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Linearized equations for J2 perturbed motion relative to an elliptical orbit

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LINEARIZED EQUATIONS FOR J2 PERTURBED MOTION RELATIVE TO AN ELLIPTICAL ORBIT

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ABSTRACT

LINEARIZED EQUATIONS FOR J2 PERTURBED MOTION RELATIVE TO AN ELLIPTICAL ORBIT

by Dennis C. Pak

A set of linearized equations was derived for the motion, relative to an elliptical reference orbit, of an object influenced by J2 perturbations. Solutions from numerical simulations were used to compare these equations and the linearized Keplerian equations to the exact equations. The inclusion of the J2 perturbations in the derived linear equations increased the accuracy of the solution significantly in the out-of-orbit-plane direction, while the accuracy within the orbit plane remained roughly unchanged. These equations could be useful in the analysis of orbital rendezvous or formation flying problems.

TABLE OF CONTENTS

TABLE OF FIGURES

Page

NOMENCLATURE

Symbol	Description
a	Semi-major diameter of reference orbit
e	Eccentricity of reference orbit
f	True anomaly of reference orbit
h	Specific angular momentum of reference orbit
i	Inclination angle of reference orbit
$\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$	Unit vectors in the directions of the orbit frame coordinate axes
$\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, $\hat{\mathbf{K}}$	Unit vectors in the directions of the ECI frame coordinate axes
J_2	Coefficient representing the effects of the Earth's oblateness in the Legendre polynomial representation of the Earth's gravitational field
μ	Gravitational coefficient
M	Mean anomaly of reference orbit
n	Mean angular rate of reference orbit
ω	Argument of periapsis of reference orbit
Ω	Longitude of the ascending node of reference orbit
$\mathbf{p}$	J_2 perturbation acceleration vector
p_x	x-component of $\mathbf{p}$ (orbit frame)
p_y	y-component of $\mathbf{p}$ (orbit frame)
p_z	z-component of $\mathbf{p}$ (orbit frame)
p_X	X-component of $\mathbf{p}$ (ECI frame)
p_Y	Y-component of $\mathbf{p}$ (ECI frame)
p_Z	Z-component of $\mathbf{p}$ (ECI frame)
$\mathbf{r}$	Position vector of spacecraft relative to orbit frame
r	Magnitude of $\mathbf{r}$
$r_x = x$	x-component of $\mathbf{r}$ (orbit frame)
$r_y = y$	y-component of $\mathbf{r}$ (orbit frame)
$r_z = z$	z-component of $\mathbf{r}$ (orbit frame)

Symbol	Description
$\dot{r}_x = \dot{x} = v_x = {}^{O}\frac{d}{dt}x$	Rate of change in time of x with respect to the orbit frame
$\dot{r}_y = \dot{y} = v_y = {}^{O}\frac{d}{dt}y$	Rate of change in time of y with respect to the orbit frame
$\dot{r}_z = \dot{z} = v_z = {}^{O}\frac{d}{dt}z$	Rate of change in time of z with respect to the orbit frame
r_X	X-component of $\mathbf{r}$ (ECI frame)
r_Y	Y-component of $\mathbf{r}$ (ECI frame)
r_Z	Z-component of $\mathbf{r}$ (ECI frame)
$\mathbf{r}_1$	Position vector of spacecraft #1 relative to orbit frame
$\mathbf{r}_2$	Position vector of spacecraft #2 relative to orbit frame
$\mathbf{R}$	Position vector of spacecraft relative to ECI frame
R	Magnitude of $\mathbf{R}$
$R_X = X$	X-component of $\mathbf{R}$ (ECI frame)
$R_Y = Y$	Y-component of $\mathbf{R}$ (ECI frame)
$R_Z = Z$	Z-component of $\mathbf{R}$ (ECI frame)
$\dot{R}_X = \dot{X} = V_X = {}^{N}\frac{d}{dt}X$	Rate of change in time of X with respect to the ECI frame
$\dot{R}_Y = \dot{Y} = V_Y = {}^{N}\frac{d}{dt}Y$	Rate of change in time of Y with respect to the ECI frame
$\dot{R}_Z = \dot{Z} = V_Z = {}^{N}\frac{d}{dt}Z$	Rate of change in time of Z with respect to the ECI frame
$\mathbf{R}_O$	Position vector of orbit frame origin relative to the ECI frame
R_O	Magnitude of $\mathbf{R}_O$
$R_{OX} = X_O$	X-component of $\mathbf{R}_O$ (ECI frame)
$R_{OY} = Y_O$	Y-component of $\mathbf{R}_O$ (ECI frame)
$R_{OZ} = Z_O$	Z-component of $\mathbf{R}_O$ (ECI frame)
$\dot{R}_{OX} = \dot{X}_O = V_{OX} = {}^{N}\frac{d}{dt}X_O$	Rate of change in time of X_O with respect to the ECI frame

Symbol	Description
$\dot{R}_{OY} = \dot{Y}_O = V_{OY} = {}^{N}\frac{d}{dt} Y_O$	Rate of change in time of Y_O with respect to the ECI frame
$\dot{R}_{OZ} = \dot{Z}_O = V_{OZ} = {}^{N}\frac{d}{dt} Z_O$	Rate of change in time of Z_O with respect to the ECI frame
R_P	Component of $\mathbf{R}$ in the perifocal system $\hat{\mathbf{P}}$-direction
R_Q	Component of $\mathbf{R}$ in the perifocal system $\hat{\mathbf{Q}}$-direction
R_W	Component of $\mathbf{R}$ in the perifocal system $\hat{\mathbf{W}}$-direction
$R_\oplus$	Mean radius of the Earth
t	Current time
t_p	Time of periapsis passage of reference orbit
θ	Argument of latitude at epoch of reference orbit
$\mathbf{V}$	Rate of change in time of $\mathbf{R}$ with respect to the ECI frame
$\mathbf{V}_O$	Rate of change in time of $\mathbf{R}_O$ with respect to the ECI frame
$V_P = {}^{N}\frac{d}{dt} R_P$	Rate of change in time of R_P with respect to the perifocal frame
$V_Q = {}^{N}\frac{d}{dt} R_Q$	Rate of change in time of R_Q with respect to the perifocal frame
$V_W = {}^{N}\frac{d}{dt} R_W$	Rate of change in time of R_W with respect to the perifocal frame
$(\ )_0$	At initial time
$(\ )_O$	Of the reference orbit
$(\ )_x$	In the x-direction (orbit frame)
$(\ )_y$	In the y-direction (orbit frame)
$(\ )_z$	In the z-direction (orbit frame)
$(\ )_X$	In the X-direction (ECI frame)
$(\ )_Y$	In the Y-direction (ECI frame)
$(\ )_Z$	In the Z-direction (ECI frame)

Symbol	Description
$^{N}(\ )$	With respect to a Newtonian (e.g., ECI) frame
$^{O}(\ )$	With respect to the orbit frame

1. INTRODUCTION

The study of the dynamics of relative motion between spacecraft customarily begins with the Clohessy-Wiltshire equations. They are a set of linearized equations for the motion, relative to a circular reference orbit, of an object in an inverse square gravity field. Melton[1] provides a method for generalizing the linear equations of motion to an elliptical orbit which enables the determination of a closed-form, time-explicit, approximate solution. Ross[2] gives a set of equations based on the C-W equations which incorporates the J_2 gravitational perturbations. He states in his introduction: "In principle, these equations can be generalized for elliptical reference orbits as described by Melton".

The objective of this thesis is to develop a set of linearized equations for the motion, relative to an elliptical reference orbit, of an object influenced by J_2 perturbations. Solutions from numerical simulations will be used to compare these equations and the equations given by Melton to the exact equations of motion. It will be determined whether the inclusion of the J_2 perturbations provides a significant increase in accuracy over Melton's equations.

1.1. Nonlinearity of the Exact Equations of Motion

The exact equations describing the motion of an object in an inverse square gravitational field follow directly from Newton's second law and his law of gravitation, and can be written in vector form as

$$\ddot{\mathbf{R}} = \frac{\mu}{R^3}\mathbf{R} \tag{1}$$

In this equation, $\mathbf{R}$ is the position vector of the object relative to the inertially fixed center of the gravitational field and μ is the gravitational coefficient. Eq. (1) can be written in component form as

$$\ddot{X} = \mu\frac{X}{\left(X^2+Y^2+Z^2\right)^{3/2}},\ \ddot{Y} = \mu\frac{Y}{\left(X^2+Y^2+Z^2\right)^{3/2}},\ \ddot{Z} = \mu\frac{Z}{\left(X^2+Y^2+Z^2\right)^{3/2}}\ .$$

The relative motion between two objects in orbit is determined simply by

$$\Delta\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1 \tag{2}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ for each of the objects is described by Eq. (1). Although the above equations may appear straightforward, they are nonlinear and can thus be difficult to analyze.

Linearized equations provide approximations to nonlinear equations. Their reduced accuracy is accepted as a tradeoff for their relative mathematical simplicity. As an alternative to Eq. (1), the motion of an object in orbit can be described relative to a reference orbit, and the equations for this relative motion can be linearized. The linearization is made possible by the assumption that the distance of the object from the reference orbit is very small compared to the size of the reference orbit itself.

1.2. Oblateness of the Earth and J_2 Perturbations

The gravitational field about a perfectly uniform, perfectly spherical mass has an inverse square relation. The earth, however, is neither perfectly uniform nor perfectly spherical, and the inverse square relation describes closely, but not exactly, the earth's actual gravitational field. The aspect of the earth's geometry that causes its gravitational field to depart most significantly from the inverse square relation is its oblateness. The earth, compared to a perfect sphere, is flatter at the poles and wider at the equator. The difference in force caused by the earth's oblateness, however, is small compared to the inverse square force itself, and can thus be treated as a perturbation. This perturbation force is referred to as the J_2 perturbation acceleration, and is given by Eq. (8.28) of Kaplan[3] as

$$\mathbf{p} = \frac{3J_2\mu R_\oplus^2}{2R^5}\left[\left(5\frac{Z^2}{R^2}-1\right)\left(X\hat{\mathbf{I}}+Y\hat{\mathbf{J}}\right)+Z\left(5\frac{Z^2}{R^2}-3\right)\hat{\mathbf{K}}\right]. \tag{3}$$

This expression gives the J_2 perturbation acceleration in the Earth Centered Inertial (ECI) coordinate directions in terms of the ECI coordinates.

Although this perturbation force is small, its effect can cause the motion of an orbiting object to depart significantly from the pure inverse square motion, or Keplerian motion. Incorporating this perturbation force into the equations provides a more accurate description of the actual motion of an orbiting object.

1.3. Derivation of Linearized Equations for Keplerian Motion

The derivation of the linearized equations of motion for a spacecraft subject to a pure inverse square gravitational field will be presented in this section. Although this derivation and its results are well known, they are the basis on which the work in this thesis is built. The derivation follows, for the most part, that which is given in Schaub[4].

Consider a rotating reference frame whose origin follows an elliptical reference orbit. One axis of this frame is aligned with the direction of the outward radial, for which there is a corresponding unit vector $\hat{\mathbf{i}}$. Another axis is aligned with the direction perpendicular to the orbit plane, for which there is a corresponding unit vector $\hat{\mathbf{k}}$. The remaining axis is normal to the other two axes and has a corresponding unit vector $\hat{\mathbf{j}}$. The position vector, relative to the center of the gravitational field, of an object in orbit can be written

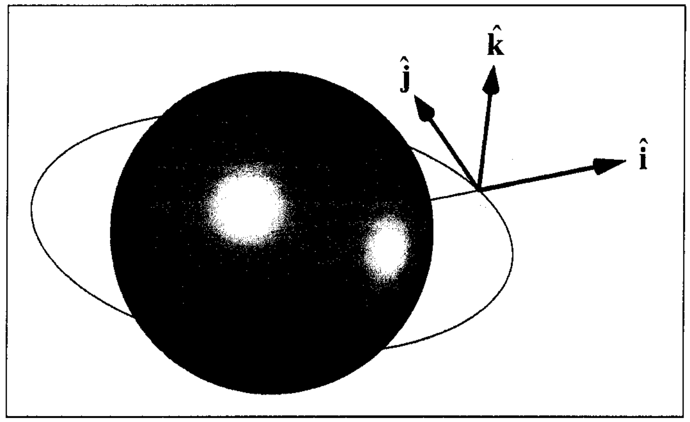

Figure 1. Orbit frame coordinate directions.

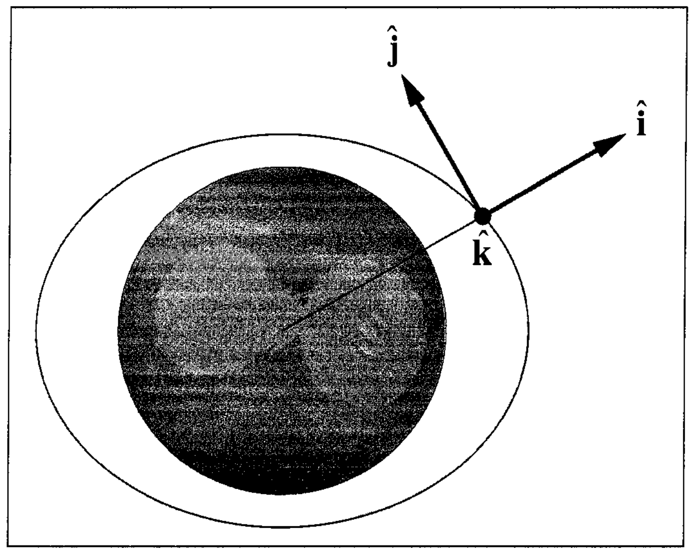

Figure 2. Orbit frame coordinate directions, view perpendicular to orbit plane.

$$\mathbf{R} = (R_O + x)\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \ . \tag{4}$$

Differentiating twice with respect to time in the inertial frame gives

$$\begin{aligned}\ddot{\mathbf{R}} = &\left[\ddot{R}_O + \ddot{x} - 2\dot{f}\dot{y} - \ddot{f}y - \dot{f}^2(R_O + x)\right]\hat{\mathbf{i}} \\ &+\left[\ddot{y} + \ddot{f}(R_O + x) + 2\dot{f}(\dot{R}_O + \dot{x}) - \dot{f}^2 y\right]\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}\end{aligned} \ , \tag{5}$$

which can be rearranged to give

$$\ddot{\mathbf{R}} = \left[\ddot{x} - 2\dot{f}\dot{y} - \ddot{f}y - \dot{f}^2 x + \left(\ddot{R}_O - \dot{f}^2 R_O\right)\right]\hat{\mathbf{i}} \\ + \left[\ddot{y} + 2\dot{f}\dot{x} + \ddot{f}x - \dot{f}^2 y + \left(R_O \ddot{f} + 2\dot{R}_O \dot{f}\right)\right]\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \quad . \tag{6}$$

Expressions will now be developed to replace the expressions in parentheses in Eq. (6). The position vector for the reference orbit, expressed in the orbit frame, is simply

$$\mathbf{R}_O = R_O \hat{\mathbf{i}} \ . \tag{7}$$

Differentiating twice with respect to time in the inertial frame gives

$$\ddot{\mathbf{R}}_O = \left(\ddot{R}_O - \dot{f}^2 R_O\right)\hat{\mathbf{i}} + \left(\dot{f}^2 R_O + 2\dot{f}\dot{R}_O\right)\hat{\mathbf{j}} \ . \tag{8}$$

In terms of the gravitational force,

$$\ddot{\mathbf{R}}_O = -\frac{\mu}{R_O^2}\hat{\mathbf{i}} \ . \tag{9}$$

Equating coefficients for $\hat{\mathbf{i}}$ in Eqs. (8) and (9) gives

$$\ddot{R}_O - \dot{f}^2 R_O = -\frac{\mu}{R_O^2} \ . \tag{10}$$

Angular momentum expressed in the orbit frame is given by

$$\mathbf{h}_O = \mathbf{R}_O \times \dot{\mathbf{R}}_O = R_O \hat{\mathbf{i}} \times \left(\dot{R}_O \hat{\mathbf{i}} + R_O \dot{f}\ \hat{\mathbf{j}}\right) = R_O^2 \dot{f}\ \hat{\mathbf{k}} \tag{11}$$

$$\Rightarrow h_O = R_O^2 \dot{f} \ . \tag{12}$$

Since angular momentum is constant,

$$0 = \frac{d}{dt} h_O = \frac{d}{dt}\left(R_O^2 \dot{f}\right) = R_O^2 \ddot{f} + 2R_O \dot{R}_O \dot{f} \tag{13}$$

$$\Rightarrow R_O \ddot{f} + 2\dot{R}_O \dot{f} = 0 \ . \tag{14}$$

Substituting Eqs. (10) and (14) into Eq. (6) gives

$$\ddot{\mathbf{R}} = \left[\ddot{x} - 2\dot{f}\dot{y} - \ddot{f}y - \dot{f}^2 x - \frac{\mu}{R_O^2}\right]\hat{\mathbf{i}} + \left[\ddot{y} + 2\dot{f}\dot{x} + \ddot{f}x - \dot{f}^2 y\right]\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \ . \tag{15}$$

The expression for $\ddot{\mathbf{R}}$ in terms of the gravitational force is

$$\ddot{\mathbf{R}} = -\frac{\mu}{R^3}\left[(R_O + x)\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right] . \tag{16}$$

Equating coefficients in Eqs. (15) and (16) produces the three second-order differential equations

$$\ddot{x} - 2\dot{f}\dot{y} - \ddot{f}y - \dot{f}^2 x - \frac{\mu}{R_O^2} = -\frac{\mu}{R^3}(R_O + x) \tag{17a}$$

$$\ddot{y} + 2\dot{f}\dot{x} + \ddot{f}x - \dot{f}^2 y = -\frac{\mu}{R^3} y \tag{17b}$$

$$\ddot{z} = -\frac{\mu}{R^3} z \ . \tag{17c}$$

Next, we will develop approximate linearized expressions for the terms on the right side of Eqs. (17). The position vector given by Eq. (4) has the magnitude

$$R = \left[(R_O + x)^2 + y^2 + z^2\right]^{1/2} , \tag{18}$$

which can be expanded and rearranged to give

$$R = R_O\left[1 + \frac{2x}{R_O} + \frac{x^2 + y^2 + z^2}{R_O^2}\right]^{1/2} . \tag{19}$$

If we assume that the distance of the object from the reference orbit is small compared to the reference orbit radial distance, $(x^2 + y^2 + z^2) << R_O^2$, then

$$R \cong R_O \left(1 + \frac{2x}{R_O}\right)^{1/2} , \tag{20}$$

which allows us to write

$$\frac{\mu}{R^3} \cong \frac{\mu}{R_O^3}\left(1 + \frac{2x}{R_O}\right)^{-3/2} . \tag{21}$$

Applying the binomial series expansion to the right side of this equation gives

$$\frac{\mu}{R^3} = \frac{\mu}{R_O^3}\left[1 - 3\frac{x}{R_O} + \frac{15}{2}\left(\frac{x}{R_O}\right)^2 + \cdots\right] . \tag{22}$$

Assuming that $x^2 << R_O^2$, we can approximate by ignoring second-order and higher terms, so that

$$\frac{\mu}{R^3} \cong \frac{\mu}{R_O^3}\left(1 - 3\frac{x}{R_O}\right) . \tag{23}$$

This allows us to write

$$-\frac{\mu}{R^3}(R_O + x) \cong -\frac{\mu}{R_O^3}\left[1 - 3\frac{x}{R_O}\right](R_O + x) = -\frac{\mu}{R_O^3}\left(R_O + x - 3x - 3\frac{x^2}{R_O}\right) \tag{24a}$$

$$-\frac{\mu}{R^3}y \cong -\frac{\mu}{R_O^3}\left[1 - 3\frac{x}{R_O}\right]y = -\frac{\mu}{R_O^3}\left(y - 3\frac{xy}{R_O}\right) \tag{24b}$$

$$-\frac{\mu}{R^3}z \cong -\frac{\mu}{R_O^3}\left[1 - 3\frac{x}{R_O}\right]z = -\frac{\mu}{R_O^3}\left(z - 3\frac{xz}{R_O}\right) . \tag{24c}$$

In Eqs. (24), if we assume that $x^2, xy, xz << R_O$, the nonlinear terms can be eliminated, giving

$$-\frac{\mu}{R^3}\left(R_O + x\right) \cong -\frac{\mu}{R_O^3}\left(R_O - 2x\right) \qquad (25a)$$

$$-\frac{\mu}{R^3}y \cong -\frac{\mu}{R_O^3}y \qquad (25b)$$

$$-\frac{\mu}{R^3}z \cong -\frac{\mu}{R_O^3}z \quad . \qquad (25c)$$

Substituting Eqs. (25) into Eqs. (17) and simplifying results in

$$\ddot{x} - \frac{2\mu}{R_O^3} - 2\dot{f}\dot{y} - \ddot{f}y - \dot{f}^2 x = 0 \qquad (26a)$$

$$\ddot{y} + \frac{\mu}{R_O^3}y + 2\dot{f}\dot{x} + \ddot{f}x - \dot{f}^2 y = 0 \qquad (26b)$$

$$\ddot{z} + \frac{\mu}{R_O^3}z = 0 \quad . \qquad (26c)$$

Eqs. (26) give the linearized equations of motion with respect to an elliptical reference orbit. These equations contain time-varying, periodic coefficients $\dot{f}, \ddot{f}$, and R_O. For a circular orbit, $\ddot{f} = 0$ and $\dot{f}^2 = \mu/R_O^3 = n^2$, reducing the equations to

$$\ddot{x} - 2n\dot{y} - 3n^2 x = 0 \qquad (27a)$$

$$\ddot{y} + 2n\dot{x} = 0 \qquad (27b)$$

$$\ddot{z} + n^2 z = 0 \quad . \qquad (27c)$$

Eqs. (27) are the Clohessy-Wiltshire equations.

1.4. J_2 Perturbed Motion Relative to a Circular Reference Orbit

Ross derived a set of linearized equations of motion for a spacecraft in a nearly circular orbit with the effects of J_2 perturbations included. The equations were derived by adding J_2 effects as perturbation terms to the Clohessy-Wiltshire equations. The primary work involved deriving, for the x, y and z directions, linearized expressions for the J_2 perturbation acceleration, in terms of x, y and z. Ross outlines the derivation of this J_2 perturbation acceleration and gives the result

$$p_x \cong \frac{3J_2 n^2 R_\oplus^2}{2R_O}\left[\left(1-\frac{7x}{R_O}\right)\left(1-3\sin^2 nt\sin^2 i\right)+\frac{x}{R_O}\left(9\cos^2 nt+9\cos^2 i\sin^2 nt-6\right)\right.$$
$$\left.+\frac{y}{R_O}\left(8\sin nt\cos nt\cos^2 i-8\sin nt\cos nt\right)-\frac{z}{R_O}\left(8\sin nt\sin i\cos i\right)\right] \tag{28}$$

$$p_y \cong \frac{3J_2 n^2 R_\oplus^2}{2R_O}\left[\left(1-\frac{7x}{R_O}\right)\sin(2nt)\sin^2 i+\frac{x}{R_O}\left(6\sin nt\cos nt\sin^2 i\right)\right.$$
$$\left.+\frac{y}{R_O}\left(7\cos^2 nt+5\cos^2 i-4-7\cos^2 nt\cos^2 i\right)-\frac{z}{R_O}\left(2\cos nt\sin i\cos i\right)\right] \tag{29}$$

$$p_z \cong \frac{3J_2 n^2 R_\oplus^2}{2R_O}\left[\left(1-\frac{7x}{R_O}\right)\sin nt\sin 2i+\frac{x}{R_O}\left(6\cos i\sin i\sin nt\right)\right.$$
$$\left.+\frac{y}{R_O}\left(2\cos nt\sin i\cos i\right)-\frac{z}{R_O}\left(7\cos^2 i+5\cos^2 nt-4-5\cos^2 nt\cos^2 i\right)\right]. \tag{30}$$

In the above equations, i is the inclination of the circular reference orbit and t is the time since passage of the ascending node.

Including these terms in the Clohessy-Wiltshire equations and rearranging produces the equations of motion in the final form

$$\ddot{\mathbf{r}} + \mathbf{C}\dot{\mathbf{r}} + [\mathbf{K} + \mathbf{P}]\mathbf{r} + \mathbf{q} = \mathbf{0} \tag{31}$$

where

$$\mathbf{C} = 2n \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{32}$$

$$\mathbf{K} = n^2 \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

$$\mathbf{P}(t) = n^2 J_R \begin{bmatrix} 12\sin^2 i \sin^2 nt - 4 & -4\sin^2 i \sin 2nt & -4\sin 2i \sin nt \\ -4\sin^2 i \sin 2nt & 1 + \sin^2 i\left(2 - 7\sin^2 nt\right) & \sin 2i \cos nt \\ -4\sin 2i \sin nt & \sin 2i \cos nt & 3 - \sin^2 i\left(2 + 5\sin^2 nt\right) \end{bmatrix} \tag{34}$$

$$\mathbf{q}(t) = n^2 J_R R_O \begin{bmatrix} 1 - 3\sin^2 nt \sin^2 i \\ \sin 2nt \sin^2 i \\ \sin nt \sin 2i \end{bmatrix} \tag{35}$$

$$J_R = \frac{3J_2 R_\oplus^2}{2R_O^2} \ . \tag{36}$$

1.5. Keplerian Motion Relative to an Elliptical Reference Orbit

Melton provides a time-explicit solution for the relative motion between elliptical orbits. The equations presented by Melton which were the basis for the development of his solution are equivalent to Eqs. (26).

$$\ddot{x} = \frac{2\mu}{R_O^3(t)}x + 2\dot{f}(t)\dot{y} + \ddot{f}(t)y + \dot{f}^2(t)x + p_x(t) \tag{37a}$$

$$\ddot{y} = -\frac{\mu}{R_O^{\,3}(t)}y - 2\dot{f}(t)\dot{x} - \ddot{f}(t)x + \dot{f}^2(t)y + p_y(t) \tag{37b}$$

$$\ddot{z} = -\frac{\mu}{R_O^{\,3}(t)}z + p_z(t) \ . \tag{37c}$$

The portions of Melton's work that are pertinent to this thesis are the expressions he provides as approximations for the time-varying coefficients in Eqs. (37).

$$\dot{f} = \frac{h}{a^2}\left[1 + 2e\cos M + \frac{e^2}{2}(1 + 5\cos 2M) + \mathrm{O}\left(e^3\right)\right] \tag{38}$$

$$\ddot{f} = -\frac{2h}{a^2}\left[en\sin M + e^2\left(n\sin 2M + 3n\cos M\sin M\right) + \mathrm{O}\left(e^3\right)\right] \tag{39}$$

$$\dot{f}^2 = \frac{h^2}{a^4}\left[1 + 4e\cos M + \frac{e^2}{2}(3 + 7\cos 2M) + \mathrm{O}\left(e^3\right)\right] \tag{40}$$

where M is the mean anomaly and is given by

$$M = n\left(t - t_p\right) \ . \tag{41}$$

Eqs. (38-40) were generated using Lagrange's generalized expansion theorem. They are truncated series expansions that are functions of e and M, and are approximate to the order of e^2.

Melton provides the equation

$$\frac{R_O}{a} = 1 - e\cos M + \frac{e^2}{2}\left(1 - \cos 2M\right) + \mathrm{O}\left(e^3\right) \ , \tag{42}$$

presumably for the purpose of providing an approximate expression for the quantity $1/R_O^3(t)$:

$$\frac{1}{R_O^3} \cong \frac{1}{a^3\left[1 - e\cos M + \frac{e^2}{2}\left(1 - \cos 2M\right)\right]^3} . \tag{43}$$

This expression would be suitable for use in a computational algorithm. However, it would not be suitable for the purposes of creating a combined algebraic expression, since it would produce a lengthy expression in the denominator. A more appropriate expression to use would be

$$\frac{1}{R_O^3} \cong \frac{1}{a^3}\left(1 + \frac{3}{2}e^2 + 3e\cos M + \frac{9}{2}e^2\cos 2M\right) . \tag{44}$$

This expression was derived by expanding the Fourier-Bessel solution for a/R_O (see Section 2.2 below).

2. DERIVATION OF EQUATIONS FOR J_2 PERTURBED MOTION RELATIVE TO AN ELLIPTICAL ORBIT

The primary work of this thesis, the derivation of the linearized equations of motion for a J_2 perturbed object relative to an elliptical reference orbit, will be presented in this section. This derivation follows the same general procedure as outlined in Ross's paper for the case of a circular reference orbit. Whereas the basis equations used by Ross were the Clohessy-Wiltshire equations, the basis equations for this derivation are Eqs (37-40) and Eq. (44).

The primary task in this derivation is to find, in terms of the orbit frame coordinate variables x, y and z, linearized expressions for the J_2 perturbation acceleration in the orbit frame coordinate directions. These expressions will be linear in x, y and z, and will have constant or time-varying periodic coefficients.

2.1. Linearized J_2 Perturbation Acceleration

The expression for the J_2 perturbation acceleration in the ECI directions in terms of the ECI coordinates that was used in Ross's paper is

$$\mathbf{p} = \frac{3J_2\mu R_\oplus^2}{2R^5}\left[\left(5\frac{Z^2}{R^2}-1\right)\left(X\hat{\mathbf{I}}+Y\hat{\mathbf{J}}\right)+Z\left(5\frac{Z^2}{R^2}-3\right)\hat{\mathbf{K}}\right] \tag{45}$$

where

$$R=\sqrt{\left(X^2+Y^2+Z^2\right)}\ .$$

This equation gives perturbation acceleration as a function of position in space, and is independent of the motion of the object under consideration. Eq. (45) can also be written in the form

$$\begin{bmatrix} p_X \\ p_Y \\ p_Z \end{bmatrix} = \frac{3J_2 \mu R_\oplus^2}{2} \frac{1}{R^7} \begin{bmatrix} 5Z^2 - \left(X^2 + Y^2 + Z^2\right)X \\ 5Z^2 - \left(X^2 + Y^2 + Z^2\right)Y \\ 5Z^2 - 3\left(X^2 + Y^2 + Z^2\right)Z \end{bmatrix}. \tag{46}$$

The first task is to linearize the quantity $1/R^7$ in Eq. (46). This can be accomplished as follows:

$$\begin{aligned} R &= \sqrt{\left(X^2 + Y^2 + Z^2\right)} \\ &= \sqrt{\left(x + R_O(t)\right)^2 + y^2 + z^2} \\ &= \sqrt{x^2 + 2xR_O(t) + \left(R_O(t)\right)^2 + y^2 + z^2} \\ &= R_O(t)\sqrt{\frac{x^2 + y^2 + z^2}{\left(R_O(t)\right)^2} + 1 + \frac{2x}{R_O(t)}} \end{aligned}$$

As in Section 1.3, the assumption that $\left(x^2 + y^2 + z^2\right) << R_O^2$ allows us to write

$$R \cong R_O(t)\sqrt{1 + \frac{2x}{R_O(t)}} \tag{47}$$

and

$$\frac{1}{R^7} \cong \frac{1}{\left(R_O(t)\right)^7}\left(1 + \frac{2x}{R_O(t)}\right)^{-7/2}. \tag{48}$$

This expression can be further simplified by using the binomial series approximation and neglecting terms containing $\left[x/R_O(t)\right]^n$, $n \geq 2$, to give

$$\frac{1}{R^7} \cong \frac{1}{R_O(t)^7}\left(1 - \frac{7x}{R_O(t)}\right). \tag{49}$$

This equation is identical to Eq. (11c) of Ross, with the exception that the radius of the reference orbit is time-varying.

The quantities X, Y and Z in Eq. (46) can be transformed by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = {}^{N}\mathbf{C}^{O} \begin{bmatrix} x + R_O(t) \\ y \\ z \end{bmatrix} \tag{50}$$

where

$${}^{N}\mathbf{C}^{O} = \left({}^{O}\mathbf{C}^{N}\right)^{T} \tag{51}$$

and

$${}^{O}\mathbf{C}^{N} = \begin{bmatrix} \cos\theta(t)\cos\Omega - \sin\theta(t)\cos i\sin\Omega & \cos\theta(t)\sin\Omega + \sin\theta(t)\cos i\cos\Omega & \sin\theta(t)\sin i \\ -\sin\theta(t)\cos\Omega - \cos\theta(t)\cos i\sin\Omega & -\sin\theta(t)\sin\Omega + \cos\theta(t)\cos i\cos\Omega & \cos\theta(t)\sin i \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{bmatrix}. \tag{52}$$

In this last equation, $\theta(t)$ is the argument of latitude at epoch,

$$\theta(t) = \omega + f(t) \ , \tag{53}$$

where ω is the argument of periapsis and $f(t)$ is the true anomaly.

Eq. (50) differs from Eq. (10) in Ross in that, once again, R_O is time-varying. Eq. (52) differs from Eq. (8) in Ross in that the quantity nt has been replaced by $\theta(t)$. Eq (52) gives a direction cosine matrix which maps vectors with components in the ECI frame into vectors with components in the orbit frame via a 3-1-3 Euler angle transformation. For the case of a circular orbit, since the angular velocity is constant, the final rotation about the body 3 axis can be given by the angular velocity multiplied by the time since passage of the ascending node. For an elliptical reference orbit, however, the angular rate is not constant and the corresponding rotation quantity cannot be stated so simply. Expressions for both $R_O(t)$ and $\theta(t)$ will be developed in Section 2.2 below.

When Eqs. (49-53) are substituted into Eq. (46) and algebraically expanded, each of the resulting terms has an n$^{\text{th}}$ order product of x, y, and z in the numerator and a quantity R_O^{4+n} in the denominator, where, for a particular term, $n = 0$, 1, 2, 3 or 4. Thus, the terms all have the form

$$C\frac{1}{R_O^4}\frac{x^b y^c z^d}{R_O^n}, \quad b+c+d=n=0,\ 1,\ 2,\ 3,\ 4$$

where *b*, *c* and *d* are zero or positive integers and *C* is some constant expression. Employing once again the assumption that the distance of the object from the reference orbit is small compared to the size of the reference orbit, we can write

$$\frac{x^b y^c z^d}{R_O^n} \cong 0\ , \quad b+c+d=n=2,\ 3,\ 4\ .$$

Thus, the perturbation acceleration expressions can be approximated by retaining only the terms for which $n = 0$ and 1. These approximate expressions are given below by Eqs. (54). Mathematica was used here and at subsequent steps in this thesis to manipulate and simplify lengthy algebraic expressions.

$p_1 =$

$$\begin{aligned}
&x\,\frac{3 J_2 R_e^2 \mu}{2 R_0^5}\left(-1+5\cos[2i]+10\cos[2\theta]\sin[i]^2\right)\left(\cos[\theta]\cos[\Omega]-\cos[i]\sin[\theta]\sin[\Omega]\right)\\
&+y\,\frac{3 J_2 R_e^2 \mu}{32 R_0^5}\left(2\cos[\Omega]\left((3+5\cos[2i])\sin[\theta]+30\sin[i]^2\sin[3\theta]\right)\right.\\
&\quad\left.+\left(15\cos[3i]\cos[\theta]+\cos[i]\left(\cos[\theta]+60\cos[3\theta]\sin[i]^2\right)\right)\sin[\Omega]\right)\\
&+z\,\frac{3 J_2 R_e^2 \mu}{16 R_0^5}\left(20\cos[\Omega]\sin[2i]\sin[2\theta]-\left((3+5\cos[2\theta])\sin[i]+30\sin[3i]\sin[\theta]^2\right)\sin[\Omega]\right)\\
&-\frac{3 J_2 R_e^2 \mu}{8 R_0^4}\left(-1+5\cos[2i]+10\cos[2\theta]\sin[i]^2\right)\left(\cos[\theta]\cos[\Omega]-\cos[i]\sin[\theta]\sin[\Omega]\right)
\end{aligned} \tag{54a}$$

$p_2 =$

$$\begin{aligned}
&x\,\frac{3 J_2 R_e^2 \mu}{2 R_0^5}\left(-1+5\cos[2i]+10\cos[2\theta]\sin[i]^2\right)\left(\cos[i]\cos[\Omega]\sin[\theta]+\cos[\theta]\sin[\Omega]\right)\\
&-y\,\frac{3 J_2 R_e^2 \mu}{8 R_0^5}\left(\cos[i]\cos[\theta]\cos[\Omega]\left(-11+15\cos[2i]+30\cos[2\theta]\sin[i]^2\right)\right.\\
&\quad\left.+\left(-9+5\cos[2i]-30\cos[2\theta]\sin[i]^2\right)\sin[\theta]\sin[\Omega]\right)\\
&+z\,\frac{3 J_2 R_e^2 \mu}{16 R_0^5}\sin[i]\left(2\cos[\Omega]\left(9-5\cos[2\theta]+30\cos[2i]\sin[\theta]^2\right)+80\cos[i]\cos[\theta]\sin[\theta]\sin[\Omega]\right)\\
&-\frac{3 J_2 R_e^2 \mu}{8 R_0^4}\left(-1+5\cos[2i]+10\cos[2\theta]\sin[i]^2\right)\left(\cos[i]\cos[\Omega]\sin[\theta]+\cos[\theta]\sin[\Omega]\right)
\end{aligned} \tag{54b}$$

$p_3 =$

$$\begin{aligned}
&x\,\frac{3 J_2 R_e^2 \mu}{8 R_0^5}\left(3\,(\sin[i]+5\sin[3i])\sin[\theta]+20\sin[i]^3\sin[3\theta]\right)\\
&-y\,\frac{9 J_2 R_e^2 \mu}{32 R_0^5}\left(15\cos[3\theta]\sin[i]+\cos[\theta]\left(\sin[i]+20\sin[3i]\sin[\theta]^2\right)\right)\\
&-z\,\frac{9 J_2 R_e^2 \mu}{16 R_0^5}\left(5\cos[3i]+\cos[i]\left(3+20\cos[2\theta]\sin[i]^2\right)\right)\\
&-\frac{3 J_2 R_e^2 \mu}{32 R_0^4}\left(3\,(\sin[i]+5\sin[3i])\sin[\theta]+20\sin[i]^3\sin[3\theta]\right)
\end{aligned} \tag{54c}$$

Eqs (54) are the linearized expressions for the perturbation acceleration in the ECI coordinate directions in terms of the orbit frame coordinates x, y, and z. These equations can be transformed to give the perturbation accelerations in the orbit frame coordinate directions with the equation

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = {}^{O}C^{N} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} . \tag{55}$$

Substituting Eqs. (52) and (54) into Eq. (55) and simplifying gives

$$\begin{aligned} p_x = J_2 \mu R_{\oplus}^2 \Bigg\{ & \left[\frac{3}{2(R_O(t))^5} \left(1 + 3\cos 2i + 6\cos 2\theta(t)\sin^2 i\right) \right] x + \left[\frac{6}{(R_O(t))^5} \left(\sin^2 i \sin 2\theta(t)\right) \right] y \\ & + \left[\frac{6}{(R_O(t))^5} \left(\sin 2i \sin \theta(t)\right) \right] z - \frac{3}{8(R_O(t))^4} \left(1 + 3\cos 2i + 6\cos 2\theta(t)\sin^2 i\right) \Bigg\} \end{aligned} \tag{56a}$$

$$\begin{aligned} p_y = J_2 \mu R_{\oplus}^2 \Bigg\{ & \left[\frac{6}{(R_O(t))^5} \left(\sin^2 i \sin 2\theta(t)\right) \right] x - \left[\frac{3}{8(R_O(t))^5} \left(1 + 3\cos 2i + 14\cos 2\theta(t)\sin^2 i\right) \right] y \\ & - \left[\frac{3}{(R_O(t))^5} \left(\cos i \cos \theta(t) \sin i\right) \right] z - \frac{3}{2(R_O(t))^4} \left(\sin 2\theta(t) \sin^2 i\right) \Bigg\} \end{aligned} \tag{56b}$$

$$\begin{aligned} p_z = J_2 \mu R_{\oplus}^2 \Bigg\{ & \left[\frac{6}{(R_O(t))^5} \left(\sin 2i \sin \theta(t)\right) \right] x - \left[\frac{3}{(R_O(t))^5} \left(\cos i \cos \theta(t) \sin i\right) \right] y \\ & - \left[\frac{3}{8(R_O(t))^5} \left(3 + 9\cos 2i + 10\cos 2\theta(t)\sin^2 i\right) \right] z - \frac{3}{(R_O(t))^4} \left(\cos i \sin i \sin \theta(t)\right) \Bigg\} . \end{aligned} \tag{56c}$$

It should be noted that Eqs. (56) do not contain Ω, the longitude of the ascending node. This makes sense, since a change in Ω corresponds to a rotation of the xyz orbit frame about the ECI Z-axis, and the J_2 perturbation accelerations are symmetrical about the ECI Z-axis. Eqs. (54), on the other hand, which give accelerations in the ECI component directions, *do* contain Ω. In the orbit frame, a change in Ω, with all other orbital elements remaining fixed, would not change the J_2 perturbation acceleration quantities. However, a change in Ω *would* change the orientation of the orbit frame with respect to the ECI frame. It follows that the perturbation acceleration quantities in the ECI frame change with Ω, and Eqs. (54) reflect this dependency.

Eqs. (56) are equivalent to Ross Eqs. (12) if n^2 is replaced by μ/a^3. The substitution of n^2 for μ/a^3 could be made in Ross since the reference orbit there is circular. Also, it should be noted that in Eqs. (56), R_O and θ are time-varying.

2.2. Expressions for Time-Varying Coefficients

Expressions will now be developed for the time-varying quantities containing θ and R_O. These expressions will be in the form of approximate truncated series which are functions of *e* and *M*. Solutions for time-varying quantities related to the motion of an elliptical orbit can be expressed as Fourier-Bessel series expansions. Taff[5] gives the following formulae:

$$\cos f = -e + 2\frac{1-e^2}{e}\sum_{n=1}^{\infty} J_n(ne)\cos(nM) \tag{57}$$

$$\sin f = \sqrt{1-e^2}\sum_{n=1}^{\infty}\frac{2}{n}J_n'(ne)\sin(nM) \tag{58}$$

$$\frac{a}{r} = 1 + 2\sum_{n=1}^{\infty} J_n(ne)\cos(nM) \ . \tag{59}$$

In the above equations, J_n are the Bessel functions of the first kind of order *n*, and $J_n'(ne) = \partial J_n(ne)/\partial e$. These Bessel functions can be expressed as series expansions, as given in Battin[6], by

$$J_n(ne) = \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{1}{2}ne\right)^{n+2j}}{j!(n+j)!} \quad . \tag{60}$$

Care should be taken not to confuse $J_2(ne)$ with the constant J_2 associated with the gravitational field. Substituting Eq. (60) into Eqs. (57-59) and carrying out the expansions for each gives

$$\cos f(t) = -e + \left(1 - \frac{9}{8}e^2\right)\cos(M) + e\cos(2M) + \frac{9}{8}e^2\cos(3M) + O(e^3) \tag{61}$$

$$\sin f(t) = \frac{b}{a}\left[\left(1 - \frac{3}{8}e^2\right)\sin M + e\sin 2M + \frac{9}{8}e^2\sin 3M\right] + O(e^3) \tag{62}$$

$$\frac{a}{r(t)} = 1 + e\cos M + e^2\cos 2M + O(e^3) \quad . \tag{63}$$

With these expressions, we can develop the expressions we require for the time-varying quantities in Eqs. (56).

The trigonometric angle-addition formulae allow us to write

$$\cos\theta(t) = \cos(\omega + f(t)) = \cos\omega\cos f(t) - \sin\omega\sin f(t) \tag{64}$$

$$\sin\theta(t) = \sin(\omega + f(t)) = \sin\omega\cos f(t) + \cos\omega\sin f(t) \quad . \tag{65}$$

Substituting Eqs. (61) and (62) into Eqs. (64) and (65) results in

$$\begin{aligned}\cos\theta \cong{}& -e\cos\omega + \left(1 - \frac{9}{8}e^2\right)\cos\omega\cos M + e\cos\omega\cos 2M + \frac{9}{8}e^2\cos\omega\cos 3M \\ &+ \frac{b}{a}\left(\frac{3}{8}e^2 - 1\right)\sin\omega\sin M - \frac{b}{a}e\sin\omega\sin 2M - \frac{9}{8}\frac{b}{a}e^2\sin\omega\sin 3M\end{aligned} \tag{66}$$

$$\sin\theta \cong -e\sin\omega + \left(1-\frac{9}{8}e^2\right)\sin\omega\cos M + e\sin\omega\cos 2M + \frac{9}{8}e^2\sin\omega\cos 3M$$
$$+\frac{b}{a}\left(1-\frac{3}{8}e^2\right)\cos\omega\sin M + \frac{b}{a}e\cos\omega\sin 2M + \frac{9}{8}\frac{b}{a}e^2\cos\omega\sin 3M \quad . \tag{67}$$

The trigonometric double-angle formulae allow us to write

$$\sin 2\theta(t) = 2\sin\theta(t)\cos\theta(t) \tag{68}$$

$$\cos 2\theta(t) = 1 - 2\sin^2\theta(t) \ . \tag{69}$$

Substituting Eqs. (66) and (67) into Eqs. (68) and (69) results in

$$\cos 2\theta \cong \frac{3}{4}e^2\cos 2\omega - 2e\cos 2\omega\cos M + \left(1-4e^2\right)\cos 2\omega\cos 2M + 2e\cos 2\omega\cos 3M$$
$$+\frac{13}{4}e^2\cos 2\omega\cos 4M + 2\frac{b}{a}e\sin 2\omega\sin M + \frac{b}{a}\left(\frac{7}{2}e^2 - 1\right)\sin 2\omega\sin 2M$$
$$-2\frac{b}{a}e\sin 2\omega\sin 3M - \frac{13}{4}\frac{b}{a}e^2\sin 2\omega\sin 4M \tag{70}$$

$$\sin 2\theta \cong \frac{3}{4}e^2\sin 2\omega - 2e\sin 2\omega\cos M + \left(1-4e^2\right)\sin 2\omega\cos 2M + 2e\sin 2\omega\cos 3M$$
$$+\frac{13}{4}e^2\sin 2\omega\cos 4M - 2\frac{b}{a}e\cos 2\omega\sin M + \frac{b}{a}\left(1-\frac{7}{2}e^2\right)\cos 2\omega\sin 2M \ .$$
$$+2\frac{b}{a}e\cos 2\omega\sin 3M + \frac{13}{4}\frac{b}{a}e^2\cos 2\omega\sin 4M \tag{71}$$

Eq. (63) allows us to write

$$\frac{1}{R_O^3} \cong \frac{1}{a^3}\left(1 + e\cos M + e^2\cos 2M\right)^3 \ . \tag{72}$$

Algebraically expanding Eq. (72) results in

$$\frac{1}{R_O^3} \cong \frac{1}{a^3}\left(1 + \frac{3}{2}e^2 + 3e\cos M + \frac{9}{2}e^2\cos 2M\right) \ . \tag{73}$$

Similarly, we can derive the expressions

$$\frac{1}{R_O^4} \cong \frac{1}{a^4}\left(1+3e^2+4e\cos M+7e^2\cos 2M\right) \tag{74}$$

$$\frac{1}{R_O^5} \cong \frac{1}{a^5}\left(1+5e^2+5e\cos M+10e^2\cos 2M\right) . \tag{75}$$

2.3. Summary of Equations of Motion

Eqs. (37-40), (56), (66), (67), (70), (71), and (73-75) collectively give the linearized equations of motion, and are summarized below.

$$\ddot{x} = \frac{2\mu}{R_O^{\,3}(t)}x + 2\dot{f}(t)\dot{y} + \ddot{f}(t)y + \dot{f}^2(t)x + p_x(t) \tag{37a}$$

$$\ddot{y} = -\frac{\mu}{R_O^{\,3}(t)}y - 2\dot{f}(t)\dot{x} - \ddot{f}(t)x + \dot{f}^2(t)y + p_y(t) \tag{37b}$$

$$\ddot{z} = -\frac{\mu}{R_O^{\,3}(t)}z + p_z(t) \ . \tag{37c}$$

$$\dot{f} = \frac{h}{a^2}\left[1+2e\cos M+\frac{e^2}{2}(1+5\cos 2M)+\mathrm{O}\left(e^3\right)\right] \tag{38}$$

$$\ddot{f} = -\frac{2h}{a^2}\left[en\sin M+e^2\left(n\sin 2M+3n\cos M\sin M\right)+\mathrm{O}\left(e^3\right)\right] \tag{39}$$

$$\dot{f}^2 = \frac{h^2}{a^4}\left[1+4e\cos M+\frac{e^2}{2}(3+7\cos 2M)+\mathrm{O}\left(e^3\right)\right] \tag{40}$$

$$\begin{aligned} p_x = J_2\mu R_\oplus^2 \Bigg\{ & \left[\frac{3}{2\left(R_O(t)\right)^5}\left(1+3\cos 2i+6\cos 2\theta(t)\sin^2 i\right)\right]x+\left[\frac{6}{\left(R_O(t)\right)^5}\left(\sin^2 i\sin 2\theta(t)\right)\right]y \\ & +\left[\frac{6}{\left(R_O(t)\right)^5}\left(\sin 2i\sin\theta(t)\right)\right]z-\frac{3}{8\left(R_O(t)\right)^4}\left(1+3\cos 2i+6\cos 2\theta(t)\sin^2 i\right)\Bigg\} \end{aligned} \tag{56a}$$

$$p_y = J_2 \mu R_\oplus^2 \left\{ \left[\frac{6}{\left(R_O(t)\right)^5} \left(\sin^2 i \sin 2\theta(t)\right) \right] x - \left[\frac{3}{8\left(R_O(t)\right)^5} \left(1 + 3\cos 2i + 14\cos 2\theta(t) \sin^2 i\right) \right] y - \left[\frac{3}{\left(R_O(t)\right)^5} \left(\cos i \cos \theta(t) \sin i\right) \right] z - \frac{3}{2\left(R_O(t)\right)^4} \left(\sin 2\theta(t) \sin^2 i\right) \right\} \tag{56b}$$

$$p_z = J_2 \mu R_\oplus^2 \left\{ \left[\frac{6}{\left(R_O(t)\right)^5} \left(\sin 2i \sin \theta(t)\right) \right] x - \left[\frac{3}{\left(R_O(t)\right)^5} \left(\cos i \cos \theta(t) \sin i\right) \right] y - \left[\frac{3}{8\left(R_O(t)\right)^5} \left(3 + 9\cos 2i + 10\cos 2\theta(t) \sin^2 i\right) \right] z - \frac{3}{\left(R_O(t)\right)^4} \left(\cos i \sin i \sin \theta(t)\right) \right\} \tag{56c}$$

$$\cos\theta \cong -e\cos\omega + \left(1 - \frac{9}{8}e^2\right)\cos\omega\cos M + e\cos\omega\cos 2M + \frac{9}{8}e^2\cos\omega\cos 3M + \frac{b}{a}\left(\frac{3}{8}e^2 - 1\right)\sin\omega\sin M - \frac{b}{a}e\sin\omega\sin 2M - \frac{9}{8}\frac{b}{a}e^2\sin\omega\sin 3M \tag{66}$$

$$\sin\theta \cong -e\sin\omega + \left(1 - \frac{9}{8}e^2\right)\sin\omega\cos M + e\sin\omega\cos 2M + \frac{9}{8}e^2\sin\omega\cos 3M + \frac{b}{a}\left(1 - \frac{3}{8}e^2\right)\cos\omega\sin M + \frac{b}{a}e\cos\omega\sin 2M + \frac{9}{8}\frac{b}{a}e^2\cos\omega\sin 3M \tag{67}$$

$$\cos 2\theta \cong \frac{3}{4}e^2\cos 2\omega - 2e\cos 2\omega\cos M + \left(1 - 4e^2\right)\cos 2\omega\cos 2M + 2e\cos 2\omega\cos 3M + \frac{13}{4}e^2\cos 2\omega\cos 4M + 2\frac{b}{a}e\sin 2\omega\sin M + \frac{b}{a}\left(\frac{7}{2}e^2 - 1\right)\sin 2\omega\sin 2M - 2\frac{b}{a}e\sin 2\omega\sin 3M - \frac{13}{4}\frac{b}{a}e^2\sin 2\omega\sin 4M \tag{70}$$

$$\sin 2\theta \cong \frac{3}{4}e^2\sin 2\omega - 2e\sin 2\omega\cos M + \left(1 - 4e^2\right)\sin 2\omega\cos 2M + 2e\sin 2\omega\cos 3M + \frac{13}{4}e^2\sin 2\omega\cos 4M - 2\frac{b}{a}e\cos 2\omega\sin M + \frac{b}{a}\left(1 - \frac{7}{2}e^2\right)\cos 2\omega\sin 2M + 2\frac{b}{a}e\cos 2\omega\sin 3M + \frac{13}{4}\frac{b}{a}e^2\cos 2\omega\sin 4M \tag{71}$$

$$\frac{1}{R_O^3} \cong \frac{1}{a^3}\left(1 + \frac{3}{2}e^2 + 3e\cos M + \frac{9}{2}e^2\cos 2M\right) \tag{73}$$

$$\frac{1}{R_O^4} \cong \frac{1}{a^4}\left(1 + 3e^2 + 4e\cos M + 7e^2\cos 2M\right) \tag{74}$$

$$\frac{1}{R_O^5} \cong \frac{1}{a^5}\left(1 + 5e^2 + 5e\cos M + 10e^2\cos 2M\right) \tag{75}$$

2.4. Equations in State-Space Form

The linearized equations of motion in Section 2.3 can be combined and represented in the form

$$\ddot{\mathbf{r}} + \mathbf{C}\dot{\mathbf{r}} + [\mathbf{K} + \mathbf{P}]\mathbf{r} + \mathbf{q} = \mathbf{0} \tag{76}$$

The expressions for the elements of **C**, **K**, **P** and **q** in the above equation are lengthy and have been placed in Appendix A.

2.5. Remarks on the Equations

It should be noted that all of the orbital elements that appear in the equations are for the reference elliptical orbit. Also, Ω does not appear in these equations since the J_2 perturbations are symmetrical about the Z-axis, as was discussed in Section 2.1.

Another important aspect to note is the fact that the J_2 perturbation accelerations contain a periodic forcing term reflected by $\mathbf{q}$ in Eq. (76). In the case of linearized equations for Keplerian motion (e.g., the Clohessy-Wiltshire equations, the equations given by Melton), an object with zero displacement and velocity in the orbit frame will continue to have zero displacement and velocity at all future times. Therefore, those equations can be used directly to analyze the relative motion between two spacecraft if it is assumed that one of the spacecraft resides on the reference orbit. In the case of Ross's equations or the equations derived in this thesis, a spacecraft at zero initial displacement and velocity will not, in general, remain at zero displacement. Thus, using these equations to analyze the relative motion between two spacecraft necessitates subtracting the solution for one spacecraft from that of the other.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors with respect to the reference orbit for each of two spacecraft. The position vector for one spacecraft with respect to the other is then given by $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The equations of motion for each spacecraft can be written using Eq (76):

$$\ddot{\mathbf{r}}_1 + \mathbf{C}\dot{\mathbf{r}}_1 + [\mathbf{K} + \mathbf{P}]\mathbf{r}_1 + \mathbf{q} = \mathbf{0} \tag{77}$$

$$\ddot{\mathbf{r}}_2 + \mathbf{C}\dot{\mathbf{r}}_2 + [\mathbf{K} + \mathbf{P}]\mathbf{r}_2 + \mathbf{q} = \mathbf{0} \ . \tag{78}$$

Subtracting Eq (77) from Eq. (78) gives

$$(\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1) + \mathbf{C}(\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1) + [\mathbf{K} + \mathbf{P}](\mathbf{r}_2 - \mathbf{r}_1) + (\mathbf{q} - \mathbf{q}) = \mathbf{0} \tag{79}$$

$$\Rightarrow \Delta\ddot{\mathbf{r}} + \mathbf{C}\Delta\dot{\mathbf{r}} + [\mathbf{K} + \mathbf{P}]\Delta\mathbf{r} = \mathbf{0} \ . \tag{80}$$

Thus, Eq. (76), with $\Delta\mathbf{r}$ substituted for $\mathbf{r}$, directly describes the relative motion between two orbiting spacecraft if $\mathbf{q}$ is set to zero.

3. NUMERICAL SIMULATIONS

Simulation code was developed in Matlab which computes numerical solutions and generates comparative plots for the relative motion between two objects in orbit. Solutions are determined with the Matlab ode45 solver, which is an implementation of the Runge-Kutta 4/5 algorithm. The following sets of equations are compared:

1. The exact equations for Keplerian motion
2. The exact equations with J_2 perturbations
3. The linearized equations for Keplerian motion (Melton)
4. The linearized equations with J_2 perturbations (derived in this thesis)

The solution for each set of equations is determined at the same equally spaced time steps over one period of the reference orbit. The solution to the exact equations with J_2 perturbations serves as the reference, and is considered to be the real motion that would actually occur in orbit. The exact equations for Keplerian motion are included so that the effect of the J_2 perturbation can be seen in the actual motion.

The inputs required by the simulation code are

1. The constant orbital elements of the reference orbit $(a,\ e,\ i,\ \Omega,\ \omega)$ and the initial true anomaly (f_0)
2. Initial position and velocity $(x_0,\ y_0,\ z_0,\ \dot{x}_0,\ \dot{y}_0,\ \dot{z}_0)$ in the orbit frame for each of the two objects
3. Geophysical constants $(\mu,\ J_2,\ R_\oplus)$

4. Numerical integration options

The subscript $(\ )_0$ indicates initial time, and should not be confused with $(\ )_O$, which indicates the reference orbit.

The forms of the exact equations that are used in the simulation are composed of ECI frame variables. Thus, the initial conditions needed for these equations must be calculated from the inputs. No such calculations are required for the linearized equations, however, since the inputs directly provide all the initial condition information needed.

After the exact equation solutions are computed, they must be transformed from the ECI frame to the orbit frame so that the linearized equation solutions can be compared against them. This transformation requires that the solution for the reference orbit also be determined at each of the time steps.

3.1. Determination of Initial Conditions for the Exact Equations

The initial position and velocity in the ECI frame must be determined from the input information. The position and velocity vectors for an elliptical orbit written in the perifocal frame are given by Bate[7] Eqs. (2.5-1) and (2.5-4):

$$\mathbf{R} = \begin{bmatrix} R_P \\ R_Q \\ R_W \end{bmatrix} = R \begin{bmatrix} \cos f \\ \sin f \\ 0 \end{bmatrix} \tag{81}$$

$$\mathbf{V} = \begin{bmatrix} V_P \\ V_Q \\ V_W \end{bmatrix} = \sqrt{\frac{\mu}{p}} \begin{bmatrix} -\sin f \\ e + \cos f \\ 0 \end{bmatrix} . \tag{82}$$

The components of a vector $\mathbf{a}$ in the perifocal frame can be transformed to the the ECI frame by

$$\begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix} = \mathbf{T} \begin{bmatrix} a_P \\ a_Q \\ a_W \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_P \\ a_Q \\ a_W \end{bmatrix} \tag{83}$$

where the elements of $\mathbf{T}$ are given in Bate by Eqs 2.6-14:

$$\begin{aligned} T_{11} &= \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\ T_{12} &= -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i \\ T_{13} &= \sin\Omega\sin i \\ T_{21} &= \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i \\ T_{22} &= -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i \quad . \\ T_{23} &= -\cos\Omega\sin i \\ T_{31} &= \sin\omega\sin i \\ T_{32} &= \cos\omega\sin i \\ T_{33} &= \cos i \end{aligned} \tag{84}$$

With Eqs. (81-83), we can write

$$(\mathbf{R}_O)_0 = \begin{bmatrix} R_{OX} \\ R_{OY} \\ R_{OZ} \end{bmatrix}_0 = \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}_0 = (R_O)_0 \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \cos f_0 \\ \sin f_0 \\ 0 \end{bmatrix} \tag{85}$$

$$(\mathbf{V}_O)_0 = \begin{bmatrix} V_{OX} \\ V_{OY} \\ V_{OZ} \end{bmatrix}_0 = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix}_0 = \sqrt{\frac{\mu}{p}} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} -\sin f_0 \\ e + \cos f_0 \\ 0 \end{bmatrix} . \tag{86}$$

The quantities p and R in the above equations can be determined by the conic section equations

$$p = a(1-e^2) \tag{87}$$

and

$$(R_O)_0 = \frac{p}{1+e\cos f_0} \ . \tag{88}$$

Eqs. (85) and (86) , with Eqs. (84), (87), and (88), can be used to determine the initial conditions for the reference orbit, $\begin{bmatrix} X_O & Y_O & Z_O & \dot{X}_O & \dot{Y}_O & \dot{Z}_O \end{bmatrix}_0$, using the input values $a,\ e,\ i,\ \Omega,\ \omega$ and f_0.

The initial position vector of the object under consideration is the sum of the position vector of the reference orbit and the relative position vector of the object

$$\mathbf{R}_0 = (\mathbf{R}_O)_0 + \mathbf{r}_0 \tag{89}$$

$$\Rightarrow \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_0 = \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}_0 + \begin{bmatrix} r_X \\ r_Y \\ r_Z \end{bmatrix}_0 \tag{90}$$

where the relative position vector in ECI coordinates can be determined by

$$\begin{bmatrix} r_X \\ r_Y \\ r_Z \end{bmatrix}_0 = {}^N\mathbf{C}^O \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}_0 = {}^N\mathbf{C}^O \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} . \tag{91}$$

The initial velocity of the object in the ECI frame is

$$\mathbf{V}_0 = \left[\frac{{}^N d}{dt}\mathbf{R}_O + \frac{{}^N d}{dt}\mathbf{r} \right]_0 = (\mathbf{V}_O)_0 + \left[\frac{{}^N d}{dt}\mathbf{r} \right]_0 \ . \tag{92}$$

The first term in the rightmost expression of Eq. (92) is given by Eq. (86). The second term can be expressed using the transport theorem as

$$\left[\frac{{}^{N}d}{dt}\mathbf{r}\right]_0 = \left[\frac{{}^{O}d}{dt}\mathbf{r}\right]_0 + {}^{N}\boldsymbol{\omega}_0^{O} \times \mathbf{r}_0 \tag{93}$$

The initial angular velocity of the orbit frame with respect to the inertial frame is

$${}^{N}\boldsymbol{\omega}_0^{O} = \frac{\mathbf{h}_O}{\left(R_O\right)_0^2} \tag{94}$$

where the angular momentum vector, $\mathbf{h}_O$, can be determined by

$$\mathbf{h}_O = \left(\mathbf{R}_O\right)_0 \times \left(\mathbf{V}_O\right)_0 \quad . \tag{95}$$

Thus, expressing the vector in the inertial frame, we can write

$${}^{N}\boldsymbol{\omega}_0^{O} = \begin{bmatrix} {}^{N}\omega_X^{O} \\ {}^{N}\omega_Y^{O} \\ {}^{N}\omega_Z^{O} \end{bmatrix}_0 = \frac{1}{\left(R_O\right)_0^2} \begin{bmatrix} R_{OX} \\ R_{OY} \\ R_{OZ} \end{bmatrix}_0 \times \begin{bmatrix} V_{OX} \\ V_{OY} \\ V_{OZ} \end{bmatrix}_0 \quad . \tag{96}$$

Finally, we can write

$$\Rightarrow \mathbf{V}_0 = \begin{bmatrix} V_{OX} \\ V_{OY} \\ V_{OZ} \end{bmatrix}_0 + {}^{N}\mathbf{C}^{O} \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} + \left(\frac{1}{\left(R_O\right)_0^2} \begin{bmatrix} R_{OX} \\ R_{OY} \\ R_{OZ} \end{bmatrix}_0 \times \begin{bmatrix} V_{OX} \\ V_{OY} \\ V_{OZ} \end{bmatrix}_0 \right) \times \left({}^{N}\mathbf{C}^{O} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right) \quad . \tag{97}$$

3.2. Determination of Time of Periapsis Passage and Mean Anomaly

The initial eccentric anomaly, E_0, can be determined from Prussing[8] Eq. (2.9):

$$E_0 = \cos^{-1}\left(\frac{e + \cos f_0}{1 + e\cos f_0}\right) . \tag{98}$$

With E_0 known and assuming that $t = 0$ at the initial time t_0, Kepler's equation can be used to determine the time of periapsis passage t_p:

$$M_0 = E_0 - e\sin E_0 \tag{99}$$

$$\Rightarrow n\left(t_0 - t_p\right) = E_0 - e\sin E_0 \tag{100}$$

$$\Rightarrow t_p = -\frac{E_0 - e\sin E_0}{n} . \tag{101}$$

With t_p known, the mean anomaly can be determined at each time step by

$$\Rightarrow M = n\left(t - t_p\right) . \tag{102}$$

3.3. Transformation of Numerical Solution of Exact Equations to Orbit Frame

The exact Keplerian and exact J_2 perturbed solutions must be transformed to the orbit frame before the linearized equation solutions can be compared against them. This transformation requires that the reference orbit argument of latitude θ and time rate of change of true anomaly $\dot{f}$ be determined at each time step.

First, the eccentricity vector for the reference orbit must be calculated. Since this vector is invariant for a Keplerian orbit, it only needs to be calculated once. It can be determined from the initial conditions using Eq. (2.4-5) in Bate:

$$\mathbf{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{R_O}\right)\mathbf{R}_O - \left(\mathbf{R}_O \cdot \mathbf{V}_O\right)\mathbf{V}_O\right] . \tag{103}$$

The remaining equations in this section are utilized for calculations at each time step. The true anomaly for the reference orbit can be determined by

$$f = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{e} \cdot \mathbf{R_O}}{eR_O}\right) & \text{if } R_O \cdot V_O \geq 0 \\ 2\pi - \cos^{-1}\left(\dfrac{\mathbf{e} \cdot \mathbf{R_O}}{eR_O}\right) & \text{if } R_O \cdot V_O < 0 \end{cases} \tag{104}$$

and the argument of latitude by

$$\theta = \omega + f \; . \tag{105}$$

The time rate of change of true anomaly for the reference orbit is given by Prussing Eq. (1.38) as

$$\dot{f} = \frac{h}{R_O^2} \; . \tag{106}$$

The position of the object with respect to the reference orbit is

$$\mathbf{r} = \mathbf{R} - \mathbf{R}_O \tag{107}$$

$$\Rightarrow \begin{bmatrix} r_X \\ r_Y \\ r_Z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix} . \tag{108}$$

The two vectors on the right side of Eq. (108) are obtained from the numerical solution. The relative position vector can be transformed to orbit coordinates by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = {}^{O}\mathbf{C}^{N} \begin{bmatrix} r_X \\ r_Y \\ r_Z \end{bmatrix} . \tag{109}$$

The relative velocity vector in the inertial frame is

$$\frac{{}^{N}d}{dt}\mathbf{r} = \mathbf{V} - \mathbf{V}_O = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} - \begin{bmatrix} \dot{X}_O \\ \dot{Y}_O \\ \dot{Z}_O \end{bmatrix} . \tag{110}$$

The two vectors on the right side of Eq. (110) are obtained from the numerical solution. Using the transport theorem once again, we can write

$$\mathbf{v} = \frac{{}^{O}d}{dt}\mathbf{r} = \frac{{}^{N}d}{dt}\mathbf{r} + {}^{O}\boldsymbol{\omega}^{N} \times \mathbf{r} \tag{111}$$

$$\Rightarrow \mathbf{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = {}^{O}\mathbf{C}^{N} \begin{bmatrix} \dot{X} - \dot{X}_O \\ \dot{Y} - \dot{Y}_O \\ \dot{Z} - \dot{Z}_O \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\dot{f} \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} . \tag{112}$$

3.4. Simulation Results

Plots were generated by the simulation code using the following inputs:

$J_2 =$	1082.63×10^{-6}	$x_{1\,0} =$	0 m	$x_{2\,0} =$	100 m
$\mu =$	3.986004×10^{14} m^3/s^2	$y_{1\,0} =$	0 m	$y_{2\,0} =$	100 m
$R_\oplus =$	6.378136×10^{6} m	$z_{1\,0} =$	0 m	$z_{2\,0} =$	100 m
$a =$	$R_\oplus + 800 \times 10^{3}$ m	$\dot{x}_{1\,0} =$	0 m/s	$\dot{x}_{2\,0} =$	10 m/s
$e =$	0.1	$\dot{y}_{1\,0} =$	0 m/s	$\dot{y}_{2\,0} =$	10 m/s
$i =$	$60°$	$\dot{z}_{1\,0} =$	0 m/s	$\dot{z}_{2\,0} =$	10 m/s
$\Omega =$	$0°$				
$\omega =$	$90°$				
$f_0 =$	$0°$				

Plots of position vs. time, velocity vs. time, path plots (e.g., x-position vs. z-position), and error plots were generated, and can be found in Appendix B.

The plot for x-position vs. time is shown in Figure 3 on page 51. In this plot, the exact Keplerian and the exact perturbed curves have diverged noticeably by the end of the orbit period. The linear equations on the other hand nearly overlap each other over the entire period. No accuracy improvement can thus be discerned from this plot.

Figure 4 on page 52 gives the plot for y-position vs. time. All four curves nearly overlap over most of the time range, until a divergence becomes apparent near the end of the orbit period. At this point, the two curves for the linearized equations continue to nearly overlap each other, as they diverge from the curves for the exact equations, which also continue to nearly overlap each other. This would seem to suggest that, at least in the y-direction, the magnitude of the J_2 perturbation is small relative to the error that exists between the exact Keplerian and the linearized Keplerian equations.

Figure 5 on page 53 shows the plot for z-position vs. time. The linear perturbed and exact perturbed curves show very close correspondence over the entire orbit period. It should be noted, however, that the Keplerian solutions do not overlap as closely as the perturbed solutions do.

The most useful aspect of the path plots is that they allow one to see the relative scale of the motion amongst the coordinate directions; axis scales for the plots were set equal. Also, the relatively large y-direction discrepancy between the exact and the linearized equations at the end of the orbit period should be noted.

The error plots show the maximum errors in the linear solutions over one orbit period for a range of eccentricities (the input eccentricity value does not have meaning for these plots). The error between each of the linear solutions and the exact perturbed solution was determined at each time step. The maximum error over the orbit period was then identified. This maximum error computation was repeated for a range of eccentricities, and the results were plotted. Figure 13 on page 61 shows curves for the maximum x-position error and Figure 14 on page 62 shows curves for the maximum y-position error. These plots seem to indicate that the equations developed in this thesis do not significantly change the x- and y-position accuracy, regardless of eccentricity.

Figure 15 on page 63 shows the maximum error over one orbit period for the z-position. This plot indicates a considerable increase in the z-position accuracy for eccentricities smaller than 0.3. For eccentricities greater than 0.3, the error is slightly greater.

The position vs. time plots, velocity vs. time plots and the error plots seem to suggest that the equations developed in this thesis provide a significant increase in accuracy in the z-direction. The accuracy in the x- and y-directions, however, appears to be roughly unchanged. Although not presented in this thesis, simulation runs for varied and different sets of initial conditions were conducted, and it can be said that the plots presented in this thesis are fairly representative of those of the greater parameter and initial condition space.

4. FUTURE WORK

The primary work in this thesis consists of the derivation of the linearized equations for J_2 perturbed motion relative to an elliptical orbit. While numerical simulations do seem to indicate that these equations provide an improvement in accuracy in the out-of-orbit-plane direction, further work would be required to determine the degree of practical benefit given by their use. Possibilities for future work include:

- Analysis of an orbital rendezvous or formation flying problem
- Design of a controller for orbital rendezvous or formation flying utilizing the Lyapunov-Floquet transformation
- Further analysis of the equations themselves to gain a better understanding of their behavior and accuracy

5. CONCLUSIONS

A set of linearized equations was derived for the motion, relative to an elliptical reference orbit, of an object influenced by J_2 perturbations. Numerically determined solutions were used to compare these equations and the linearized Keplerian equations to the exact equations. The inclusion of the J_2 perturbations in the derived equations increased the accuracy of the solution significantly in the out-of-orbit-plane direction, while the accuracy within the orbit plane remained roughly unchanged.

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APPENDIX A: C, K, P AND q FOR STATE-SPACE FORM

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{11} = 0$$

$$C_{12} = \left(h/a^2\right)\left\{2 + 4e\cos M + e^2\left[5\cos 2M + 1\right]\right\}$$

$$C_{13} = 0$$

$$C_{12} = -\left(h/a^2\right)\left\{2 + 4e\cos M + e^2\left[5\cos 2M + 1\right]\right\}$$

$$C_{22} = 0$$

$$C_{23} = 0$$

$$C_{31} = 0$$

$$C_{32} = 0$$

$$C_{33} = 0$$

$$\mathbf{K} = a^4 \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

$$K_{11} = e\left(4h^2 + 6\mu a\right)\cos M + e^2\left(7h^2 + 9\mu a\right)\cos 2M + \left(1 + 3e^2\right)h^2 + \left(2 + 3e^2\right)\mu a$$

$$K_{12} = -2ea^2 hn \sin M - 5e^2 a^2 hn \sin 2M$$

$$K_{13} = 0$$

$$K_{21} = 2ea^2 hn \sin M + 5e^2 a^2 hn \sin 2M$$

$$K_{22} = e\left(4h^2 - 3\mu a\right)\cos M + (1/2)e^2\left(14h^2 - 9\mu a\right)\cos 2M$$
$$+\left(1 + 3e^2\right)h^2 - (1/2)\mu a\left(2 + 3e^2\right)$$

$$K_{23} = 0$$

$$K_{31} = 0$$

$$K_{32} = 0$$

$$K_{33} = -3\mu ea \cos M - (9/2)\mu e^2 a \cos 2M - (1/2)\left(2 + 3e^2\right)\mu a$$

$$\mathbf{P} = \frac{J_2 \mu R_\oplus^2}{a^5} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

$$P_{jk} = c_1 \cos M + c_2 \cos 2M + c_3 \cos 3M + c_4 \cos 4M$$
$$+ c_5 \sin M + c_6 \sin 2M + c_7 \sin 3M + c_8 \sin 4M + c_9$$

P_{11}:

$$c_1 = \frac{3}{2} e \left(5 + 15 \text{Cos}[2\,i] + 3 \text{Cos}[2\,\omega] \text{Sin}[i]^2\right)$$

$$c_2 = 15 e^2 + 45 e^2 \text{Cos}[2\,i] + 9 \left(1 + e^2\right) \text{Cos}[2\,\omega] \text{Sin}[i]^2$$

$$c_3 = \frac{81}{2} e \text{Cos}[2\,\omega] \text{Sin}[i]^2$$

$$c_4 = \frac{477}{4} e^2 \text{Cos}[2\,\omega] \text{Sin}[i]^2$$

$$c_5 = -9 e \sqrt{1 - e^2} \text{Cos}[\omega] \text{Sin}[i]^2 \text{Sin}[\omega]$$

$$c_6 = -9 \sqrt{1 - e^2} \left(2 + 3 e^2\right) \text{Cos}[\omega] \text{Sin}[i]^2 \text{Sin}[\omega]$$

$$c_7 = -81 e \sqrt{1 - e^2} \text{Cos}[\omega] \text{Sin}[i]^2 \text{Sin}[\omega]$$

$$c_8 = -\frac{477}{2} e^2 \sqrt{1 - e^2} \text{Cos}[\omega] \text{Sin}[i]^2 \text{Sin}[\omega]$$

$$c_9 = \frac{3}{4} \left(2 + 10 e^2 + 6 \left(1 + 5 e^2\right) \text{Cos}[2\,i] + 9 e^2 \text{Cos}[2\,\omega] \text{Sin}[i]^2\right)$$

P_{12}:

$c_1 = 6\, e \, \mathrm{Cos}[\omega]\, \mathrm{Sin}[i]^2\, \mathrm{Sin}[\omega]$

$c_2 = 6\, (1 + e^2)\, \mathrm{Sin}[i]^2\, \mathrm{Sin}[2\,\omega]$

$c_3 = 27\, e\, \mathrm{Sin}[i]^2\, \mathrm{Sin}[2\,\omega]$

$c_4 = 159\, e^2\, \mathrm{Cos}[\omega]\, \mathrm{Sin}[i]^2\, \mathrm{Sin}[\omega]$

$c_5 = 3\, e\, \sqrt{1 - e^2}\, \mathrm{Cos}[2\,\omega]\, \mathrm{Sin}[i]^2$

$c_6 = 3\, \sqrt{1 - e^2}\, (2 + 3\, e^2)\, \mathrm{Cos}[2\,\omega]\, \mathrm{Sin}[i]^2$

$c_7 = 27\, e\, \sqrt{1 - e^2}\, \mathrm{Cos}[2\,\omega]\, \mathrm{Sin}[i]^2$

$c_8 = \frac{159}{2}\, e^2\, \sqrt{1 - e^2}\, \mathrm{Cos}[2\,\omega]\, \mathrm{Sin}[i]^2$

$c_9 = 9\, e^2\, \mathrm{Cos}[\omega]\, \mathrm{Sin}[i]^2\, \mathrm{Sin}[\omega]$

P_{13}:

$c_1 = \frac{3}{4}\, (8 + 51\, e^2)\, \mathrm{Sin}[2\,i]\, \mathrm{Sin}[\omega]$

$c_2 = 21\, e\, \mathrm{Sin}[2\,i]\, \mathrm{Sin}[\omega]$

$c_3 = \frac{207}{4}\, e^2\, \mathrm{Sin}[2\,i]\, \mathrm{Sin}[\omega]$

$c_4 = 0$

$c_5 = \frac{3}{4}\, \sqrt{1 - e^2}\, (8 + 17\, e^2)\, \mathrm{Cos}[\omega]\, \mathrm{Sin}[2\,i]$

$c_6 = 21\, e\, \sqrt{1 - e^2}\, \mathrm{Cos}[\omega]\, \mathrm{Sin}[2\,i]$

$c_7 = \frac{207}{4}\, e^2\, \sqrt{1 - e^2}\, \mathrm{Cos}[\omega]\, \mathrm{Sin}[2\,i]$

$c_8 = 0$

$c_9 = 9\, e\, \mathrm{Sin}[2\,i]\, \mathrm{Sin}[\omega]$

P_{21}:

$c_1 = 6\,e\,Cos[\omega]\,Sin[i]^2\,Sin[\omega]$
$c_2 = 6\,(1 + e^2)\,Sin[i]^2\,Sin[2\,\omega]$
$c_3 = 27\,e\,Sin[i]^2\,Sin[2\,\omega]$
$c_4 = 159\,e^2\,Cos[\omega]\,Sin[i]^2\,Sin[\omega]$
$c_5 = 3\,e\,\sqrt{1 - e^2}\,Cos[2\,\omega]\,Sin[i]^2$
$c_6 = 3\,\sqrt{1 - e^2}\,(2 + 3\,e^2)\,Cos[2\,\omega]\,Sin[i]^2$
$c_7 = 27\,e\,\sqrt{1 - e^2}\,Cos[2\,\omega]\,Sin[i]^2$
$c_8 = \frac{159}{2}\,e^2\,\sqrt{1 - e^2}\,Cos[2\,\omega]\,Sin[i]^2$
$c_9 = 9\,e^2\,Cos[\omega]\,Sin[i]^2\,Sin[\omega]$

P_{22}:

$c_1 = -\frac{3}{8}\,e\,(5 + 15\,Cos[2\,i] + 7\,Cos[2\,\omega]\,Sin[i]^2)$
$c_2 = -\frac{3}{4}\,(5\,e^2 + 15\,e^2\,Cos[2\,i] + 7\,(1 + e^2)\,Cos[2\,\omega]\,Sin[i]^2)$
$c_3 = -\frac{189}{8}\,e\,Cos[2\,\omega]\,Sin[i]^2$
$c_4 = -\frac{1113}{16}\,e^2\,Cos[2\,\omega]\,Sin[i]^2$
$c_5 = \frac{21}{4}\,e\,\sqrt{1 - e^2}\,Cos[\omega]\,Sin[i]^2\,Sin[\omega]$
$c_6 = \frac{21}{8}\,\sqrt{1 - e^2}\,(2 + 3\,e^2)\,Sin[i]^2\,Sin[2\,\omega]$
$c_7 = \frac{189}{4}\,e\,\sqrt{1 - e^2}\,Cos[\omega]\,Sin[i]^2\,Sin[\omega]$
$c_8 = \frac{1113}{8}\,e^2\,\sqrt{1 - e^2}\,Cos[\omega]\,Sin[i]^2\,Sin[\omega]$
$c_9 = -\frac{3}{16}\,(2 + 10\,e^2 + 6\,(1 + 5\,e^2)\,Cos[2\,i] + 21\,e^2\,Cos[2\,\omega]\,Sin[i]^2)$

P_{23}:

$$c_1 = -\frac{3}{8}\left(8 + 51 e^2\right) \text{Cos}[i]\,\text{Cos}[\omega]\,\text{Sin}[i]$$

$$c_2 = -\frac{21}{2}\, e\, \text{Cos}[i]\,\text{Cos}[\omega]\,\text{Sin}[i]$$

$$c_3 = -\frac{207}{8}\, e^2\, \text{Cos}[i]\,\text{Cos}[\omega]\,\text{Sin}[i]$$

$$c_4 = 0$$

$$c_5 = \frac{3}{8}\sqrt{1 - e^2}\left(8 + 17 e^2\right) \text{Cos}[i]\,\text{Sin}[i]\,\text{Sin}[\omega]$$

$$c_6 = \frac{21}{2}\, e\sqrt{1 - e^2}\, \text{Cos}[i]\,\text{Sin}[i]\,\text{Sin}[\omega]$$

$$c_7 = \frac{207}{8}\, e^2\sqrt{1 - e^2}\, \text{Cos}[i]\,\text{Sin}[i]\,\text{Sin}[\omega]$$

$$c_8 = 0$$

$$c_9 = -\frac{9}{2}\, e\, \text{Cos}[i]\,\text{Cos}[\omega]\,\text{Sin}[i]$$

P_{31}:

$$c_1 = \frac{3}{4}\left(8 + 51 e^2\right) \text{Sin}[2\,i]\,\text{Sin}[\omega]$$

$$c_2 = 21\, e\, \text{Sin}[2\,i]\,\text{Sin}[\omega]$$

$$c_3 = \frac{207}{4}\, e^2\, \text{Sin}[2\,i]\,\text{Sin}[\omega]$$

$$c_4 = 0$$

$$c_5 = \frac{3}{4}\sqrt{1 - e^2}\left(8 + 17 e^2\right) \text{Cos}[\omega]\,\text{Sin}[2\,i]$$

$$c_6 = 21\, e\sqrt{1 - e^2}\, \text{Cos}[\omega]\,\text{Sin}[2\,i]$$

$$c_7 = \frac{207}{4}\, e^2\sqrt{1 - e^2}\, \text{Cos}[\omega]\,\text{Sin}[2\,i]$$

$$c_8 = 0$$

$$c_9 = 9\, e\, \text{Sin}[2\,i]\,\text{Sin}[\omega]$$

P_{32}:

$$c_1 = -\frac{3}{8}\left(8 + 51 e^2\right) \text{Cos}[i]\,\text{Cos}[\omega]\,\text{Sin}[i]$$
$$c_2 = -\frac{21}{2}\, e\, \text{Cos}[i]\,\text{Cos}[\omega]\,\text{Sin}[i]$$
$$c_3 = -\frac{207}{8}\, e^2\, \text{Cos}[i]\,\text{Cos}[\omega]\,\text{Sin}[i]$$
$$c_4 = 0$$
$$c_5 = \frac{3}{8}\sqrt{1 - e^2}\left(8 + 17 e^2\right) \text{Cos}[i]\,\text{Sin}[i]\,\text{Sin}[\omega]$$
$$c_6 = \frac{21}{2}\, e\sqrt{1 - e^2}\, \text{Cos}[i]\,\text{Sin}[i]\,\text{Sin}[\omega]$$
$$c_7 = \frac{207}{8}\, e^2\sqrt{1 - e^2}\, \text{Cos}[i]\,\text{Sin}[i]\,\text{Sin}[\omega]$$
$$c_8 = 0$$
$$c_9 = -\frac{9}{2}\, e\, \text{Cos}[i]\,\text{Cos}[\omega]\,\text{Sin}[i]$$

P_{33}:

$$c_1 = -\frac{15}{8}\, e\left(3 + 9\,\text{Cos}[2\,i] + \text{Cos}[2\,\omega]\,\text{Sin}[i]^2\right)$$
$$c_2 = -\frac{15}{4}\left(3 e^2 + 9 e^2\,\text{Cos}[2\,i] + \left(1 + e^2\right) \text{Cos}[2\,\omega]\,\text{Sin}[i]^2\right)$$
$$c_3 = -\frac{135}{8}\, e\, \text{Cos}[2\,\omega]\,\text{Sin}[i]^2$$
$$c_4 = -\frac{795}{16}\, e^2\, \text{Cos}[2\,\omega]\,\text{Sin}[i]^2$$
$$c_5 = \frac{15}{4}\, e\sqrt{1 - e^2}\, \text{Cos}[\omega]\,\text{Sin}[i]^2\,\text{Sin}[\omega]$$
$$c_6 = \frac{15}{8}\sqrt{1 - e^2}\left(2 + 3 e^2\right) \text{Sin}[i]^2\,\text{Sin}[2\,\omega]$$
$$c_7 = \frac{135}{4}\, e\sqrt{1 - e^2}\, \text{Cos}[\omega]\,\text{Sin}[i]^2\,\text{Sin}[\omega]$$
$$c_8 = \frac{795}{8}\, e^2\sqrt{1 - e^2}\, \text{Cos}[\omega]\,\text{Sin}[i]^2\,\text{Sin}[\omega]$$
$$c_9 = -\frac{9}{16}\left(2 + 10 e^2 + 6\left(1 + 5 e^2\right) \text{Cos}[2\,i] + 5 e^2\,\text{Cos}[2\,\omega]\,\text{Sin}[i]^2\right)$$

$$\mathbf{q} = \frac{J_2 \mu R_\oplus^2}{a^4} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$q_j = c_1 \cos M + c_2 \cos 2M + c_3 \cos 3M + c_4 \cos 4M$$
$$+ c_5 \sin M + c_6 \sin 2M + c_7 \sin 3M + c_8 \sin 4M + c_9$$

q_1:

$$c_1 = -\frac{3}{2} e \, (1 + 3 \, \text{Cos}[2\,i])$$
$$c_2 = -\frac{3}{8} \left(7 e^2 + 21 e^2 \, \text{Cos}[2\,i] - 6 \left(-1 + e^2\right) \text{Cos}[2\,\omega] \, \text{Sin}[i]^2\right)$$
$$c_3 = -9 \, e \, \text{Cos}[2\,\omega] \, \text{Sin}[i]^2$$
$$c_4 = -\frac{387}{16} e^2 \, \text{Cos}[2\,\omega] \, \text{Sin}[i]^2$$
$$c_5 = 0$$
$$c_6 = -\frac{9}{8} \sqrt{1 - e^2} \left(-2 + e^2\right) \text{Sin}[i]^2 \, \text{Sin}[2\,\omega]$$
$$c_7 = 9 \, e \sqrt{1 - e^2} \, \text{Sin}[i]^2 \, \text{Sin}[2\,\omega]$$
$$c_8 = \frac{387}{8} e^2 \sqrt{1 - e^2} \, \text{Cos}[\omega] \, \text{Sin}[i]^2 \, \text{Sin}[\omega]$$
$$c_9 = -\frac{3}{16} \left(2 + 6 e^2 + 6 \left(1 + 3 e^2\right) \text{Cos}[2\,i] + 3 e^2 \, \text{Cos}[2\,\omega] \, \text{Sin}[i]^2\right)$$

q_2:

$$c_1 = 0$$
$$c_2 = 3\left(-1+e^2\right) \text{Cos}[\omega]\, \text{Sin}[i]^2\, \text{Sin}[\omega]$$
$$c_3 = -6\, e\, \text{Sin}[i]^2\, \text{Sin}[2\,\omega]$$
$$c_4 = -\frac{129}{4}\, e^2\, \text{Cos}[\omega]\, \text{Sin}[i]^2\, \text{Sin}[\omega]$$
$$c_5 = 0$$
$$c_6 = \frac{3}{4}\sqrt{1-e^2}\left(-2+e^2\right) \text{Cos}[2\,\omega]\, \text{Sin}[i]^2$$
$$c_7 = -6\, e\sqrt{1-e^2}\, \text{Cos}[2\,\omega]\, \text{Sin}[i]^2$$
$$c_8 = -\frac{129}{8}\, e^2\sqrt{1-e^2}\, \text{Cos}[2\,\omega]\, \text{Sin}[i]^2$$
$$c_9 = -\frac{3}{4}\, e^2\, \text{Cos}[\omega]\, \text{Sin}[i]^2\, \text{Sin}[\omega]$$

q_3:

$$c_1 = -\frac{3}{8}\left(8+27\, e^2\right) \text{Cos}[i]\, \text{Sin}[i]\, \text{Sin}[\omega]$$
$$c_2 = -9\, e\, \text{Cos}[i]\, \text{Sin}[i]\, \text{Sin}[\omega]$$
$$c_3 = -\frac{159}{8}\, e^2\, \text{Cos}[i]\, \text{Sin}[i]\, \text{Sin}[\omega]$$
$$c_4 = 0$$
$$c_5 = -\frac{3}{8}\sqrt{1-e^2}\left(8+9\, e^2\right) \text{Cos}[i]\, \text{Cos}[\omega]\, \text{Sin}[i]$$
$$c_6 = -9\, e\sqrt{1-e^2}\, \text{Cos}[i]\, \text{Cos}[\omega]\, \text{Sin}[i]$$
$$c_7 = -\frac{159}{8}\, e^2\sqrt{1-e^2}\, \text{Cos}[i]\, \text{Cos}[\omega]\, \text{Sin}[i]$$
$$c_8 = 0$$
$$c_9 = -3\, e\, \text{Cos}[i]\, \text{Sin}[i]\, \text{Sin}[\omega]$$

APPENDIX B: SIMULATION PLOTS

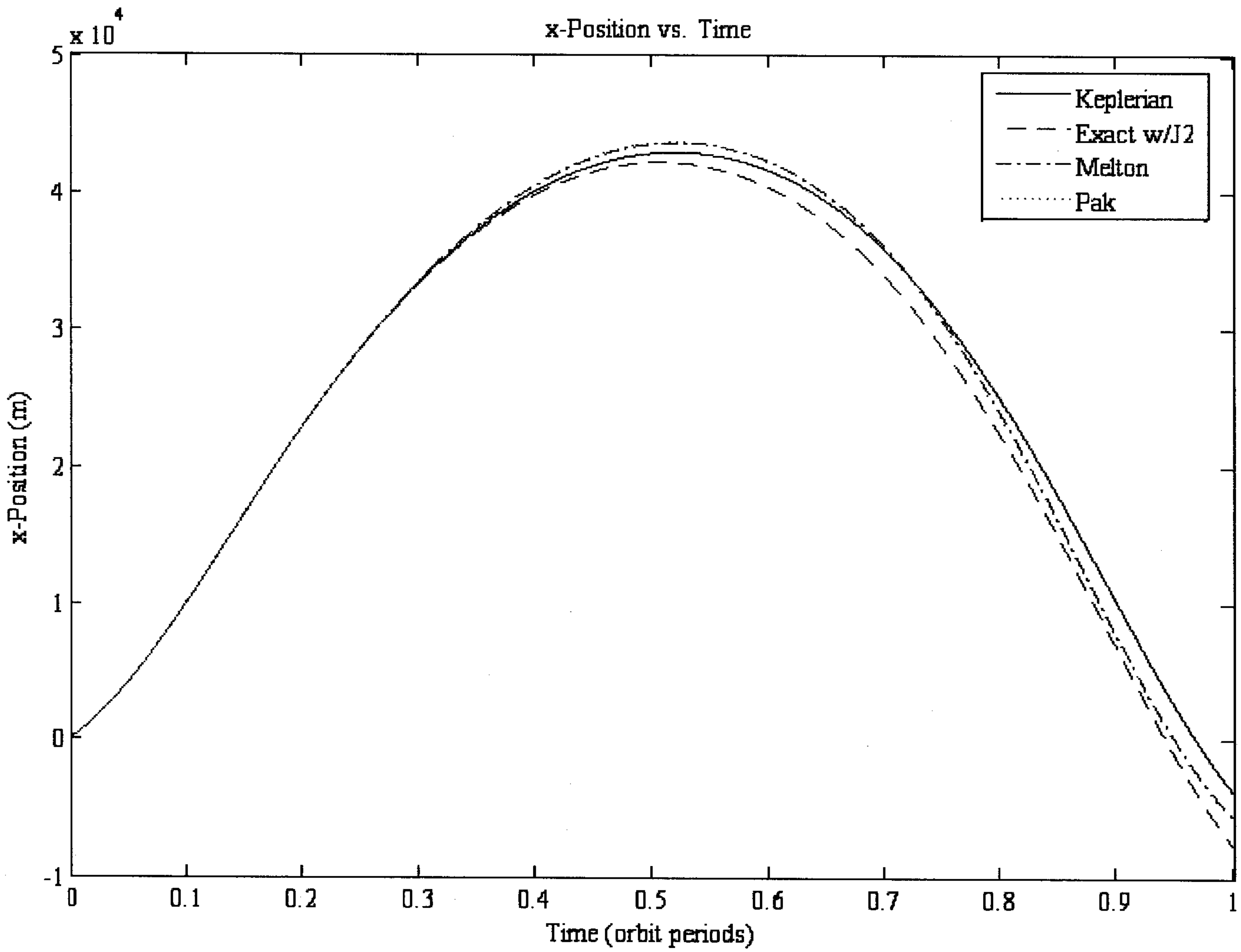

Figure 3. *x*-Position vs. Time

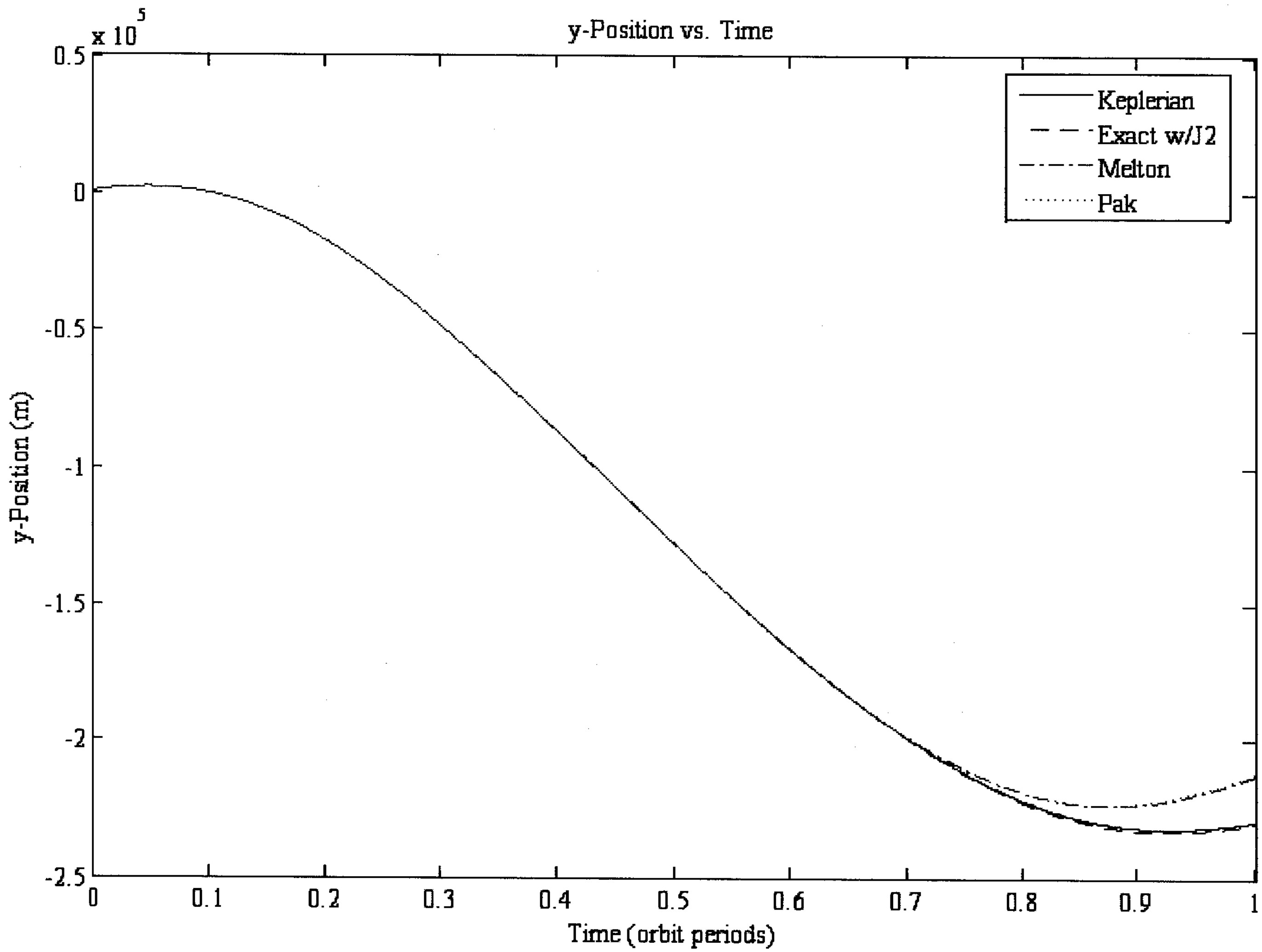

Figure 4. ***y*-Position vs. Time**

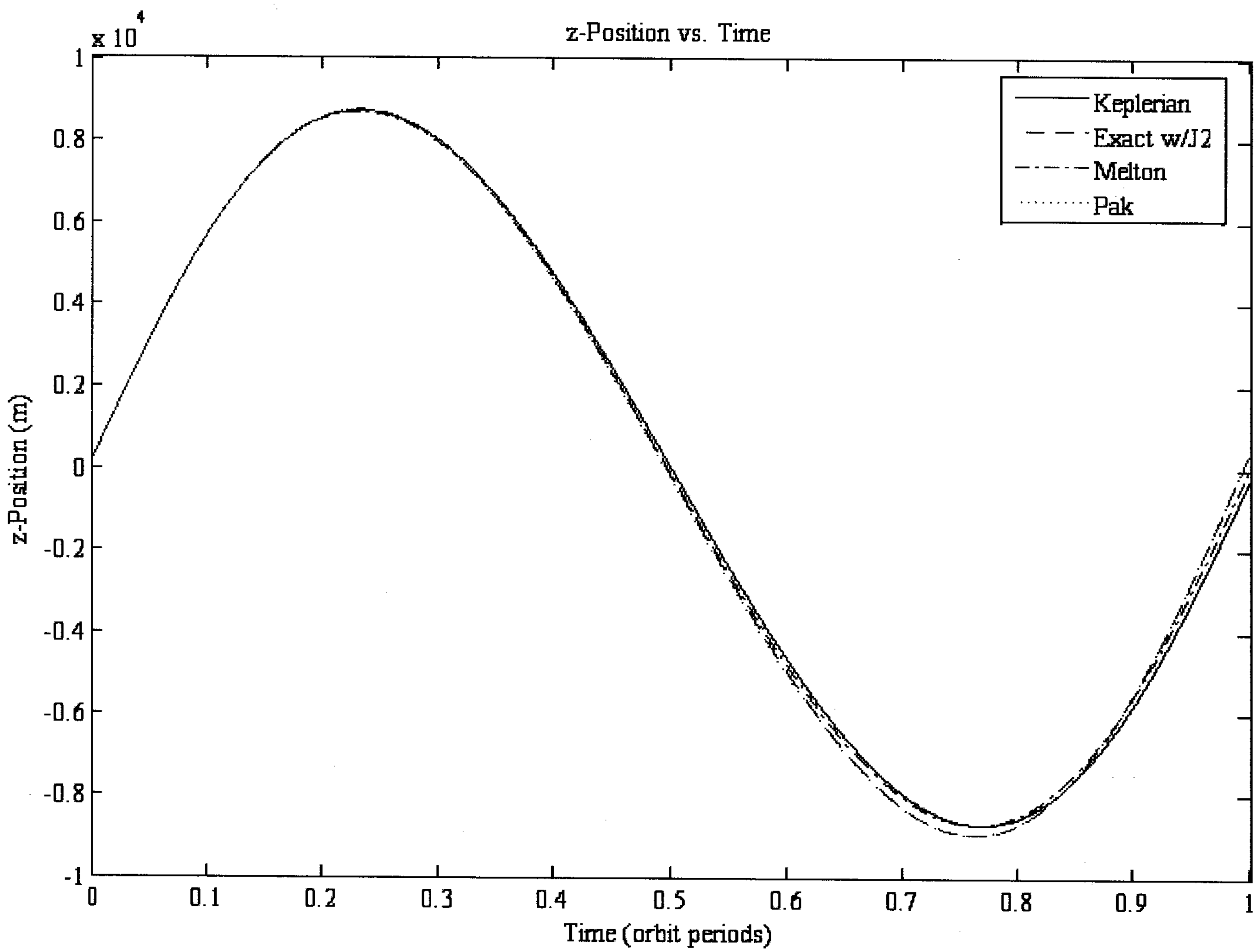

Figure 5. ***z*-Position vs. Time**

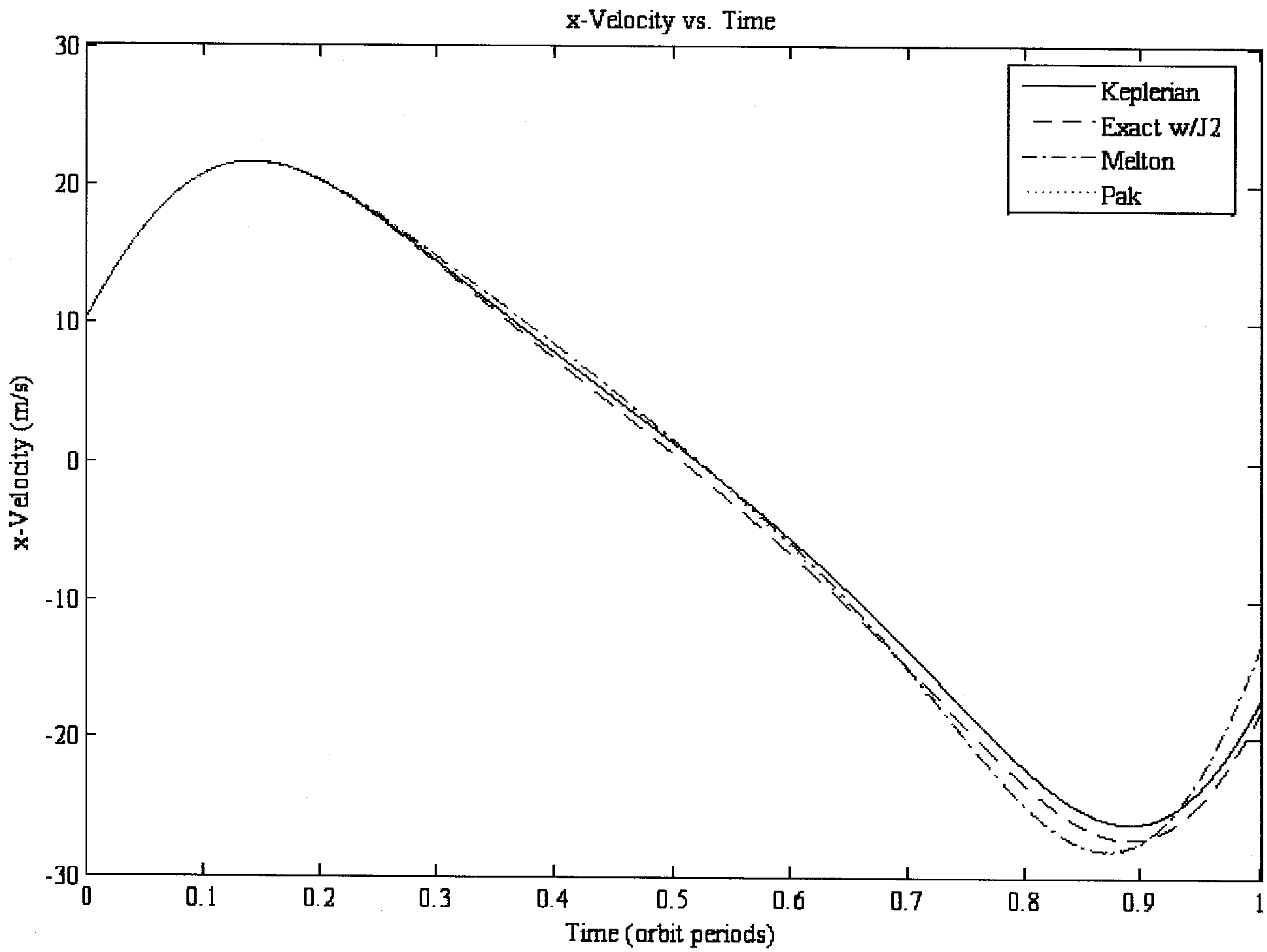

Figure 6. ***x*-Velocity vs. Time**

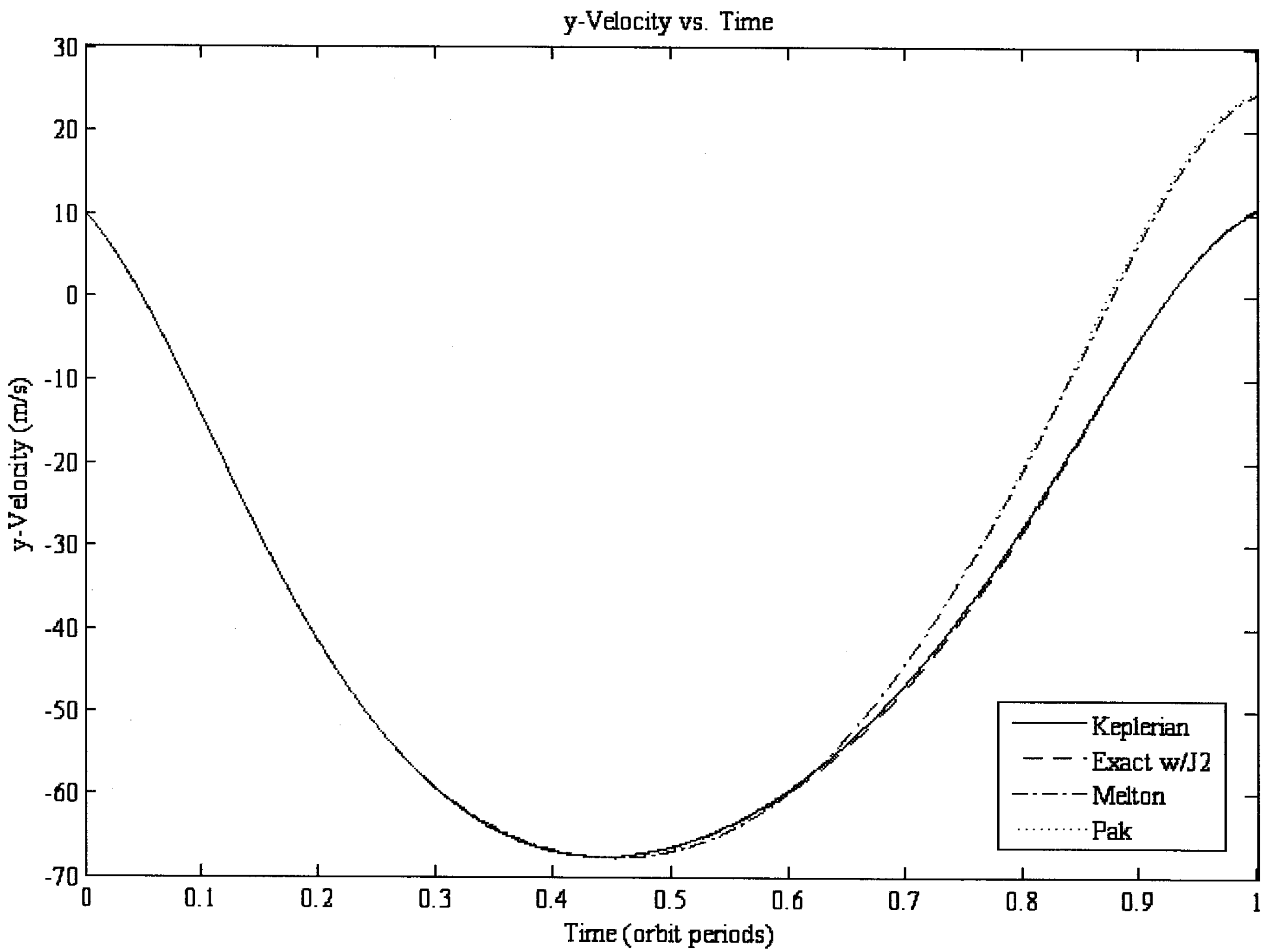

Figure 7. ***y*-Velocity vs. Time**

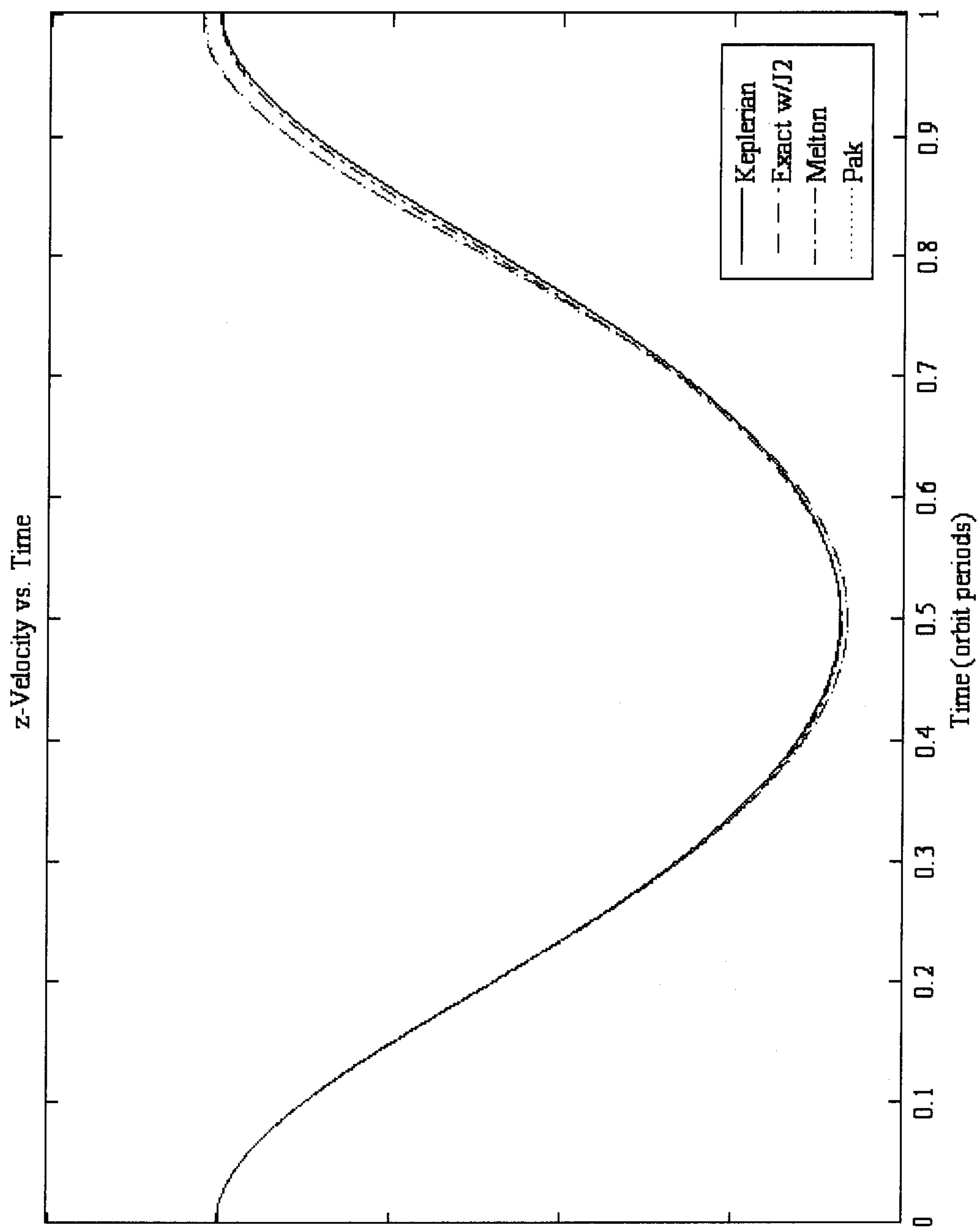

Figure 8. z-Velocity vs. Time

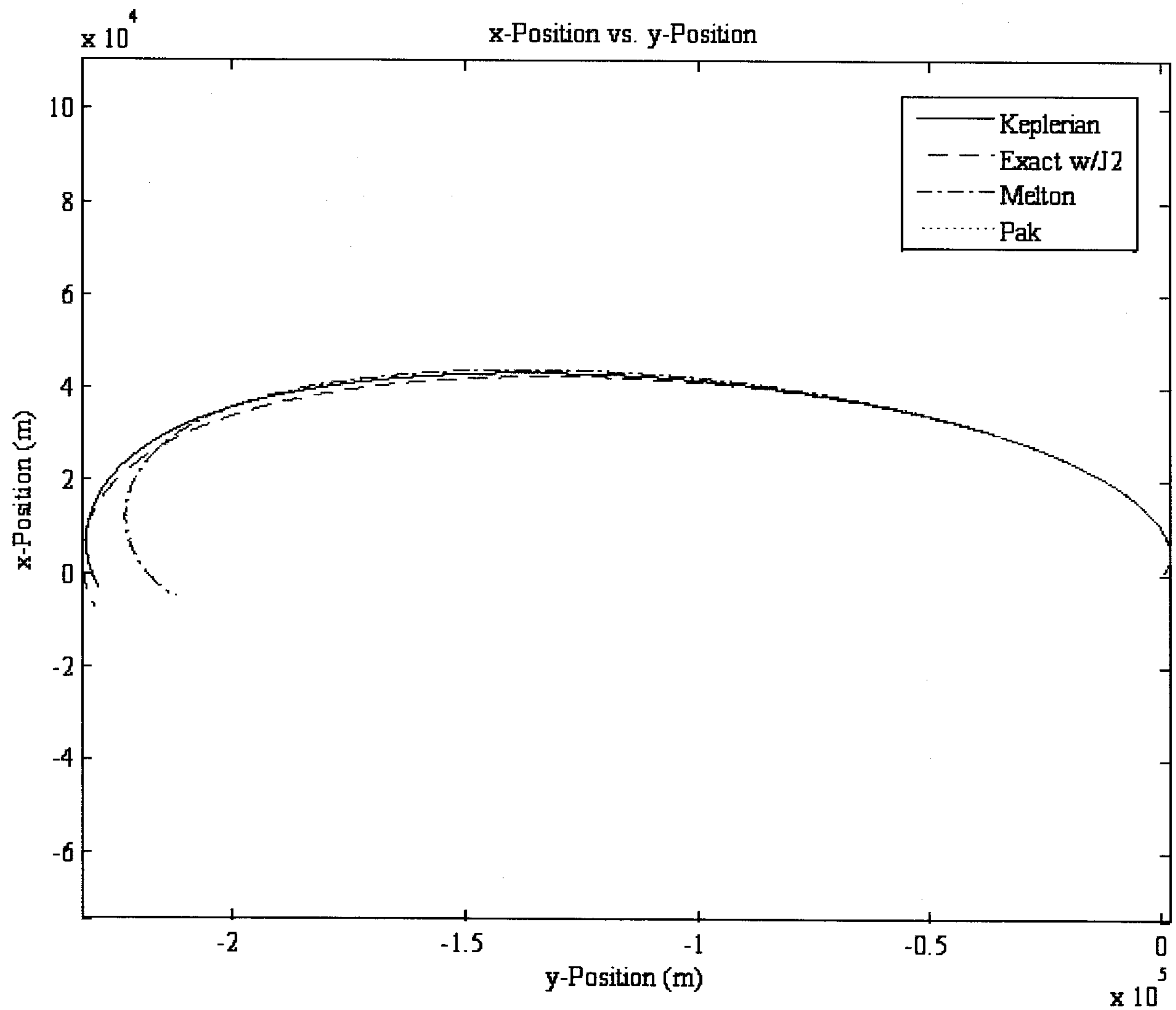

Figure 9. ***x*-Position vs. *y*-Position**

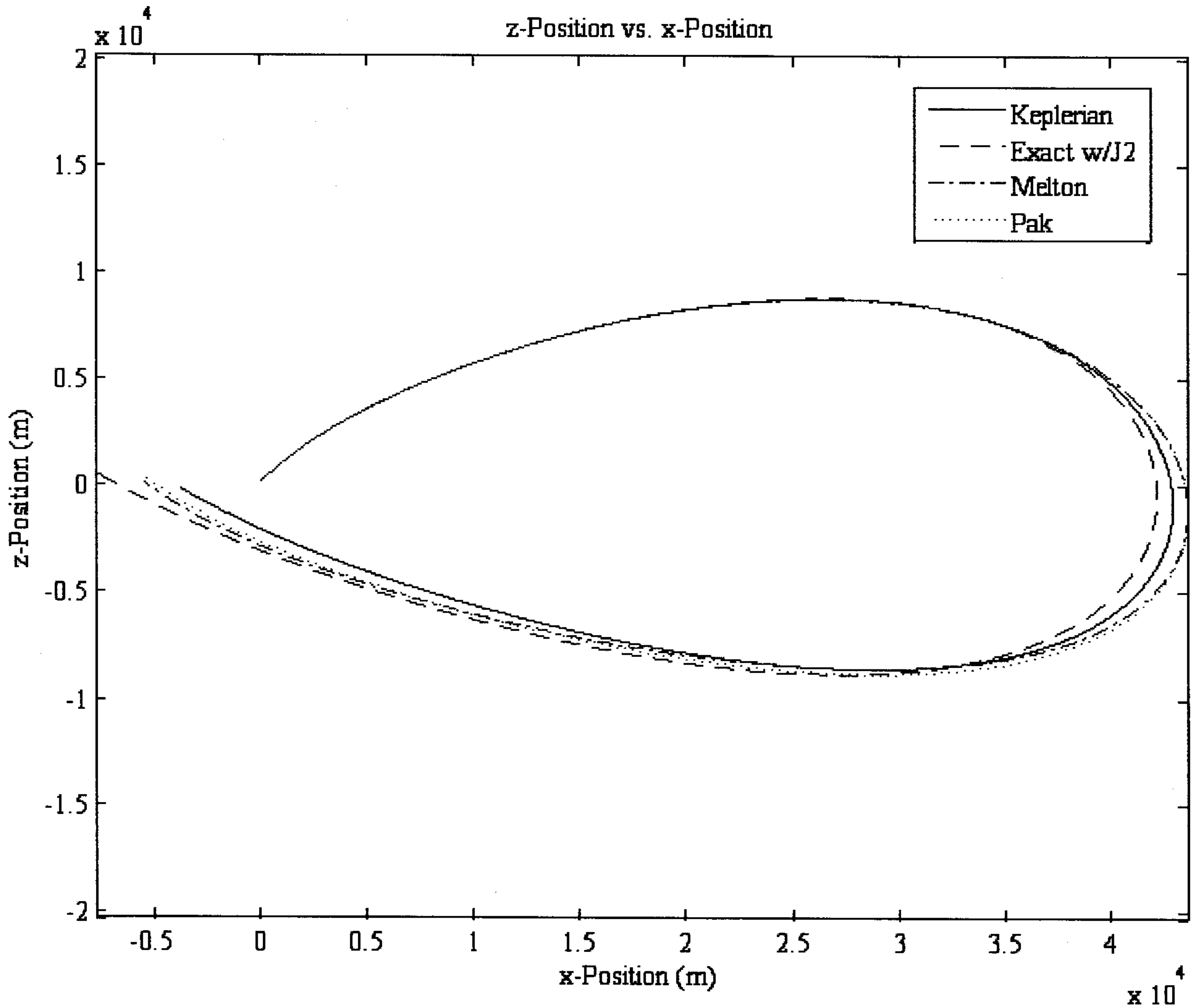

Figure 10. ***z*-Position vs. *x*-Position**

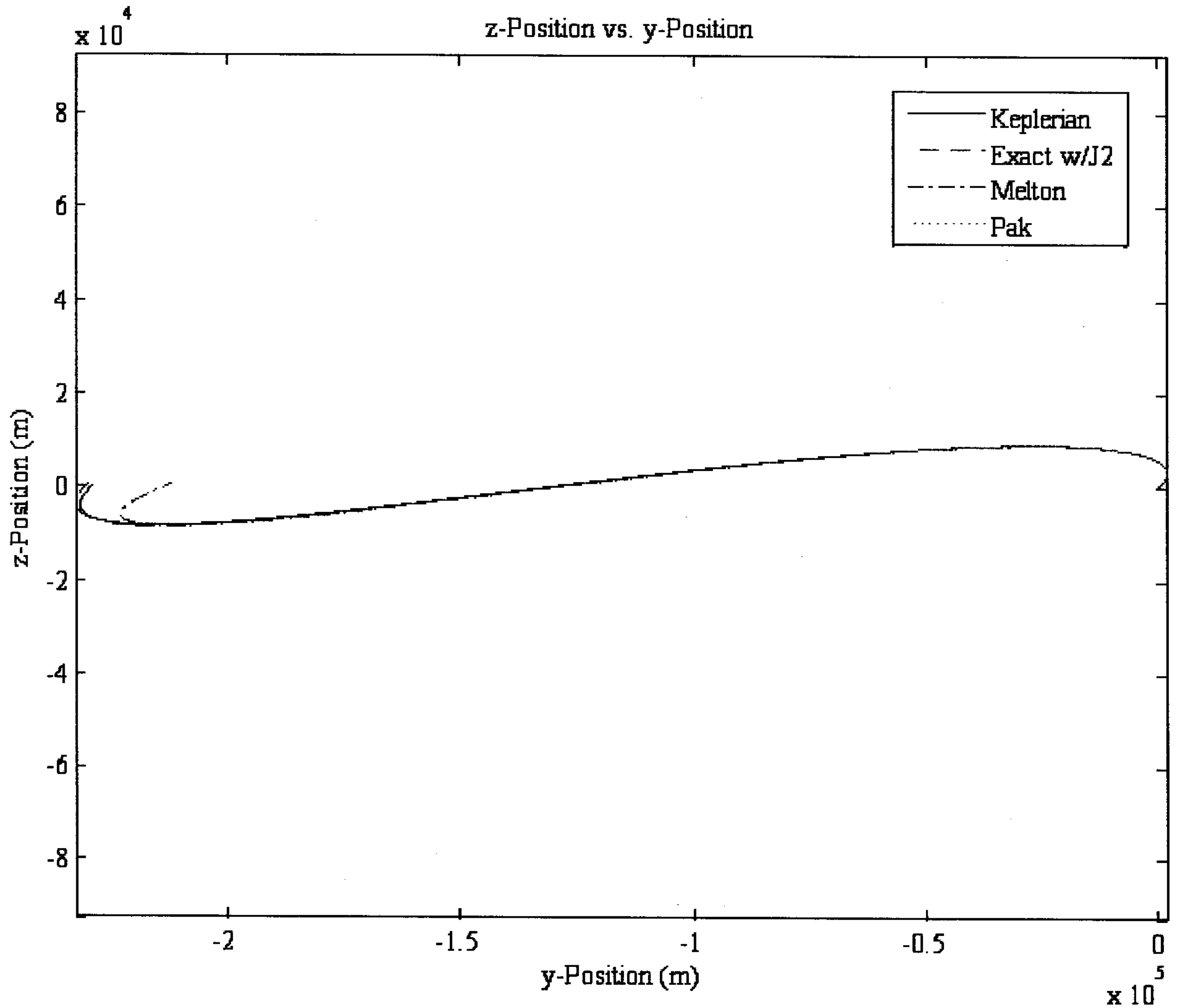

Figure 11. ***z*-Position vs. *y*-Position**

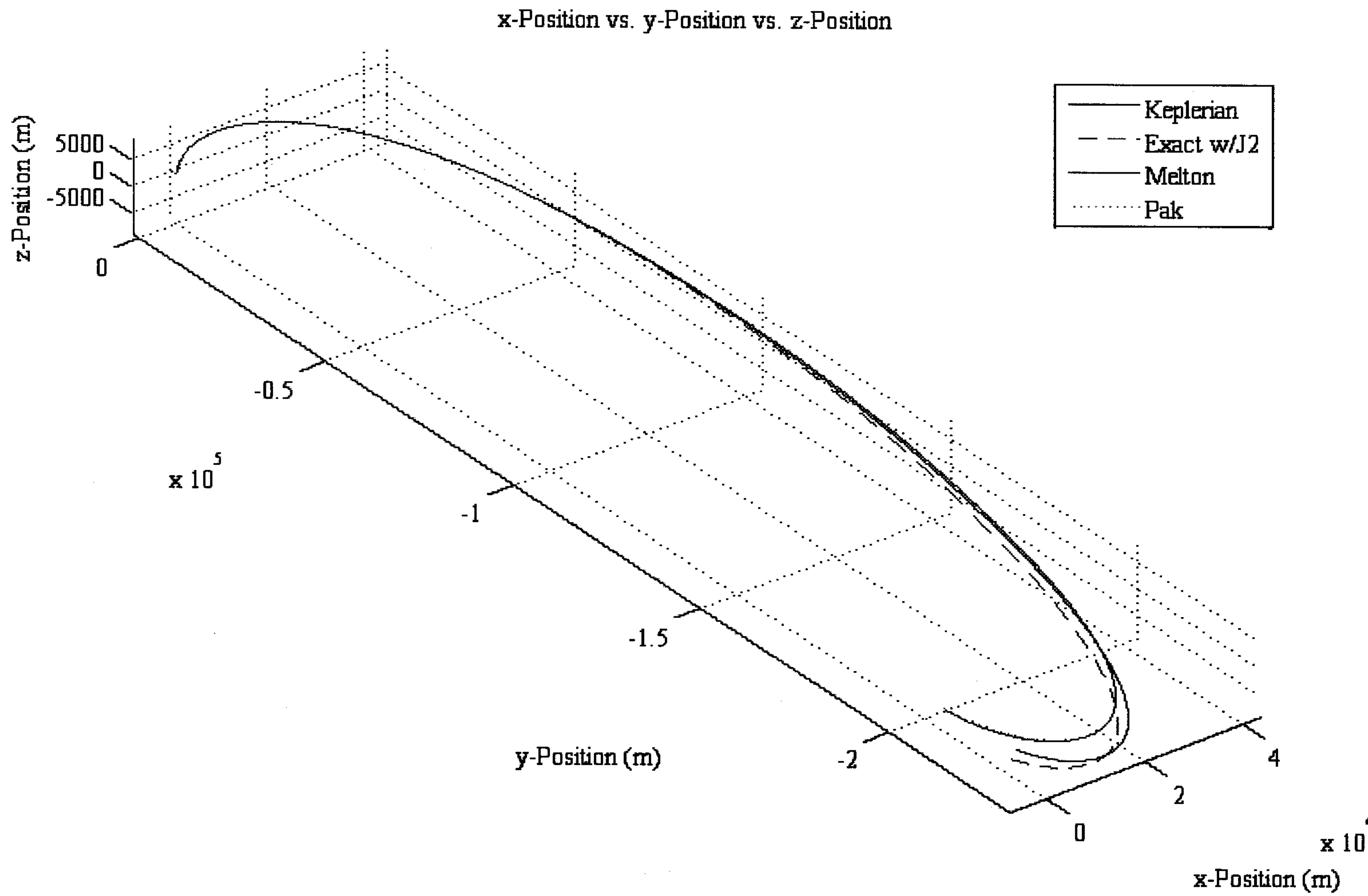

Figure 12. ***x*-Position vs. *y*-Position vs. *z*-Position**

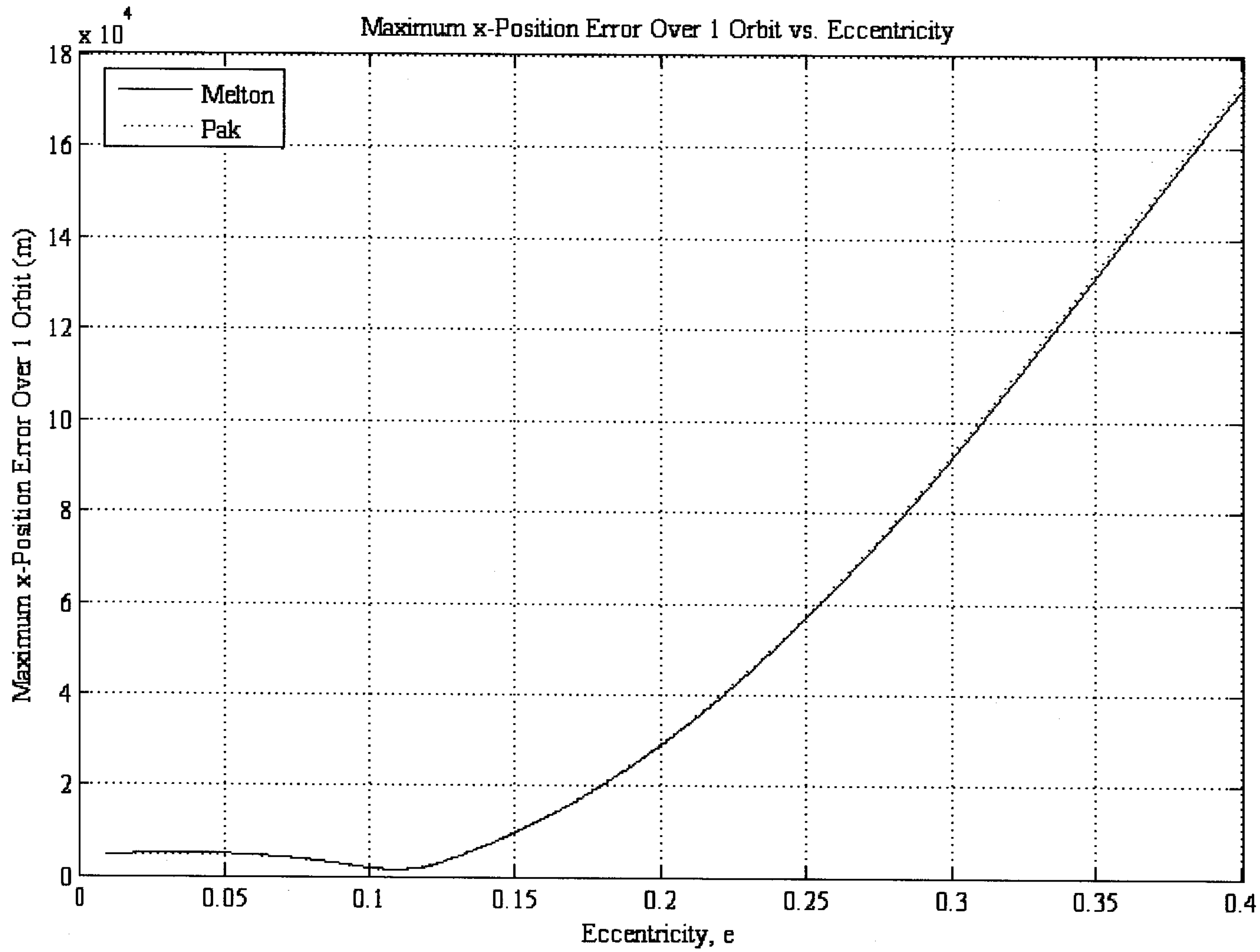

Figure 13. Maximum *x*-Position Error vs. Eccentricity

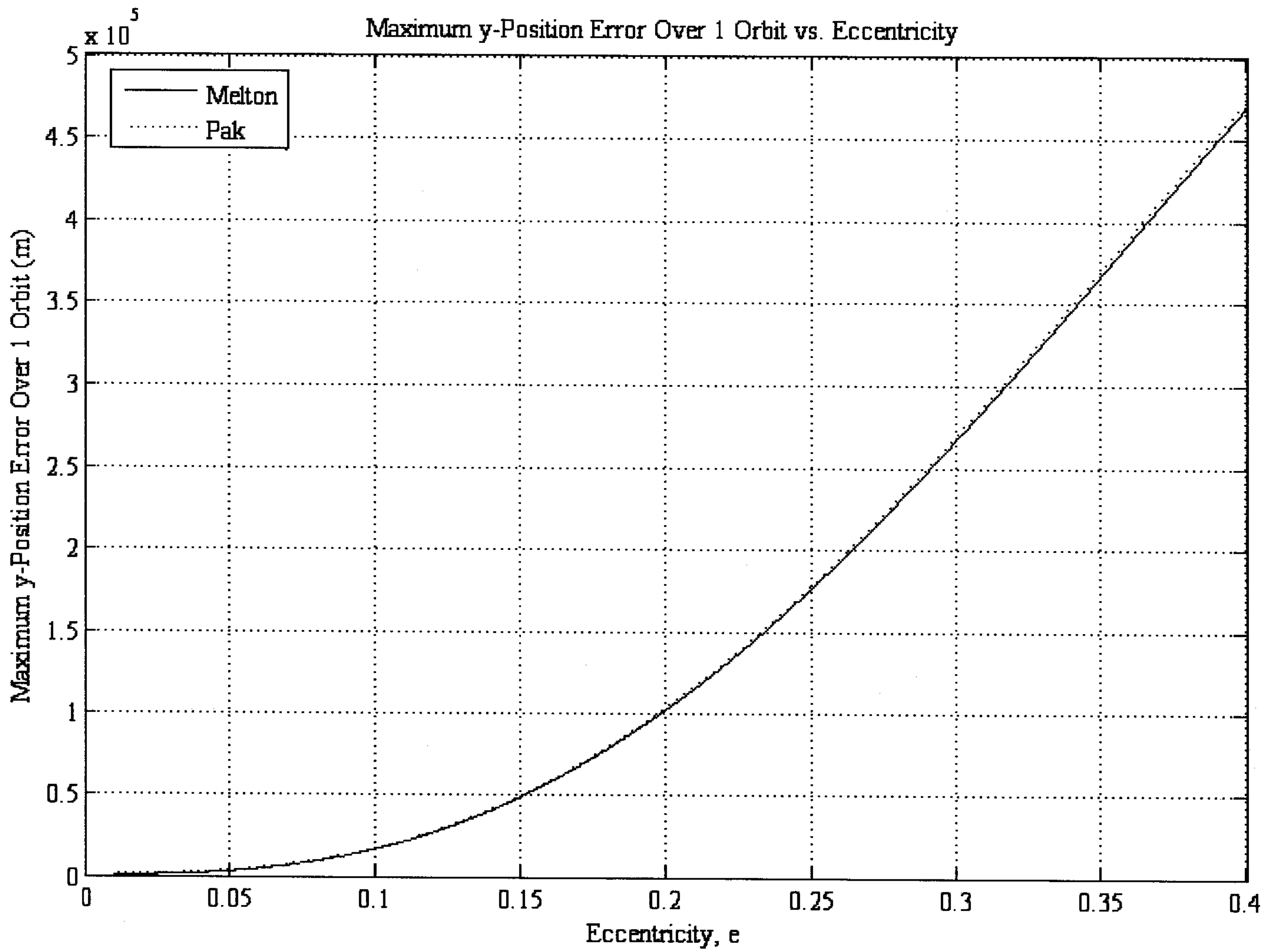

Figure 14. Maximum *y*-Position Error vs. Eccentricity

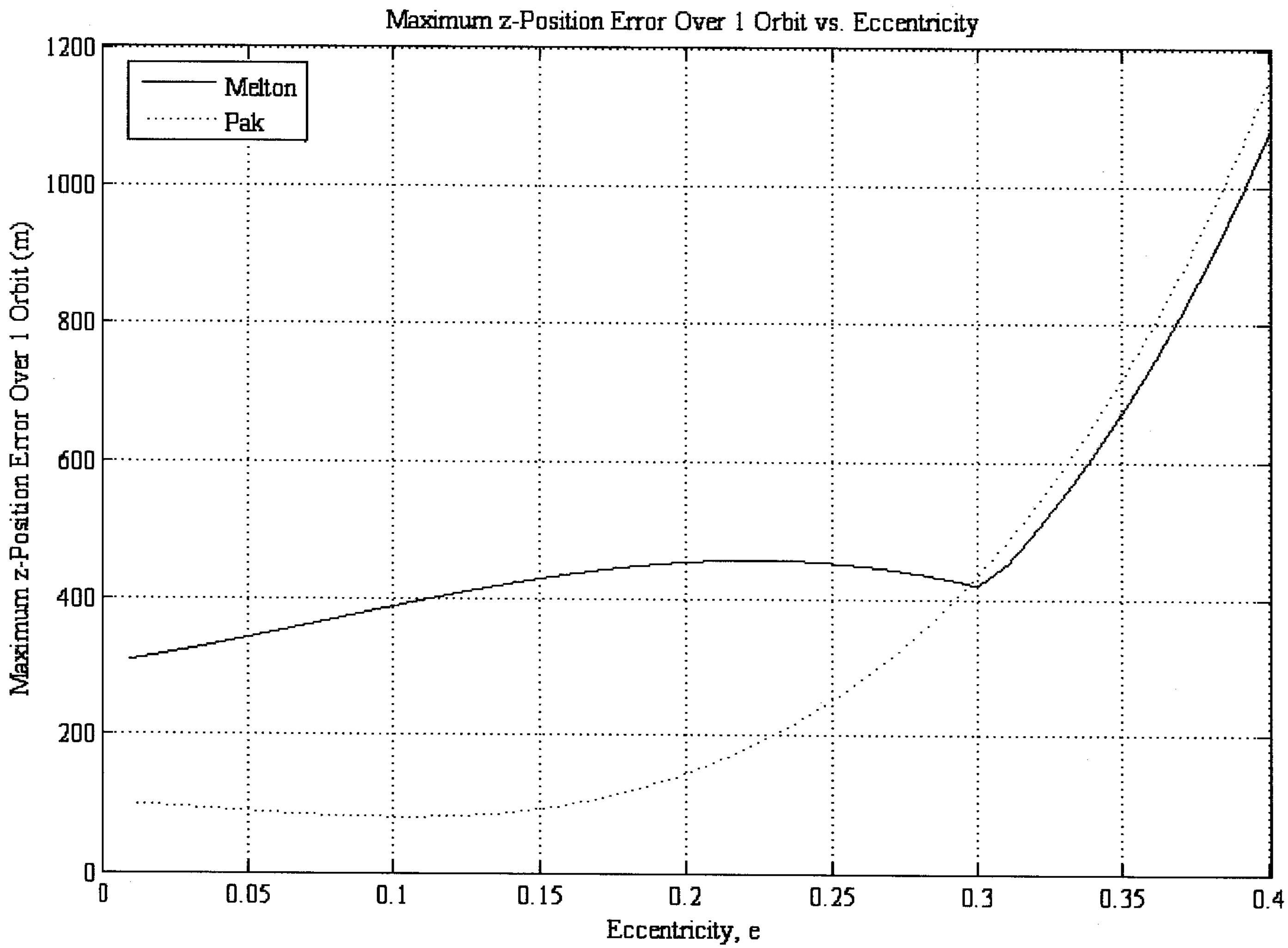

Figure 15. Maximum z-Position Error vs. Eccentricity

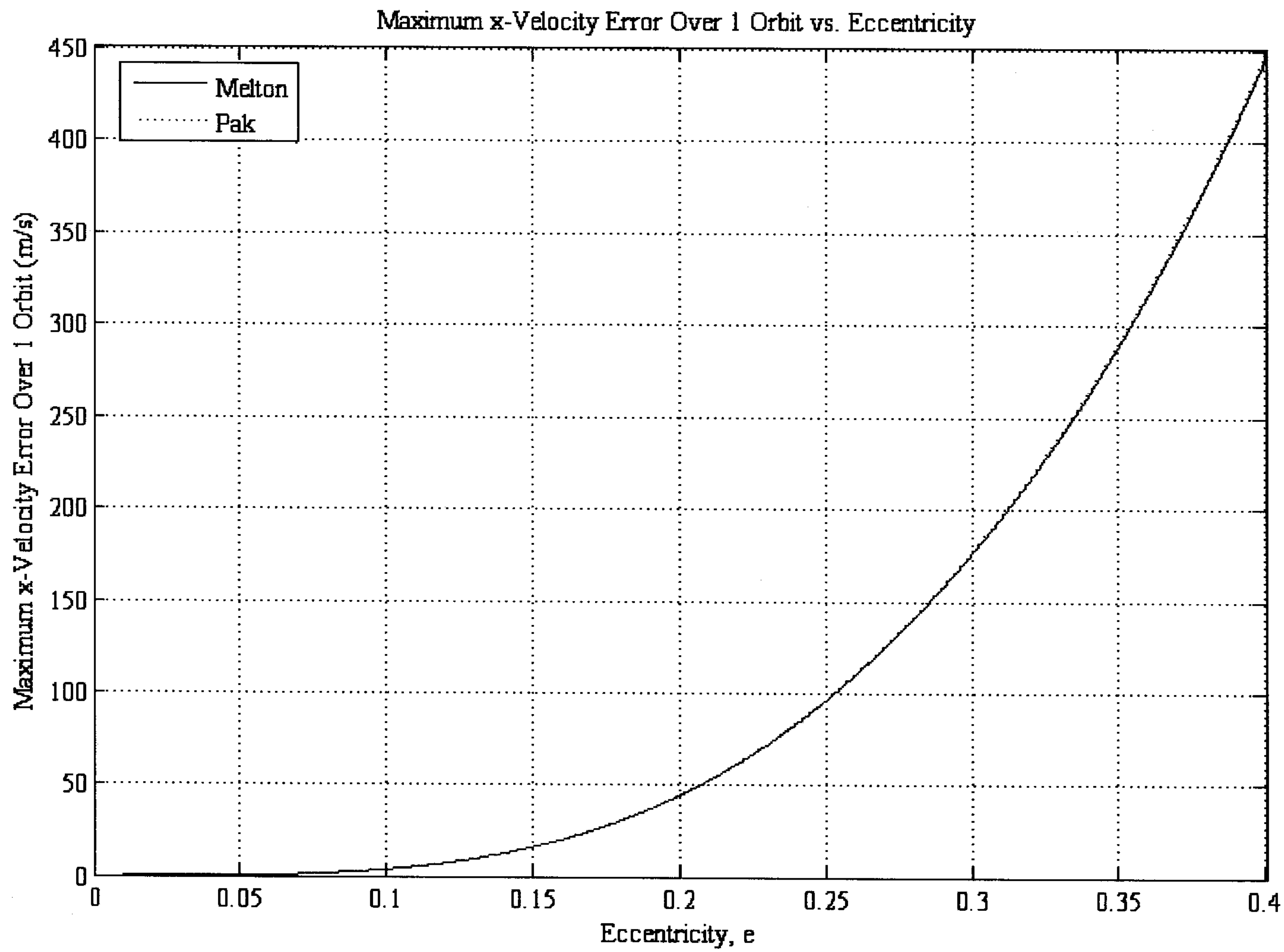

Figure 16. Maximum x-Velocity Error vs. Eccentricity

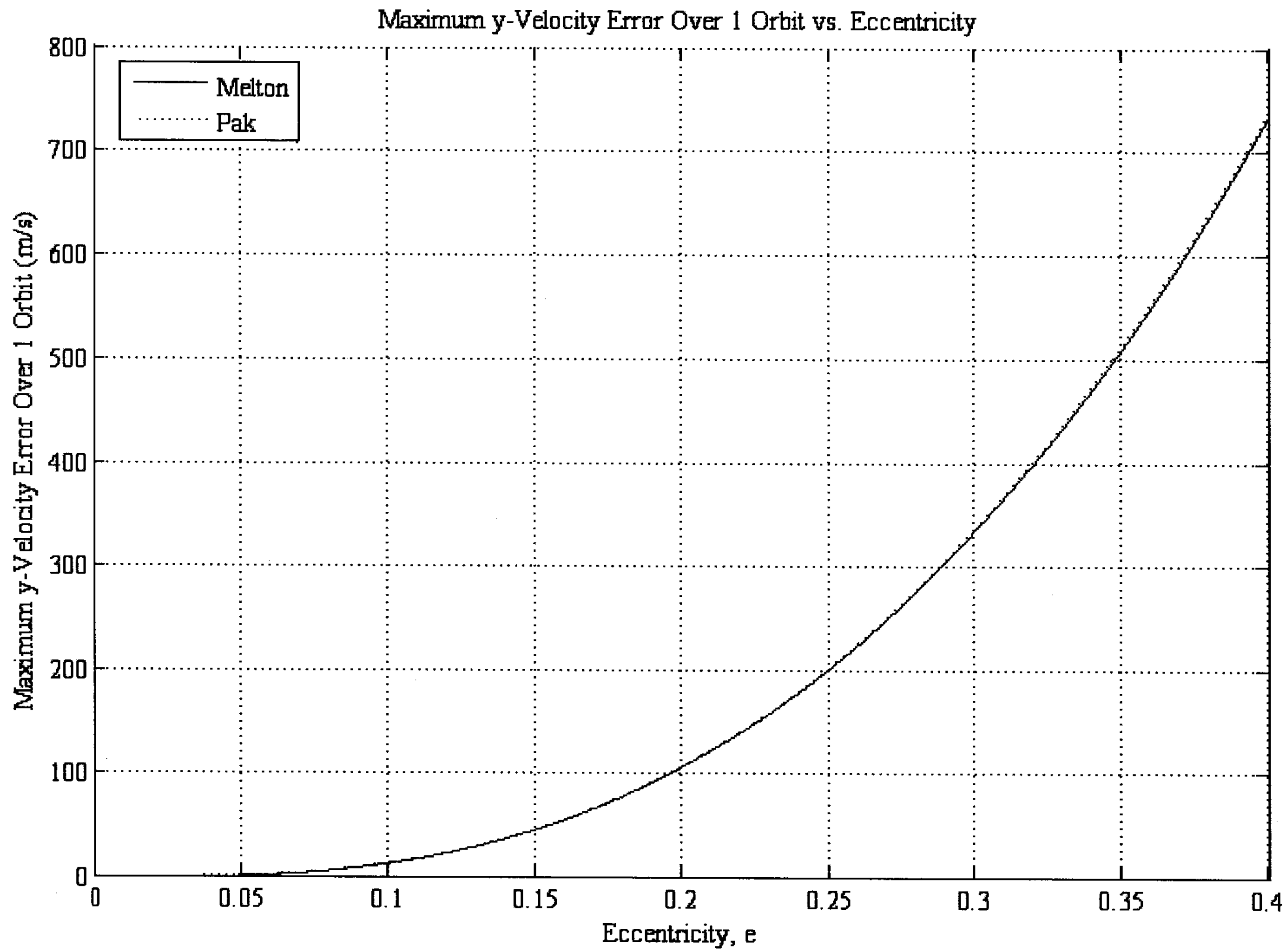

Figure 17. Maximum *y*-Velocity Error vs. Eccentricity

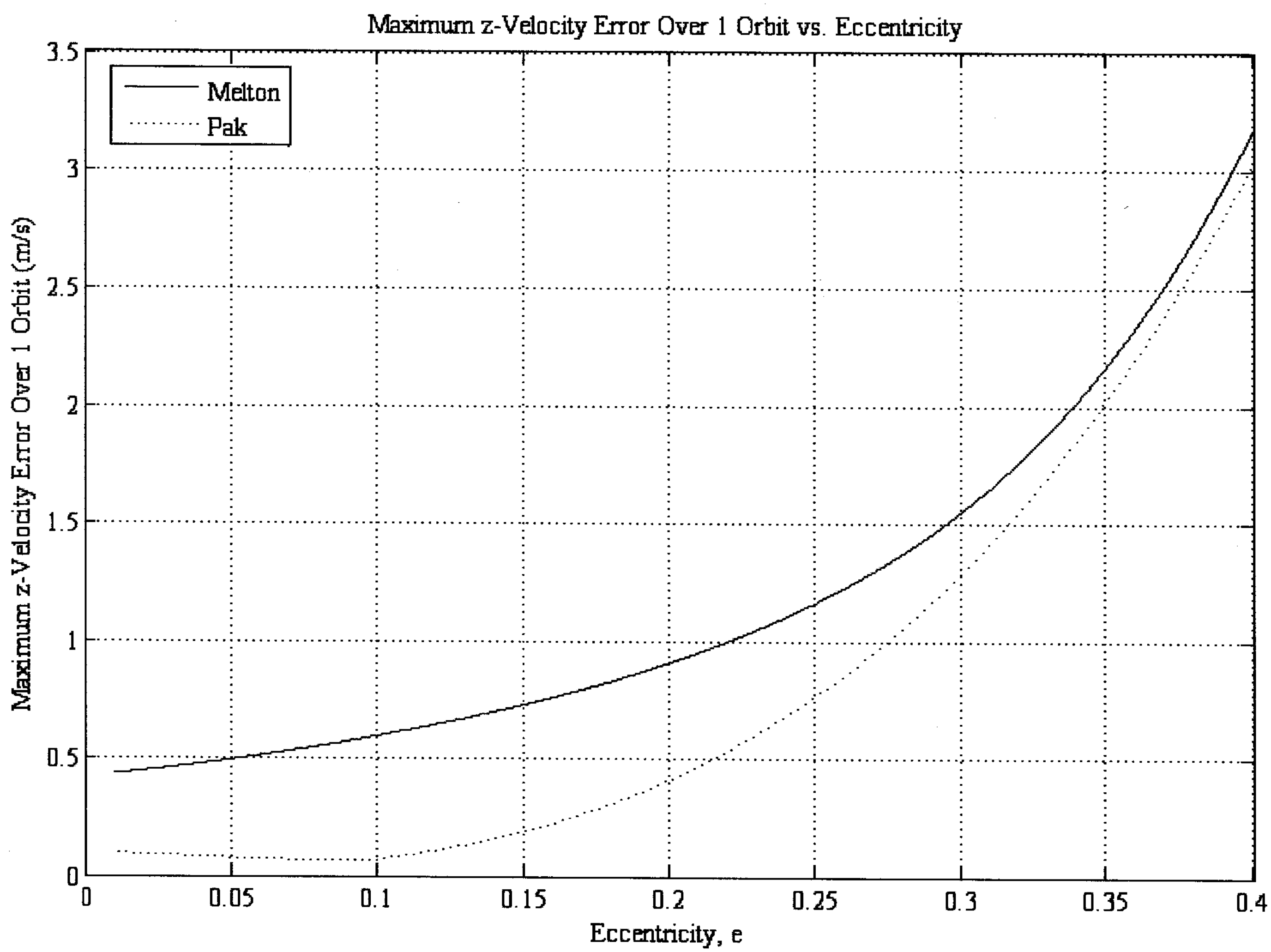

Figure 18. Maximum *z*-Velocity Error vs. Eccentricity

APPENDIX C: MATLAB CODE – SCRIPT FOR POSITION VS. TIME AND PATH PLOTS

```
global e  r1_o_init  v1_o_init  r2_o_init  v2_o_init
global t  T  x_o  x_J2_o  x_Melton_o  x_Pak_o

inputs;
mainCalcs;

tNorm = t/T;
legendStrings = {'Keplerian','Exact w/J2','Melton','Pak'};

titleStrings = {'x-Position','y-Position','z-Position',...
    'x-Velocity','y-Velocity','z-Velocity'};
ylabelStrings = {'x-Position (m)','y-Position (m)','z-Position (m)',...
    'x-Velocity (m/s)','y-Velocity (m/s)','z-Velocity (m/s)'};
for j = 1:6
    figure;
    plot(tNorm, x_o(:,j), '-',...
        tNorm, x_J2_o(:,j), '--',...
        tNorm, x_Melton_o(:,j), '-.',...
        tNorm, x_Pak_o(:,j), ':');
    axis auto;
    title([char(titleStrings(j)),' vs. Time']);
    legend(legendStrings);
    xlabel('Time (orbit periods)'); ylabel(ylabelStrings(j));
end

XY = [ 2,1 ; 1,3 ; 2,3 ];
xlabelStrings = {'y-Position','x-Position','y-Position'};
ylabelStrings = {'x-Position','z-Position','z-Position'};
for j = 1:3
    figure;
    plot(x_o(:,XY(j,1)),         x_o(:,XY(j,2)), '-',...
        x_J2_o(:,XY(j,1)),       x_J2_o(:,XY(j,2)), '--',...
        x_Melton_o(:,XY(j,1)),   x_Melton_o(:,XY(j,2)), '-.',...
        x_Pak_o(:,XY(j,1)),      x_Pak_o(:,XY(j,2)), ':' );
    axis equal;
    title([char(ylabelStrings(j)),' vs. ',char(xlabelStrings(j))]);
    legend(legendStrings);
    xlabel([char(xlabelStrings(j)),' (m)']);
    ylabel([char(ylabelStrings(j)),' (m)']);
end

figure;
plot3(...
    x_o(:,1),           x_o(:,2),           x_o(:,3), '-',...
    x_J2_o(:,1),        x_J2_o(:,2),        x_J2_o(:,3), '--',...
    x_Melton_o(:,1),    x_Melton_o(:,2),    x_Melton_o(:,3),...
    x_Pak_o(:,1),       x_Pak_o(:,2),       x_Pak_o(:,3), ':');
axis equal;
xlabel('x-Position (m)'); ylabel('y-Position (m)');
zlabel('z-Position (m)');
title('x-Position vs. y-Position vs. z-Position');
legend(legendStrings);
grid on;
```

APPENDIX D: MATLAB CODE – SCRIPT FOR ERROR PLOTS

```
global e  r1_o_init  v1_o_init  r2_o_init  v2_o_init numberOfOrbits
global x_J2_o  x_Melton_o  x_Pak_o

inputs();

% ECCENTRICITY VALUES
eValues = [0.01:0.01:0.4];

for k = 1:length(eValues)

    % OVERRIDE e VALUE FROM INPUTS
    e = eValues(k);

    mainCalcs();

    Melton_error(k,:) = max(abs(x_Melton_o - x_J2_o));
    Pak_error(k,:) = max(abs(x_Pak_o - x_J2_o));

end

yLabelStrings1 = {'x-Position', 'y-Position', 'z-Position',...
    'x-Velocity', 'y-Velocity', 'z-Velocity'};
yLabelStrings2 = {'(m)', '(m)', '(m)', '(m/s)', '(m/s)', '(m/s)'};
if numberOfOrbits == 1
    orbitString = 'Orbit';
else
    orbitString = 'Orbits';
end
for q = 1:6
    figure;
    plot(eValues, Melton_error(:,q), '-',...
        eValues, Pak_error(:,q), ':');
    title(['Maximum ',char(yLabelStrings1(q)),' Error Over ',...
            num2str(numberOfOrbits),' ',orbitString,...
            ' vs. Eccentricity']);
    legend('Melton', 'Pak', 'Location', 'NorthWest');
    xlabel('Eccentricity, e');
    ylabel(['Maximum ',char(yLabelStrings1(q)),' Error Over ',...
            num2str(numberOfOrbits),' ',orbitString,' ',...
            char(yLabelStrings2(q))]);
    grid on;
end
```

APPENDIX E: MATLAB CODE – COMMON FUNCTIONS

```
function inputs

global miu  J2  Re  a  e  i  raan  argp  f_init  r1_o_init  v1_o_init...
    r2_o_init  v2_o_init  numberOfOrbits  timeIncrement  options

%********************************
%***** INPUTS
%********************************

% GEOPHYSICAL CONSTANTS
miu = 3.986004415e14;           % From Vallado
J2 = 1082.6269e-6;              % From Vallado
Re = 6.3781363e6;               % From Vallado

% ORBITAL ELEMENTS OF REFERENCE ORBIT
a = Re + 800e3;
e = 0.1;
i = 60/180*pi;
raan = 0/180*pi;
argp = 90/180*pi;

% INITIAL TRUE ANOMALY OF REFERENCE ORBIT
f_init = 0/180*pi;

% INITIAL POSITION AND VELOCITY OF SPACECRAFT 1 AND 2 WRT REFERENCE
ORBIT
r1_o_init = [   0           0           0       ];
v1_o_init = [   0           0           0       ];
r2_o_init = [   100         100         100     ];
v2_o_init = [   10          10          10      ];

% NUMBER OF ORBITS
numberOfOrbits = 1;

% NUMERICAL INTEGRATION OPTIONS
timeIncrement = 10;
options = odeset('RelTol',1e-7,'AbsTol',1e-10);
```

```
function mainCalcs

%*********************************
%***** NOTATION
%*********************************

% Ro    ->  Ref. orbit position         [Rx Ry Rz]
% Vo    ->  Ref. orbit velocity         [Vx Vy Vz]
% Xo    ->  Ref. orbit state vector     [Rx Ry Rz Vx Vy Vz]

% R1    ->  S/C 1 position
% V1    ->  S/C 1 velocity
% X1    ->  S/C 1 state vector

% r1_eci->  S/C 1 position relative to orbit, ECI coord.
% v1_eci->  S/C 1 time derivative wrt orbit frame of r1, ECI coord.
% x1_eci->  [r1_eci  v1_eci]

% r1_o  ->  S/C 1 position relative to orbit, orbit coord.
% v1_o  ->  S/C 1 time derivative wrt orbit frame of r1, orbit coord.
% x1_o  ->  [r1_o  v1_o]

% r_eci ->  r2_eci - r1_eci
% v_eci ->  v2_eci - v1_eci
% x_eci ->  x2_eci - x1_eci

% r_o   ->  r2_o - r1_o
% v_o   ->  v2_o - v1_o
% x_o   ->  x2_o - x1_o

global miu  J2  Re  a  e  i  raan  argp  f_init  r1_o_init  v1_o_init...
    r2_o_init  v2_o_init  numberOfOrbits  timeIncrement  options

global t  T  x_o  x_J2_o  x_Melton_o  x_Pak_o       % LinearVsExact.m
global t  X2_J2  x2_Pak_o  theta  Xo               % RaanChange.m
global x_J2_o  x_Melton_o  x_Pak_o                 % ErrorPlots.m
global n  t  tp  argp  theta  Ro                   % Expansions.m

%*********************************
%***** CALCULATIONS FOR REFERENCE ORBIT
%*********************************

% ADDITIONAL ORBITAL CONSTANTS FOR REFERENCE ORBIT
n = sqrt(miu/a^3);
T = 2*pi/n;
p = a*(1 - e^2);
h = sqrt(p*miu);

% TIME OF PERIAPSIS PASSAGE FOR REFERENCE ORBIT (FOR INITIAL TIME t=0)
E = acos( (e + cos(f_init))/(1 + e*cos(f_init)) );
tp = -(E - e*sin(E))/n;

% INITIAL ARGUMENT OF LATITUDE FOR REFERENCE ORBIT
theta_init = argp + f_init;
```

```
% INITIAL REF. ORBIT STATE VECTOR (ECI VECTORS, ECI FRAME)
Ro_mag_init = a*(1 - e^2)/(1 + e*cos(f_init));
RR =[cos(raan)*cos(argp)-sin(raan)*sin(argp)*cos(i),...
    -cos(raan)*sin(argp)-sin(raan)*cos(argp)*cos(i),...
    sin(raan)*sin(i);...
    sin(raan)*cos(argp)+cos(raan)*sin(argp)*cos(i),...
    -sin(raan)*sin(argp)+cos(raan)*cos(argp)*cos(i),...
    -cos(raan)*sin(i);...
     sin(argp)*sin(i), cos(argp)*sin(i), cos(i)];
Ro_init = ( RR*(Ro_mag_init*[cos(f_init); sin(f_init); 0]) )';
Vo_init = ( RR*(sqrt(miu/p)*[-sin(f_init); e+cos(f_init);0]) )';
Xo_init = [Ro_init Vo_init];

% REF. ORBIT ANGULAR MOMENTUM VECTOR
h_vector = cross(Ro_init, Vo_init);

% INITIAL TRUE ANOMALY RATE (ORBIT FRAME ROTATION RATE)
fdot_init = h_vector/norm(Ro_init)^2;

% REF. ORBIT ECCENTRICITY
e_vector = (1/miu)*( (norm(Vo_init)^2 - miu/norm(Ro_init))*Ro_init...
    - dot(Ro_init, Vo_init)*Vo_init );

%*********************************
%***** INITIAL CONDITIONS FOR SPACECRAFT: ECI VECTORS, ECI FRAME)
%*********************************

% INITIAL S/C #1 STATE VECTOR
r1_eci_init = ( Orbit2ECI(raan, i, theta_init)*(r1_o_init') )';
v1_eci_init = ( Orbit2ECI(raan, i, theta_init)*(v1_o_init') )';
Rs1_init = Ro_init + r1_eci_init;
eci_v_eci_init = v1_eci_init + cross(fdot_init, r1_eci_init);
Vs1_init = Vo_init + eci_v_eci_init;
X1_init = [Rs1_init Vs1_init];

% INITIAL S/C #2 STATE VECTOR
r2_eci_init = ( Orbit2ECI(raan, i, theta_init)*(r2_o_init') )';
v2_eci_init = ( Orbit2ECI(raan, i, theta_init)*(v2_o_init') )';
Rs2_init = Ro_init + r2_eci_init;
eci_v2_eci_init = v2_eci_init + cross(fdot_init, r2_eci_init);
Vs2_init = Vo_init + eci_v2_eci_init;
X2_init = [Rs2_init Vs2_init];

%*********************************
%***** INITIAL CONDITIONS FOR SPACECRAFT: REL. VECTORS, ORBIT FRAME)
%*********************************

x1_o_init = [ r1_o_init  v1_o_init ];
x2_o_init = [ r2_o_init  v2_o_init ];

%*********************************
%***** TIME VALUES FOR NUMERICAL INTEGRATION
%*********************************
```

```
t = [0:timeIncrement:(numberOfOrbits*T)];

%*********************************
%***** REFERENCE ORBIT
%*********************************

[t1,Xo] = ode45(@exact_odefun, t, Xo_init, options, [miu]);

Ro = [Xo(:,1), Xo(:,2), Xo(:,3)];
Vo = [Xo(:,4), Xo(:,5), Xo(:,6)];

% TRUE ANOMALY (f), TRUE ANOMALY RATE (fdot),
% AND ARGUMENT OF LATITUDE (theta) FOR EACH TIME VALUE
for j = 1:length(t)
    if dot(Ro(j,:), Vo(j,:)) >= 0
        f(j) = acos( dot(e_vector, Ro(j,:))/(e*norm(Ro(j,:))) );
    else
        f(j) = 2*pi - acos( dot(e_vector, Ro(j,:))/(e*norm(Ro(j,:))) );
    end
    fdot(j) = h/norm(Ro(j,:))^2;
end

% LINE IMMEDIATELY BELOW REMOVES MINISCULE IMAGINARY COMPONENTS
% INTRODUCED BY NUMERICAL ROUNDOFF ERROR IN ARGUMENT FOR acos
f = real(f);

theta = argp + f;

%*********************************
%***** EXACT SOLUTION, KEPLERIAN
%*********************************

[t1,X1] = ode45(@exact_odefun,t,X1_init,options,[miu]);
[t2,X2] = ode45(@exact_odefun,t,X2_init,options,[miu]);

x_eci = X2 - X1;

r_eci       = [x_eci(:,1) x_eci(:,2) x_eci(:,3)];
eci_v_eci   = [x_eci(:,4) x_eci(:,5) x_eci(:,6)];

for j = 1:length(t)
    r_o(j,:) = ( ECI2Orbit(raan, i, theta(j))*(r_eci(j,:)') )';
    eci_v_o = ECI2Orbit(raan, i, theta(j))*eci_v_eci(j,:)';
    v_o(j,:) = ( eci_v_o + cross([0 0 -fdot(j)], r_o(j,:))' )';
end
x_o = [r_o  v_o];

%*********************************
%***** EXACT SOLUTION: WITH J2 PERTURBATIONS
%*********************************

[t1,X1_J2] = ode45(@exactwJ2_odefun,t,X1_init,options,[miu,J2,Re]);
```

```
[t2,X2_J2] = ode45(@exactwJ2_odefun,t,X2_init,options,[miu,J2,Re]);

x_J2_eci = X2_J2 - X1_J2;

r_J2_eci        = [x_J2_eci(:,1) x_J2_eci(:,2) x_J2_eci(:,3)];
eci_v_J2_eci    = [x_J2_eci(:,4) x_J2_eci(:,5) x_J2_eci(:,6)];

for j = 1:length(t)
    r_J2_o(j,:) = ( ECI2Orbit(raan, i, theta(j))*r_J2_eci(j,:)' )';
    eci_v_J2_o = ECI2Orbit(raan, i, theta(j))*eci_v_J2_eci(j,:)';
    v_J2_o(j,:) = ( eci_v_J2_o + cross([0 0 -fdot(j)], r_J2_o(j,:))' )';
end
x_J2_o = [r_J2_o  v_J2_o];

%*********************************
%***** MELTON'S EQUATIONS
%*********************************

MeltonArgs = [a,e,n,tp,h,miu];
[t1,x1_Melton_o] =
ode45(@melton_odefun,t,x1_o_init,options,MeltonArgs);
[t2,x2_Melton_o] =
ode45(@melton_odefun,t,x2_o_init,options,MeltonArgs);

x_Melton_o = x2_Melton_o - x1_Melton_o;

r_Melton_o = [x_Melton_o(:,1) x_Melton_o(:,2) x_Melton_o(:,3)];
v_Melton_o = [x_Melton_o(:,4) x_Melton_o(:,5) x_Melton_o(:,6)];

%*********************************
%***** PAK'S EQUATIONS
%*********************************

PakArgs = [a,e,n,tp,h,miu,J2,argp,Re,i];
[t1,x1_Pak_o] = ode45(@pak_odefun,t,x1_o_init,options,PakArgs);
[t2,x2_Pak_o] = ode45(@pak_odefun,t,x2_o_init,options,PakArgs);

x_Pak_o = x2_Pak_o - x1_Pak_o;

r_Pak_o = [x_Pak_o(:,1) x_Pak_o(:,2) x_Pak_o(:,3)];
v_Pak_o = [x_Pak_o(:,4) x_Pak_o(:,5) x_Pak_o(:,6)];
```

```
function C = ECI2Orbit(Om, i, theta)

C = [cos(theta)*cos(Om)-sin(theta)*cos(i)*sin(Om)...
     cos(theta)*sin(Om)+sin(theta)*cos(i)*cos(Om)...
     sin(theta)*sin(i);...
     -sin(theta)*cos(Om)-cos(theta)*cos(i)*sin(Om)...
     -sin(theta)*sin(Om)+cos(theta)*cos(i)*cos(Om)...
     cos(theta)*sin(i);...
     sin(i)*sin(Om), -sin(i)*cos(Om), cos(i)];

return


function C = Orbit2ECI(Om, i, theta)

C = [cos(theta)*cos(Om)-sin(theta)*cos(i)*sin(Om)...
     -sin(theta)*cos(Om)-cos(theta)*cos(i)*sin(Om)...
     sin(i)*sin(Om);...
     cos(theta)*sin(Om)+sin(theta)*cos(i)*cos(Om)...
     -sin(theta)*sin(Om)+cos(theta)*cos(i)*cos(Om)...
     -sin(i)*cos(Om);...
     sin(theta)*sin(i), cos(theta)*sin(i), cos(i)];

return
```

APPENDIX F: MATHEMATICA CODE – MAIN DERIVATION

- Direction cosine matrix

```
OCN = {{Cos[θ] Cos[Ω] - Sin[θ] Cos[i] Sin[Ω],
    Cos[θ] Sin[Ω] + Sin[θ] Cos[i] Cos[Ω],
    Sin[θ] Sin[i]},
   {-Sin[θ] Cos[Ω] - Cos[θ] Cos[i] Sin[Ω],
    -Sin[θ] Sin[Ω] + Cos[θ] Cos[i] Cos[Ω],
    Cos[θ] Sin[i]},
   {Sin[i] Sin[Ω], -Sin[i] Cos[Ω], Cos[i]}};
```

- ECI coordinate variables in terms of orbit coordinate variables

```
NCO = Transpose[OCN];

XYZ = NCO.{x + R0, y, z};

X = XYZ[[1]];
Y = XYZ[[2]];
Z = XYZ[[3]];
```

- Expression for J2 perturbation acceleration from Kaplan

$$\text{pGCI} = \left\{\text{Expand}\left[\frac{3\, J_2\, \mu\, R_e\verb|^|2}{2}\left(\frac{1 - 7\, x / R_0}{R_0\verb|^|7}\right)(5\, Z\verb|^|2 - (X\verb|^|2 + Y\verb|^|2 + Z\verb|^|2))\, X\right],\right.$$
$$\text{Expand}\left[\frac{3\, J_2\, \mu\, R_e\verb|^|2}{2}\left(\frac{1 - 7\, x / R_0}{R_0\verb|^|7}\right)(5\, Z\verb|^|2 - (X\verb|^|2 + Y\verb|^|2 + Z\verb|^|2))\, Y\right],$$
$$\left.\text{Expand}\left[\frac{3\, J_2\, \mu\, R_e\verb|^|2}{2}\left(\frac{1 - 7\, x / R_0}{R_0\verb|^|7}\right)(5\, Z\verb|^|2 - 3\, (X\verb|^|2 + Y\verb|^|2 + Z\verb|^|2))\, Z\right]\right\};$$

- Linearize equations

$$\text{pGCILinear} = \{0, 0, 0\};$$
$$\text{For}[j = 1,\ j \le 3,\ j{+}{+},$$
$$\quad \text{pGCILinear}[[j]] = \frac{1}{R_0^5}\,\text{Coefficient}\left[\text{pGCI}[[j]],\ \frac{1}{R_0^5}\right] +$$
$$\qquad \frac{1}{R_0^4}\,\text{Coefficient}\left[\text{pGCI}[[j]],\ \frac{1}{R_0^4}\right]$$
$$]$$

- Create matrix whose columns contain coefficients for x, y, and z and constants

```
f[x_, y_] = 0;
pGCILinearCoefxyzc = Array[f, {3, 4}];
pGCILinearTermsxyz = {0, 0, 0};
xyzList = {x, y, z};
For[j = 1, j ≤ 3, j++,
 For[k = 1, k ≤ 3, k++,
  pGCILinearCoefxyzc[[j, k]] = Coefficient[pGCILinear[[j]], xyzList[[k]]];
  pGCILinearTermsxyz[[j]] =
   pGCILinearTermsxyz[[j]] + xyzList[[k]] * pGCILinearCoefxyzc[[j, k]];
 ];
 pGCILinearCoefxyzc[[j, 4]] = pGCILinear[[j]] - pGCILinearTermsxyz[[j]];
]
```

- Simplify, show output

```
pGCILinearCoefxyzc = FullSimplify[pGCILinearCoefxyzc];

pGCILinearCoefxyzc[[1, All]]
```

$$\left\{\frac{1}{2 R_0^5}\left(3\mu\left(-1+5\,\mathrm{Cos}[2\,i]+10\,\mathrm{Cos}[2\,\theta]\,\mathrm{Sin}[i]^2\right)\left(\mathrm{Cos}[\theta]\,\mathrm{Cos}[\Omega]-\mathrm{Cos}[i]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\Omega]\right)J_2\,R_e^2\right),\right.$$

$$\frac{1}{32 R_0^5}\left(3\mu\left(2\,\mathrm{Cos}[\Omega]\left(\left(3+5\,\mathrm{Cos}[2\,i]\right)\mathrm{Sin}[\theta]+30\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[3\,\theta]\right)+\left(15\,\mathrm{Cos}[3\,i]\,\mathrm{Cos}[\theta]+\mathrm{Cos}[i]\left(\mathrm{Cos}[\theta]+60\,\mathrm{Cos}[3\,\theta]\,\mathrm{Sin}[i]^2\right)\right)\mathrm{Sin}[\Omega]\right)J_2\,R_e^2\right),$$

$$\frac{1}{16 R_0^5}\left(3\mu\left(20\,\mathrm{Cos}[\Omega]\,\mathrm{Sin}[2\,i]\,\mathrm{Sin}[2\,\theta]-\left(\left(3+5\,\mathrm{Cos}[2\,\theta]\right)\mathrm{Sin}[i]+30\,\mathrm{Sin}[3\,i]\,\mathrm{Sin}[\theta]^2\right)\mathrm{Sin}[\Omega]\right)J_2\,R_e^2\right),$$

$$\left.-\frac{1}{8 R_0^4}\left(3\mu\left(-1+5\,\mathrm{Cos}[2\,i]+10\,\mathrm{Cos}[2\,\theta]\,\mathrm{Sin}[i]^2\right)\left(\mathrm{Cos}[\theta]\,\mathrm{Cos}[\Omega]-\mathrm{Cos}[i]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\Omega]\right)J_2\,R_e^2\right)\right\}$$

```
pGCILinearCoefxyzc[[2, All]]
```

$$\left\{\frac{1}{2 R_0^5} (3 \mu (-1 + 5 \mathrm{Cos}[2 i] + 10 \mathrm{Cos}[2 \theta] \mathrm{Sin}[i]^2) (\mathrm{Cos}[i] \mathrm{Cos}[\Omega] \mathrm{Sin}[\theta] + \mathrm{Cos}[\theta] \mathrm{Sin}[\Omega]) J_2 R_e^2),\right.$$
$$-\frac{1}{8 R_0^5} (3 \mu (\mathrm{Cos}[i] \mathrm{Cos}[\theta] \mathrm{Cos}[\Omega] (-11 + 15 \mathrm{Cos}[2 i] + 30 \mathrm{Cos}[2 \theta] \mathrm{Sin}[i]^2) + (-9 + 5 \mathrm{Cos}[2 i] - 30 \mathrm{Cos}[2 \theta] \mathrm{Sin}[i]^2) \mathrm{Sin}[\theta] \mathrm{Sin}[\Omega]) J_2 R_e^2),$$
$$\frac{1}{16 R_0^5} (3 \mu \mathrm{Sin}[i] (2 \mathrm{Cos}[\Omega] (9 - 5 \mathrm{Cos}[2 \theta] + 30 \mathrm{Cos}[2 i] \mathrm{Sin}[\theta]^2) + 80 \mathrm{Cos}[i] \mathrm{Cos}[\theta] \mathrm{Sin}[\theta] \mathrm{Sin}[\Omega]) J_2 R_e^2),$$
$$\left.-\frac{1}{8 R_0^4} (3 \mu (-1 + 5 \mathrm{Cos}[2 i] + 10 \mathrm{Cos}[2 \theta] \mathrm{Sin}[i]^2) (\mathrm{Cos}[i] \mathrm{Cos}[\Omega] \mathrm{Sin}[\theta] + \mathrm{Cos}[\theta] \mathrm{Sin}[\Omega]) J_2 R_e^2)\right\}$$

```
pGCILinearCoefxyzc[[3, All]]
```

$$\left\{\frac{3 \mu (3 (\mathrm{Sin}[i] + 5 \mathrm{Sin}[3 i]) \mathrm{Sin}[\theta] + 20 \mathrm{Sin}[i]^3 \mathrm{Sin}[3 \theta]) J_2 R_e^2}{8 R_0^5},\right.$$
$$-\frac{9 \mu (15 \mathrm{Cos}[3 \theta] \mathrm{Sin}[i] + \mathrm{Cos}[\theta] (\mathrm{Sin}[i] + 20 \mathrm{Sin}[3 i] \mathrm{Sin}[\theta]^2)) J_2 R_e^2}{32 R_0^5},$$
$$-\frac{9 \mu (5 \mathrm{Cos}[3 i] + \mathrm{Cos}[i] (3 + 20 \mathrm{Cos}[2 \theta] \mathrm{Sin}[i]^2)) J_2 R_e^2}{16 R_0^5},$$
$$\left.-\frac{3 \mu (3 (\mathrm{Sin}[i] + 5 \mathrm{Sin}[3 i]) \mathrm{Sin}[\theta] + 20 \mathrm{Sin}[i]^3 \mathrm{Sin}[3 \theta]) J_2 R_e^2}{32 R_0^4}\right\}$$

- **Transform ECI components into orbit components, simplify, show output**

```
pOLinearCoefxyzc = FullSimplify[Expand[OCN.pGCILinearCoefxyzc]];
pOLinearCoefxyzc[[1, All]]
```

$$\left\{\frac{3 \mu (1 + 3 \mathrm{Cos}[2 i] + 6 \mathrm{Cos}[2 \theta] \mathrm{Sin}[i]^2) J_2 R_e^2}{2 R_0^5}, \frac{6 \mu \mathrm{Sin}[i]^2 \mathrm{Sin}[2 \theta] J_2 R_e^2}{R_0^5}, \frac{6 \mu \mathrm{Sin}[2 i] \mathrm{Sin}[\theta] J_2 R_e^2}{R_0^5}, -\frac{3 \mu (1 + 3 \mathrm{Cos}[2 i] + 6 \mathrm{Cos}[2 \theta] \mathrm{Sin}[i]^2) J_2 R_e^2}{8 R_0^4}\right\}$$

```
pOLinearCoefxyzc[[2, All]]
```

$$\left\{\frac{6 \mu \mathrm{Sin}[i]^2 \mathrm{Sin}[2 \theta] J_2 R_e^2}{R_0^5}, -\frac{3 \mu (1 + 3 \mathrm{Cos}[2 i] + 14 \mathrm{Cos}[2 \theta] \mathrm{Sin}[i]^2) J_2 R_e^2}{8 R_0^5}, -\frac{3 \mu \mathrm{Cos}[i] \mathrm{Cos}[\theta] \mathrm{Sin}[i] J_2 R_e^2}{R_0^5}, -\frac{3 \mu \mathrm{Cos}[\theta] \mathrm{Sin}[i]^2 \mathrm{Sin}[\theta] J_2 R_e^2}{R_0^4}\right\}$$

```
pOLinearCoefxyzc[[3, All]]
```

$$\left\{\frac{6\,\mu\,\mathrm{Sin}[2\,i]\,\mathrm{Sin}[\theta]\,J_2\,R_e^2}{R_0^5},\ -\frac{3\,\mu\,\mathrm{Cos}[i]\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[i]\,J_2\,R_e^2}{R_0^5},\ -\frac{3\,\mu\,(3+9\,\mathrm{Cos}[2\,i]+10\,\mathrm{Cos}[2\,\theta]\,\mathrm{Sin}[i]^2)\,J_2\,R_e^2}{8\,R_0^5},\ -\frac{3\,\mu\,\mathrm{Cos}[i]\,\mathrm{Sin}[i]\,\mathrm{Sin}[\theta]\,J_2\,R_e^2}{R_0^4}\right\}$$

- Create placeholders for trig functions for i

```
pOLinearCoefxyzc =
  pOLinearCoefxyzc /. {Cos[i] → cosi, Sin[i] → sini, Cos[2 i] → cosTwoi,
    Sin[2 i] → sinTwoi};
```

- Series expansions for time-varying quantities (expressions brought in from other notebooks).

$$\mathrm{cosTheta} = -e\,\mathrm{Cos}[\omega] + e\,\mathrm{Cos}[2\,M]\,\mathrm{Cos}[\omega] + \frac{9}{8}\,e^2\,\mathrm{Cos}[3\,M]\,\mathrm{Cos}[\omega] + \mathrm{Cos}[M]\left(\mathrm{Cos}[\omega] - \frac{9}{8}\,e^2\,\mathrm{Cos}[\omega]\right) + \frac{1}{8}\sqrt{1-e^2}\,(-8+3\,e^2)\,\mathrm{Sin}[M]\,\mathrm{Sin}[\omega] - e\sqrt{1-e^2}\,\mathrm{Sin}[2\,M]\,\mathrm{Sin}[\omega] - \frac{9}{8}\,e^2\sqrt{1-e^2}\,\mathrm{Sin}[3\,M]\,\mathrm{Sin}[\omega];$$

$$\mathrm{sinTheta} = \frac{1}{8}\,(8-3\,e^2)\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[M] + e\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[2\,M] + \frac{9}{8}\,e^2\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[3\,M] - e\,\mathrm{Sin}[\omega] + e\,\mathrm{Cos}[2\,M]\,\mathrm{Sin}[\omega] + \frac{9}{8}\,e^2\,\mathrm{Cos}[3\,M]\,\mathrm{Sin}[\omega] + \mathrm{Cos}[M]\left(\mathrm{Sin}[\omega] - \frac{9}{8}\,e^2\,\mathrm{Sin}[\omega]\right);$$

$$\mathrm{cosTwoTheta} = \frac{3}{4}\,e^2\,\mathrm{Cos}[2\,\omega] - 2\,e\,\mathrm{Cos}[M]\,\mathrm{Cos}[2\,\omega] + (1-4\,e^2)\,\mathrm{Cos}[2\,M]\,\mathrm{Cos}[2\,\omega] + 2\,e\,\mathrm{Cos}[3\,M]\,\mathrm{Cos}[2\,\omega] + \frac{13}{4}\,e^2\,\mathrm{Cos}[4\,M]\,\mathrm{Cos}[2\,\omega] + 4\,e\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[M]\,\mathrm{Sin}[\omega] + \sqrt{1-e^2}\,(-2+7\,e^2)\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[2\,M]\,\mathrm{Sin}[\omega] - 4\,e\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[3\,M]\,\mathrm{Sin}[\omega] - \frac{13}{2}\,e^2\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[4\,M]\,\mathrm{Sin}[\omega];$$

$$\mathrm{sinTwoTheta} = -2\,e\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[M] + \frac{1}{2}\,(2-7\,e^2)\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[2\,M] + 2\,e\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[3\,M] + \frac{13}{4}\,e^2\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[4\,M] + \frac{3}{2}\,e^2\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[\omega] - 4\,e\,\mathrm{Cos}[M]\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[\omega] + 4\,e\,\mathrm{Cos}[3\,M]\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[\omega] + \frac{13}{2}\,e^2\,\mathrm{Cos}[4\,M]\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[\omega] + (1-4\,e^2)\,\mathrm{Cos}[2\,M]\,\mathrm{Sin}[2\,\omega];$$

$$\mathrm{oneOverRFourth} = \left(1+3\,e^2+4\,e\,\mathrm{Cos}[M]+7\,e^2\,\mathrm{Cos}[2\,M]\right)/a^4;$$

$$\mathrm{oneOverRFifth} = \left(1+5\,e^2+5\,e\,\mathrm{Cos}[M]+10\,e^2\,\mathrm{Cos}[2\,M]\right)/a^5;$$

- Create place holders in series expansions for trig functions that are not functions of M.

```
cosTheta = cosTheta /. {Cos[ω] → cosw, Sin[ω] → sinw};
sinTheta = sinTheta /. {Cos[ω] → cosw, Sin[ω] → sinw};
cosTwoTheta =
  cosTwoTheta /. {Cos[ω] → cosw, Sin[ω] → sinw, Cos[2 ω] → cosTwow,
    Sin[2 ω] → sinTwow};
sinTwoTheta =
  sinTwoTheta /. {Cos[ω] → cosw, Sin[ω] → sinw, Cos[2 ω] → cosTwow,
    Sin[2 ω] → sinTwow};
```

- Insert series expansions, trig reduce, eliminate terms of order greater than e^2, collect

```
pOLinearCoefxyzc =
  pOLinearCoefxyzc /. {Cos[θ] → cosTheta, Sin[θ] → sinTheta,
    Cos[2 θ] → cosTwoTheta, Sin[2 θ] → sinTwoTheta, 1/R0^4 → oneOverRFourth,
    1/R0^5 → oneOverRFifth};

pOLinearCoefxyzc = TrigReduce[Expand[pOLinearCoefxyzc]];

pOLinearCoefxyzc = Coefficient[pOLinearCoefxyzc, e, 0] +
   e*Coefficient[pOLinearCoefxyzc, e, 1] +
   e^2*Coefficient[pOLinearCoefxyzc, e, 2];

trigFuncList = {Cos[M], Cos[2 M], Cos[3 M], Cos[4 M], Sin[M], Sin[2 M],
   Sin[3 M], Sin[4 M]};

pOLinearCoefxyzc = Collect[pOLinearCoefxyzc, trigFuncList];
```

- Create data structure with elements for each cosnM and sinnM term

```
f[x_, y_, z_] = 0;
pOLinearCoefxyzcCoefTrignM = Array[f, {3, 4, 9}];
f[x_, y_] = 0;
pOLinearCoefxyzcTermsTrignM = Array[f, {3, 4}];
For[j = 1, j ≤ 3, j++,
 For[k = 1, k ≤ 4, k++,
  For[l = 1, l ≤ 8, l++,
   pOLinearCoefxyzcCoefTrignM[[j, k, l]] =
    Coefficient[pOLinearCoefxyzc[[j, k]], trigFuncList[[l]]];
   pOLinearCoefxyzcTermsTrignM[[j, k]] =
    pOLinearCoefxyzcTermsTrignM[[j, k]] +
     pOLinearCoefxyzcCoefTrignM[[j, k, l]] * trigFuncList[[l]]
  ];
  pOLinearCoefxyzcCoefTrignM[[j, k, 9]] =
   pOLinearCoefxyzc[[j, k]] - pOLinearCoefxyzcTermsTrignM[[j, k]]
 ]
]
```

- Replace the placeholders with actual functions, full simplify.

```
pOLinearCoefxyzcCoefTrignM =
  pOLinearCoefxyzcCoefTrignM /.
   {cosi → Cos[i], sini → Sin[i], cosTwoi → Cos[2 i], sinTwoi → Sin[2 i],
    cosw → Cos[ω], sinw → Sin[ω], cosTwow → Cos[2 ω], sinTwow → Sin[2 ω]};

pOLinearCoefxyzcCoefTrignM = FullSimplify[pOLinearCoefxyzcCoefTrignM];
```

- Split data structure into P and q

```
PCoefTrignM = {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}};
qCoefTrignM = {0, 0, 0};

For[j = 1, j ≤ 3, j++,
 For[k = 1, k ≤ 3, k++,
  PCoefTrignM[[j, k]] = pOLinearCoefxyzcCoefTrignM[[j, k, All]]
 ];
 qCoefTrignM[[j]] = pOLinearCoefxyzcCoefTrignM[[j, 4, All]]
]
```

Simplify[Expand[PCoefTrignM[[1, 1]] * (a^5 / (μ J_2 R_e^2))]]

$$\left\{\frac{3}{2}\,e\,(5+15\,\mathrm{Cos}[2\,i]+3\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2),\right.$$
$$15\,e^2+45\,e^2\,\mathrm{Cos}[2\,i]+9\,(1+e^2)\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ \frac{81}{2}\,e\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,$$
$$\frac{477}{4}\,e^2\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ -9\,e\,\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],$$
$$-9\,\sqrt{1-e^2}\,(2+3\,e^2)\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],$$
$$-81\,e\,\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],\ -\frac{477}{2}\,e^2\,\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],$$
$$\left.\frac{3}{4}\,(2+10\,e^2+6\,(1+5\,e^2)\,\mathrm{Cos}[2\,i]+9\,e^2\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2)\right\}$$

Simplify[Expand[PCoefTrignM[[1, 2]] * (a^5 / (μ J_2 R_e^2))]]

$$\left\{6\,e\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],\ 6\,(1+e^2)\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[2\,\omega],\right.$$
$$27\,e\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[2\,\omega],\ 159\,e^2\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],\ 3\,e\,\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,$$
$$3\,\sqrt{1-e^2}\,(2+3\,e^2)\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ 27\,e\,\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,$$
$$\left.\frac{159}{2}\,e^2\,\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ 9\,e^2\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega]\right\}$$

Simplify[Expand[PCoefTrignM[[1, 3]] * (a^5 / (μ J_2 R_e^2))]]

$$\left\{\frac{3}{4}\,(8+51\,e^2)\,\mathrm{Sin}[2\,i]\,\mathrm{Sin}[\omega],\ 21\,e\,\mathrm{Sin}[2\,i]\,\mathrm{Sin}[\omega],\ \frac{207}{4}\,e^2\,\mathrm{Sin}[2\,i]\,\mathrm{Sin}[\omega],\right.$$
$$0,\ \frac{3}{4}\,\sqrt{1-e^2}\,(8+17\,e^2)\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[2\,i],\ 21\,e\,\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[2\,i],$$
$$\left.\frac{207}{4}\,e^2\,\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[2\,i],\ 0,\ 9\,e\,\mathrm{Sin}[2\,i]\,\mathrm{Sin}[\omega]\right\}$$

Simplify[Expand[PCoefTrignM[[2, 1]] * (a^5 / (μ J_2 R_e^2))]]

$$\left\{6\,e\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],\ 6\,(1+e^2)\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[2\,\omega],\right.$$
$$27\,e\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[2\,\omega],\ 159\,e^2\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],\ 3\,e\,\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,$$
$$3\,\sqrt{1-e^2}\,(2+3\,e^2)\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ 27\,e\,\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,$$
$$\left.\frac{159}{2}\,e^2\,\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ 9\,e^2\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega]\right\}$$

Simplify[Expand[PCoefTrignM[[2, 2]] * (a^5 / (μ J2 Re^2))]]

$$\left\{-\frac{3}{8} e \left(5+15 \operatorname{Cos}[2 i]+7 \operatorname{Cos}[2 \omega] \operatorname{Sin}[i]^{2}\right),\right.$$
$$-\frac{3}{4}\left(5 e^{2}+15 e^{2} \operatorname{Cos}[2 i]+7\left(1+e^{2}\right) \operatorname{Cos}[2 \omega] \operatorname{Sin}[i]^{2}\right),$$
$$-\frac{189}{8} e \operatorname{Cos}[2 \omega] \operatorname{Sin}[i]^{2},\ -\frac{1113}{16} e^{2} \operatorname{Cos}[2 \omega] \operatorname{Sin}[i]^{2},$$
$$\frac{21}{4} e \sqrt{1-e^{2}} \operatorname{Cos}[\omega] \operatorname{Sin}[i]^{2} \operatorname{Sin}[\omega],\ \frac{21}{8} \sqrt{1-e^{2}}\left(2+3 e^{2}\right) \operatorname{Sin}[i]^{2} \operatorname{Sin}[2 \omega],$$
$$\frac{189}{4} e \sqrt{1-e^{2}} \operatorname{Cos}[\omega] \operatorname{Sin}[i]^{2} \operatorname{Sin}[\omega],\ \frac{1113}{8} e^{2} \sqrt{1-e^{2}} \operatorname{Cos}[\omega] \operatorname{Sin}[i]^{2} \operatorname{Sin}[\omega],$$
$$\left.-\frac{3}{16}\left(2+10 e^{2}+6\left(1+5 e^{2}\right) \operatorname{Cos}[2 i]+21 e^{2} \operatorname{Cos}[2 \omega] \operatorname{Sin}[i]^{2}\right)\right\}$$

Simplify[Expand[PCoefTrignM[[2, 3]] * (a^5 / (μ J2 Re^2))]]

$$\left\{-\frac{3}{8}\left(8+51 e^{2}\right) \operatorname{Cos}[i] \operatorname{Cos}[\omega] \operatorname{Sin}[i],\right.$$
$$-\frac{21}{2} e \operatorname{Cos}[i] \operatorname{Cos}[\omega] \operatorname{Sin}[i],\ -\frac{207}{8} e^{2} \operatorname{Cos}[i] \operatorname{Cos}[\omega] \operatorname{Sin}[i],\ 0,$$
$$\frac{3}{8} \sqrt{1-e^{2}}\left(8+17 e^{2}\right) \operatorname{Cos}[i] \operatorname{Sin}[i] \operatorname{Sin}[\omega],\ \frac{21}{2} e \sqrt{1-e^{2}} \operatorname{Cos}[i] \operatorname{Sin}[i] \operatorname{Sin}[\omega],$$
$$\left.\frac{207}{8} e^{2} \sqrt{1-e^{2}} \operatorname{Cos}[i] \operatorname{Sin}[i] \operatorname{Sin}[\omega],\ 0,\ -\frac{9}{2} e \operatorname{Cos}[i] \operatorname{Cos}[\omega] \operatorname{Sin}[i]\right\}$$

Simplify[Expand[PCoefTrignM[[3, 1]] * (a^5 / (μ J2 Re^2))]]

$$\left\{\frac{3}{4}\left(8+51 e^{2}\right) \operatorname{Sin}[2 i] \operatorname{Sin}[\omega],\ 21 e \operatorname{Sin}[2 i] \operatorname{Sin}[\omega],\ \frac{207}{4} e^{2} \operatorname{Sin}[2 i] \operatorname{Sin}[\omega],\right.$$
$$0,\ \frac{3}{4} \sqrt{1-e^{2}}\left(8+17 e^{2}\right) \operatorname{Cos}[\omega] \operatorname{Sin}[2 i],\ 21 e \sqrt{1-e^{2}} \operatorname{Cos}[\omega] \operatorname{Sin}[2 i],$$
$$\left.\frac{207}{4} e^{2} \sqrt{1-e^{2}} \operatorname{Cos}[\omega] \operatorname{Sin}[2 i],\ 0,\ 9 e \operatorname{Sin}[2 i] \operatorname{Sin}[\omega]\right\}$$

Simplify[Expand[PCoefTrignM[[3, 2]] * (a^5 / (μ J2 Re^2))]]

$$\left\{-\frac{3}{8}\left(8+51 e^{2}\right) \operatorname{Cos}[i] \operatorname{Cos}[\omega] \operatorname{Sin}[i],\right.$$
$$-\frac{21}{2} e \operatorname{Cos}[i] \operatorname{Cos}[\omega] \operatorname{Sin}[i],\ -\frac{207}{8} e^{2} \operatorname{Cos}[i] \operatorname{Cos}[\omega] \operatorname{Sin}[i],\ 0,$$
$$\frac{3}{8} \sqrt{1-e^{2}}\left(8+17 e^{2}\right) \operatorname{Cos}[i] \operatorname{Sin}[i] \operatorname{Sin}[\omega],\ \frac{21}{2} e \sqrt{1-e^{2}} \operatorname{Cos}[i] \operatorname{Sin}[i] \operatorname{Sin}[\omega],$$
$$\left.\frac{207}{8} e^{2} \sqrt{1-e^{2}} \operatorname{Cos}[i] \operatorname{Sin}[i] \operatorname{Sin}[\omega],\ 0,\ -\frac{9}{2} e \operatorname{Cos}[i] \operatorname{Cos}[\omega] \operatorname{Sin}[i]\right\}$$

Simplify[Expand[PCoefTrignM[[3, 3]] * (a^5 / (μ J_2 R_e^2))]]

$$\Big\{-\frac{15}{8}\,e\,(3+9\,\mathrm{Cos}[2\,i]+\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2),$$
$$-\frac{15}{4}\,(3\,e^2+9\,e^2\,\mathrm{Cos}[2\,i]+(1+e^2)\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2),$$
$$-\frac{135}{8}\,e\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ -\frac{795}{16}\,e^2\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,$$
$$\frac{15}{4}\,e\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],\ \frac{15}{8}\sqrt{1-e^2}\,(2+3\,e^2)\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[2\,\omega],$$
$$\frac{135}{4}\,e\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],\ \frac{795}{8}\,e^2\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],$$
$$-\frac{9}{16}\,(2+10\,e^2+6\,(1+5\,e^2)\,\mathrm{Cos}[2\,i]+5\,e^2\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2)\Big\}$$

Simplify[Expand[qCoefTrignM[[1]] * (a^4 / (μ J_2 R_e^2))]]

$$\Big\{-\frac{3}{2}\,e\,(1+3\,\mathrm{Cos}[2\,i]),\ -\frac{3}{8}\,(7\,e^2+21\,e^2\,\mathrm{Cos}[2\,i]-6\,(-1+e^2)\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2),$$
$$-9\,e\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ -\frac{387}{16}\,e^2\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,$$
$$0,\ -\frac{9}{8}\sqrt{1-e^2}\,(-2+e^2)\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[2\,\omega],$$
$$9\,e\sqrt{1-e^2}\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[2\,\omega],\ \frac{387}{8}\,e^2\sqrt{1-e^2}\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],$$
$$-\frac{3}{16}\,(2+6\,e^2+6\,(1+3\,e^2)\,\mathrm{Cos}[2\,i]+3\,e^2\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2)\Big\}$$

Simplify[Expand[qCoefTrignM[[2]] * (a^4 / (μ J_2 R_e^2))]]

$$\Big\{0,\ 3\,(-1+e^2)\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],$$
$$-6\,e\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[2\,\omega],\ -\frac{129}{4}\,e^2\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega],\ 0,$$
$$\frac{3}{4}\sqrt{1-e^2}\,(-2+e^2)\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ -6\,e\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,$$
$$-\frac{129}{8}\,e^2\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[i]^2,\ -\frac{3}{4}\,e^2\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i]^2\,\mathrm{Sin}[\omega]\Big\}$$

Simplify[Expand[qCoefTrignM[[3]] * (a^4 / (μ J_2 R_e^2))]]

$$\Big\{-\frac{3}{8}\,(8+27\,e^2)\,\mathrm{Cos}[i]\,\mathrm{Sin}[i]\,\mathrm{Sin}[\omega],$$
$$-9\,e\,\mathrm{Cos}[i]\,\mathrm{Sin}[i]\,\mathrm{Sin}[\omega],\ -\frac{159}{8}\,e^2\,\mathrm{Cos}[i]\,\mathrm{Sin}[i]\,\mathrm{Sin}[\omega],\ 0,$$
$$-\frac{3}{8}\sqrt{1-e^2}\,(8+9\,e^2)\,\mathrm{Cos}[i]\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i],\ -9\,e\sqrt{1-e^2}\,\mathrm{Cos}[i]\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i],$$
$$-\frac{159}{8}\,e^2\sqrt{1-e^2}\,\mathrm{Cos}[i]\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[i],\ 0,\ -3\,e\,\mathrm{Cos}[i]\,\mathrm{Sin}[i]\,\mathrm{Sin}[\omega]\Big\}$$

- **K Matrix**

$$K = \left\{\left\{\frac{2\mu}{R_0{}^3} + \text{fdot}^2,\ \text{fdotdot},\ 0\right\},\ \left\{-\text{fdotdot},\ -\frac{\mu}{R_0{}^3} + \text{fdot}^2,\ 0\right\},\ \left\{0,\ 0,\ -\frac{\mu}{R_0{}^3}\right\}\right\};$$

$$\text{oneOverRThird} = \left(1 + \frac{3\,e^2}{2} + 3\,e\,\text{Cos}[M] + \frac{9}{2}\,e^2\,\text{Cos}[2\,M]\right) \Big/ a^3;$$

$$\text{fdotSquared} = \frac{h^2}{a^4}\,\left(1 + 4\,e\,\text{Cos}[M] + e^2\,(3 + 7\,\text{Cos}[2\,M])\right);$$

$$\text{fdoubledot} = \frac{-2\,h}{a^2}\,\left(e\,n\,\text{Sin}[M] + e^2\,n\,\text{Sin}[2\,M] + 3\,e^2\,n\,\text{Cos}[M]\,\text{Sin}[M]\right);$$

$$K = K\ /.\ \left\{\frac{1}{R_0{}^3} \to \text{oneOverRThird},\ \text{fdot}^2 \to \text{fdotSquared},\ \text{fdotdot} \to \text{fdoubledot}\right\};$$

```
K = TrigReduce[Expand[K]];

trigFuncList = {Cos[M], Cos[2 M], Cos[3 M], Cos[4 M], Sin[M], Sin[2 M],
   Sin[3 M], Sin[4 M]};
f[x_, y_, z_] = 0;
KCoefTrignM = Array[f, {3, 3, 9}];
f[x_, y_] = 0;
KTermsTrignM = Array[f, {3, 3}];
For[j = 1, j ≤ 3, j++,
 For[k = 1, k ≤ 3, k++,
  For[l = 1, l ≤ 8, l++,
   KCoefTrignM[[j, k, l]] = Coefficient[K[[j, k]], trigFuncList[[l]]];
   KTermsTrignM[[j, k]] =
    KTermsTrignM[[j, k]] + KCoefTrignM[[j, k, l]] * trigFuncList[[l]]
  ];
  KCoefTrignM[[j, k, 9]] = K[[j, k]] - KTermsTrignM[[j, k]]
 ]
]

KCoefTrignM = FullSimplify[KCoefTrignM];
```

$$\text{Simplify}\left[\text{Expand}\left[\text{KCoefTrignM} * a^4\right]\right]$$

$$\left\{\{\{e\,(4\,h^2 + 6\,a\,\mu),\ e^2\,(7\,h^2 + 9\,a\,\mu),\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ (1 + 3\,e^2)\,h^2 + a\,(2 + 3\,e^2)\,\mu\},\right.$$
$$\{0,\ 0,\ 0,\ 0,\ -2\,a^2\,e\,h\,n,\ -5\,a^2\,e^2\,h\,n,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}\},$$
$$\left\{\{0,\ 0,\ 0,\ 0,\ 2\,a^2\,e\,h\,n,\ 5\,a^2\,e^2\,h\,n,\ 0,\ 0,\ 0\},\ \left\{e\,(4\,h^2 - 3\,a\,\mu),\ \frac{1}{2}\,e^2\,(14\,h^2 - 9\,a\,\mu),\right.\right.$$
$$\left.\left.0,\ 0,\ 0,\ 0,\ 0,\ 0,\ (1 + 3\,e^2)\,h^2 - \frac{1}{2}\,a\,(2 + 3\,e^2)\,\mu\right\},\ \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}\right\},$$
$$\left\{\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\},\right.$$
$$\left.\left.\left\{-3\,a\,e\,\mu,\ -\frac{9}{2}\,a\,e^2\,\mu,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{1}{2}\,a\,(2 + 3\,e^2)\,\mu\right\}\right\}\right\}$$

APPENDIX G: MATHEMATICA CODE – EXPRESSIONS FOR TIME-VARYING QUANTITIES

▪ Determine and Simplify $\cos\nu$

$$\text{cosNu} = -e + 2\,\frac{1-e^2}{e}\sum_{n=1}^{5}\left(\sum_{j=0}^{5}\left((-1)^j\,\frac{\left(\frac{1}{2}\,n\,e\right)^{n+2j}}{j!\,(n+j)!}\right)\text{Cos}[n\,M]\right);$$

```
cosNu = Expand[cosNu];
cosNu = Coefficient[cosNu, e, 0] + e*Coefficient[cosNu, e, 1] +
   e^2*Coefficient[cosNu, e, 2];
cosNu = Collect[cosNu, Table[Cos[n M], {n, 1, 5}]]
```

$$-e + \left(1 - \frac{9\,e^2}{8}\right)\text{Cos}[M] + e\,\text{Cos}[2\,M] + \frac{9}{8}\,e^2\,\text{Cos}[3\,M]$$

▪ Determine and Simplify $\sin\nu$

$$\text{sinNu} = \left(1-e^2\right)^{1/2}\sum_{n=1}^{5}\left(\left(\frac{2}{n}\right)D\left[\sum_{j=0}^{5}\left((-1)^j\,\frac{\left(\frac{1}{2}\,n\,e\right)^{n+2j}}{j!\,(n+j)!}\right),\,e\right]\text{Sin}[n\,M]\right);$$

```
sinNu = Expand[sinNu];
sinNu = Coefficient[sinNu, e, 0] + e*Coefficient[sinNu, e, 1] +
   e^2*Coefficient[sinNu, e, 2];
sinNu = Collect[sinNu, Table[Sin[n M], {n, 1, 5}]]
```

$$\left(\sqrt{1-e^2} - \frac{3}{8}\,e^2\sqrt{1-e^2}\right)\text{Sin}[M] + e\sqrt{1-e^2}\,\text{Sin}[2\,M] + \frac{9}{8}\,e^2\sqrt{1-e^2}\,\text{Sin}[3\,M]$$

▪ Determine and Simplify $\cos\theta$

```
cosTheta = cosw*cosNu - sinw*sinNu;

cosTheta = Expand[cosTheta];

Dummy =  FullSimplify[Coefficient[cosTheta, Cos[M]]]*Cos[M] +
   FullSimplify[Coefficient[cosTheta, Cos[2 M]]]*Cos[2 M] +
   FullSimplify[Coefficient[cosTheta, Cos[3 M]]]*Cos[3 M] +
   FullSimplify[Coefficient[cosTheta, Sin[M]]]*Sin[M] +
   FullSimplify[Coefficient[cosTheta, Sin[2 M]]]*Sin[2 M] +
   FullSimplify[Coefficient[cosTheta, Sin[3 M]]]*Sin[3 M];

Dummy2 = Simplify[cosTheta - Dummy];

cosTheta = Dummy + Dummy2;
```

```
cosTheta = cosTheta /. {cosw → Cos[ω], sinw → Sin[ω]}
```

$$-e\,Cos[\omega] + e\,Cos[2M]\,Cos[\omega] + \frac{9}{8}e^2\,Cos[3M]\,Cos[\omega] + Cos[M]\left(Cos[\omega] - \frac{9}{8}e^2\,Cos[\omega]\right) + \frac{1}{8}\sqrt{1-e^2}\,(-8+3e^2)\,Sin[M]\,Sin[\omega] - e\sqrt{1-e^2}\,Sin[2M]\,Sin[\omega] - \frac{9}{8}e^2\sqrt{1-e^2}\,Sin[3M]\,Sin[\omega]$$

▪ Determine and Simplify sinθ

```
sinTheta = sinw * cosNu + cosw * sinNu;

sinTheta = Expand[sinTheta];

Dummy = FullSimplify[Coefficient[sinTheta, Cos[M]]] * Cos[M] +
   FullSimplify[Coefficient[sinTheta, Cos[2 M]]] * Cos[2 M] +
   FullSimplify[Coefficient[sinTheta, Cos[3 M]]] * Cos[3 M] +
   FullSimplify[Coefficient[sinTheta, Sin[M]]] * Sin[M] +
   FullSimplify[Coefficient[sinTheta, Sin[2 M]]] * Sin[2 M] +
   FullSimplify[Coefficient[sinTheta, Sin[3 M]]] * Sin[3 M];

Dummy2 = Simplify[sinTheta - Dummy];

sinTheta = Dummy + Dummy2;

sinTheta = sinTheta /. {cosw → Cos[ω], sinw → Sin[ω]}
```

$$\frac{1}{8}(8-3e^2)\sqrt{1-e^2}\,Cos[\omega]\,Sin[M] + e\sqrt{1-e^2}\,Cos[\omega]\,Sin[2M] + \frac{9}{8}e^2\sqrt{1-e^2}\,Cos[\omega]\,Sin[3M] - e\,Sin[\omega] + e\,Cos[2M]\,Sin[\omega] + \frac{9}{8}e^2\,Cos[3M]\,Sin[\omega] + Cos[M]\left(Sin[\omega] - \frac{9}{8}e^2\,Sin[\omega]\right)$$

▪ Determine and Simplify cos2θ

```
cosTheta = cosTheta /. {Cos[ω] → cosw, Sin[ω] → sinw};
sinTheta = sinTheta /. {Cos[ω] → cosw, Sin[ω] → sinw};

cosTwoTheta = Expand[1 - 2 * sinTheta ^2];

cosTwoTheta =  Coefficient[cosTwoTheta, e, 0] +
   e * Coefficient[cosTwoTheta, e, 1] +
   e^2 * Coefficient[cosTwoTheta, e, 2];

cosTwoTheta = TrigReduce[cosTwoTheta];
```

```
cosTwoTheta = Collect[cosTwoTheta,
   Join[Table[Cos[n M], {n, 1, 5}], Table[Sin[n M], {n, 1, 5}]]];

cosTwoTheta = cosTwoTheta /. {cosw → Cos[ω], sinw → Sin[ω]};

Dummy = FullSimplify[Coefficient[cosTwoTheta, Cos[M]]] * Cos[M] +
   FullSimplify[Coefficient[cosTwoTheta, Cos[2 M]]] * Cos[2 M] +
   FullSimplify[Coefficient[cosTwoTheta, Cos[3 M]]] * Cos[3 M] +
   FullSimplify[Coefficient[cosTwoTheta, Cos[4 M]]] * Cos[4 M] +
   FullSimplify[Coefficient[cosTwoTheta, Sin[M]]] * Sin[M] +
   FullSimplify[Coefficient[cosTwoTheta, Sin[2 M]]] * Sin[2 M] +
   FullSimplify[Coefficient[cosTwoTheta, Sin[3 M]]] * Sin[3 M] +
   FullSimplify[Coefficient[cosTwoTheta, Sin[4 M]]] * Sin[4 M];

Dummy2 = FullSimplify[cosTwoTheta - Dummy];

cosTwoTheta = Dummy + Dummy2
```

$$\frac{3}{4} e^2 \mathrm{Cos}[2\omega] - 2 e \mathrm{Cos}[M] \mathrm{Cos}[2\omega] + (1 - 4 e^2) \mathrm{Cos}[2M] \mathrm{Cos}[2\omega] - 2 e \mathrm{Cos}[3M] \mathrm{Cos}[2\omega] + \frac{13}{4} e^2 \mathrm{Cos}[4M] \mathrm{Cos}[2\omega] + 4 e \sqrt{1-e^2} \mathrm{Cos}[\omega] \mathrm{Sin}[M] \mathrm{Sin}[\omega] + \sqrt{1-e^2} (-2 + 7 e^2) \mathrm{Cos}[\omega] \mathrm{Sin}[2M] \mathrm{Sin}[\omega] - 4 e \sqrt{1-e^2} \mathrm{Cos}[\omega] \mathrm{Sin}[3M] \mathrm{Sin}[\omega] - \frac{13}{2} e^2 \sqrt{1-e^2} \mathrm{Cos}[\omega] \mathrm{Sin}[4M] \mathrm{Sin}[\omega]$$

▪ Determine and Simplify sin2θ

```
sinTwoTheta = Expand[2 sinTheta * cosTheta];

sinTwoTheta = Coefficient[sinTwoTheta, e, 0] +
   e * Coefficient[sinTwoTheta, e, 1] +
   e^2 * Coefficient[sinTwoTheta, e, 2];

sinTwoTheta = TrigReduce[sinTwoTheta];

sinTwoTheta = Collect[sinTwoTheta,
   Join[Table[Cos[n M], {n, 1, 5}], Table[Sin[n M], {n, 1, 5}]]];

sinTwoTheta = sinTwoTheta /. {cosw → Cos[ω], sinw → Sin[ω]};
```

```
Dummy = FullSimplify[Coefficient[sinTwoTheta, Cos[M]]] * Cos[M] +
   FullSimplify[Coefficient[sinTwoTheta, Cos[2 M]]] * Cos[2 M] +
   FullSimplify[Coefficient[sinTwoTheta, Cos[3 M]]] * Cos[3 M] +
   FullSimplify[Coefficient[sinTwoTheta, Cos[4 M]]] * Cos[4 M] +
   FullSimplify[Coefficient[sinTwoTheta, Sin[M]]] * Sin[M] +
   FullSimplify[Coefficient[sinTwoTheta, Sin[2 M]]] * Sin[2 M] +
   FullSimplify[Coefficient[sinTwoTheta, Sin[3 M]]] * Sin[3 M] +
   FullSimplify[Coefficient[sinTwoTheta, Sin[4 M]]] * Sin[4 M];

Dummy2 = FullSimplify[sinTwoTheta - Dummy];

sinTwoTheta = Dummy + Dummy2
```

$$-2\,e\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[M]+\frac{1}{2}\left(2-7\,e^2\right)\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[2\,M]+$$
$$2\,e\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[3\,M]+\frac{13}{4}\,e^2\sqrt{1-e^2}\,\mathrm{Cos}[2\,\omega]\,\mathrm{Sin}[4\,M]+$$
$$\frac{3}{2}\,e^2\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[\omega]-4\,e\,\mathrm{Cos}[M]\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[\omega]+4\,e\,\mathrm{Cos}[3\,M]\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[\omega]+$$
$$\frac{13}{2}\,e^2\,\mathrm{Cos}[4\,M]\,\mathrm{Cos}[\omega]\,\mathrm{Sin}[\omega]+\left(1-4\,e^2\right)\mathrm{Cos}[2\,M]\,\mathrm{Sin}[2\,\omega]$$

■ Determine and Simplify $\frac{a}{r}$

$$\text{aOverr} = 1 + 2\sum_{n=1}^{5}\left(\sum_{j=0}^{5}\left((-1)^j\,\frac{\left(\frac{1}{2}\,n\,e\right)^{n+2j}}{j!\,(n+j)!}\right)\text{Cos}[n\,M]\right);$$

```
aOverr = Collect[aOverr, Table[Cos[n M], {n, 1, 5}]];
```

```
aOverr = Coefficient[aOverr, e, 0] + e*Coefficient[aOverr, e, 1] +
  e^2*Coefficient[aOverr, e, 2]
```

$1 + e\,\text{Cos}[M] + e^2\,\text{Cos}[2\,M]$

■ Determine and Simplify $\frac{a^2}{r^2}$

```
a2overR2 = Expand[aOverr^2];
```

```
a2overR2 = TrigReduce[a2overR2];
```

```
a2overR2 = Coefficient[a2overR2, e, 0] + e*Coefficient[a2overR2, e, 1] +
  e^2 Coefficient[a2overR2, e, 2]
```

$1 + 2\,e\,\text{Cos}[M] + \frac{1}{2}\,e^2\,(1 + 5\,\text{Cos}[2\,M])$

■ Determine and Simplify $\frac{a^3}{r^3}$

```
a3overR3 = Expand[aOverr^3];
```

```
a3overR3 = TrigReduce[a3overR3];
```

```
a3overR3 = Coefficient[a3overR3, e, 0] + e*Coefficient[a3overR3, e, 1] +
  e^2 Coefficient[a3overR3, e, 2];
```

```
a3overR3 = Collect[a3overR3, {Cos[M], Cos[2 M]}]
```

$1 + \frac{3\,e^2}{2} + 3\,e\,\text{Cos}[M] + \frac{9}{2}\,e^2\,\text{Cos}[2\,M]$

■ Determine and Simplify $\frac{a^4}{r^4}$

```
a4overR4 = Expand[aOverr^4];

a4overR4 = TrigReduce[a4overR4];

a4overR4 = Coefficient[a4overR4, e, 0] + e*Coefficient[a4overR4, e, 1] +
   e^2*Coefficient[a4overR4, e, 2];

a4overR4 = Collect[a4overR4, {Cos[M], Cos[2 M]}]
```

$1 + 3 e^2 + 4 e \text{Cos}[M] + 7 e^2 \text{Cos}[2 M]$

■ Determine and Simplify $\frac{a^5}{r^5}$

```
a5overR5 = Expand[aOverr^5];

a5overR5 = TrigReduce[a5overR5];

a5overR5 = Coefficient[a5overR5, e, 0] + e*Coefficient[a5overR5, e, 1] +
   e^2*Coefficient[a5overR5, e, 2];

a5overR5 = Collect[a5overR5, {Cos[M], Cos[2 M]}]
```

$1 + 5 e^2 + 5 e \text{Cos}[M] + 10 e^2 \text{Cos}[2 M]$