf{A}\cdot\overline{\mathbf{n}} \\ \mathbf{B}\cdot\mathbf{e}_1 & \mathbf{B}\cdot\mathbf{e}_2 & \mathbf{B}\cdot\mathbf{e}_3 & \mathbf{B}\cdot\mathbf{n} & \mathbf{B}\cdot\overline{\mathbf{n}} \\ \mathbf{C}\cdot\mathbf{e}_1 & \mathbf{C}\cdot\mathbf{e}_2 & \mathbf{C}\cdot\mathbf{e}_3 & \mathbf{C}\cdot\mathbf{n} & \mathbf{C}\cdot\overline{\mathbf{n}} \\ \mathbf{D}\cdot\mathbf{e}_1 & \mathbf{D}\cdot\mathbf{e}_2 & \mathbf{D}\cdot\mathbf{e}_3 & \mathbf{D}\cdot\mathbf{n} & \mathbf{D}\cdot\overline{\mathbf{n}} \end{bmatrix} = 0.$$

By substituting all the inner products from the standard form Eq. (4.10) into the matrix, we get

$$\det \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 & -2 & 2\mathbf{x}^2 \\ 2a_1 & 2a_2 & 2a_3 & -2 & 2\mathbf{a}^2 \\ 2b_1 & 2b_2 & 2b_3 & -2 & 2\mathbf{b}^2 \\ 2c_1 & 2c_2 & 2c_3 & -2 & 2\mathbf{c}^2 \\ 2d_1 & 2d_2 & 2d_3 & -2 & 2\mathbf{d}^2 \end{bmatrix} = 0,$$

where subscripts, as in x_1, x_2, x_3, represent the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ components of the vector $\mathbf{x} \in \mathbb{R}^3$.

We can factor out a 2 from each row and rearrange the columns of the matrix without impacting the results to get

$$\det \begin{bmatrix} \mathbf{x}^2 & x_1 & x_2 & x_3 & -1 \\ \mathbf{a}^2 & a_1 & a_2 & a_3 & -1 \\ \mathbf{b}^2 & b_1 & b_2 & b_3 & -1 \\ \mathbf{c}^2 & c_1 & c_2 & c_3 & -1 \\ \mathbf{d}^2 & d_1 & d_2 & d_3 & -1 \end{bmatrix} = 0, \tag{4.55}$$

Equation (4.55) is in the standard form for representing a sphere in $\mathbb{R}^3$ with a determinant [Che08, p. 3-15]. The variable $\mathbf{x}$ represents the locus of all points on a sphere passing through points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$.

We note that both spheres and planes are represented by quadvectors in the conformal model. In fact, planes can be viewed as spheres with infinite radius. If the four points that define a sphere are coplanar, then the sphere is a plane, i.e.,

$$\mathbf{L}_s \wedge \mathbf{n} = 0 \quad \Rightarrow \quad \text{plane.}$$

This explains why the earlier derivation of a plane through $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ led to the quadvector $\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{n}$, which explicitly includes the point at infinity.

The center of the sphere is the point

$$\text{Center:} \quad \boxed{\mathbf{P}_0 = \mathbf{L}_s \mathbf{n} \mathbf{L}_s}. \tag{4.56}$$

The operation $\mathbf{L}_s \mathbf{M} \mathbf{L}_s$ reflects any object $\mathbf{M}$ in the sphere $\mathbf{L}_s$. Eq. (4.56) says that the center of a sphere is the image of the point at infinity under a reflection in a sphere.

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APPENDIX A

NOTATIONAL CONVENTIONS USED IN THE LITERATURE

The subject of geometric algebra is new enough that standards have not yet been established. This causes difficulties when perusing literature written by different authors. Table A.1 provides a comparison of some of the notational conventions used by different authors.

Table A.1: Notational Conventions Used in the Popular Literature

Property	**Hestenes [Hes03]**	**Dorst, Fontijne & Mann [DFM07]**	**Doran & Lasenby [DL03]**
orthonormal vectors	$\sigma_1, \sigma_2, \sigma_3$	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	e_1, e_2, e_3
pseudoscalar	$\mathbf{i}$	$\mathbf{I}$	I
dual	$\mathbf{i}A$	$\mathbf{A}_k^* = \mathbf{A}_k \rfloor \mathbf{I}_n^{-1}$	$A^* = IA$
reversion	$A^\dagger$	$A^\sim$ or $\widetilde{A}$	$A^\dagger$
infinity vector	$\mathbf{e}$	∞	n
origin vector	$\mathbf{e}_0$	o	$\overline{\mathbf{n}}$
conformal map	$x = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e + e_0$	$p = \mathbf{p} + \frac{1}{2}\mathbf{p}^2\infty + \mathrm{o}$	$X = 2x + x^2\mathbf{n} - \overline{\mathbf{n}}$

Property	**Vince [Vin08]**	**Macdonald [Mac10] & [Mac12]**
orthonormal vectors	e_1, e_2, e_3	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
pseudoscalar	I	$\mathbf{i}$ (2D) & $\mathbf{I}$ (3D)
dual	$A^* = IA$	$\mathbf{A}^* = \mathbf{A}\mathbf{I}^{-1}$
reversion	$A^\sim$ or $\widetilde{A}$	$\mathbf{A}^\sim$ or $\widetilde{\mathbf{A}}$
infinity vector	n	e_∞
origin vector	$\overline{n}$	e_o
conformal map	$X = \frac{1}{2}(2x + x^2 n - \overline{n})$	$p = e_o + \mathbf{p} + \frac{1}{2}\mathbf{p}^2 e_\infty$

APPENDIX B

FORMULAS AND IDENTITIES OF GEOMETRIC ALGEBRA

B.1 Axioms and Definitions of Geometric Algebra

In this section we present a set of axioms and definitions for geometric algebra. This section closely follows the development presented in Hestenes and Sobczyk [HS84]. For the sake of simplicity and ease of application, we have not attempted to eliminate all redundancy from the axioms.

Definition B.1.1 (**Multivector**).

An element of the **geometric algebra** $\mathcal{G}$ *is called a* **multivector**.

We assume that $\mathcal{G}$ is closed under addition and geometric multiplication. The sum or geometric product of any pair of multivectors is a unique multivector.

Axiom B.1.2.

Let A, B, and C be multivectors, then the geometric sum *and* product *of* *multivectors have the following properties:*

$$\textbf{A1}: \quad A + B = B + A \qquad \textit{addition is commutative,} \tag{B.1}$$

$$\textbf{A2}: \quad (A + B) + C = A + (B + C) \qquad \textit{addition is associative,} \tag{B.2}$$

$$(AB)C = A(BC) \qquad \textit{multiplication is associative,} \tag{B.3}$$

$$\textbf{A3}: \quad A(B + C) = AB + AC \qquad \textit{left distributive,} \tag{B.4}$$

$$(B + C)A = BA + CA \qquad \textit{right distributive,} \tag{B.5}$$

$$\textbf{A4}: \quad A + 0 = A \qquad \exists \textit{ a unique additive identity 0,} \tag{B.6}$$

$$1A = A \qquad \exists \textit{ a unique multiplicative identity 1,} \tag{B.7}$$

$$\textbf{A5}: \quad A + (-A) = 0 \qquad \forall A, \exists \textit{ a unique additive inverse } -A. \tag{B.8}$$

Geometric algebra is different from other associative algebras in that a multivector is assumed to be composed of different types of objects that we classify by *grade*. We assume that any multivector A can be written as the sum

$$A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \cdots = \sum_k \langle A\rangle_k. \tag{B.9}$$

Definition B.1.3 (*k*-vector).

The quantity $\langle A\rangle_k$ is called the **k-vector** *part of A. If $A = \langle A\rangle_k$ for some positive integer k, then A is said to be* **homogeneous** *of grade k and is called a* **k-vector**. *We use the terms* **scalar**, **vector**, **bivector**, *and* **trivector** *as synonyms for* **0-vector**, **1-vector**, **2-vector**, *and* **3-vector**, *respectively.*

Axiom B.1.4.

The **grade operator** $\langle\ \rangle_k$ *has the following properties*

$$\textbf{G1}: \quad \langle A + B\rangle_k = \langle A\rangle_k + \langle B\rangle_k, \tag{B.10}$$

$$\textbf{G2}: \quad \langle sA\rangle_k = s\langle A\rangle_k = \langle A\rangle_k s, \qquad \textit{where } s \textit{ is a scalar (0-vector),} \tag{B.11}$$

$$\textbf{G3}: \quad \langle\langle A\rangle_k\rangle_k = \langle A\rangle_k. \tag{B.12}$$

Axioms (B.10) and (B.11) imply that the space $\mathcal{G}^k$ of all k-vectors is a linear subspace of $\mathcal{G}$. Axiom (B.11) also implies that scalars compose a commutative subalgebra of $\mathcal{G}$. The space $\mathcal{G}^0$ of all scalars is identical with the set of real numbers $\mathbb{R}$. We regard the wider definition of the scalars (e.g. the complex numbers) as entirely unnecessary and unfavorable to the purposes of geometric algebra. The scalar grade operator is assigned the special notation

$$\langle M \rangle \equiv \langle M \rangle_0. \tag{B.13}$$

The grade operator is a projection operator, so $\langle A \rangle_k$ can be regarded as the projection of A into the space $\mathcal{G}^k$.

Axiom B.1.5.

Multiplication of vectors is related to scalars by the assumption that the square of any nonzero vector $\mathbf{u}$ *is equal to the square of a unique positive*[1] *scalar* $|\mathbf{u}|$ *called the magnitude of* $\mathbf{u}$*, that is*

$$\mathbf{u}\mathbf{u} = \mathbf{u}^2 = \langle \mathbf{u}^2 \rangle_0 = |\mathbf{u}|^2 > 0. \tag{B.14}$$

Definition B.1.6 (*k*-blade).

A multivector A_k *is called a* ***k*-blade** *or a* **simple *k*-vector** *if and only if it can be factored into a product of* k *anticommuting vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$*, that is*

$$A_k = \mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_k, \tag{B.15}$$

$$\textit{where} \quad \mathbf{u}_i \mathbf{u}_j = -\mathbf{u}_j \mathbf{u}_i \quad \textit{for } i, j = 1, 2, \ldots, k \textit{ and } i \neq j. \tag{B.16}$$

Any ***k*-vector** *can be expressed as the sum of* k*-blades. We use a bold font to emphasize that a variable is a* k*-blade as apposed to a* k*-vector or multivector.*

[1] Note that this axiom must be modified when working with spaces that are not positive definite, i.e., spaces with negative or mixed signature.

Axiom B.1.7.

For every nonzero k-blade $\mathbf{A}_k$, there exists a nonzero vector $\mathbf{u}$ in $\mathcal{G}$, such that $\mathbf{A}_k\mathbf{u}$ is an $(k+1)$-blade.

This guarantees the existence of nontrivial blades of every finite grade. It implies that $\mathcal{G}$ is is a linear space of infinite dimension.

B.2 Notation

With the variety of different types of objects in geometric algebra, it is important to be consistent with the notation used. In the formulas and identities that follow, we use the notation shown below.

scalar - lower-case letters, e.g. a, b, c, s.

vector - lower-case, bold characters, e.g. $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

blade - (when grade is of no consequence) upper-case, bold character, e.g. $\mathbf{B}, \mathbf{A}$.

k**-blade** - upper-case, bold character with subscript indicating grade, e.g. $\mathbf{B}_k, \mathbf{A}_k$.

k**-vector** - upper-case characters with subscript indicating grade, e.g. A_k, B_k.

multivector - upper-case character with no subscript, e.g. A, B.

B.3 Useful Formulas and Identities

In navigating the literature on geometric algebra, it is useful to have a list of formulas and identities arranged for easy reference. This section provides such a list. In the following, n is the dimension of the space we are modeling, i.e., $\mathbb{R}^n$.

B.3.1 The Grade Operator

The quantity $\langle A \rangle_k$ is called the *k-vector* part of the multivector A.

The *grade operator* $\langle\ \rangle_k$ has the following properties:

$$\langle A + B \rangle_k = \langle A \rangle_k + \langle B \rangle_k, \tag{B.17}$$

$$\langle sA \rangle_k = s\,\langle A \rangle_k = \langle A \rangle_k\, s, \qquad \text{where } s \text{ is a scalar (0-vector)}, \tag{B.18}$$

$$\langle \langle A \rangle_k \rangle_k = \langle A \rangle_k. \tag{B.19}$$

B.3.2 Reversion

In algebraic computations, it is often desirable to reorder the factors in a product. As such, we introduce the operation of *reversion* defined by the equations

$$\text{scalar reverse:} \qquad \widetilde{s} = s, \tag{B.20}$$

$$\text{vector reverse:} \qquad \widetilde{\mathbf{u}} = \mathbf{u}, \tag{B.21}$$

$$\text{vector product reverse:} \qquad (\mathbf{u}_1\mathbf{u}_2 \ldots \mathbf{u}_{k-1}\mathbf{u}_k)^\sim = \mathbf{u}_k\,\mathbf{u}_{k-1} \ldots \mathbf{u}_2\,\mathbf{u}_1, \tag{B.22}$$

$$\text{multivector product reverse:} \qquad (AB)^\sim = \widetilde{B}\widetilde{A}, \tag{B.23}$$

$$\text{multivector sum reverse:} \qquad (A+B)^\sim = \widetilde{A} + \widetilde{B}. \tag{B.24}$$

B.3.3 Norm (in Positive Definite Spaces)

The norm $|\mathbf{v}|$ of a vector $\mathbf{v}$ in an inner-product space is given by $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$. Consider the standard basis for $\mathbb{G}^3, \mathfrak{B} = \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{123}\}$. Let $\mathbf{e}_J$ represent one of these basis elements and a_J represent the scalar coefficient of the $\mathbf{e}_J$ element in the multivector M, i.e.,

$$M = a_0\,1 + a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3 + a_{12}\,\mathbf{e}_{12} + a_{23}\,\mathbf{e}_{23} + a_{31}\,\mathbf{e}_{31} + a_{123}\,\mathbf{e}_{123}.$$

The norm is defined by the equations

$$\text{vector:} \qquad |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}, \tag{B.25}$$

$$\text{k-blade:} \qquad |\mathbf{B}_k| = |\mathbf{b}_1||\mathbf{b}_2|\cdots|\mathbf{b}_k|, \tag{B.26}$$

$$\text{k-vector:} \qquad |A_k| = \sqrt{A_k \cdot A_k}, \tag{B.27}$$

$$\text{multivector:} \qquad |M| = \sqrt{\sum_J |a_J|^2} = \sqrt{\sum_{k=0}^{n} |\langle M\rangle_k|^2}. \tag{B.28}$$

Properties of the norm.

$$\text{If } \mathbf{v} \neq 0, \text{ then } |\mathbf{v}| > 0, \qquad \text{positive definite} \tag{B.29}$$

$$|a\mathbf{v}| = |a||\mathbf{v}|, \qquad \text{scalar multiple} \tag{B.30}$$

$$|\mathbf{u}\cdot\mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|, \qquad \text{Cauchy-Schwarz inequality} \tag{B.31}$$

$$|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|. \qquad \text{triangle inequality} \tag{B.32}$$

B.3.4 Inverse

Every non-null k-blade $\mathbf{B}_k$ has an inverse.

$$\text{vector inverse:} \qquad \mathbf{u}^{-1} = \frac{\mathbf{u}}{|\mathbf{u}|^2} \tag{B.33}$$

$$\text{blade inverse:} \qquad \mathbf{B}_k^{-1} = \frac{\widetilde{\mathbf{B}}_k}{|\mathbf{B}_k|^2} \tag{B.34}$$

B.3.5 Geometric Product

The geometric product is the fundamental operation of geometric algebra. The geometric product of two vectors has two components, a scalar (the inner product) and a bivector (the outer product).

vector-vector: $$\mathbf{u}\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v} \tag{B.35}$$

vector-bivector: $$\mathbf{u}B_2 = \mathbf{u}\cdot B_2 + \mathbf{u}\wedge B_2 \tag{B.36}$$

vector-k-vector: $$\mathbf{u}B_k = \mathbf{u}\cdot B_k + \mathbf{u}\wedge B_k \tag{B.37}$$

j-vector-k-vector: $$\begin{aligned} A_jB_k = {} & \langle A_jB_k\rangle_{|j-k|} + \langle A_jB_k\rangle_{|j-k|+2} + \cdots \\ & + \langle A_jB_k\rangle_{j+k-2} + \langle \mathbf{A}_jB_k\rangle_{j+k}. \end{aligned} \tag{B.38}$$

Properties of the geometric product:

$$\mathbf{v}^2 = \mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2, \quad \text{norm squared} \tag{B.39}$$

$$\mathbf{u}\mathbf{v} = \mathbf{u}\cdot\mathbf{v}, \quad \text{if } \mathbf{u}\parallel\mathbf{v}, \quad \text{parallel} \tag{B.40}$$

$$\mathbf{u}\mathbf{v} = \mathbf{u}\wedge\mathbf{v} = -\mathbf{v}\wedge\mathbf{u}, \quad \text{if } \mathbf{u}\perp\mathbf{v}, \quad \text{perpendicular} \tag{B.41}$$

$$(a\mathbf{u})\,\mathbf{v} = a\,(\mathbf{u}\mathbf{v}), \quad \text{homogeneous} \tag{B.42}$$

$$\mathbf{u}\,(a\mathbf{v}) = a\,(\mathbf{u}\mathbf{v}), \tag{B.43}$$

$$(\mathbf{u}+\mathbf{v})\,\mathbf{w} = \mathbf{u}\mathbf{w} + \mathbf{v}\mathbf{w}, \quad \text{distributive} \tag{B.44}$$

$$\mathbf{w}\,(\mathbf{u}+\mathbf{v}) = \mathbf{w}\mathbf{u} + \mathbf{w}\mathbf{v}, \tag{B.45}$$

$$\mathbf{u}\,(\mathbf{v}\mathbf{w}) = (\mathbf{u}\mathbf{v})\,\mathbf{w}. \quad \text{associative} \tag{B.46}$$

B.3.6 Inner Product

The inner product is a *grade-lowering* operation, i.e., if A_j is a j-vector and B_k is a k-vector with $j \leq k$, then $A_j \cdot B_k$ is a multivector of grade $k - j$.

scalar-scalar: $$a \cdot b = 0 \tag{B.47}$$

scalar-vector: $$a \cdot \mathbf{v} = 0 \tag{B.48}$$

vector-vector: $$\mathbf{u} \cdot \mathbf{v} = \tfrac{1}{2}(\mathbf{uv} + \mathbf{vu}), \tag{B.49}$$

vector-vector: $$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta, \qquad 0 \leq \theta \leq \pi \tag{B.50}$$

vector-bivector: $$\mathbf{u} \cdot B_2 = \tfrac{1}{2}(\mathbf{u}B_2 - B_2\mathbf{u}) = -B_2 \cdot \mathbf{u}, \tag{B.51}$$

vector-k-vector: $$\mathbf{u} \cdot B_k = \tfrac{1}{2}(\mathbf{u}B_k + (-1)^{k+1}B_k\mathbf{u}) = \langle \mathbf{u}B_k \rangle_{k-1}, \tag{B.52}$$

j-vector-k-vector: $$A_j \cdot B_k = \begin{cases} \langle A_j B_k \rangle_{|k-j|}, & \text{if } j, k > 0, \\ 0, & \text{if } j = 0 \text{ or } k = 0, \end{cases} \tag{B.53}$$

multivector-multivector: $$A \cdot B = \sum_{j=0}^{n} \sum_{k=0}^{n} \langle A \rangle_j \cdot \langle B \rangle_k. \tag{B.54}$$

Properties of the inner product:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}, \qquad \text{commutative} \tag{B.55}$$

$$(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}), \qquad \text{homogeneous} \tag{B.56}$$

$$\mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v}),$$

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}, \qquad \text{distributive} \tag{B.57}$$

$$\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v},$$

$$\text{If } \mathbf{v} \neq 0, \text{ then } \mathbf{v} \cdot \mathbf{v} > 0. \qquad \text{for positive definite spaces} \tag{B.58}$$

B.3.7 Outer Product

The outer product is a *grade-raising* operation, i.e., if A_j is a j-vector and B_k is a k-vector, then $A_j \wedge B_k$ is a multivector of grade $j+k$.

scalar-scalar: $$a \wedge b = a\,b, \tag{B.59}$$

scalar-vector: $$a \wedge \mathbf{v} = a\,\mathbf{v}, \tag{B.60}$$

vector-vector: $$\mathbf{u} \wedge \mathbf{v} = \tfrac{1}{2}(\mathbf{uv} - \mathbf{vu}), \tag{B.61}$$

vector-vector: $$\mathbf{u} \wedge \mathbf{v} = |\mathbf{u}||\mathbf{v}|\,\mathbf{I}\sin\theta, \qquad 0 \le \theta \le \pi \tag{B.62}$$

vector-bivector: $$\mathbf{u} \wedge B_2 = \tfrac{1}{2}(\mathbf{u}B_2 + B_2\mathbf{u}) = B_2 \wedge \mathbf{u}, \tag{B.63}$$

vector-k-vector: $$\mathbf{u} \wedge B_k = \tfrac{1}{2}(\mathbf{u}B_k - (-1)^{k+1}B_k\mathbf{u}) = \langle \mathbf{u}B_k \rangle_{k+1}, \tag{B.64}$$

j-vector-k-vector: $$A_j \wedge B_k = \begin{cases} \langle A_j B_k \rangle_{j+k}, & \text{if } j+k \le n, \\ 0, & \text{if } j+k > n, \end{cases} \tag{B.65}$$

multivector-multivector: $$A \wedge B = \sum_{j=0}^{n} \sum_{k=0}^{n} \langle A \rangle_j \wedge \langle B \rangle_k, \tag{B.66}$$

arbitrary number of vectors: $$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_r = \frac{1}{r!} \sum (-1)^{\epsilon}\, \mathbf{v}_{k_1}\mathbf{v}_{k_2} \cdots \mathbf{v}_{k_r}, \tag{B.67}$$

where the sum runs over every permutation $k_1, \ldots, k_r$ of $1, \ldots, r$, and ϵ is $+1$ or -1 as the permutation $k_1, \ldots, k_r$ is an even or odd respectively.

Properties of the outer product:

$$\mathbf{v} \wedge \mathbf{v} = 0, \qquad \text{parallel} \qquad \text{(B.68)}$$

$$\mathbf{v} \wedge \mathbf{u} = -(\mathbf{u} \wedge \mathbf{v}), \qquad \text{anticommutative} \qquad \text{(B.69)}$$

$$(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}), \qquad \text{associative} \qquad \text{(B.70)}$$

$$(a\mathbf{u}) \wedge \mathbf{v} = a(\mathbf{u} \wedge \mathbf{v}), \qquad \text{homogeneous} \qquad \text{(B.71)}$$

$$\mathbf{u} \wedge (a\mathbf{v}) = a(\mathbf{u} \wedge \mathbf{v}),$$

$$(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w}, \qquad \text{distributive} \qquad \text{(B.72)}$$

$$\mathbf{w} \wedge (\mathbf{u} + \mathbf{v}) = \mathbf{w} \wedge \mathbf{u} + \mathbf{w} \wedge \mathbf{v}.$$

B.3.8 Orthogonal Test

In an inner-product space, the inner product can be used as a test for orthogonality.

$$\text{vector-vector:} \quad \mathbf{u} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{u} \perp \mathbf{v}. \qquad \text{(B.73)}$$

$$\text{vector-vector:} \quad \mathbf{v}\mathbf{u} = -\mathbf{u}\mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} \perp \mathbf{v}. \qquad \text{(B.74)}$$

$$\text{vector-blade:} \quad \mathbf{u} \cdot \mathbf{B}_k = 0 \quad \Leftrightarrow \quad \mathbf{u} \perp \mathbf{B}_k. \qquad \text{(B.75)}$$

$$\text{blade-blade:} \quad \mathbf{A}_j \cdot \mathbf{B}_k = 0 \quad \Leftrightarrow \quad \begin{cases} \text{there is a nonzero vector} \\ \text{in } \mathbf{A}_k \text{ orthogonal to } \mathbf{B}_k. \end{cases} \qquad \text{(B.76)}$$

B.3.9 Parallel Test (Subspace Membership Text)

The outer product can be used to test for subspace membership.

$$\text{vector-vector:} \quad \mathbf{u} \wedge \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{u} \parallel \mathbf{v}. \tag{B.77}$$

$$\text{vector-vector:} \quad \mathbf{u}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} \parallel \mathbf{v}. \tag{B.78}$$

$$\text{vector-}k\text{-blade:} \quad \mathbf{u} \wedge \mathbf{B}_k = 0 \quad \Leftrightarrow \quad \mathbf{u} \in \mathbf{B}_k. \tag{B.79}$$

$$\text{vector-}k\text{-blade:} \quad \mathbf{u} \cdot \mathbf{B}_k^* = 0 \quad \Leftrightarrow \quad \mathbf{u} \in \mathbf{B}_k. \tag{B.80}$$

$$j\text{-blade-}k\text{-blade:} \quad \mathbf{A}_j \wedge \mathbf{B}_k = 0 \Leftrightarrow \quad \mathbf{A}_j \subseteq \mathbf{B}_k \qquad \text{for } j < k \tag{B.81}$$

B.3.10 Duality

Let $\mathbf{I}$ be the pseudoscalar for the space, e.g. for $\mathbb{R}^3$, $\mathbf{I} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ and $\mathbf{I}^{-1} = \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1$. Then the dual M^* of the multivector M is

$$M^* = M\mathbf{I}^{-1}. \tag{B.82}$$

Properties of Duality:

$$j\text{-vector-}k\text{-vector:} \qquad (A_j \cdot B_k)^* = A_j \wedge B_k^*, \tag{B.83}$$

$$j\text{-vector-}k\text{-vector:} \qquad (A_j \wedge B_k)^* = A_j \cdot B_k^*. \tag{B.84}$$

B.3.11 Projection and Rejection

Formulas for determining the component of a vector $\mathbf{u}$ in a subspace $\mathbf{B}_k$ (*projection*) and perpendicular to a subspace (*rejection*) are

$$\text{vector-}k\text{-blade projection:} \qquad \mathbf{u}_\parallel = (\mathbf{u} \cdot \mathbf{B}_k)/\mathbf{B}_k \tag{B.85}$$

$$\text{vector-}k\text{-blade rejection:} \qquad \mathbf{u}_\perp = (\mathbf{u} \wedge \mathbf{B}_k)/\mathbf{B}_k \tag{B.86}$$

B.3.12 Other Useful Identities

$$A \cdot (B \cdot C) = (A \wedge B) \cdot C \tag{B.87}$$

$$\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v})\mathbf{w} - (\mathbf{u} \cdot \mathbf{w})\mathbf{v} \tag{B.88}$$

$$\text{Jacobi Identity:} \quad \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) + \mathbf{v} \cdot (\mathbf{w} \wedge \mathbf{u}) + \mathbf{w} \cdot (\mathbf{u} \wedge \mathbf{v}) = 0 \tag{B.89}$$

B.4 Derivations of Formulas

This section provides some derivations of the formulas and assertions presented in the previous section.

B.4.1 $|a\mathbf{v}| = |a|\,|\mathbf{v}|$

$$|a\mathbf{v}|^2 = (a\mathbf{v}) \cdot (a\mathbf{v}) = a^2(\mathbf{v} \cdot \mathbf{v}) = (|a|\,|\mathbf{v}|)^2,$$

$$\text{therefore,} \quad |a\mathbf{v}| = |a|\,|\mathbf{v}|.$$

B.4.2 $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|$ (Cauchy-Schwarz inequality)

This inequality holds for all positive definite spaces.

Case 1: Let $\mathbf{v} = \mathbf{0}$, then the inequality becomes $|\mathbf{u} \cdot \mathbf{0}| \leq |\mathbf{u}|\,|\mathbf{0}|$. This is true because $\mathbf{u} \cdot \mathbf{0} = \mathbf{0}$.

Case 2: Let $\mathbf{v} \neq \mathbf{0}$. For every scalar λ,

$$0 \leq |\mathbf{u} - \lambda\mathbf{v}|^2 = (\mathbf{u} - \lambda\mathbf{v}) \cdot (\mathbf{u} - \lambda\mathbf{v}) = |\mathbf{u}|^2 - 2\lambda(\mathbf{u} \cdot \mathbf{v}) + \lambda^2|\mathbf{v}|^2.$$

Let $\lambda = (\mathbf{u} \cdot \mathbf{v})/|\mathbf{v}|^2$, thus

$$
\begin{aligned}
0 \leq |\mathbf{u}|^2 - 2\lambda(\mathbf{u} \cdot \mathbf{v}) + \lambda^2|\mathbf{v}|^2 &= |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v})^2/|\mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2|\mathbf{v}|^2/|\mathbf{v}|^4 \\
&= |\mathbf{u}|^2 - (\mathbf{u} \cdot \mathbf{v})^2/|\mathbf{v}|^2 \\
0 \leq |\mathbf{u}|^2\,|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2 & \\
0 \leq |\mathbf{u}|^2\,|\mathbf{v}|^2 - |\mathbf{u} \cdot \mathbf{v}|^2 & \\
|\mathbf{u} \cdot \mathbf{v}|^2 \leq |\mathbf{u}|^2\,|\mathbf{v}|^2 & \\
|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|. &
\end{aligned}
$$

B.4.3 $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ (triangle inequality)

This inequality holds for all positive definite spaces. The proof uses the Cauchy-Schwarz inequality from above.

$$
\begin{aligned}
|\mathbf{u} + \mathbf{v}|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\
&= |\mathbf{u}|^2 + 2\,|\mathbf{u} \cdot \mathbf{v}| + |\mathbf{v}|^2 \\
&\leq |\mathbf{u}|^2 + 2\,|\mathbf{u}|\,|\mathbf{v}| + |\mathbf{v}|^2 \\
&= (|\mathbf{u}| + |\mathbf{v}|)^2 \\
\text{therefore,} \quad |\mathbf{u} + \mathbf{v}| &\leq |\mathbf{u}| + |\mathbf{v}|.
\end{aligned}
$$

B.4.4 $\mathbf{u}^{-1} = \mathbf{u}/|\mathbf{u}|^2$ (vector inverse)

If $\mathbf{u} \neq 0$, then $\mathbf{uu} = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\wedge\mathbf{u} = |\mathbf{u}|^2 + 0 = |\mathbf{u}|^2$. Thus, we have

$$1 = \frac{\mathbf{uu}}{\mathbf{uu}} = \frac{\mathbf{uu}}{|\mathbf{u}|^2} = \mathbf{u}\left(\frac{\mathbf{u}}{|\mathbf{u}|^2}\right) = \left(\frac{\mathbf{u}}{|\mathbf{u}|^2}\right)\mathbf{u}.$$

$$\text{Therefore,} \quad \mathbf{u}^{-1} = \frac{\mathbf{u}}{|\mathbf{u}|^2}.$$

B.4.5 $\mathbf{u}\cdot\mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu})$ (inner product)

Method 1:

$$\begin{aligned}
2(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} \\
&= \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} \\
&= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) - \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} \\
&= (\mathbf{u}+\mathbf{v})^2 - \mathbf{u}^2 - \mathbf{v}^2 \\
&= \mathbf{u}^2 + \mathbf{uv} + \mathbf{vu} + \mathbf{v}^2 - \mathbf{u}^2 - \mathbf{v}^2 \\
&= \mathbf{uv} + \mathbf{vu}.
\end{aligned}$$

$$\text{Therefore,} \qquad \mathbf{u}\cdot\mathbf{v} = \tfrac{1}{2}(\mathbf{uv} + \mathbf{vu}).$$

Method 2:

$$\mathbf{uv} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}, \qquad \mathbf{vu} = \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\wedge\mathbf{u} = \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\wedge\mathbf{v}.$$

$$\mathbf{uv} + \mathbf{vu} = 2(\mathbf{u}\cdot\mathbf{v}).$$

$$\text{Therefore,} \qquad \mathbf{u}\cdot\mathbf{v} = \tfrac{1}{2}(\mathbf{uv} + \mathbf{vu}).$$

B.4.6 $\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu})$ **(outer product)**

Following Method 2 above:

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}, \qquad \mathbf{vu} = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \wedge \mathbf{u} = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \wedge \mathbf{v}.$$

$$\mathbf{uv} - \mathbf{vu} = 2(\mathbf{u} \wedge \mathbf{v}).$$

$$\text{Therefore,} \qquad \mathbf{u} \wedge \mathbf{v} = \tfrac{1}{2}(\mathbf{uv} - \mathbf{vu}).$$

B.4.7 $\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w} - (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v}$

The following identity is used in this derivation:

$$\mathbf{u} \cdot \mathbf{v} = \tfrac{1}{2}(\mathbf{uv} + \mathbf{vu}) \quad \Leftrightarrow \quad \mathbf{uv} = -\mathbf{vu} + 2\mathbf{u} \cdot \mathbf{v}.$$

Beginning with the geometric product of three vectors and using the identity from above twice, we get

$$\begin{aligned} \mathbf{uvw} &= (\mathbf{uv})\mathbf{w} \\ &= (-\mathbf{vu} + 2\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\ &= -\mathbf{vuw} + 2(\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\ &= -\mathbf{v}(\mathbf{uw}) + 2(\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\ &= -\mathbf{v}(-\mathbf{wu} + 2\mathbf{u} \cdot \mathbf{w}) + 2(\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\ &= \mathbf{vwu} - 2(\mathbf{u} \cdot \mathbf{w})\mathbf{v} + 2(\mathbf{u} \cdot \mathbf{v})\mathbf{w}. \end{aligned}$$

Rearranging terms and dividing by 2, we get

$$\begin{aligned} (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w} - (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} &= \tfrac{1}{2}[\mathbf{uvw} - \mathbf{vwu}] \\ &= \tfrac{1}{2}[\mathbf{u}(\mathbf{vw}) - (\mathbf{vw})\mathbf{u}] \end{aligned}$$

$$= \tfrac{1}{2}[\mathbf{u}(\mathbf{v}\cdot\mathbf{w}+\mathbf{v}\wedge\mathbf{w})-(\mathbf{v}\cdot\mathbf{w}+\mathbf{v}\wedge\mathbf{w})\mathbf{u}]$$

$$= \tfrac{1}{2}[\mathbf{u}(\mathbf{v}\cdot\mathbf{w})+\mathbf{u}(\mathbf{v}\wedge\mathbf{w})-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}-(\mathbf{v}\wedge\mathbf{w})\mathbf{u}]$$

$$\stackrel{5}{=} \tfrac{1}{2}[(\mathbf{v}\cdot\mathbf{w})\mathbf{u}+\mathbf{u}(\mathbf{v}\wedge\mathbf{w})-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}-(\mathbf{v}\wedge\mathbf{w})\mathbf{u}]$$

$$= \tfrac{1}{2}[\mathbf{u}(\mathbf{v}\wedge\mathbf{w})-(\mathbf{v}\wedge\mathbf{w})\mathbf{u}]$$

$$\stackrel{7}{=} \tfrac{1}{2}[\mathbf{u}\cdot(\mathbf{v}\wedge\mathbf{w})+\mathbf{u}\wedge(\mathbf{v}\wedge\mathbf{w})-(\mathbf{v}\wedge\mathbf{w})\cdot\mathbf{u}-(\mathbf{v}\wedge\mathbf{w})\wedge\mathbf{u}]$$

$$\stackrel{8}{=} \tfrac{1}{2}[\mathbf{u}\cdot(\mathbf{v}\wedge\mathbf{w})+\mathbf{u}\wedge(\mathbf{v}\wedge\mathbf{w})+\mathbf{u}\cdot(\mathbf{v}\wedge\mathbf{w})-\mathbf{u}\wedge(\mathbf{v}\wedge\mathbf{w})]$$

$$= \tfrac{1}{2}[2\,\mathbf{u}\cdot(\mathbf{v}\wedge\mathbf{w})]$$

$$= \mathbf{u}\cdot(\mathbf{v}\wedge\mathbf{w}).$$

$$\therefore \quad \mathbf{u}\cdot(\mathbf{v}\wedge\mathbf{w}) = (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}-(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}. \tag{B.90}$$

Here is a description of some of the steps in the derivation.

Step 5: $\mathbf{v}\cdot\mathbf{w}$ is a scalar so we can bring it in front of $\mathbf{u}$.

Step 7: We expand the geometric product of a vector and a bivector to the sum of their inner product and outer product.

Step 8: The inner product of a vector and a bivector is anticommutative.

The outer product of a vector and a bivector is commutative.

B.4.8 $\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) + \mathbf{v} \cdot (\mathbf{w} \wedge \mathbf{u}) + \mathbf{w} \cdot (\mathbf{u} \wedge \mathbf{v}) = 0$ (Jacobi Identity)

We expand each term in the expression using Eq. (B.90).

$$\begin{aligned}
&\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) + \mathbf{v} \cdot (\mathbf{w} \wedge \mathbf{u}) + \mathbf{w} \cdot (\mathbf{u} \wedge \mathbf{v}) \\
&= [(\mathbf{u} \cdot \mathbf{v})\,\mathbf{w} - (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v}] + [(\mathbf{v} \cdot \mathbf{w})\,\mathbf{u} - (\mathbf{v} \cdot \mathbf{u})\,\mathbf{w}] + [(\mathbf{w} \cdot \mathbf{u})\,\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\,\mathbf{u}] \\
&= (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w} - (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} + (\mathbf{v} \cdot \mathbf{w})\,\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w} + (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\,\mathbf{u} \\
&= 0
\end{aligned}$$

$$\therefore \quad \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) + \mathbf{v} \cdot (\mathbf{w} \wedge \mathbf{u}) + \mathbf{w} \cdot (\mathbf{u} \wedge \mathbf{v}) = 0. \tag{B.91}$$

Note that in the next to last step, we used the fact that the inner product of two vectors is commutative.

APPENDIX C

GLOSSARY

$\mathbb{R}^n$ – a vector space of dimension n over the field $\mathbb{R}$ of real numbers with a Euclidean signature.

$\mathbb{R}^{p,q}$ – a vector space of dimension $n = p + q$ with p "positive dimensions" and q "negative dimensions" over the field $\mathbb{R}$ of real numbers with signature (p, q).

$\mathbb{G}^n$ – a geometric algebra over $\mathbb{R}^n$.

$\mathbb{G}^{p,q}$ – a geometric algebra over $\mathbb{R}^{p,q}$.

$\bigwedge^k \mathbb{R}^n$ – the grade k subspace of $\mathbb{G}^n$, i.e., the subspace of k-vectors.

AB – the *geometric product* of multivectors A and B.

$A \cdot B$ – the *inner product* of multivectors A and B.

$A \wedge B$ – the *outer product* of multivectors A and B.

M^{-1} – the *inverse* of the multivector M.

$\langle M \rangle_k$ – the *k-vector* part of the multivector M.

$\langle M \rangle$ – the 0-*vector* (scalar) part $\langle M \rangle_0$ of the multivector M.

$\widetilde{M}$ – the *reversion* of the multivector M.

M^* – the *dual* of the multivector M.

bivector – the outer product of two vectors, also called a 2-blade. We may think of a bivector $\mathbf{B} = \mathbf{u} \wedge \mathbf{v}$ as an oriented area element with attitude, orientation, and norm. A bivector is zero if either vector is zero or if the two vectors are parallel. A bivector is not the same as a 2-vector. A *2-vector* is a linear combination of bivectors (2-blades). Bivectors are blades in $\mathbb{G}^3$.

blade – A multivector M_k is called a **k-*blade*** or a ***simple* k-*vector*** if and only if it can be factored into a product of k anticommuting vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$, that is

$$M_k = \mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_k,$$

$$\text{where} \quad \mathbf{u}_i \mathbf{u}_j = -\mathbf{u}_j \mathbf{u}_i \quad \text{for } i, j = 1, 2, \ldots, k \text{ and } i \neq j.$$

A k-blade is an outer product of k vectors that span a k dimensional subspace of $\mathbb{R}^n$. Blades are a generalization of the concept of scalars and vectors to include simple bivectors, trivectors, and so on. A *0-blade* is a scalar.

dual – for a given geometric algebra $\mathbb{G}^n$ with a unit pseudoscalar $\mathbf{I}$, the *dual* of a multivector M, denoted by M^*, is defined as $M^* = M\mathbf{I}^{-1}$. If $\mathbf{B}_k$ is a blade of grade k, then the dual of $\mathbf{B}_k$ represents the *orthogonal complement* in $\mathbb{R}^n$ of the subspace represented by $\mathbf{B}_k$. For example, if $\mathbf{B} \in \mathbb{G}^3$ and $\mathbf{B} = \mathbf{e}_1 \wedge \mathbf{e}_2$, then $\mathbf{B}^* = \mathbf{e}_3$. The dual, $\mathbf{B}_k^*$, has a grade of $n - k$ where n is the grade of the pseudoscalar for the space. If $\mathbf{B}_k$ represents a subspace U, then $\mathbf{B}_k^*$ represents $U^\perp$.

grade – the number of vector factors in a nonzero k-blade. We denote it by the $grade()$ symbol, e.g., $grade(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k) = k$.

invertible – a vector is considered to be *invertible* if it is not the zero vector $\mathbf{0}$

or a null vector.

k-blade – see *blade*.

k-vector – any linear combination of *k-blades*. For example, $\mathbf{B}_1 = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3$ is a 2-vector in $\mathbb{G}^3$. In $\mathbb{G}^3$, 2-vectors are always 2-blades. This is not true in higher dimensions. In $\mathbb{G}^4$, the 2-blades $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_3\mathbf{e}_4$ represent subspaces, but the 2-vector $\mathbf{B}_2 = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4$ is not a blade and represents no subspace.

multivector – an element of the vector space $\mathbb{G}^n$. A multivector M of grade n is the sum of $n+1$ k-vectors, i.e., $M =$ 0-vector + 1-vector + ... + n-vector.

null vector – a vector whose inner product with itself is zero. Null vectors show up when we work with spaces of mixed signature (both positive and negative), as with the conformal model. Null vectors are not necessarily the zero vector. However, the zero vector is a null vector. Null vectors are not invertible.

pseudoscalar – an n-vector in $\mathbb{G}^n$. A *unit pseudoscalar* is denoted by $\mathbf{I}$. In Euclidean 3-space the unit pseudoscalar $\mathbf{I} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. In Euclidean 2-space the unit pseudoscalar $\mathbf{I}_{(2)} = \mathbf{e}_1 \wedge \mathbf{e}_2$. The name pseudoscalar comes from the study of dual spaces where the pseudoscalar plays the role that the scalar does in normal space.

reversion – an unary operation on multivectors obtained by reversing each of its components. The reversion of a k-blade $\mathbf{B}$, denoted by $\widetilde{\mathbf{B}}$, is just all the vectors of $\mathbf{B}$ in reverse order, viz.,

$$\begin{aligned} \text{If} \quad & \mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k, \\ \text{then} \quad & \widetilde{\mathbf{B}} = \mathbf{b}_k \wedge \mathbf{b}_{k-1} \wedge \cdots \wedge \mathbf{b}_1. \end{aligned}$$

For multivectors A and B, reversion has the following properties:

$$(A+B)^{\sim} = \widetilde{A} + \widetilde{B}, \qquad (AB)^{\sim} = \widetilde{B}\widetilde{A},$$

$$\langle \widetilde{A} \rangle_0 = \langle A \rangle_0, \qquad \langle \widetilde{A} \rangle_1 = \langle A \rangle_1.$$

Note that $\widetilde{\mathbf{I}} = \mathbf{I}^{-1}$. The notation of the tilde is chosen to be reminiscent of an editor's notation for an interchange of terms. Some literature denotes reversion with a dagger superscript, e.g., $M^{\dagger}$.

rotor – a geometric product of an even number of invertible unit vectors. Rotors have the useful property that $R^{-1} = \widetilde{R}$ (where $\widetilde{R}$ is the reversion of R, see *reversion*). Examples of rotors are $\mathbf{uv}, \mathbf{vw}, \mathbf{uw}$, where $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$. Rotors can always be expressed as exponentials, e.g., $\mathbf{uv} = e^{\mathbf{I}_{(2)}\theta}$, where $\mathbf{I}_{(2)}$ is the unit pseudoscalar in the $\mathbf{uv}$-plane and θ is the angle from $\mathbf{u}$ to $\mathbf{v}$. Rotors are even unit versors.

signature – if an n-dimensional vector space has p independent unit vectors satisfying $\mathbf{e}_i \cdot \mathbf{e}_i = 1$, and q independent unit vectors satisfying $\mathbf{e}_i \cdot \mathbf{e}_i = -1$, then it is customary to to say that the space has p "positive dimensions" and q "negative dimensions" (with $n = p + q$). Rather than $\mathbb{R}^n$, we then write $\mathbb{R}^{p,q}$ and call $(p,\, q)$ the *signature* of the space. An n-dimensional Euclidean space is then written as $\mathbb{R}^{n,0}$, the representational space for the conformal model as $\mathbb{R}^{n+1,1}$, and an n-dimensional Minkowski space as $\mathbb{R}^{n-1,1}$ [DFM07, p. 586].

trivector – the outer product of three vectors, also called a 3-blade. We may think of a trivector $\mathbf{T} = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ as an oriented volume element with attitude, orientation, and norm. A trivector is zero if any of the vectors are zero or if all three vectors are coplanar. A trivector is not the same as a

3-vector. A *3-vector* is a linear combination of trivectors (3-blades).

versor – a geometric product of invertible vectors. Examples of versors are $\mathbf{u}, \mathbf{uv}, \mathbf{uvw}$, etc. A versor is called *even* or *odd* depending upon whether it has an even or odd number of vector terms.