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Tree decomposition of graphs

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TREE DECOMPOSITION OF GRAPHS

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ABSTRACT

TREE DECOMPOSITION OF GRAPHS

by Tanya Boboricken

In this thesis we decompose cubic graphs, complete bipartite graphs, and certain complete tripartite graphs into the minimum number of trees $\tau(G)$.

For cubic graphs we show that any 2-connected cubic graph G of order p can be decomposed into $\tau(G) \leq \lfloor \frac{p}{4} \rfloor$ trees. Also, we show that for every $p \geq 8$, and $2 \leq k \leq \lfloor \frac{p}{4} \rfloor$, there exists a 2-connected cubic graph of order p such that $\tau(G) = k$.

We give Beineke's 1964 decomposition of complete bipartite graphs into $\tau(K_{m,n})$ trees where the sizes of any two trees used in the decomposition differ by at most one. We then extend Beineke's result by decomposing $K_{m,n}$ into $\tau(K_{m,n})$ trees such that all but perhaps one are spanning trees.

For complete tripartite graphs $K_{m,n,p}$ we find upper and lower bounds for $\tau(K_{m,n,p})$. We then decompose a few families of $K_{m,n,p}$ into $\tau(K_{m,n,p})$ trees and state a conjecture.

ACKNOWLEDGMENT

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Chapter 1

Definitions and Notations

A *graph* G is a pair of sets (V,E) where V is finite and nonempty, and E is a set of unordered pairs of elements of V. The elements of V are called the *vertices* (singular is vertex) of G and the elements of E are called the *edges* of G. The *empty graph* is a pair of sets (V,E) where $V=E=\emptyset$. The *trivial graph* consists of one vertex and no edges. A *nontrivial graph* has at least two vertices. We will write V(G) for the set of vertices of G and E(G) for the set of edges of G. We let $|V(G)|$ denote the number of vertices in G, and $|E(G)|$ denote the number of edges in G. We say a graph G has *order* $|V(G)|$ and *size* $|E(G)|$. Two graphs that have the same structure, differing only in the way they are drawn or labeled, are said to be *isomorphic.* We use the notation $G_1 = G_2$ to denote that the graphs G_1 and G_2 are isomorphic. Two graphs G_1 and G_2 are said to be *nearly equal* in size if $|\,|E(G_1)| - |E(G_2)|\,| \leq 1$.

If $\{v,w\} = vw \in E(G)$, we call v and w *adjacent vertices* and say that v and w are *joined* by the edge vw. We call v and w the *endvertices* of the edge vw. We say that v and w are *incident to* the edge vw, and that vw is *incident to* the vertices v and w. If uv, vw $\in E(G)$, we call uw and vw *adjacent edges.* The *degree of a vertex v*, denoted *deg(v),* in a graph G is the number of edges of G incident to v. The *neighborhood of a vertex v*, denoted *N(v)*, is the set of vertices adjacent to v. Thus, $|N(v)| = deg(v)$. A graph is said to be *regular of degree r* if every vertex has degree equal to r. A regular graph of degree 3 is called *cubic.*

A u-v *walk* is an alternating sequence of vertices and edges, beginning with vertex u and ending with vertex v, where each edge is incident to the vertices immediately before and after it in the sequence. We normally list just the consecutive vertices in the walk as the edges are implied. A u-v *path* is a u-v walk with no repeated vertices. A *cycle* is a walk $u_1 u_2 \dots u_n u_1$ whose initial vertex is the same as the terminal vertex and where each of the vertices $u_1, \dots, u_n$ are distinct. An *n-cycle* is a cycle with n edges. A 3-cycle is called a *triangle.* A vertex u is said to be *connected* to a vertex v if there exists a u-v path in G. A graph G is *connected* if every two of its vertices are connected, otherwise we say G is *disconnected.* The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G results in a disconnected or trivial graph. A graph is said to be *i-connected,* $i \geq 1$, if $\kappa(G) \geq i$.

A *subgraph* of a graph G is a graph H such that every vertex of H is a vertex of G, and every edge of H is an edge of G. If X is a nonempty subset of the vertex set V(G) of a graph G, then the *subgraph* $\langle X \rangle$ *of G induced by X* is the graph with vertex set X and whose edge set consists of edges of G incident with two elements of X. Similarly, if S is a nonempty subset of E(G), then the *subgraph* $\langle S \rangle$ *induced by S* is the graph whose vertex set consists of those vertices in G incident with at least one edge of S and whose edge set is S.

A *component* of a graph G is a connected subgraph of G not properly contained in any other connected subgraph of G. A disconnected graph consists of more than one component. A graph without a cycle is called *acyclic.* We consider the empty graph to be acyclic. A *spanning forest* of a graph G is an acyclic subgraph of G that contains all the vertices of G. It does not have to be connected. A *tree* is a connected acyclic graph. A *spanning tree* is a connected spanning forest. A connected graph with p vertices has a spanning tree with p-1 edges.

Let G_1 and G_2 be graphs. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. A graph G is said to be *decomposable* into subgraphs $H_1, H_2, \ldots, H_t$ if any two subgraphs H_i and H_j have no edges in common, and the union of all the subgraphs H_i is G. More formally, a *decomposition* of G is a collection of subgraphs of G such that each $H_i = \langle E_i \rangle$ for some subset E_i of E(G) where E_i is a partition of E(G). If G is the edge-disjoint union of $H_1, H_2, \ldots, H_t$ and at least t-1 of the H_j are spanning trees, then we say that G is decomposable into *nearly all* spanning trees. A *factor* of a graph G is a spanning subgraph of G. A factor of a graph such that all the degrees equal one is called a *1-factor.* A graph is called *1-factorable* if it is decomposable into 1-factors.

The least integer k such that $k \geq a$ is denoted by $\lceil a \rceil$; the greatest integer k such that $k \leq a$ is denoted by $\lfloor a \rfloor$. The minimum number of forests whose union is G is denoted by a(G). The minimum number of trees whose union is G is denoted by $\tau(G)$.

The *complete bipartite graphs*, denoted $K_{m,n}$, are the graphs whose vertices can be partitioned into two sets, called *partite* sets, $U=\{u_1, u_2, \ldots, u_m\}$ and $V=\{1,2,3,\ldots,n\}$, such that every vertex in one set is adjacent to every vertex in the other set, but no vertices in the same set are adjacent. The vertices in the set U will be depicted with dark points, vertices in V with light points. We will assume that $m \leq n$. The edges will be referred to by their endvertices.

The *complete tripartite graphs*, denoted $K_{m,n,p}$, are the graphs whose vertices can be partitioned into three partite sets: $U=\{u_1, u_2, \ldots, u_m\}$, $V=\{v_1, v_2, \ldots, v_n\}$, and $W=\{w_1, w_2, \ldots, w_p\}$ depicted with dark points, light points, and dark centered points, respectively, such that all possible edges between different type vertices exist, and there are no edges with both endvertices of the same type. We assume that $m \leq n \leq p$. Here also the edges will be referred to by their endvertices.

Chapter 2

Introduction

Decompositions of graphs into various subgraphs with some specified property is a popular topic in graph theory. There are essentially two types of decompositions: one decomposing the vertex set V(G) of the graph G, and the other decomposing the edge set E(G) of the graph G. In this thesis we are concerned with the decomposition of the edge set of the complete bipartite and tripartite graphs and of cubic graphs into the minimum number of sets such that each set induces a tree. This decomposition is related to arboricity which has been widely studied.

The *arboricity* of a nonempty graph G, is the minimum number of subsets into which E(G) can be partitioned so that each subset induces an acyclic subgraph. An equivalent definition of arboricity of a graph G is the minimum number of spanning forests, denoted a(G), whose union is G.

In 1960, Nash-Williams [12] determined the arboricity of any graph. Nash-Williams proved that given a nontrivial graph G with p vertices and q edges where q_n is the maximum number of edges in any subgraph with n vertices, then $a(G) = \max_n \left\lceil \frac{q_n}{n-1} \right\rceil$. The fact that $a(G) \geq \max_n \left\lceil \frac{q_n}{n-1} \right\rceil$ can be shown as follows. Since G has p vertices, the maximum number of edges in any spanning forest is p-1. Hence, the minimum possible number of spanning forests required to decompose G is at least $\frac{q}{p-1}$. By definition, a(G) is the minimum number of spanning forests necessary to decompose G, from which it follows that $a(G) \geq \frac{q}{p-1}$. Now, for any subgraph H of G, $a(G) \geq a(H)$. Therefore, $a(G) \geq \max_n \left\lceil \frac{q_n}{n-1} \right\rceil$.

Although Nash-William's result gives us the value of a(G), his proof does not provide a construction for the decomposition of the graph G into a(G) forests. In 1964, Beineke [3] provided a decomposition of the complete bipartite graph into a(G) forests. Beineke's decomposition is actually stronger than he states as his method decomposes the complete bipartite graph into trees. In this thesis we extend Beineke's results by characterizing the trees used in his decomposition and by providing another decomposition of the complete bipartite graphs where each of the trees, except possibly one, is a spanning tree. We also provide a decomposition of several families of the complete tripartite graphs.

We became interested in this topic as a result of decomposing cubic graphs into the minimum number of trees. In 1992, Boboricken and Valdés [4] showed that any 2-connected cubic graph can be decomposed into $\left\lfloor \frac{p}{4} \right\rfloor$ or fewer trees. Furthermore, Boboricken and Valdés showed that for every even $p \geq 8$, and for every k, $2 \leq k \leq \left\lfloor \frac{p}{4} \right\rfloor$, there exists a 2-connected cubic graph G with p vertices such that $\tau(G) = k$. Similarly, Boboricken and Valdés showed that any 3-connected cubic graph with $p \geq 12$ vertices can be decomposed into $\lfloor \frac{p}{6} \rfloor$ or fewer trees; and for every even $p \geq 12$, and for every k, $2 \leq k \leq \lfloor \frac{p}{6} \rfloor$, there exists a 3-connected cubic graph G with p vertices such that $\tau(G) = k$. We include Boboricken and Valdés' results concerning 2-connected cubic graphs in Chapter 3.

After decomposing cubic graphs into trees, we thought that it would be interesting to decompose the complete bipartite graphs into trees. We independently showed that the minimum number of trees required to decompose the complete bipartite graphs $K_{m,n}$ is $\lceil \frac{mn}{m+n-1} \rceil$. However, before we formally wrote the proof, we found a reference to Beineke's paper, "Decompositions of Complete Graphs into Forests," in the text book, Graph Theory, by Frank Harary [8]. We suspected that he might have decomposed the complete bipartite graphs into the minimum number of trees. After reading Beineke's paper we learned that Beineke had indeed solved this problem in 1964. Surprisingly, our original method was very similar to Beineke's method. Most of the results that follow in Chapter 4 are proved in Beineke's paper.

In Chapter 5 we decompose $K_{m,n}$ into $\tau(K_{m,n})$ trees where all but at most one tree is a spanning tree. In Chapter 6 we decompose several families of the complete tripartite graphs into $\tau(K_{m,n,p})$ trees.

Other work concerning tree decomposition include decomposing the maximal planar and maximum projective planar graphs into the minimum number of trees [10,13,15]. In particular, maximal planar graphs can be decomposed into three edge-disjoint trees. At the

24th South-Eastern International Conference on Combinatorics, Graph Theory, and Computing, held in February, 1993, Pippert [7] discussed his work with Chilakamarri, Hamberger, and Weakley on attempting to determine which $K_{m,n}$ can be decomposed into edge-disjoint paths of each length from one to m+n. Also, Balakrishnan and Kumar [1] recently considered the problem of decomposing $K_{m,n,p}$ into copies of various specified subgraphs.

Chapter 3

Tree Decomposition of 2-Connected Cubic Graphs

In this chapter we show that any 2-connected cubic graph of order p can be decomposed into $\lfloor \frac{p}{4} \rfloor$ or fewer trees. We find it useful to introduce the notion of a k-tree coloring.

A decomposition of a graph into trees will be called a *k-tree coloring* if the edges of the graph can be partitioned into k sets in such a way that the graph induced by the edges of each set is a tree and k is minimal. It will be shown that any 2-connected cubic graph with $p \geq 8$ vertices is k-tree colorable where $k \leq \lfloor \frac{p}{4} \rfloor$. Also, for every even $p \geq 8$ and for every $2 \leq k \leq \lfloor \frac{p}{4} \rfloor$, there exists a 2-connected cubic graph with p vertices that is k-tree colorable.

Throughout this chapter, we will consider cubic graphs where p represents the number of vertices. Remember that in a graph, two vertices are not allowed to be joined by more than one edge.

3.1 Decomposition of Cubic Graphs into $\tau(G) \leq \lfloor \frac{p}{4} \rfloor$ trees

We begin with the following four definitions.

Definition 3.1.1. A *multigraph* is similar to a graph but more than one edge can join two vertices.

Definition 3.1.2. In a multigraph, *multiple edges* are formed when more than one edge joins two vertices.

Definition 3.1.3. Two graphs are said to be *homeomorphic* if they can both be obtained from the same graph by a sequence of subdivisions of edges. The graphs shown in Figure 3.1 are homeomorphic to each other.

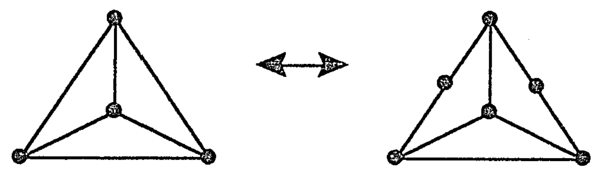

Figure 3.1

Definition 3.1.4. A *critical edge* e in a 2-connected cubic graph G is an edge whose removal results in a graph with connectivity one. An example of a critical edge e is given in Figure 3.2.

Definition 3.1.5. A *removable vertex* v is a vertex whose removal from a 2-connected cubic graph G does not result in a graph of connectivity one. An example of a removable vertex v is given in Figure 3.2.

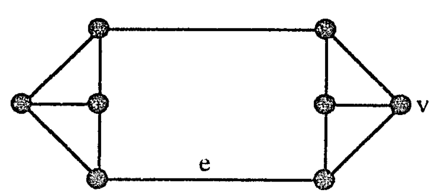

Figure 3.2

The following theorem and three lemmas are required in the proof of Theorem 3.1.9.

Theorem 3.1.6 (Whitney). A nontrivial graph G is k-connected if and only if for each pair u, v of distinct vertices there are at least k internally disjoint u-v paths in G.

Proof. A proof is given in [6] on page 159.

Lemma 3.1.7. Every two-connected cubic graph G has a removable vertex.

Proof. Call the maximal connected subgraphs of G, which contain no critical edges, blocks. Contract each of these blocks to a vertex. The resulting graph will be a multigraph in which every edge is a critical edge and in which every vertex is of even degree; i.e., a graph in which each edge is on exactly one cycle, and in which no cycle is a loop. There must be some vertex which is on exactly one cycle. Consider the block to which it corresponds. This block consists of at least four vertices, only two of which are of degree 2. So, at least two vertices are not adjacent to critical edges and therefore are removable. ◦

Lemma 3.1.8. Let G be a cubic graph that contains a triangle and which is k-tree colorable. Then the subdivision of any two edges of a triangle and the joining of the two new vertices with an edge does not change the tree colorability.

Proof. Say the three vertices of the triangle are w, x, and y. Let wx be subdivided with new vertex x' and edge wy be subdivided with new vertex y'. (See Figure 3.1.3)

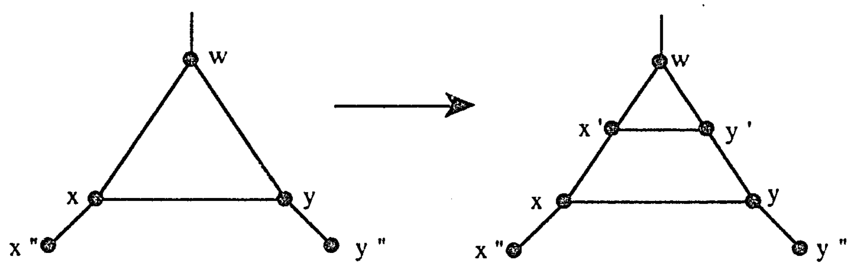

Figure 3.1.3

We consider two cases:

Case 1. Edges wx and wy are assigned different colors, say red and blue, respectively. Then edges wx' and x'x and x'y' can be colored red and wy' and y'y can be colored blue. The red and blue subgraphs are still trees as required. (See Figure 3.1.3.)

Case 2. Edges wx and wy have the same color, say red. Edge xy cannot be red, thus let it be colored blue. In this case let x" be the third vertex adjacent to x and let y" be the third vertex adjacent to y. (See Figure 3.1.3) We then consider two subcases.

Subcase I. Edges xx" and yy" are not both red. Without loss of generality, assume yy" is not red. (See Figure 3.1.4.) Here color wx', x'x, and wy' with red and x'y', y'y with blue. By doing this, the blue tree now has two additional edges and the red graph is still a tree.

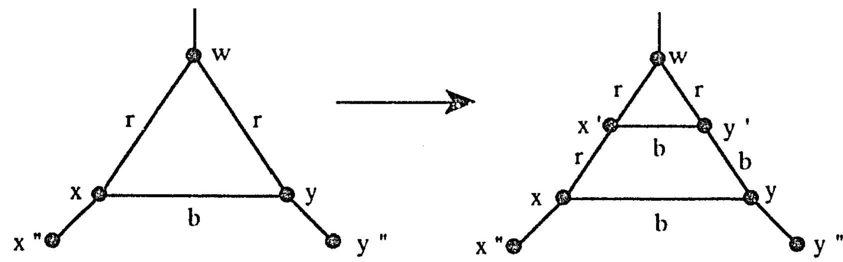

Figure 3.1.4

Subcase II. Edges xx" and yy" are both red. (See Figure 3.1.5) Here color wx', x'x, and wy' with red and color x'y' and y'y with blue. After doing this the red graph is disconnected. In the component of the red graph which contains y we recolor each edge with blue. In the resulting graph both the red and blue graphs are trees. •

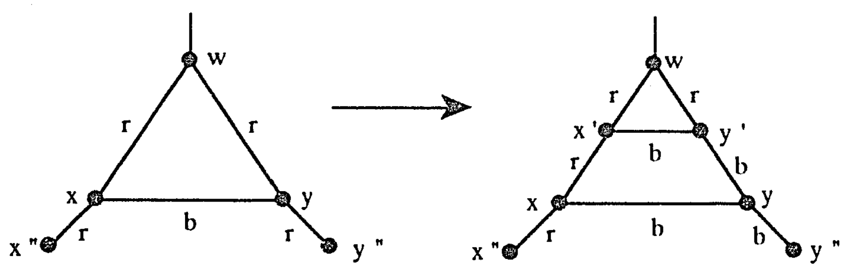

Figure 3.1.5

Let $G-v$ denote the graph G after the deletion of the vertex v and all of its incident edges.

Theorem 3.1.9. Any 2-connected cubic graph, $p \geq 8$, can be k-tree colored, $k \leq \lfloor \frac{p}{4} \rfloor$.

Proof. The proof is by induction on the number of vertices. All cubic graphs with p vertices, $4 \leq p \leq 14$, can be found in [5]. There are only five 2-connected cubic graphs with 8 vertices and each one is 2-tree colorable as shown in Figure 3.1.6. In Figure 3.1.6, the solid line represents one color, say red, and the dashed line represents another color, say blue. Similarly, there are eighteen 2-connected cubic graphs with 10 vertices and each one is 2-tree colorable.

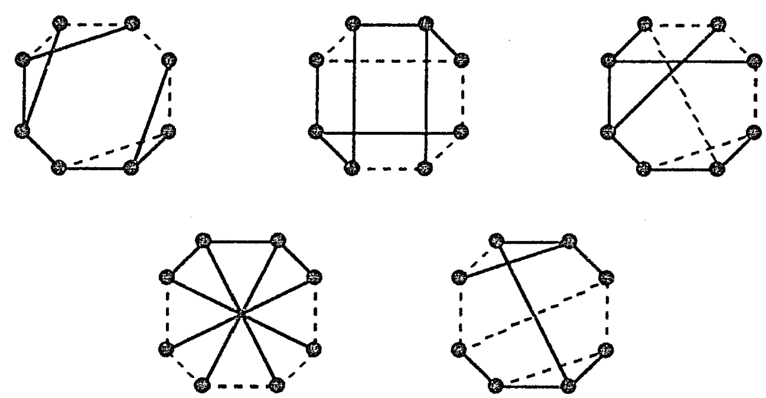

Figure 3.1.6

Next, assume any 2-connected cubic graph with k vertices, $6 < k < p$, can be m-tree colored, $m \leq \lfloor \frac{k}{4} \rfloor$. Let G be a cubic graph with $p \geq 12$ vertices. By Lemma 3.1.7, there exists a removable vertex v in G. Consider the cubic graph or cubic multigraph G* homeomorphic to G-v. It has p-4 vertices.

If G* has no multiple edges, then, by the inductive hypothesis, we can m-tree color G* where $m \leq \lfloor \frac{p-4}{4} \rfloor$. In G, color the edges

G **G***

Case 1.

Case 2.

Case 3.

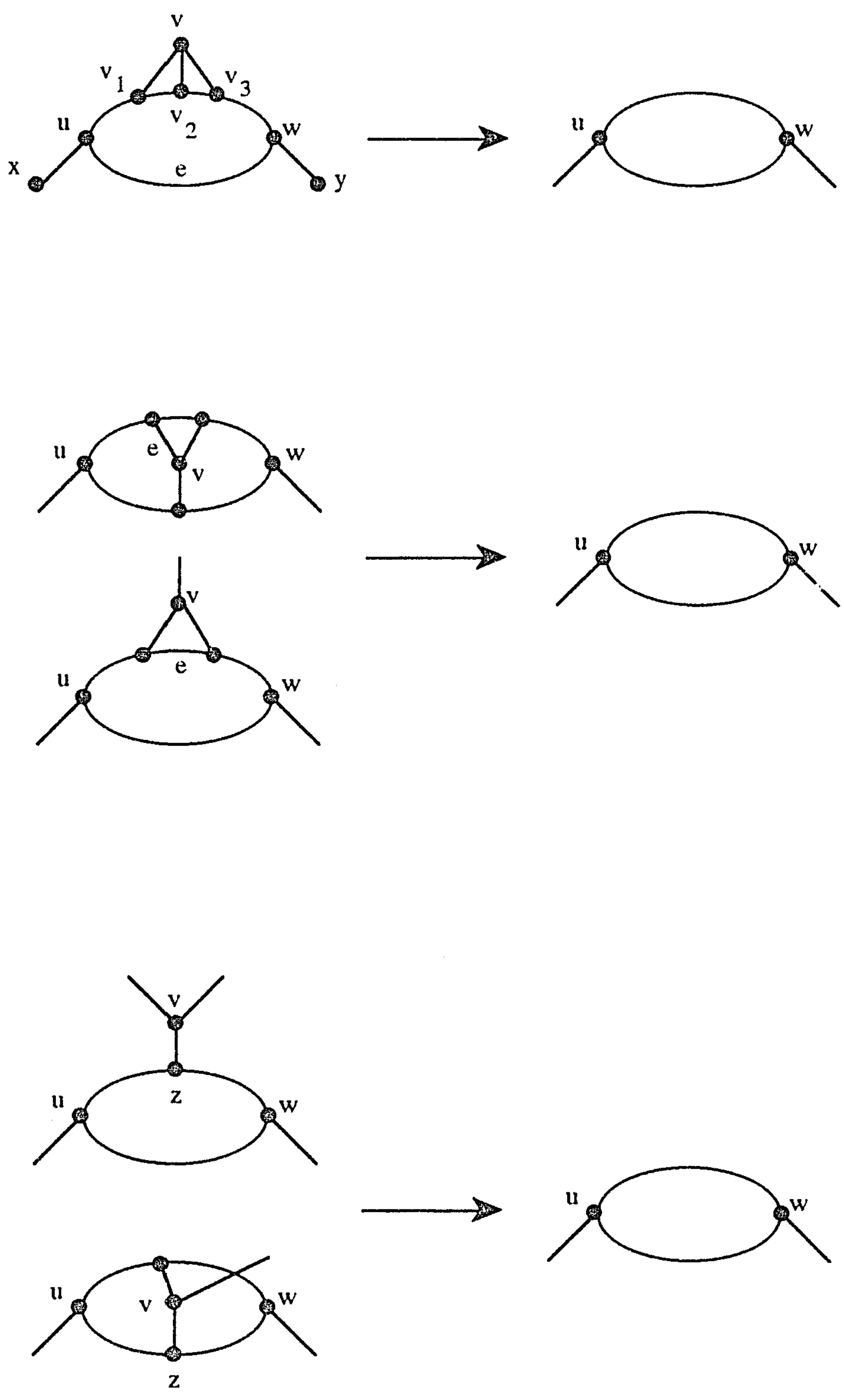

Figure 3.1.7

incident to v with a color not assigned in G* and color the edges common to G* and G as they were colored in G*. Furthermore, each edge of G which is formed from a subdivision of an edge e in G* is assigned the color of e. Thus we have partitioned E(G) into m+1 trees. Therefore G can be k-tree colored, $k \leq m+1 \leq \lfloor \frac{p}{4} \rfloor$.

There are three cases where multiple edges are formed in G*. These cases are shown in Figure 3.1.7. We will treat each of these three cases separately.

Case 1. (See Figure 3.1.7.) In this case, edge e is not critical. Consider the cubic graph G**, which is homeomorphic to G-e. G** has p-2 vertices. Therefore G** is k-tree colorable, $k \leq \lfloor \frac{p-2}{4} \rfloor$. There are two subcases to consider.

Subcase I. If the edges v_1x and v_3y (See Figure 3.1.9) are colored differently in G**, say v_1x is red and v_3y is blue, then in G color v_1u, ux, and e red and color v_3w and wy blue. Color the remaining edge in G as they were colored in G**.

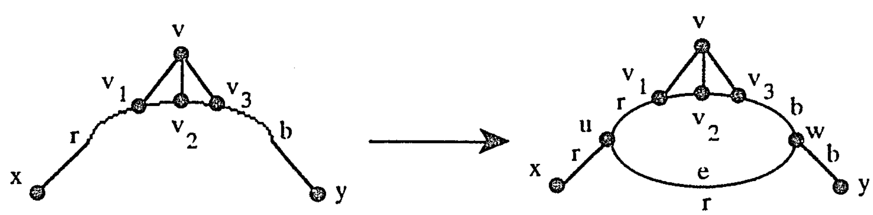

Figure 3.1.9

<u>Subcase II.</u> If v_1x and v_3y are both colored the same color, say red, then either (A) there is a red x-y path containing v_3 or (B) there is a red x-y path which does not contain v_3.

(A) If one of vv_3 or v_2v_3 is not red, without loss of generality, say vv_3 is blue. Then color v_3w and uw blue and v_1u, ux, and wy red. (See Figure 3.1.10.) In doing this we have disconnected the red tree. Color the remaining edges as they were assigned in G**. Next, color the component of the red tree that contains y with blue. Now the red and blue graphs are both trees.

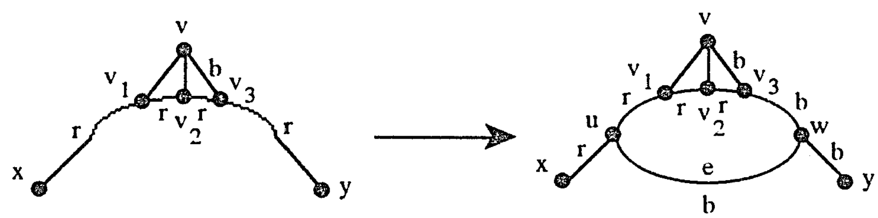

Figure 3.1.10

If both vv_3 and v_2v_3 are red, then vv_2 is not red, say it is blue. Then exactly one edge of vv_1 or v_1v_2 is red; otherwise, we have a red 4-cycle. (See Figure 3.1.11.) Therefore, without loss of generality, say v_1v is not red. Interchange the colors assigned to vv_3 and vv_2. Next, color v_3w, and uw blue, and v_1u,ux, and wy red. Color the remaining edges as they were colored in G**, except color the component of the red tree that contains y blue. The red and blue graphs are still trees.

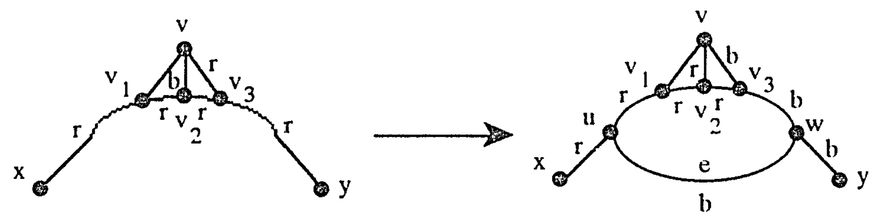

Figure 3.1.11

(B) If neither vv_3 nor v_2v_3 is red then say vv_3 is blue. (See Figure 3.1.12.) Color v_3w and uw blue and color v_1u, ux, and wy red. Color the remaining edges as they were colored in G**. Both the red and the blue graphs are still trees.

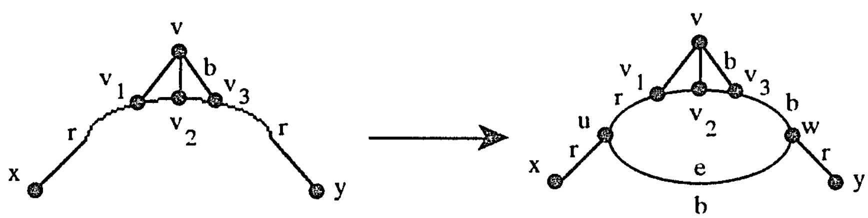

Figure 3.1.12

If both vv_3 and v_2v_3 are red, then v_1v and v_1v_2 are both not red else there is a red cycle. However, by symmetry, this was handled in the preceding paragraph.

If exactly one of vv_3 or v_2v_3 is red, say vv_3 is red and v_2v_3 is blue. Now by symmetry and the preceding two paragraphs exactly one of vv_1 or v_1v_2 is red. In fact, v_1v_2 is red for otherwise there is a red cycle. Similarly, vv_2 is not red. See Figure 3.1.13. Now recolor $v v_1$ with red. Color v_3w, uw with blue and v_1u, ux, wy with red.

Color the remaining edges as they were colored in G**. The resulting red and blue graphs are still trees, as required. Thus case 1 is proved.

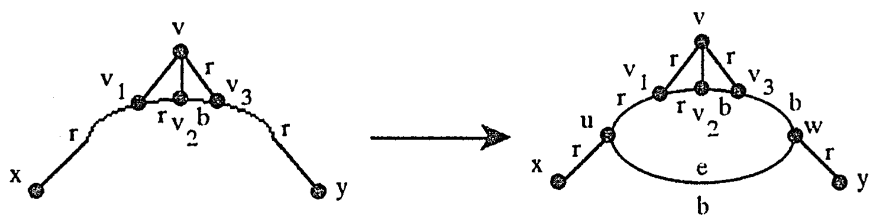

Figure 3.1.13

Case 2. Remove e and by the inductive hypothesis on $k \leq \lfloor \frac{p-2}{4} \rfloor$, k-tree color, the cubic graph G*** which is homeomorphic to G-e. Replace e after subdividing the appropriate edges of G***. The resulting graph is k-tree colorable by Lemma 3.1.8. Thus case 2 is proved.

Case 3. Since v is removable, G-v is 2-connected. It follows from Whitney's theorem, Theorem 3.1.6, that G-z is 2-connected and hence z is a removable vertex. If no multiple edges are formed in cubic G**** which is homeomorphic to G-z, then k-tree color, G**** using the inductive hypothesis on $k \leq \lfloor \frac{p-4}{4} \rfloor$, and then in G color the edges incident to z with a color not assigned in G****. Color the edges common to G**** and G as they were colored in G****. Furthermore, each edge of G which is formed from a subdivision of an edge e in G**** is assigned the color of e. If multiple edges are formed by the removal of z (see Figure 3.1.14) then remove e and proceed as in

Case 2. Thus Case 3 is proved. Therefore, any 2-connected cubic graph, $p \geq 8$, can be k-tree colored, $k \leq \lfloor \frac{p}{4} \rfloor$. •

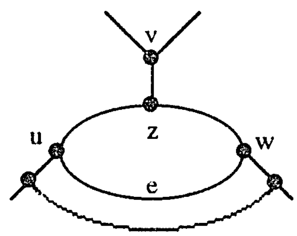

Figure 3.1.14

3.2 Examples of k-tree colorable cubic graphs of order $p \geq 8$, for every k, $2 \leq k \leq \lfloor \frac{p}{4} \rfloor$

In the previous section we showed that all 2-connected cubic graphs of order p are k-tree colorable where $k \leq \lfloor \frac{p}{4} \rfloor$. In this section we show that the upper bound is tight, by giving an example of a family of 2-connected cubic graphs that require $\lfloor \frac{p}{4} \rfloor$ tree colors. In addition, we show that for every even $p \geq 8$, and for every k, $2 \leq k \leq \lfloor \frac{p}{4} \rfloor$, there exists a 2-connected cubic graph with p vertices that require k tree colors.

Theorem 3.2.1. For every even $p \geq 8$, and for every k, $2 \leq k \leq \lfloor \frac{p}{4} \rfloor$, there exists a 2-connected cubic graph $G(p,k)$ with p vertices that is k-tree colorable.

Proof. Form the graph $G(p,k)$, $p \geq 8$, $2 \leq k \leq \lfloor \frac{p}{4} \rfloor$, as follows: For $2 \leq i \leq k$, we have four distinct vertices $v_{i,1}$, $v_{i,2}$, $v_{i,3}$, $v_{i,4}$, and five edges $v_{i,2}v_{i,1}$, $v_{i,2}v_{i,3}$, $v_{i,2}v_{i,4}$, $v_{i,1}v_{i,3}$, $v_{i,3}v_{i,4}$. (See Figure 3.2.1)

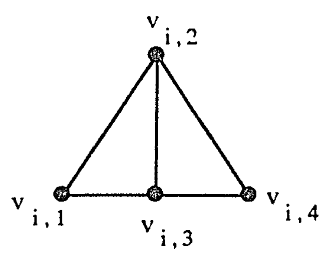

Figure 3.2.1

We have additional vertices $v_{1,1}$, $v_{1,2}$, ..., $v_{1,s}$ where $s = p - 4(k-1)$. These s additional vertices are joined by edges $v_{1,j}v_{1,j+1}$ for j even and $1 \leq j \leq s-2$, edges $v_{1,j}v_{1,j+2}$ for $1 \leq j \leq s-2$, and edges $v_{1,1}v_{1,2}$, $v_{1,s-1}v_{1,s}$. (See Figure 3.2.2.)

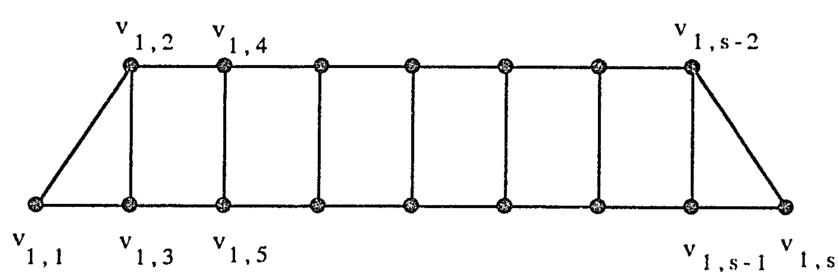

Figure 3.2.2

Graph G(p,k) consists of all vertices and edges listed above, and the edges $v_{1,s}v_{2,1}$, $v_{2,4}v_{3,1}$, $v_{3,4}v_{4,1}$, ..., $v_{k,4}v_{1,1}$. Each of these last k edges are called *joins*. We illustrate G(26,2) and G(26,6) in Figures 3.2.3 and 3.2.4.

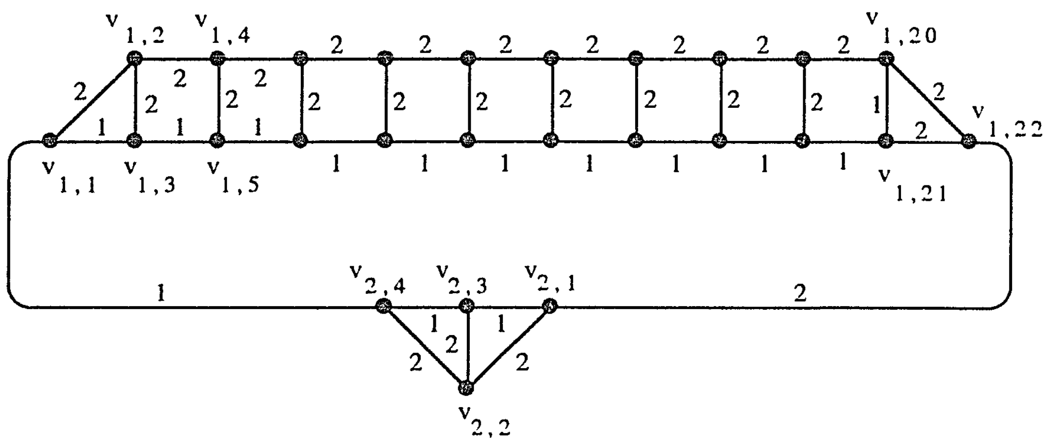

Figure 3.2.3

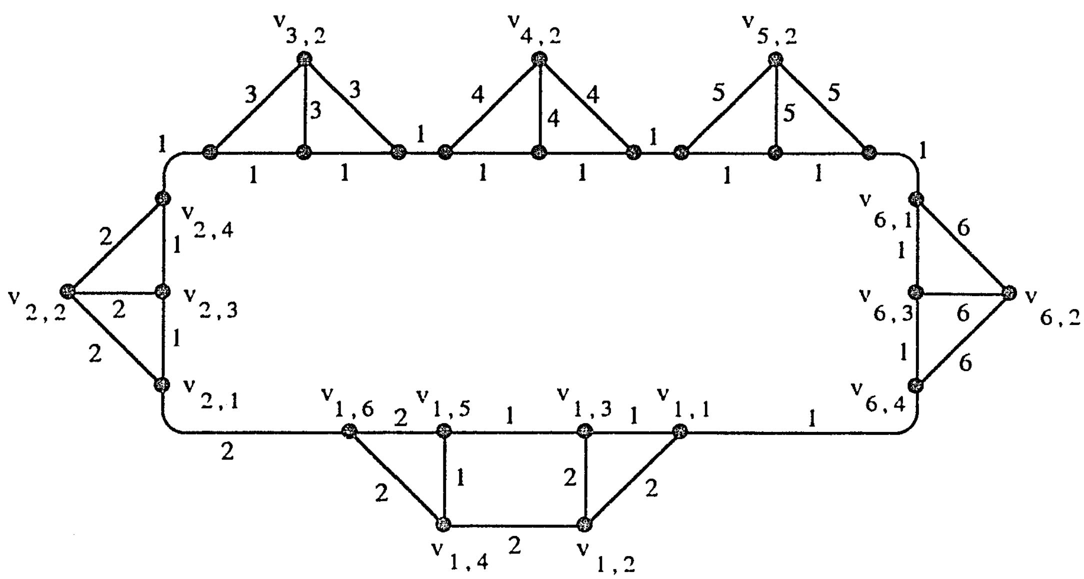

Figure 3.2.4

For $1 \leq i \leq k$, the subgraph induced by all vertices with first subscript i is called a *hump*. Graph G(p,k) is cubic. 2-connected with order p. We show that it is k-tree colorable. Suppose we have a tree coloring of G(p,k) with exactly $g \leq k$ colors $c_1, \ldots, c_g$ are used on the joins. If c_i is used on ℓ joins, then the joins must be consecutive and the $\ell - 1$ humps in between require at least one additional color per hump which is not used outside the hump. Let t_i equal the number of joins of color c_i. Since G(p,k) has k joins, $k = t_1 + \ldots + t_g$. $\tau(G(p,k)) \geq g + (t_1 - 1) + \ldots + (t_g - 1) = t_1 + \ldots + t_g$. So, $\tau(G(p,k)) \geq k$. But G(p,k) can be tree colored with k colors as follows. In the humps, for $2 < i \leq k$, the color edges $v_{i,2}v_{i,1}$, $v_{i,2}v_{i,3}$, $v_{i,2}v_{i,4}$, with the color i, and color the remaining edges with 1. Color join $v_{1,s}v_{2,1}$ with color 2, and the remaining joins are colored with 1. For j odd and $1 \leq j \leq s-3$, color edges $v_{1,j}v_{1,j+2}$ with color 1, color edge $v_{1,s-2}v_{1,s-1}$ with 1, and color the remaining edges of the form $v_{1,x}v_{1,y}$ with 2. Thus, G(p,k) can be tree colored with k colors. Therefore, $\tau(G(p,k)) = k$. So G(p,k) has p vertices and is k-tree colorable. •

Boboricken and Valdés [4] also showed that 3-connected cubic graphs can be k-tree colored where $k \leq \lfloor \frac{p}{6} \rfloor$. We direct the interested reader to [4], where the proof can be found.

Chapter 4

Tree Decomposition of the Complete Bipartite Graphs into Trees of Nearly Equal Size

From Nash-Williams' famous result given on page 5, we know that $\tau(K_{m,n}) \geq a(K_{m,n}) \geq \left\lceil \frac{mn}{m+n-1} \right\rceil$. In this chapter we show that $\tau(K_{m,n}) = a(K_{m,n}) = \left\lceil \frac{mn}{m+n-1} \right\rceil$, and that the trees used in the decomposition can be made to differ in size by at most one edge.

4.1 Preliminary Results

Lemma 4.1.1. Let m, n be positive integers, $m \leq n$, then $\tau(K_{m,n}) \geq \left\lceil \frac{mn}{m+n-1} \right\rceil$.

Proof. $K_{m,n}$ has mn edges. The maximum number of edges of a tree contained in $K_{m,n}$ is m+n-1. Hence, the minimum number of trees required to decompose $K_{m,n}$ is at least $\frac{mn}{m+n-1}$. Since $\tau(K_{m,n})$ is an integer, it follows that $\tau(K_{m,n}) \geq \left\lceil \frac{mn}{m+n-1} \right\rceil$. •

Lemma 4.1.2. Let m>1 be an integer. The function $f(x) = \left\lceil \frac{mx}{m+x-1} \right\rceil$ is a nondecreasing function for all $x > 0$.

Proof. Given that $0 < x_1 < x_2$ and $m > 1$,

it follows that $x_1(m^2-m) < x_2(m^2-m)$.

Hence, $m^2x_1 - mx_1 + mx_1x_2 < m^2x_2 - mx_2 + mx_1x_2$

and $(mx_1)(m+x_2-1) < mx_2(m+x_1-1)$. So, $\frac{m x_1}{m+x_1-1} < \frac{m x_2}{m+x_2-1}$.

This implies $\left\lceil \frac{m x_1}{m+x_1-1} \right\rceil \leq \left\lceil \frac{m x_2}{m+x_2-1} \right\rceil$. Thus, $f(x_1) \leq f(x_2)$. •

Lemma 4.1.3. Let x,y be positive real numbers.
Then, $\lceil x+y \rceil - \lceil y \rceil = \lfloor x \rfloor$ or $\lceil x \rceil$.

Proof. First, $\lceil x+y \rceil - \lceil y \rceil \leq \lceil x \rceil + \lceil y \rceil - \lceil y \rceil = \lceil x \rceil$.
Next, $\lceil x+y \rceil - \lceil y \rceil \geq \lfloor x \rfloor + \lceil y \rceil - \lceil y \rceil = \lfloor x \rfloor$. So, $\lfloor x \rfloor \leq \lceil x+y \rceil - \lceil y \rceil \leq \lceil x \rceil$.
Since $\lceil x+y \rceil - \lceil y \rceil$ is an integer, it follows that $\lceil x+y \rceil - \lceil y \rceil = \lfloor x \rfloor$ or $\lceil x \rceil$. •

Lemma 4.1.4. $\tau(K_{m,n}) = m$ when $n > (m-1)^2$. Furthermore, the trees used in the decomposition can be made to be the same size.

Proof. $K_{m,n}$ can be decomposed into m trees isomorphic to $K_{1,n}$. Therefore, $\tau(K_{m,n}) \leq m$. $\tau(K_{m,n}) \geq \left\lceil \frac{m n}{m+n-1} \right\rceil$ by Lemma 4.1.1. When $n \geq (m-1)^2+1$, then by Lemma 4.1.2, we know that $\left\lceil \frac{m n}{m+n-1} \right\rceil \geq \left\lceil \frac{m(m^2-2m+2)}{m^2-m+1} \right\rceil = \left\lceil m - 1 + \frac{1}{m^2-m+1} \right\rceil = m$. Thus, when $n > (m-1)^2$, $\tau(K_{m,n}) = m$. The trees used in the decomposition are each of size n as they are each isomorphic to $K_{1,n}$. •

4.2 Construction of Array A

To show that $\tau(K_{m,n}) = \left\lceil \frac{m n}{m+n-1} \right\rceil$, $n \leq (m-1)^2$, Beineke shows that $\tau(K_{m,n}) \leq \left\lceil \frac{m n}{m+n-1} \right\rceil$. It then follows from Lemma 4.1.1 that $\tau(K_{m,n}) = \left\lceil \frac{m n}{m+n-1} \right\rceil$.

Beineke's argument proceeds essentially as follows. First, we let $t = \left\lceil \frac{m\,n}{m+n-1} \right\rceil$. Then, we define an m by t array A whose cells contain finite sequences of consecutive positive integers modulo n. We let $a(i,j) = \left\lceil (i+j)(\frac{n}{t}) \right\rceil - \left\lceil (i+j-1)(\frac{n}{t}) \right\rceil$ be the length of the sequence in the (i,j) cell, $1 \le i \le m$, $1 \le j \le t$, of A. We place a(i,j) consecutive integers, modulo n, into the (i,j) cell in the following way: The entries in the first row are consecutive positive integers, modulo n, starting at 1. Thus the entries in cell (1,1) are 1,2,...,a(1,1), the entries in cell (1,2) are a(1,1)+1,..., a(1,1)+a(1,2); and so on. Similarly, each row i is filled with consecutive integers, modulo n, where the first integer in row i, $2 \le i \le m$, is the last integer in cell (i-1,1).

An example of the array A and its corresponding decomposition is given in Example 4.2.1.

Example 4.2.1: The complete bipartite graph $K_{5,9}$ can be decomposed into 4 trees. We form array A, with $t = \left\lceil \frac{45}{13} \right\rceil = 4$ columns, and show the decomposition that corresponds to the array A.

$a(1,1) = \left\lceil 2(\frac{9}{4}) \right\rceil - \left\lceil \frac{9}{4} \right\rceil = 5 - 3 = 2.$

$a(2,1) = a(1,2) = \left\lceil 3(\frac{9}{4}) \right\rceil - \left\lceil 2(\frac{9}{4}) \right\rceil = 7 - 5 = 2.$

$a(3,1) = a(2,2) = a(1,3) = \left\lceil 4(\frac{9}{4}) \right\rceil - \left\lceil 3(\frac{9}{4}) \right\rceil = 9 - 7 = 2.$

$a(4,1) = a(3,2) = a(2,3) = a(1,4) = \left\lceil 5(\frac{9}{4}) \right\rceil - \left\lceil 4(\frac{9}{4}) \right\rceil = 12 - 9 = 3.$

$a(5,1) = a(4,2) = a(3,3) = a(2,4) = \left\lceil 6(\frac{9}{4}) \right\rceil - \left\lceil 5(\frac{9}{4}) \right\rceil = 14 - 12 = 2.$

$a(5,2) = a(4,3) = a(3,4) = \lceil 7(\frac{9}{4}) \rceil - \lceil 6(\frac{9}{4}) \rceil = 16 - 14 = 2.$

$a(5,3) = a(4,4) = \lceil 8(\frac{9}{4}) \rceil - \lceil 7(\frac{9}{4}) \rceil = 18 - 16 = 2.$

$a(5,4) = \lceil 9(\frac{9}{4}) \rceil - \lceil 8(\frac{9}{4}) \rceil = 21 - 18 = 3.$

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \\ \end{array}$$

$$A = \begin{array}{c|cccc|} & T_1 & T_2 & T_3 & T_4 \\ u_1 & 12 & 34 & 56 & 789 \\ u_2 & 23 & 45 & 678 & 91 \\ u_3 & 34 & 567 & 89 & 12 \\ u_4 & 456 & 78 & 91 & 23 \\ u_5 & 67 & 89 & 12 & 345 \end{array}$$

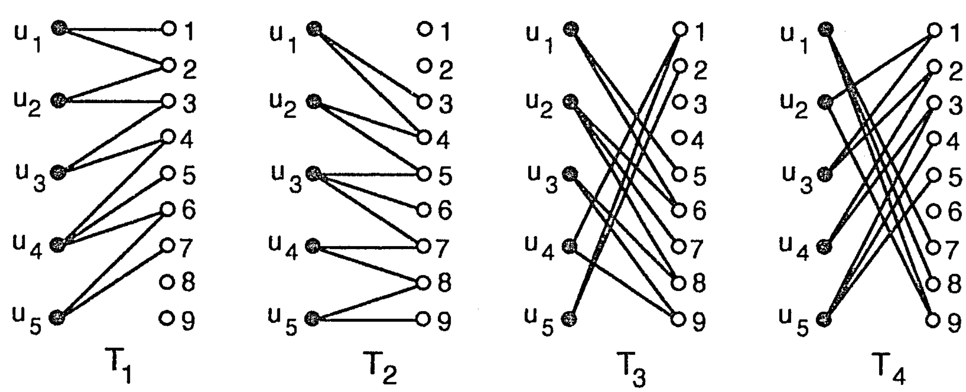

Figure 4.2.1

The columns in the array A correspond to the trees used in the decomposition of $K_{5,9}$. The entries in the i'th row correspond to endvertices in V of edges incident to vertex u_i. In Figure 4.2.1 each tree T_j is shown together with the vertices of $K_{5,9}$ which are not in T_j. Notice that $T_1 \cup T_2 \cup T_3 \cup T_4 = K_{5,9}$. Also notice that T_1, T_2, and T_3

each have 11 edges and T_4 has 12 edges. We will prove that, in general, $\bigcup_{i=1}^{t} T_i = K_{m,n}$ and that $\left| |E(T_i)| - |E(T_j)| \right| \leq 1$ for $1 \leq i, j \leq t$.

4.3 Properties of Array A

Lemma 4.3.1. The array A has the following four properties:

I) The entries in each row are the n consecutive integers modulo n.

II) The first entry in cell (i,j) is the last entry in cell (i-1,j) for $2 \leq i \leq m$ and $1 \leq j \leq t$.

III) In each column, if the first entry of all cells except the first is excluded, the remaining entries are consecutive integers modulo n and there are at most n of them.

IV) The number of entries in any two columns differ by at most one.

Proof.

I) Since the terms being summed telescope, for each i,

$$\sum_{j=1}^{t} a(i,j) = \left\lceil (i+t)\left(\frac{n}{t}\right) \right\rceil - \left\lceil (i)\left(\frac{n}{t}\right) \right\rceil = n + \left\lceil (i)\left(\frac{n}{t}\right) \right\rceil - \left\lceil (i)\left(\frac{n}{t}\right) \right\rceil = n.$$

This summation shows that each row contains exactly n integers. The fact that these integers are consecutive follows from the construction of array A. This proves that A has property (I).

II) We use induction on j. By our definition of array A, the first entry in cell(i,1) is the last entry in cell (i-1,1), $2 \leq i \leq m-1$. That is (II) is true for $j = 1$. We assume that (II) is true for all

$i = 2,\ldots,m$ and some fixed j, $2 \le j \le t$. Let $a(i,j) = a$. Then by definition $a(i-1,j+1) = a(i,j) = a$. Let the last entry in cell $(i-1,j)$ $= x$. Then by inductive assumption, the first entry in cell (i,j) is x. See Figure 4.3.2. Since cell $(i-1,j+1)$ contains a consecutive integers starting with $x+1$, its last entry is $x+a$. Furthermore, since cell (i,j) also contains a consecutive integers and cell $(i,j+1)$ begins with the next consecutive integer, the first entry in cell $(i,j+1)$ is $x+a$. Therefore the last entry in cell $(i-1,j+1)$ is the first entry in cell $(i,j+1)$ and (II) is proved.

cell (i-1,j)	cell (i-1,j+1)
(..., x)	(x+1,...,x+a)
cell (i,j)	**cell(i,j+1)**
(x,...,$x+a-1$)	(x+a, ...)

Figure 4.3.1

III) Again, since the terms telescope, for each j, and by Lemma 4.1.3, $\sum_{i=1}^{m} a(i,j) = \lceil (m+j)(\frac{n}{t}) \rceil - \lceil (j)(\frac{n}{t}) \rceil = (\lceil \frac{m\,n}{t} \rceil$ or $\lfloor \frac{m\,n}{t} \rfloor)$ $\le \left\lceil \frac{m\,n}{(\frac{m\,n}{m+n-1})} \right\rceil = m+n-1.$ Subtracting the m-1 entries, corresponding to the first integers in each cell appearing in the preceding cell, we have no more than n entries remaining in column j. The fact that these are consecutive residue classes follows from (II). Therefore, A also has property (III).

IV) Since the number of entries in each column = $\lceil \frac{mn}{t} \rceil$ or $\lfloor \frac{mn}{t} \rfloor$, as shown in the proof of (III), we see that the number of entries in any two columns differ by at most one. Thus, A has property (IV). •

4.4 Decomposition of $K_{m,n}$ into trees of nearly equal size

Theorem 4.4.1. For the complete bipartite graph $K_{m,n}$, $\tau(K_{m,n}) = t = \left\lceil \frac{mn}{m+n-1} \right\rceil$, and the trees used in the decomposition can be selected so that their sizes differ by at most one.

Proof. Let m and n be given and let $t = \left\lceil \frac{mn}{m+n-1} \right\rceil$. If m = 1, then the graph is already a tree. If $n > (m-1)^2$, then $\tau(K_{m,n}) = m$, by Lemma 4.1.4. In this case, $K_{m,n}$ can be decomposed into m copies of the graph $K_{1,n}$. Hence we may assume $2 \le m \le n \le (m-1)^2$. We define t graphs $T_1, T_2, \ldots, T_t$ using the t columns of the array A. Specifically T_j is the subgraph induced by the edge set $\{u_i x \mid x \text{ is in cell } (i,j),\ i = 1,\ldots,m\}$ The fact that T_j is acyclic follows from Lemma 4.3.1(III) as no number is repeated in a column, except in the first entry of all cells after the initial cell in each column. Thus, no cycle is formed. It is apparent from Lemma 4.3.1(II) that each T_j is connected and hence a tree. Also, $\bigcup_{i=1}^{t} T_j = K_{m,n}$ follows from Lemma 4.3.1(I), since it implies that each u_i is adjacent to each h. Therefore $\tau(K_{m,n})$ is at most t. However, $\tau(K_{m,n}) \ge \left\lceil \frac{mn}{m+n-1} \right\rceil = t$ by Lemma

4.1.1. Hence, $\tau(K_{m,n}) = \left\lceil \frac{m\,n}{m+n-1} \right\rceil = t$. From Lemma 4.3.1(IV) we see that $\sum_{i=1}^{m} a(i,j) = \left(\left\lceil \frac{m\,n}{t} \right\rceil \text{ or } \left\lfloor \frac{m\,n}{t} \right\rfloor \right)$. Therefore, $\left|\, |E(T_j)| - |E(T_k)| \,\right| \le 1$, for $1 \le j, k \le t$. Thus, the theorem is proved; that is, $K_{m,n}$ can be decomposed into t trees such that all trees in the decomposition are nearly equal in size. •

We thought it would be interesting to decompose $K_{m,n}$ into t trees such that all but at most one of the trees is a spanning tree. These results follow in Chapter 5.

Chapter 5

Tree Decomposition of the Complete Bipartite Graphs into Nearly All Spanning Trees.

We saw in Chapter 4 that $K_{m,n}$ can be decomposed into $\tau(K_{m,n}) = t = \lceil \frac{mn}{m+n-1} \rceil$ trees where the trees differ in size by at most one edge. In this chapter we show that $K_{m,n}$ can be decomposed into t trees where at least t-l trees are spanning trees. We start by decomposing $K_{m,n}$ when $n \geq (m-1)^2$ into spanning trees, but first we prove the following.

Lemma 5.0.1. For $n > (m-1)^2$, $\tau(K_{m,n}) = m$. For $n = (m-1)^2$, $\tau(K_{m,n}) = m-1$. For $n \leq (m-1)^2$, $\tau(K_{m,n}) \leq m-1$.

Proof. For $n > (m-1)^2$, $\tau(K_{m,n}) = m$ by Lemma 4.1.4. For $n = (m-1)^2$, by Theorem 4.4.1, $\tau(K_{m,n}) = \lceil (\frac{m(m-1)^2}{m+(m-1)^2-1}) \rceil = \lceil (\frac{m(m-1)^2}{m(m-1)}) \rceil = m-1$. For $n < (m-1)^2$, using Theorem 4.4.1 and Lemma 4.1.2, we have that $\tau(K_{m,n}) = t = \lceil (\frac{mn}{m+n-1}) \rceil \leq \lceil (\frac{m(m-1)^2}{m+(m-1)^2-1}) \rceil = \lceil (\frac{m(m-1)^2}{m^2-m}) \rceil = m-1$. •

5.1 Tree Decomposition of $K_{m,n}$, $n \geq (m-1)^2$ into nearly all spanning trees

Lemma 5.1.1. $K_{m,n}$, $n \geq (m-1)^2$, can be decomposed into t-1 spanning trees and one additional tree.

Proof: For $n = (m-1)^2$, $\tau(K_{m,n}) = m-1$ by Lemma 5.0.1. Since $\frac{m(m-1)^2}{m+(m-1)^2-1} = m-1$, the method used in Theorem 4.4.1 will decompose $K_{m,(m-1)^2}$ into m-1 spanning trees. For $n > (m-1)^2$, $\tau(K_{m,n}) = m$ by Lemma 5.0.1. Therefore, we construct the following m by m array D whose cells contain finite sequences of positive integers. Array D is shown in Figure 5.1.1 on page 34. The entries in the i'th row represent endvertices of edges incident to u_i, and the columns correspond to trees, T_1 through T_m, used in the decomposition. We will show that the union of T_1 through T_m is $K_{m,n}$, and that indeed each T_j, $1 \leq j \leq m$ is a tree. The notation $a \rightarrow b$ in cell (i,j) means that the integers a through b inclusive are listed. This in turn means that vertex u_i is adjacent to vertices a through b inclusive in subgraph T_j. Similarly, the notation $\overline{a \rightarrow b}$ in cell (i,j) means the numbers 1 through n except a through b inclusive are listed. This in turn means that in T_j, u_i is adjacent to all vertices in V with the exception of vertices a through b.

To prove that the array D represents a decomposition of $K_{m,n}$ into m trees where at least (m-1) trees are spanning trees, we verify:

(I) the first (m-1) T_j's are spanning trees and T_m is a tree, and

(II) $\bigcup_{i=1}^{m} T_i = K_{m,n}$.

	T_1	T_2	T_3	...	T_i	...	T_{m-1}	T_m
u_1	$\overline{1 \rightarrow (m-2)}$	1	2	...	$(i-1)$	...	$(m-2)$	-
u_2	$1(m-2)+1$	$\overline{(m-2)+1 \rightarrow 2(m-2)}$	$1(m-2)+2$	...	$(m-2)+(i-1)$	...	$2(m-2)$	-
u_3	$2(m-2)+1$	$2(m-2)+2$	$\overline{2(m-2)+1 \rightarrow 3(m-2)}$	...	$2(m-2)+(i-1)$	...	$3(m-2)$	-
...	...	...	...	...	...	...	...	...
u_i	$(i-1)(m-2)+1$	$(i-1)(m-2)+2$	$(i-1)(m-2)+3$	...	$\overline{(i-1)(m-2)+1 \rightarrow i(m-2)}$	...	$i(m-2)$	-
...	...	...	...	...	...	...	...	...
u_{m-1}	$(m-2)^2+1$	$(m-2)^2+2$	$(m-2)^2+3$	...	$(m-2)^2+i$	...	$\overline{(m-2)^2+1 \rightarrow (m-1)(m-2)}$	-
u_m	$1 \rightarrow (m-2)\&$ $(m-1)(m-2)+1$	$1(m-2)+1 \rightarrow 2(m-2)$ $\&(m-1)(m-2)+2$	$2(m-2)+1 \rightarrow 3(m-2)$ $\&(m-1)(m-2)+3$	...	$(i-1)(m-2)+1 \rightarrow i(m-2)$ $\&(m-1)(m-2)+i$	...	$(m-2)^2+1 \rightarrow (m-1)(m-2)$ $\&(m-1)^2$	$(m-1)^2+1 \rightarrow n$

Figure 5.1.1: Array D associated with $K_{m,n}$ where $n > (m-1)^2$.

I) Clearly T_m is a tree consisting of one vertex adjacent to exactly $n-(m-1)^2$ other vertices. To show that the first (m-1) columns correspond to spanning trees, we show that all the vertices $u_1,\dots,u_m$ and $v_1,\dots,v_n$ are included and are not isolated vertices, and that there are m+n-1 edges. Under these conditions the resulting graph must be a spanning tree. Consider column j, $1 \le j \le m-1$. First look at rows j and m, between these two rows all of the numbers 1 through n are listed, and there are n+1 entries which correspond to n+1 edges. The other m-2 rows each have one entry, so there are an additional m-2 edges. Thus, there are m+n-1 entries in column j which represent m+n-1 edges in T_j. Since none of the rows in column j are empty, all of the vertices $u_1,\dots,u_m$ are in T_j. Similarly, since all of the numbers 1 through n are listed in rows j and m, all of the vertices $v_1,\dots,v_n$ are in T_j. Therefore, T_j has m+n vertices and m+n-1 edges. So, T_j is a spanning tree for $1 \le j \le m-1$.

II) To see that $\bigcup_{j=1}^{m} T_j = K_{m,n}$ consider row i, $1 \le i \le m-1$. We must show that there are exactly n distinct entries. In row i the entries in columns 1,2,...,i-1,i+1,...,m-1 are simply the missing entries from column i. Thus all entries 1 through n exist. In row m it is clear that all entries 1 through n exist. Therefore, $\bigcup_{i=1}^{m} T_i = K_{m,n}$. •

Notice that we can decompose $K_{m,(m-1)^2}$ into m-1 spanning trees using the array D as column m will be empty in this case.

Example 5.1.2. We show the decomposition of $K_{5,20}$ into 4 spanning trees and one tree with 4 edges. We use the decomposition given in array D, where the entries in row i represent endvertices in V of edges incident with u_i, and columns correspond to the trees used in the decomposition.

D =

	T_1	T_2	T_3	T_4	T_5
u_1	4-20	1	2	3	-
u_2	4	1-3&7-20	5	6	-
u_3	7	8	1-6&10-20	9	-
u_4	10	11	12	1-9&13-20	-
u_5	1-3&13	4-6&14	7-9&15	10-12&16	17-20

5.2 Preliminary results for $K_{m,n}$ when $n < (m-1)^2$

Before we begin to decompose $K_{m,n}$, $n < (m-1)^2$ into t trees where at least t-1 trees are spanning trees, we need some preliminary results.

Lemma 5.2.1. Given $b(i,j) = \lceil (i+j)(\frac{m+n-2}{m-1}) \rceil - \lceil (i+j-1)(\frac{m+n-2}{m-1}) \rceil$, $t = \lceil \frac{mn}{m+n-1} \rceil$, and $1 \le r < m \le n$, then

(I) for $1 \le i \le (m-1)$ and $j \le t-1$,

$$b(i,j) = \left(\lfloor (\frac{m+n-2}{m-1}) \rfloor \text{ or } \lceil (\frac{m+n-2}{m-1}) \rceil \right) \ge 2.$$

(II)
$$\sum_{j=1}^{r} b(i,j) = \lceil (i+r)(\frac{m+n-2}{m-1}) \rceil - \lceil (i)(\frac{m+n-2}{m-1}) \rceil$$
$$= \lfloor (r)(\frac{m+n-2}{m-1}) \rfloor \text{ or } \lceil (r)(\frac{m+n-2}{m-1}) \rceil \text{ for } i,\ 1 \le i \le m-1.$$

(III) $\max_{1 \le i \le m-1} \{ \sum_{j=1}^{r} b(i,j) \} = \sum_{j=1}^{r} b(m-1,j) = \lceil (r)(\frac{m+n-2}{m-1}) \rceil$.

(IV) $\sum_{j=1}^{t-2} b(i,j) < n$, for i, $1 \le i \le m-1$.

(V) $\sum_{j=1}^{t-1} b(i,j) \ge n-2$, for i, $1 \le i \le m-1$.

(VI) for j, $1 \le j \le t-1$, $\sum_{i=1}^{m-1} b(i,j) = m+n-2$.

Proof.

(I) $b(i,j) = (\lfloor (\frac{m+n-2}{m-1}) \rfloor$ or $\lceil (\frac{m+n-2}{m-1}) \rceil)$ by Lemma 4.1.3. Since $2 \le m \le n$, it follows that $\lfloor (\frac{m+n-2}{m-1}) \rfloor \ge 2$. Therefore $b(i,j) = (\lfloor (\frac{m+n-2}{m-1}) \rfloor$ or $\lceil (\frac{m+n-2}{m-1}) \rceil) \ge 2$.

(II) Since the terms being summed telescope, it follows that $\sum_{j=1}^{r} b(i,j) = \lceil (i+r)(\frac{m+n-2}{m-1}) \rceil - \lceil (i)(\frac{m+n-2}{m-1}) \rceil$. By Lemma 4.1.3, $\lceil (i+r)(\frac{m+n-2}{m-1}) \rceil - \lceil (i)(\frac{m+n-2}{m-1}) \rceil = \lfloor (r)(\frac{m+n-2}{m-1}) \rfloor$ or $\lceil (r)(\frac{m+n-2}{m-1}) \rceil$ for i, $1 \le i \le m-1$.

(III) By (II), we know that $\sum_{j=1}^{r} b(i,j) = \lfloor (r)(\frac{m+n-2}{m-1}) \rfloor$ or $\lceil (r)(\frac{m+n-2}{m-1}) \rceil$. Therefore, $\max_{1 \le i \le m-1} \sum_{j=1}^{r} b(i,j) \le \lceil (r)(\frac{m+n-2}{m-1}) \rceil$. Also by (II) we know that $\sum_{j=1}^{r} b(m-1,j) = \lceil (m-1+r)(\frac{m+n-2}{m-1}) \rceil - \lceil (m-1)(\frac{m+n-2}{m-1}) \rceil$

$= \lceil (m-1)(\frac{m+n-2}{m-1}) + r(\frac{m+n-2}{m-1}) \rceil - \lceil (m-1)(\frac{m+n-2}{m-1}) \rceil$

$= \lceil m+n-2 + (r)(\frac{m+n-2}{m-1}) \rceil - \lceil m+n-2 \rceil = \lceil (r)(\frac{m+n-2}{m-1}) \rceil$.

Hence, $\max_{1 \le i \le m-1} \{ \sum_{j=1}^{r} b(i,j) \} = \lceil (r)(\frac{m+n-2}{m-1}) \rceil = \sum_{j=1}^{r} b(m-1,j)$.

(IV) We know from (II) that

$\sum_{j=1}^{t-2} b(i,j) = \lceil (i+t-2)(\frac{m+n-2}{m-1}) \rceil - \lceil (i)(\frac{m+n-2}{m-1}) \rceil$ for i, $1 \le i \le m-1$.

and that, $\sum_{j=1}^{t-2} b(i,j) \le \lceil (t-2)(\frac{m+n-2}{m-1}) \rceil = \lceil (\lceil \frac{mn}{m+n-1} \rceil - 2)(\frac{m+n-2}{m-1}) \rceil$

$\le \lceil (\frac{mn}{m+n-1})(\frac{m+n-2}{m-1}) - (\frac{m+n-2}{m-1}) \rceil$

$< \lceil (\frac{mn}{m+n-1})(\frac{m+n-1}{m-1}) - (\frac{m+n-2}{m-1}) \rceil = \lceil \frac{mn-m-n+2}{m-1} \rceil = \lceil n-1+ \frac{1}{m-1} \rceil$

$\le n$. Therefore, $\sum_{j=1}^{t-2} b(i,j) < n$, for $1 \le i \le (m-1)$.

(V) We know from (III) that

$\sum_{j=1}^{t-1} b(m-1,j) = \lceil (t-1)(\frac{m+n-2}{m-1}) \rceil = \lceil \lceil \frac{mn}{m+n-1} - 1 \rceil (\frac{m+n-2}{m-1}) \rceil$

$\ge \lceil (\frac{mn-m-n+1}{m+n-1})(\frac{m+n-2}{m-1}) \rceil = \lceil (\frac{(m-1)(n-1)}{m+n-1})(\frac{m+n-1}{m-1} + \frac{-1}{m-1}) \rceil$

$= \lceil (n-1) - \frac{n-1}{m+n-1} \rceil = n-1$ for $m > 1$.

So, $\sum_{j=1}^{t-1} b(m-1,j) \ge n-1$. Since, for $1 \le i \le m-1$, the sum of the first $r = t-1$ terms of any two rows, excluding row m, differ by at most 1 by (II), it follows that $\sum_{j=1}^{t-1} b(i,j) \ge n-2$ for $1 \le i \le m-1$.

(VI) Since the terms being summed telescope,

$\sum_{i=1}^{m-1} b(i,j) = \lceil (m-1+j)(\frac{m+n-2}{m-1}) \rceil - \lceil (j)(\frac{m+n-2}{m-1}) \rceil$

$= \lceil (m+n-2) + (j)(\frac{m+n-2}{m-1}) \rceil - \lceil (j)(\frac{m+n-2}{m-1}) \rceil = (m+n-2)$.

Thus (VI) is proved. ∘

Next, we use the preliminary results to construct an array B which we use to find a decomposition of $K_{m,n}$ into trees, nearly all of which are spanning trees.

5.3 Construction of Array B

To decompose $K_{m,n}$, $n < (m-1)^2$, into t trees where at least t-1 trees are spanning trees, we start by defining an m by t array B whose cells contain finite sequences of consecutive positive integers modulo n. We let $b(i,j) = \lceil (i+j)(\frac{m+n-2}{m-1}) \rceil - \lceil (i+j-1)(\frac{m+n-2}{m-1}) \rceil$ for $1 \le i \le m-1$, and $1 \le j \le t$. We place b(i,j) consecutive integers, modulo n, into the (i,j) cell for each pair (i,j), $1 \le i \le m-1$, $1 \le j \le t-2$, in the following way: The entries in the first row are consecutive positive integers, modulo n, starting at 1. Thus the entries in cell (1,1) are 1,2,...,b(1,1), the entries in cell (1,2) are b(1,1)+1,..., b(1,1)+b(1,2); and so on. Similarly, each row i is filled with consecutive integers, modulo n. The first integer in row i, $2 \le i \le m-1$, is the last integer in cell (i-1,1), and each cell (i,j), $1 \le j \le t-2$, receives b(i,j) integers. At this point the first m-1 rows of the first t-2 columns are filled. The fact that each row has less than n entries at this stage is shown by Lemma 5.2.1(IV). In order to complete the array, we find it useful to calculate $\sum_{j=1}^{t-1} b(m-1,j)$. We choose i = m-1 because we showed, in Lemma 5.2.1(III), that the sum $\sum_{j=1}^{t-1} b(i,j)$ is largest when i = m-1. To complete the definition of array B, we consider two cases.

<u>Case 1.</u> If $\sum_{j=1}^{t-1} b(m-1,j) > n$, then place the numbers 1 through n in cell (m,t-1) and calculate the last two columns as follows: In column t-1, for $1 \le i \le m-1$, the i'th cell begins with one more than the last entry in the i'th cell of column t-2 and continues modulo n until one less than the first entry in the i'th cell in column 1. Column t is left empty. We illustrate this case in Example 5.3.1.

<u>Case 2.</u> If $\sum_{j=1}^{t-1} b(m-1,j) \le n$, then place the numbers 1 through n in cell (m,t), and calculate the last two columns as follows: In column (t-1) use the same formula and process as used for the preceding columns. That is each cell (i,t-1) receives b(i,t-1) consecutive integers modulo n, where the first integer is one more, modulo n, than the last integer in cell (i,t-2). In column t, for $1 \le i \le m-1$, the i'th cell begins with one more than the last entry in the i'th cell of column t-1 and continues modulo n until one less than the first entry in the i'th cell in column 1. We illustrate this case in Example 5.3.2.

We can view b(i,j) as the capacity of cell (i,j). For $1 \le i \le m-1$ and $1 \le j \le t-2$, each cell is filled to capacity. When $\sum_{j=1}^{t-1} b(m-1,j) \le n$, then each cell (i,t-1), $1 \le i \le m-1$, is also filled to capacity. When $\sum_{j=1}^{t-1} b(m-1,j) > n$, some cells in column t-1 are not filled to capacity.

Also, in this case, cells (i,t-1) and (i+1,t-1) may not have a common element.

We can view the entries in cell (i,j) as endvertices in V of edges incident with u_i, and we can view columns as defining subgraphs $H_1, \dots, H_t$ used in a decomposition. Specifically, H_j is the graph induced by the edge set $\{u_i p \mid \text{where } p \text{ is in cell } (i,j),\ 1 \leq i \leq m\}$. We find it convenient to label column j in array B with H_j, and row i with u_i. In the subsequent figures we show H_j together with vertices of $K_{m,n}$ not included in H_j. The decomposition is not yet into spanning trees. The next step is to use this decomposition to obtain a decomposition into spanning trees. The decomposition at this stage is shown in Examples 5.3.1 and 5.3.2.

Example 5.3.1: We show the decomposition of $K_{5,7}$ given by the array B. $\tau(K_{5,7}) = 4$, so there will be 4 columns in the array B.

For $i = 4$, $\sum_{j=1}^{3} b(i,j) = \lceil (5-1+3)(\frac{10}{4}) \rceil - \lceil (5-1)(\frac{10}{4}) \rceil = 8 > n = 7.$

So, column 4 will be empty, and column 3 is calculated separately.

$b(1,1) = \lceil 2(\frac{10}{4}) \rceil - \lceil (\frac{10}{4}) \rceil = 5 - 3 = 2.$

$b(2,1) = b(1,2) = \lceil 3(\frac{10}{4}) \rceil - \lceil 2(\frac{10}{4}) \rceil = 8 - 5 = 3.$

$b(3,1) = b(2,2) = \lceil 4(\frac{10}{4}) \rceil - \lceil 3(\frac{10}{4}) \rceil = 10 - 8 = 2.$

$b(4,1) = b(3,2) = \lceil 5(\frac{10}{4}) \rceil - \lceil 4(\frac{10}{4}) \rceil = 13 - 10 = 3.$

$b(4,2) = \lceil 6(\frac{10}{4}) \rceil - \lceil 5(\frac{10}{4}) \rceil = 15 - 13 = 2.$

B =

	H_1	H_2	H_3	H_4
u_1	1 2	3 4 5	6 7	-
u_2	2 3 4	5 6	7 1	-
u_3	4 5	6 7 1	2 3	-
u_4	5 6 7	1 2	3 4	-
u_5	-	-	1→7	-

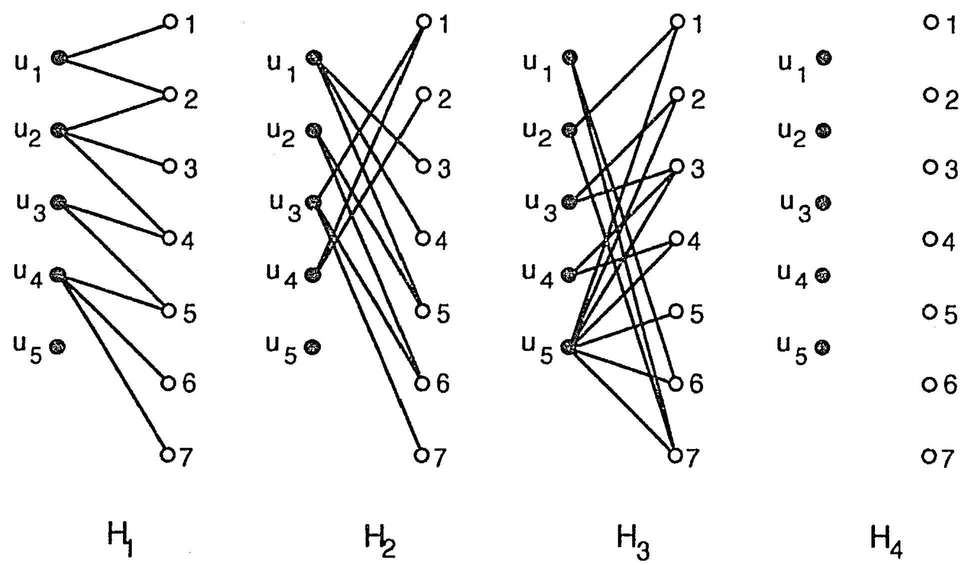

Figure 5.3.1

Example 5.3.2: We show the decomposition of $K_{6,8}$ given by array B. $\tau(K_{6,8}) = 4$, so there will be 4 columns in array B.

For $i = 5$, $\sum_{j=1}^{3} b(i,j) = \lceil (6-1+3)(\frac{12}{5}) \rceil - \lceil (6-1)(\frac{12}{5}) \rceil = 8 = n$.

Therefore, column 4 is nonempty and is calculated separately.

$b(1,1) = \lceil 2(\frac{12}{5}) \rceil - \lceil \frac{12}{5} \rceil = 5 - 3 = 2.$

$b(2,1) = b(1,2) = \lceil 3(\frac{12}{5}) \rceil - \lceil 2(\frac{12}{5}) \rceil = 8 - 5 = 3.$

$b(3,1) = b(2,2) = b(1,3) = \lceil 4(\frac{12}{5}) \rceil - \lceil 3(\frac{12}{5}) \rceil = 10 - 8 = 2.$

$b(4,1) = b(3,2) = b(2,3) = \lceil 5(\frac{12}{5}) \rceil - \lceil 4(\frac{12}{5}) \rceil = 12 - 10 = 2.$

$b(5,1) = b(4,2) = b(3,3) = \lceil 6(\frac{12}{5}) \rceil - \lceil 5(\frac{12}{5}) \rceil = 15 - 12 = 3.$

$b(5,2) = b(4,3) = \lceil 7(\frac{12}{5}) \rceil - \lceil 6(\frac{12}{5}) \rceil = 17 - 15 = 2.$

$b(5,3) = \lceil 8(\frac{12}{5}) \rceil - \lceil 7(\frac{12}{5}) \rceil = 20 - 17 = 3.$

$$B = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}\begin{array}{cccc} H_1 & H_2 & H_3 & H_4 \\ \hline 1\,2 & 3\,4\,5 & 6\,7 & 8 \\ 2\,3\,4 & 5\,6 & 7\,8 & 1 \\ 4\,5 & 6\,7 & 8\,1\,2 & 3 \\ 5\,6 & 7\,8\,1 & 2\,3 & 4 \\ 6\,7\,8 & 1\,2 & 3\,4\,5 & - \\ - & - & - & 1 \rightarrow 8 \end{array}$$

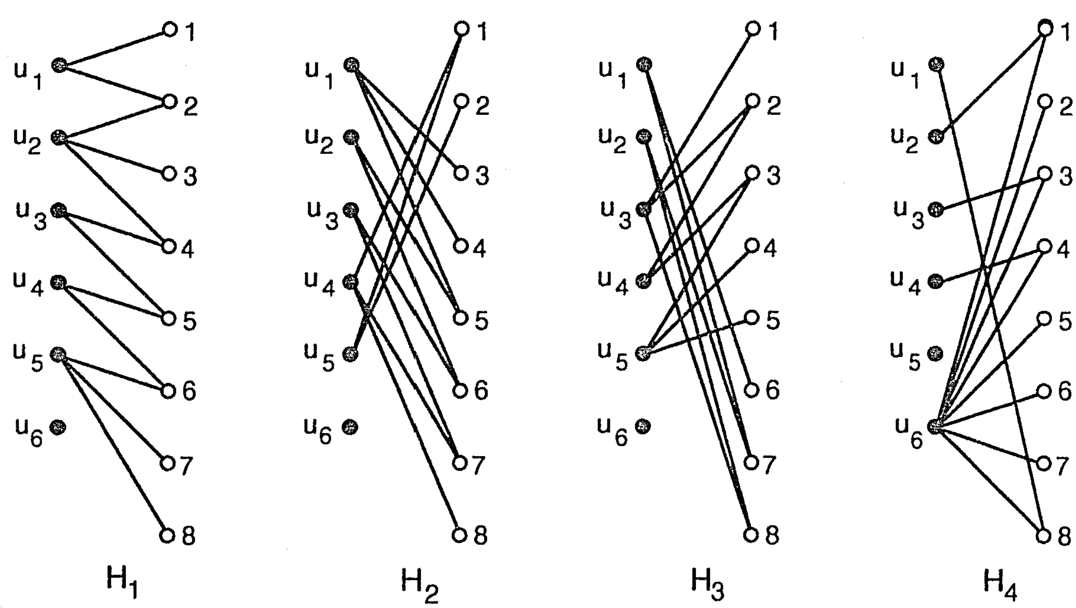

Figure 5.3.2

5.4 Properties of Array B

Let $\ell = (t-1)$ if $\sum_{j=1}^{t-1} b(m-1,j) \le n$ or $(t-2)$ if $\sum_{j=1}^{t-1} b(m-1,j) > n$.

Theorem 5.4.1. The array B has the following four properties:

(I) The entries in each row are the n consecutive integers modulo n.

(II) For $1 \le j \le \ell$, $\sum_{i=1}^{m-1} b(i,j) = m+n-2$. Therefore, in each column j, $1 \le j \le \ell$, there are $m+n-2$ entries.

(III) The first entry in cell (i,j) is the last entry in cell (i-1,j) for $2 \le i \le m-1$ and $1 \le j \le \ell$.

(IV) In each column j, $j \le \ell$, if the last entry of each cell (i,j), $1 \le i \le m-1$, is excluded, then the remaining entries are the n consecutive integers modulo n.

Proof.

(I) By the construction of array B, and Lemma 5.2.1(IV), it is apparent that each row has n entries. The first row has consecutive integers 1,2,...,n and by construction each subsequent row has entries which are consecutive integers modulo n. Thus (I) is proved.

(II) This result follows from the definition of array B and Lemma 5.2.1(IV). Thus (II) is proved.

(III) We use induction on j. By our definition of array B, the first entry in cell(i,1) is the last entry in cell (i-1,1), $2 \le i \le m-1$.

That is (III) is true for $j = 1$. We assume that (III) is true for all $i = 2,\dots,m-1$ and some fixed j, $2 \le j \le \ell$. Let $b(i,j) = b$. Then by definition $b(i-1,j+1) = b(i,j) = b$. Let the last entry in cell $(i-1,j) = x$. Then by inductive assumption, the first entry in cell (i,j) is x. Also, the last entry in cell $(i-1,j+1) = x+b$ and cells $(i-1,j+1)$ and (i,j) have the same number of entries. Therefore the first entry in cell $(i,j+1)$ is $x+b$. (See Figure 5.4.1.) Thus (III) is proved.

cell (i-1,j)	cell (i-1,j+1)
$(\dots, \quad x)$	$(x+1,\dots,x+b)$
cell (i,j)	cell(i,j+1)
$(x,\dots,x+b-1)$	$(x+b, \dots \quad)$

Figure 5.4.1

(IV) By (III), the first entry in cell (i,j) is the last entry in cell $(i-1,j)$ for $1 \le i \le m-1$, and $1 \le j \le \ell$, and there are $m+n-2$ entries by (II). Therefore, after excluding the $m-2$ repeated entries, there are n entries. By the choice of the first entry in each cell, the entries are the n consecutive integers modulo n. Thus (IV) is proved. ∎

5.5 Properties of the subgraphs defined by array B

Lemma 5.5.1. The subgraphs $H_1, \ldots, H_\ell$ are all trees of order $m+n-1$.

Proof. The graphs are connected by Lemma 5.4.1(III) since the last entry in cell (i-1,j) equals the first entry in cell (i,j) for $2 \le i \le m-1$, and $1 \le j \le \ell$. The fact that each H_j has m+n-1 vertices follows from Theorem 5.4.1(IV) and the fact that u_m is the only vertex in U not adjacent to an element in V. Since, according to Theorem 5.4.1(II), each column j, $1 \le j \le \ell$, has m+n-2 entries, it follows that each H_j has exactly m+n-2 edges. Therefore, since each H_j has m+n-1 vertices and m+n-2 edges, it follows that each H_j, $1 \le j \le \ell$, is a tree. ∘

To make H_j, $1 \le j \le \ell$, a spanning tree, an edge of the form $u_m k$ must be added to H_j.

Let H_{t-1}^* and H_t^* be the subgraphs induced by the edges of H_{t-1} and H_t, respectively, excluding the edges, if any, incident to u_m.

It is possible for H_{t-1}^* to be disconnected. An example of this case is shown in Example 5.3.1.

Lemma 5.5.2. The subgraphs H_{t-1}^* and H_t^* are acyclic.

Proof. There are two cases to consider depending whether or not column t has any entries.

(I) If column t has no entries, then H_t^* is acyclic. Also, if column t has no entries, then $\sum_{j=1}^{t-1} b(m-1,j) > n$ by construction. So cell (m-1,t-1) has fewer than b(m-1,t-1) entries. Since the total number of elements in any two rows differ by at most one entry by Lemma 5.2.1(II), $\sum_{j=1}^{t-1} b(i,j) \geq n$ for $1 \leq i \leq m-2$. Hence each cell (i,t-1), $1 \leq i \leq m-2$, has fewer than or equal to b(i,t-1) entries. If each cell has b(i,t-1) entries, then H_{t-1}^* has m+n-2 edges by Lemma 5.2.1(VI). The vertices $u_1,\ldots,u_{m-1}$ are all included in H_{t-1}^* as no row in column t-1 is empty by 5.2.1(IV). Also, if each cell (i,t-1) has b(i,t-1) entries, then the first entry in cell (i+1,t-1) is the last entry in cell (i,t-1) for $1 \leq i \leq m-2$ since cell (i+1,t-2) also has b(i,t-1) entries. After eliminating the m-2 repeated entries, the remaining n entries are the n consecutive integers modulo n. So all the vertices $v_1,\ldots,v_n$ are included in H_{t-1}^*. Hence H_{t-1}^* has m+n-1 vertices and m+n-2 edges, and thus is a tree. If some cell (i,t-1) has fewer than b(i,t-1) entries, then H_{t-1}^* is still acyclic.

(II) If column t has some entries, then $\sum_{j=1}^{t-1} b(m-1,j) \leq n$, so H_{t-1}^* is a tree and hence acyclic by Lemma 5.5.1. If each cell (i,t), $1 \leq i \leq m-1$ has b(i,t) entries, H_t^* is a tree by the same argument as (I). Again, since removing edges from a tree cannot form cycles, it follows that H_t^* is acyclic. Thus, it follows from (I) and (II) that both H_{t-1}^* and H_t^* are acyclic. ∎

5.6 Decomposition of $K_{m,n}$ into nearly all spanning trees when $2 \leq n < (m-1)^2$ and $\sum_{j=1}^{t-1} b(m-1,j) > n$

Lemma 5.6.1. Let $2 \leq n < (m-1)^2$. If $\sum_{j=1}^{t-1} b(m-1,j) > n$, then $K_{m,n}$ can be decomposed into t trees such that at least t-1 of them are spanning trees.

Proof. Since $\sum_{j=1}^{t-1} b(m-1,j) > n$, then by construction column t is empty. Therefore H_t is the empty graph. By Lemma 5.2.1(IV), for $1 \leq i \leq m-1$, each cell (i,t-1) of column t-1 is nonempty, and thus each u_i is in H_{t-1}. Note H^*_{t-1} is acyclic by Lemma 5.5.2. H^*_{t-1} has no more than m-1 components as each edge has an endvertex in the set U. Say H^*_{t-1} has $p \leq m-1$ components. Then, we need p edges of the form $u_m k$ to connect these components to u_m. For each component select an edge to join that component to u_m. Place the selected edges into H^*_{t-1}. Thus, we have changed H^*_{t-1} into a tree. Next, add an appropriate set of edges, if any, of the form $u_m k$ to change H^*_{t-1} into a spanning tree. Now we argue by contradiction that there are enough edges of the form $u_m k$ remaining to change the first t-2 trees into spanning trees as well. So, for the sake of obtaining a contradiction, assume there are not enough edges of the form $u_m k$ remaining to change the first t-2 trees into spanning trees. Place one edge of the form $u_m k$ into each tree until the edges of the form $u_m k$ run out. In this fashion, $K_{m,n}$ has been decomposed into t-1 trees. This contradicts the fact that t is the minimum number of trees required to decompose $K_{m,n}$. Hence, there must be enough edges to change $H_1, \ldots, H_{t-1}$ into spanning trees and there must be at least one edge of the form $u_m k$ to place in the tree H_t. We place all of the remaining edges of the form $u_m k$ into H_t. •

5.7 Decomposition of $K_{m,n}$ into nearly all spanning trees when $2m-3 \leq n < (m-1)^2$ and $\sum_{j=1}^{t-1} b(m-1,j) \leq n$

We showed in Lemma 5.6.1 that $K_{m,n}$ can be decomposed into t trees such that at least t-1 of them are spanning trees when $\sum_{j=1}^{t-1} b(m-1,j) > n$. Therefore, to decompose $K_{m,n}$ into almost all spanning trees, we need only consider the case where $\sum_{j=1}^{t-1} b(m-1,j) \leq n$.

The next step is to use the array B and the subgraphs $H_1, \ldots, H_t$ to form a decomposition into new subgraphs $T_1, \ldots, T_t$ of $K_{m,n}$ where all of the T_i are trees and at least t-1 of the T_i are spanning trees.

Before continuing, we consider the array B and subgraphs $H_1, \ldots, H_t$ for $K_{6,9}$ given in Example 5.7.1.

Example 5.7.1. We show the array B associated with $K_{6,9}$.

$$B = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{cccc} H_1 & H_2 & H_3 & H_4 \\ \hline 123 & 45 & 678 & 9 \\ 34 & 567 & 89 & 12 \\ 456 & 78 & 912 & 3 \\ 67 & 891 & 234 & 5 \\ 789 & 123 & 45 & 6 \\ - & - & - & 1 \to 9 \end{array}$$

Notice there exists a cycle in H_4. The cycle is $1u_2 2\, u_6 1$. In order to make the last subgraph a tree, one of the edges $u_6 1$ or $u_6 2$ will need to be moved to an earlier subgraph. We also need to move two other edges of the form $u_6 k$ to the other two earlier subgraphs in order to make them spanning trees. These changes are given in Example 5.7.2.

Example 5.7.2. The following array D represents a decomposition of $K_{6,9}$ into 3 spanning trees with 14 edges and 1 tree with 12 edges.

$$D = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \\ \hline 123 & 45 & 678 & 9 \\ 34 & 567 & 89 & 12 \\ 456 & 78 & 912 & 3 \\ 67 & 891 & 234 & 5 \\ 789 & 123 & 45 & 6 \\ 2 & 4 & 7 & 135689 \end{array}$$

Definition 5.7.3. Let C denote the least number of edges of the form $u_m k$ whose removal from H_t eliminates all cycles.

A 4-cycle is formed in H_t when two integers are in cell (i,t), $1 \leq i \leq m-1$. For example, say rs is in the k'th cell of H_t, then the 4-cycle $ru_k s u_m r$ is formed.

Lemma 5.7.4. Given $2 \leq m \leq n < (m-1)^2$, $\sum_{j=1}^{t-1} b(m-1,j) \leq n$, and the array B associated with $K_{m,n}$. Let d equal the number of cells (i,t), $1 \leq i \leq m-1$, with exactly two entries. Then C = d.

Proof. By Lemma 5.2.1(V), the last column of array B has at most 2 elements in any cell (i,t), $1 \leq i \leq m-1$. By Lemma 5.2.1(II), the number of elements in any two cells (i,t), (j,t), $1 \leq i, j \leq m-1$, differ by at most one. Therefore if any cell (i,t), $1 \leq i \leq m-1$, contains 2 elements then no cell (j,t), $1 \leq j \leq m-1$, has zero elements. If all cells (i,t), $1 \leq i \leq m-1$, have 0 or 1 element then no cycle is formed in H_t. To see this fact, say cell (i,t) has 1 element, then $\deg(u_i) = 1$ and therefore u_i is not part of a cycle. So, cycles are only formed when cell (i,t) has 1 or 2 elements for all i, $1 \leq i \leq m-1$. Therefore, H_t has order m+n as u_m is adjacent to vertices 1,...,n in V by the construction of array B, and $u_1, \ldots, u_{m-1}$ are all vertices in H_t as there are no blank cells in column t of array B. The number of edges in H_t equals (m-1) + d + n = n+m-1+d as the first m-1 cells of column t have at least 1 entry, d cells have 2 entries, and cell (m,t) has n entries. Subgraph H_t^* is acyclic by Lemma 5.5.2. Therefore all cycles in H_t contain edges incident to u_m. By the construction of array B, $n > d$. So we can make H_t into a tree of order m+n by removing d edges of the form $u_m p$. Removing fewer than d edges of the form $u_m p$ will leave at least one cycle in H_t. Therefore, $C = d$. •

Next, we show that $C \le t-1$.

Lemma 5.7.5. Given $2 \le m \le n < (m-1)^2$ and $\sum_{j=1}^{t-1} b(m-1,j) \le n$, then $C \le t-1$; that is, H_t can be made into a tree by moving t-1 or fewer edges of the form $u_m k$ to the first t-1 trees.

Proof. Since $\sum_{j=1}^{t-1} b(m-1,j) \le n$, the integers 1 through n are listed in cell (m,t). Therefore all of the edges of the form $u_m k$ are placed into the subgraph H_t. This construction often forms cycles in H_t as was shown in Example 5.7.1. We need to show that $C \le t-1$. By Lemma 5.2.1(V), we see that the last column of array B has at most two elements in each cell (i,t), $1 \le i \le m-1$. For the sake of obtaining a contradiction, we assume that $C > t-1$. It follows from Lemma 5.7.4 that $|E(H_t^*)| = (m-1) + C$. Now we count the number of edges excluding the edges incident with u_m. The number of edges in the first t-1 trees plus the number of edges in H_t^* equals mn-n. So, $(m+n-2)(t-1) + (m-1) + C = mn-n$. Since $C > t-1$, it follows that $(m+n-2)(t-1) + (m-1) + (t-1) < (m+n-2)(t-1) + (m-1) + C = mn-n$. Therefore, $(m+n-2)(t-1) + (m-1) + (t-1) < mn-n$.

But, $(m+n-2)(t-1)+(m-1)+(t-1)=(m+n-1)(t-1)+(m-1)$

$= (m+n-1)\lceil \frac{mn}{m+n-1} \rceil - (m+n-1) + (m-1) \geq mn - n$. We have reached a contradiction. Hence, $C \leq t-1$. ∘

Lemma 5.7.6. When $\sum_{j=1}^{t-1} b(m-1,j) \leq n$ and $2m-3 \leq n < (m-1)^2$, then $K_{m,n}$ is decomposable into t trees where nearly all of them are spanning trees.

Proof. By Lemma 5.2.1(V), the last column of array B has at most 2 elements in any cell (i,t), $1 \leq i \leq m-1$. Let d equal the number of cells (i,t), $1 \leq i \leq m-1$, with 2 elements. Then $d \leq t-1$ by Lemmas 5.7.4 and 5.7.5. Let $k = (t-1)-d$. Thus there are at most $(m-1) + d$ distinct integers listed in the first m-1 rows of column t. We know that $n \geq 2m-3 = (m-1) + (m-2) \geq (m-1) + (t-1)$ by Lemma 5.0.1. So,

$$n \geq (m-1) + d + k. \tag{5.7.1}$$

Since there are d cells (i,t), $1 \leq i \leq m-1$, with 2 entries, then there are d edges of the form $u_m p$ that must be removed from H_t to eliminate cycles by Lemma 5.7.4 and by the definition of C. Remove these edges from H_t and place one edge into each H_j, $1 \leq j \leq d \leq t-1$. Now the first d H_j's are spanning trees. From inequality (5.7.1) we have that there are also at least $k \geq 0$ distinct integers that are not listed in the first m-1 rows of column t. Each of these integers, e_i, $1 \leq i \leq k$, represents an edge $u_m e_i$ whose removal from H_t forms a trivial component. Remove the edges $u_m e_i$, $1 \leq i \leq k$, from H_t. The remaining edges of H_t induce a tree. Place one edge $u_m e_i$ into each H_j, $d < j \leq t-1$. Thus each H_j, $1 \leq j \leq t-1$ is a spanning tree and the resulting H_t is a tree. Hence, when $2m-3 \leq n < (m-1)^2$, $K_{m,n}$ is decomposable into t trees where nearly all of the trees are spanning trees. •

Definition 5.7.7 Let M be the maximum number of edges of the form $u_m k$ that can be removed from H_t without forming nontrivial components in H_t.

In Lemma 5.7.6 we essentially showed that $M \geq t-1$ when $2m-3 \leq n < (m-1)^2$. Unfortunately, when $n < 2m-3$, we are not guaranteed that $M \geq t-1$. For instance, $M < t-1$ for H_t associated with $K_{5,5}$. Other examples include $K_{10,12}$, $K_{10,13}$, $K_{12,15}$, and $K_{20,28}$. The array B for $K_{5,5}$ is given in Example 5.8.4 and the array B for $K_{12,15}$ is given in Example 5.8.6.

5.8 Decomposition of $K_{m,n}$ into nearly all spanning trees when $2 \leq m \leq n < 2m-3$ and $\sum_{j=1}^{t-1} b(m-1,j) \leq n$

Lemma 5.8.1. Consider $K_{m,n}$ where $t = \tau(K_{m,n})$, $m \leq n < 2m-3$, and $\sum_{j=1}^{t-1} b(m-1,j) = n$. Let b = number of cells in column t with 0 entries, let $q = m-b-1$, and let $z = t-n+m-b-2$. Then $z \leq q-2$.

Proof. By Lemma 5.0.1, $t \le m\text{-}1 \le n\text{-}1$. So, $z = t\text{-}n\text{+}m\text{-}b\text{-}2 \le m\text{-}b\text{-}3 = q\text{-}2$. ∎

Lemma 5.8.2. When $2 \le m \le n \le 2m\text{-}3$, and $\sum_{j=1}^{t-1} b(m-1,j) = n$, we can decompose $K_{m,n}$ into t-1 spanning trees and one additional tree.

Proof. Cell (m-1,t) has no entry since $\sum_{j=1}^{t-1} b(m-1,j) = n$. Therefore, cell (i,t), $1 \le i \le m\text{-}1$, has 0 or 1 element by Lemma 5.2.1 (II and III). Let q represent the number of cells in column t with 1 entry, and b represent the number of cells in column t with 0 entries. It follows that $q\text{+}b = m\text{-}1$ since cell (m,t) has n entries. Let $S = \{s_1, s_2, \ldots, s_{q-1}, s_q\}$ be the ordered set of singleton entries in column t where s_1 is in row i_1, s_2 is in row i_2, ..., s_q is in row i_q and $i_1 < i_2 < \ldots < i_q < m\text{-}1$. The singleton entries $s_1, \ldots, s_q$ are all distinct since for $1 \le r \le q$, $s_r + 1$ equals the first entry in cell $(i_r, 1)$ by the construction of array B. Furthermore, the first entry in each cell (i,1), $1 \le i \le m\text{-}1$, is distinct as each cell has at least two entries by Lemma 5.2.1(I) and if we exclude the last entry of each cell (i,1), $1 \le i \le m\text{-}2$, the remaining entries are the n consecutive integers 1,2,...,n by Theorem 5.4.1(IV) and the construction of column 1. Hence, the first entry in cell (i,1), $1 \le i < m\text{-}1$, is less than the first entry in cell (j,1), $1 \le i < j \le m\text{-}1$. So if $s_1 \ne n$, then $s_1 < s_2 < \ldots < s_q < n$. If $s_1 = n$, then, similarly, $s_2 < \ldots < s_q < s_1$. Since the graph induced by the edges of H_t with the removal of an edge $u_m s_j$, $1 \le j \le q$, results in a disconnected graph, while the graph induced by the edges of H_t with the removal of some edges of the form $u_m p$, $p \neq s_j$, results in a tree, it follows that $M = n-q$. See Figure 5.8.1.

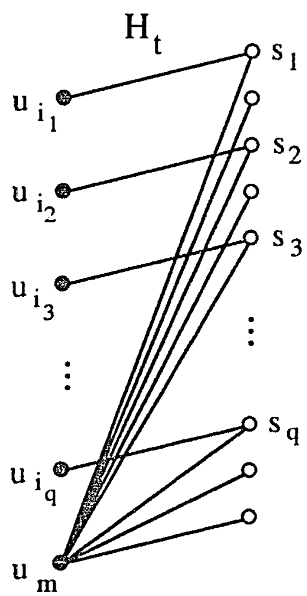

Figure 5.8.1

Also, since $q = m-b-1$, $M = n-m+b+1$. If $M \geq t-1$, we move an edge of the form $u_m p$, where $p \neq s_j$ for $j = 1,2,\ldots,q$, to each of $H_1, \ldots, H_{t-1}$. Adding an edge $u_m p$ to H_j, $1 \leq j \leq t-1$, makes H_j a spanning tree. The edges were chosen so that the remaining edges of H_t induce a tree. So, we may assume $M < t-1$. Let $z = (t-1) - M = t-n+m-b-2$. We need to reduce the number of edges of the form $u_{i_j}s_j$ to gain z edges of the form $u_m s_j$, $1 \leq j \leq q$, that can be removed from H_t. To do this we perform z or z+1 interchanges of pairs of edges between H_t and the other H_k's. For $j = 1,2,\ldots$ we exchange a pair of edges incident to s_j between H_t and some H_{k_j} until in H_t $\deg(u_{i_q}) + \deg(u_{i_{q-1}}) = z+1$ or $z+2$.

We know that each of $s_1, s_2, \ldots, s_{q-1}$ are in row i_q by Theorem 5.4.1(I). Since $m \le n \le 2m-3$, each cell(i_q, j), $1 \le j \le t-1$, has 2 or 3 entries by Lemma 5.2.1(I). So at most 3 of the s_r are in any one cell in row i_q. For $j = 1,2,\ldots,q-1$, s_j is adjacent to u_{i_j} in H_t and $u_{i_q}s_j$ is in H_{k_j} for some $k_j < t$, where H_{k_j} is a tree of order $m+n-1$ by Lemma 5.5.1. We consider the sequence $s_1, s_2, \ldots, s_{q-1}$. We first consider all s_j which occur as the first element in their cell in row i_q, then those which occur as the middle element, and then those which occur as the last element in their cell in row i_q.

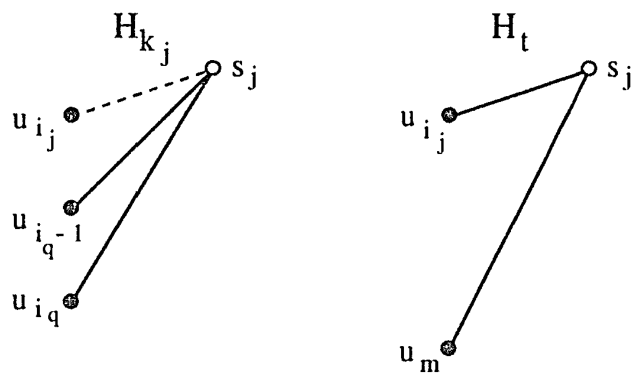

Figure 5.8.2

If s_j is the first entry in cell (i_q, k_j), then s_j is necessarily the last entry in cell (i_q-1, k_j). So edge $u_{i_q-1}s_j$ is also in H_{k_j} and in H_{k_j}, $\deg(s_j) = 2$. (See Figure 5.8.2.) From the definition of array B it follows that tree H_{k_j} has a path from u_{i_j} to u_{i_q} which contains the edges $u_{i_q-1}s_j$ and $u_{i_q}s_j$. Thus there is a u_{i_j}- u_{i_q-1} path which does not contain s_j. Move edge $u_{i_j}s_j$ from H_t to H_{k_j}. This creates a cycle $s_j u_{i_j}$- $u_{i_q-1}s_j$ in H_{k_j}. Move edge $u_{i_q-1}s_j$ from H_{k_j} to H_t. (The interchange of edges between H_{k_j} and H_t is shown in Figure 5.8.3.) This exchange leaves H_{k_j} a tree of order m+n-1, and in H_t we have increased $\deg(u_{i_q-1})$ by 1 and made u_{i_j} into an isolated vertex. The edges of H_t still induce a connected graph.

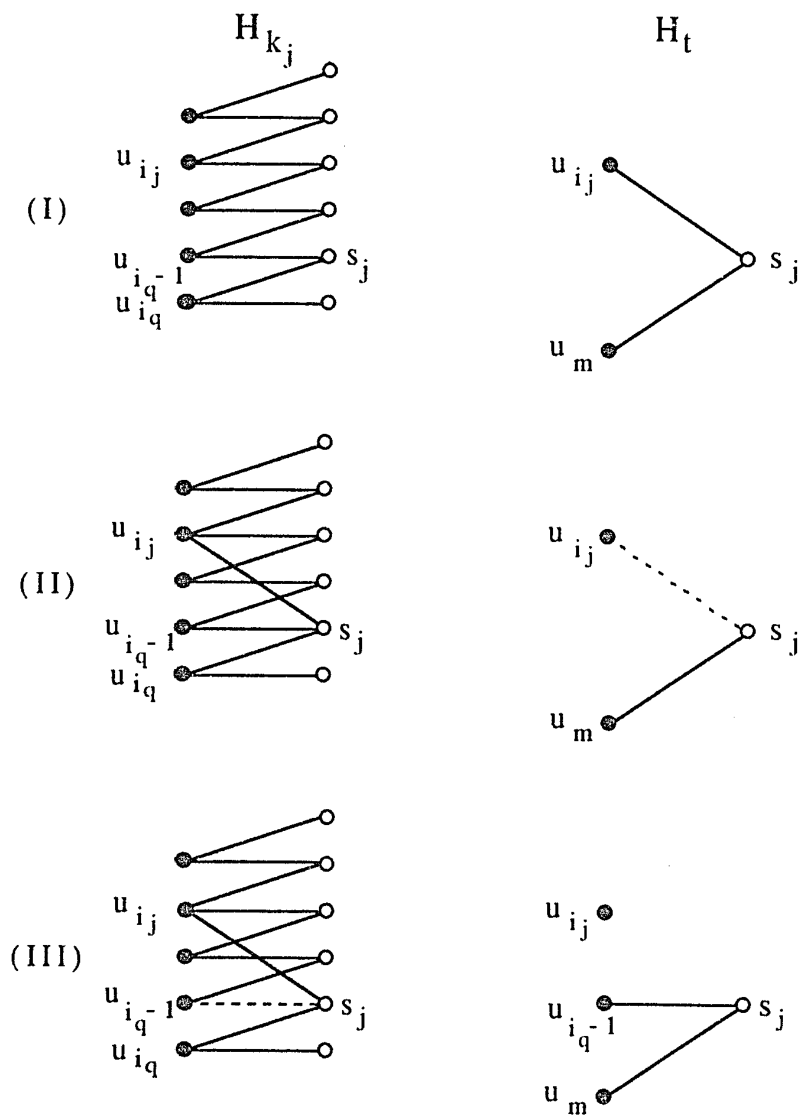

Figure 5.8.3

If s_j is the middle entry of cell (i_q, k_j), then in H_{k_j} $\deg(s_j) = 1$. There exists a u_{i_j}- s_j path in H_{k_j} which must contain edge $u_{i_q}s_j$ as $\deg(s_j) = 1$. Move edge $u_{i_j}s_j$ from H_t to H_{k_j}. This creates cycle $s_j u_{i_j}$- s_j in H_{k_j}. Move edge $u_{i_q}s_j$ from H_{k_j} to H_t. This leaves H_{k_j} a tree of order m+n-1. In H_t, we have increased $\deg(u_{i_q})$ by 1 and made u_{i_j} into an isolated vertex. Again, the remaining edges in H_t induce a connected graph.

If s_j is the last entry in cell (i_q, k_j), then s_j is the first entry in cell (i_q+1, k_j). So edge $u_{i_q+1}s_j$ is also in H_{k_j} and in H_{k_j} $\deg(s_j) = 2$. At most two exchanges have been made between tree H_{k_j} and H_t. If s_{j-1} was the middle entry of cell (i_q, k_j) then $\deg(s_j) = 1$, so s_{j-1} will not be contained in a $u_{i_j}s_j$ path.

If no exchange has been made using the first entry of cell (i_q, k_j), then, again by the definition of array B, there exists a u_{i_j}- u_{i_q+1} path which must contain $u_{i_q}s_j u_{i_q+1}$. (See Figure 5.8.4.) Hence there exists a u_{i_j}- u_{i_q} path that does not contain s_j. Move edge $u_{i_j}s_j$ from H_t to H_{k_j}. This creates a cycle $s_j u_{i_j}$- $u_{i_q}s_j$ in H_{k_j}. Move edge $u_{i_q}s_j$ from H_{k_j} to H_t. This leaves H_{k_j} a tree of order m+n-1. In H_t, we have increased $\deg(u_{i_q})$ by 1 and made u_{i_j} into an isolated vertex. Thus the edges of H_t still induce a connected graph.

If an exchange has been made using the first entry of cell (i_q, k_j), then that entry is s_{j-1} or s_{j-2}. Call it $s_{j'}$. Then $u_{i_{j'}}s_{j'}$ is an edge in H_{k_j} because of the previous exchange. (See Figure 5.8.5.) In H_{k_j}, there also exists a u_{i_j}- $u_{i_{j'}}$ path that does not contain s_j as s_j is only adjacent to u_{i_q} and u_{i_q+1} and $i_{j'} < i_j < \ldots < i_q < i_q+1$ by the construction of S. Form path u_{i_j}- $u_{i_{j'}}s_{j'}u_{i_q}s_j$. Now add edge $u_{i_j}s_j$ from H_t to H_{k_j}. This addition forms cycle $s_ju_{i_j}$- $u_{i_{j'}}s_{j'}u_{i_q}s_j$. Move edge $u_{i_q}s_j$ from H_{k_j} to H_t. H_{k_j} is now a tree of order m+n-1. In H_t, we have increased $\deg(u_{i_q})$ by 1, and made u_{i_j} into an isolated vertex. Again, the edges of H_t induce a connected graph.

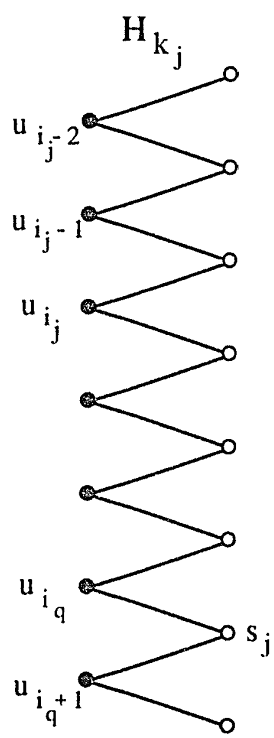

Figure 5.8.4

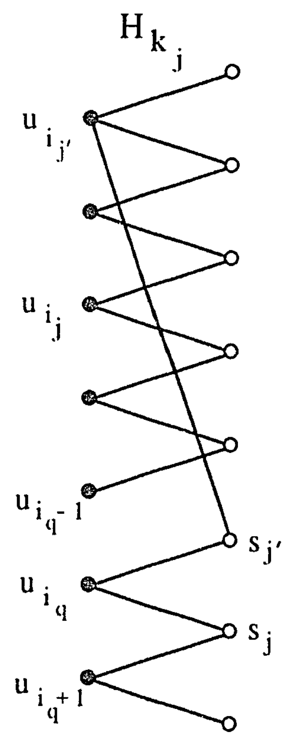

Figure 5.8.5

Perform z edge exchanges unless in so doing $\deg(u_{i_q-1})$ has changed from 0 to positive. In which case do z+1 exchanges. The z exchanges can be done because $z \leq q-2$ by Lemma 5.8.1. The z+1 exchanges can be done because $z+1 \leq q-1$ as cell (i_q-1,t) is empty. We consider 2 cases:

<u>Case 1.</u> Suppose z exchanges have been made and in H_t the current $\deg(u_{i_q-1}) = 0$. Consider the subgraph of H_t induced by $\{u_{i_q}, u_m\} \cup N(u_{i_q})$. It is isomorphic to $K_{2,z+1}$. Hence we can remove z edges incident to u_m without disconnecting H_t. Therefore, there are $z+M = t-1$ edges that can be removed from H_t without forming two nontrivial components. We remove the z edges of the form $u_m s_j$, $1 \leq j \leq z$, where $z \leq q-2$ by Lemma 5.8.1, and M edges of the form $u_m p$, $p \neq s_j$, $1 \leq j \leq q$, from H_t. The remaining edges of H_t induce a tree. We use these t-1 edges to make $H_1, \ldots, H_{t-1}$ into spanning trees and the graph induced by the remaining edges of H_t is a tree.

<u>Case 2.</u> Suppose z+1 exchanges or z exchanges with current $\deg(u_{i_q-1}) \geq 1$ in H_t have been made. In this case, in H_t, $\deg(u_{i_q-1}) + \deg(u_{i_q}) = z+2$. Say $\deg(u_{i_q}) = a$ for some a, $1 \leq a \leq z+1$, then $\deg(u_{i_q-1}) = z+2-a$. Consider the subgraph of H_t induced by $\{u_{i_q}, u_m\} \cup N(u_{i_q})$. It is isomorphic to $K_{2,a}$. Hence we can remove a-1 edges incident to u_m. Similarly, consider the subgraph of H_t induced by $\{u_{i_q-1}, u_m\} \cup N(u_{i_q-1})$. It is isomorphic to $K_{2,z+2-a}$. Hence we can remove z+1-a edges incident to u_m without disconnecting H_t. Therefore there are $a-1+z+1-a+M = t-1$ edges in H_t that can be removed without forming nontrivial components in H_t. We remove z edges of the form $u_m s_j$, $1 \leq j \leq z+1$, where $z+1 \leq q-1$ by Lemma 5.8.1, leaving one edge $u_m s_r$ where $u_{i_q-1} s_r$ is an edge in H_t and leaving another edge, $u_m s_q$, where $u_{i_q} s_q$ is also an edge in H_t. We also remove M edges of the form $u_m p$, $p \neq s_j$, $1 \leq j \leq q$, from H_t. We use these $z+M = t-1$ edges to make $H_1, \ldots, H_{t-1}$ into spanning trees. The graph induced by the remaining edges of H_t is a tree.

Therefore, we are able to decompose $K_{m,n}$, $m \le n < 2m-3$ into t-1 spanning trees and one additional tree when $\sum_{j=1}^{t-1} b(m-1,j) = n$. •

Lemma 5.8.3. When $2 \le m \le n < 2m-3$ and $\sum_{j=1}^{t-1} b(m-1,j) < n$ then $K_{m,n}$ is decomposable into t trees such that at least t-1 of them are spanning trees.

Proof. From $2 \le m \le n < 2m-3$, it follows that $m \ge 4$. Cell (m-1,t) has 1 entry by the proof of Lemma 5.2.1(V). Therefore cell (i,t), $1 \le i \le m-2$, has 1 or 2 entries by Lemma 5.2.1(II and III). Let q represent the number of cells in column t with one entry, and d represent the number of cells in column t with 2 entries. It follows that q+d = m-1 since cell (m,t) has $n \ge 4$ entries. As in Lemma 5.8.2, let $S = \{s_1, s_2, \ldots, s_q\}$ be the ordered set of singleton entries in column t where s_1 is in row i_1, s_2 is in row i_2, ..., s_q is in row i_q and $i_1 < i_2 < \ldots < i_q = m-1$. The singleton entries $s_1, \ldots, s_q$ are all distinct by the construction of array B. The construction of array B guarantees that if $s_1 \ne n$, then $s_1 < s_2 < \ldots < s_q < n-1$ because cell (m-1,1) definitely contains n-1 and n. If $s_1 = n$, then, similarly, $s_2 < \ldots < s_q < s_1 = n$.

<u>Claim:</u> In array B, at least one integer k, $1 \le k \le n$, is not listed in the first m-1 rows of column t.

<u>Proof of Claim:</u> We know from Lemma 5.2.1(I) and the fact that $m \le n < 2m-3$ that cells (1,1) and (m-1,1) have 2 or 3 entries. From Lemma 5.2.1(III), we know that the maximum number of entries in column 1 occurs in cell (m-1,1).

If cell (m-1,1) has 3 entries, then by the construction of array B, they are n-2, n-1, n. So s_q cannot equal n-2. The integer n-2 cannot be in cell (1,t) as that would mean cell (1,t) has entries n-2, n-1, n, and cell (1,t) has at most 2 entries by Lemma 5.2.1(V). By the construction of array B, n-2 cannot be in any other cell (i,t), $2 \leq i \leq m-2$. Therefore, if cell (m-1,1) has three entries, then the integer n-2 is not listed in the first m-1 rows of column t.

If cell (m-1,1) has two entries, then by the construction of array B, they are n-1, n. Furthermore, all cells (i,j), $1 \leq i \leq m-1$ and $1 \leq j \leq t-1$, have 2 entries since cell (m-1,1) has $2 = \left\lceil \frac{m+n-2}{m-1} \right\rceil$ entries by 5.2.1(III), and each cell (i,j), $1 \leq i \leq m-1$, $1 \leq j \leq t-1$, has $\left\lfloor \frac{m+n-2}{m-1} \right\rfloor$ or $\left\lceil \frac{m+n-2}{m-1} \right\rceil \geq 2$ entries by Lemma 5.2.1(I). Therefore, $\sum_{j=1}^{t-1} b(i,j) = n-1$ for all $1 \leq i \leq m-1$ since $\sum_{j=1}^{t-1} b(m-1,j) = 2(t-1) = n-1$. Hence each cell (i,t), $1 \leq i \leq m-1$, has exactly 1 entry. Thus $s_1 = n$. The remaining s_j, $2 \leq j \leq q$, are less than n-1. Hence, if cell (m-1,1) has 2 entries, the integer n-1 is not listed in the first m-1 row of column t in array B. Therefore the claim is proved.

Let r equal the number of integers not listed in the first m-1 rows of column t. By the claim, $r \geq 1$ as either n-1 or n-2 is not listed in the first m-1 rows of column t. We know $C \leq t-1$ by Lemma 5.7.5.

If $C+r \geq t-1$, select C edges of the form $u_m p$ in such a way as to remove all cycles in H_t. Next select (t-1) - C edges of the form $u_m k$ such that k is not listed in the first m-1 rows of column t. The removal of these edges from H_t will form trivial components in H_t.

At this point t-1 edges have been selected. Place one of the selected edges in each H_j, $1 \leq j \leq t-1$. Adding an edge of the form $u_m p$ to H_j, $1 \leq j \leq t-1$, makes H_j a spanning tree. The selected edges were chosen in such a way that the remaining edges of H_t induce a tree.

If $C+r < t-1$, then by Lemma 5.7.4 and the fact that $r \geq 1$, we have that $C+r \geq d+1$. So $t-1 > C+r \geq d+1$. Let $z = (t-1) - (C+r)$. Using the technique used in the proof of Lemma 5.8.2, move z edges $u_{i_j} s_j$, $1 \leq j \leq z$, to the other H_k's, $k < t$, and move z edges of the form $u_{i_q} s_j$ or $u_{i_q - 1} s_j$, $1 \leq j \leq z$, to H_t from the H_k's. The H_k's remain trees of order m+n-1, and the edges of H_t induce a connected graph. The fact that $z \leq q-2$ can be shown as follows. By Lemma 5.0.1, we know $t-1 \leq m-2$. This implies $t-1-m+q \leq q-2$. Also, $z = (t-1) - (C+r)$ $\leq (t-1) - (d+1) = t-1-m+1+q-1 = t-1-m+q$. Therefore, $z \leq q-2$. Before exchanging the z edges, in H_t $\deg(u_{i_q}) = 1$ and $\deg(u_{i_q - 1}) = 1$ or 2. Therefore, after exchanging z pairs of edges, in H_t $\deg(u_{i_q}) + \deg(u_{i_q - 1}) = z+2$ or $z+3$. All of the edges $u_m s_j$, $1 \leq j \leq z$, can now be removed without forming a nontrivial component in H_t as the edges $u_m s_q$ and $u_{i_q} s_q$ are still in H_t and the edges ($u_{i_q - 1} s_{q-1}$ and $u_m s_{q-1}$) or ($u_{i_q - 1} e$, $e \neq s_j$, $1 \leq j \leq q$, and $u_m e$) are also still in H_t. (See Figure 5.8.6.) We select C edges of the form $u_m p$ whose removal from H_t before exchanging edges destroy all cycles in H_t. We select r edges of the form $u_m k$ where k is not listed in the first m-1 rows of column t, and the z edges of the form $u_m s_j$, $1 \leq j \leq z$. Thus we have selected $C+r+z = t-1$ edges incident to u_m. Place one of the selected edges in each H_j, $1 \leq j \leq t-1$. The addition of an edge incident to u_m

changes H_j into a spanning tree. The edges were selected in such a way that the remaining edges of H_t induce a tree.

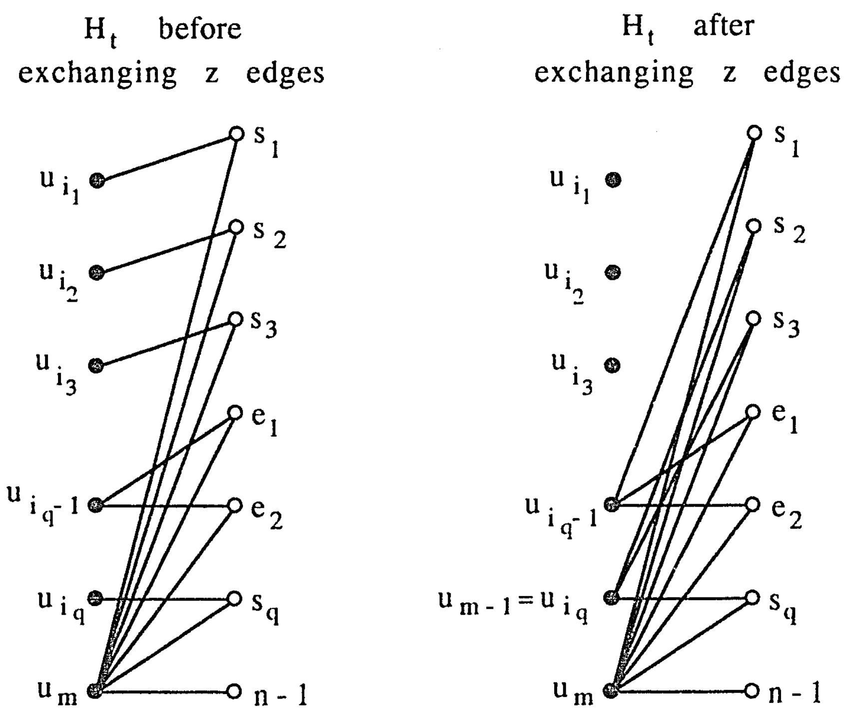

Figure 5.8.6

Therefore, when $\sum_{j=1}^{t-1} b(m-1,j) < n$ and $2 \leq m \leq n \leq 2m-3$, $K_{m,n}$ is decomposable into t trees where at least t-1 of the trees are spanning trees. ●

Example 5.8.4. We show the array B associated with $K_{5,5}$.

$$B = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{ccc} H_1 & H_2 & H_3 \\ \end{array}$$

$$B = \begin{array}{c|ccc|} & H_1 & H_2 & H_3 \\ \hline u_1 & 1\,2 & 3\,4 & 5 \\ u_2 & 2\,3 & 4\,5 & 1 \\ u_3 & 3\,4 & 5\,1 & 2 \\ u_4 & 4\,5 & 1\,2 & 3 \\ u_5 & - & - & 1 \to 5 \end{array}$$

Notice that there are no cycles in H_3, but edge $u_5 4$ is the only edge of the form $u_m p$ that can be removed from H_3 without forming two nontrivial components. Therefore, $C = 0$ and $M = 1$. However, we need to move 2 edges of the form $u_5 p$ from H_3 to make H_1 and H_2 spanning trees.

Notice edge $u_4 5$ is in H_1. Therefore add edge $u_1 5$ to H_1 forming cycle $5u_1 2\, u_2 3\, u_3 4\, u_4 5$ in H_1. Next, remove $u_4 5$ from H_1 and place it in H_3. H_1 is now a tree of order 9. The subgraph H_3 now contains the cycle $3u_4 5\, u_5 3$. Thus, in H_3, $C=1$ and $M=2$ as we can remove the edge $u_5 4$ and either $u_5 3$ or $u_5 5$ from H_3 without forming nontrivial components.

We give a decomposition of $K_{5,5}$ into 2 spanning trees with 9 edges and 1 tree with 7 edges in Example 5.8.5.

Example 5.8.5. The array D shown below represents a decomposition of $K_{5,5}$ into 2 spanning trees and one tree with 7 edges.

D =

	T_1	T_2	T_3
u_1	1 25	3 4	-
u_2	2 3	4 5	1
u_3	3 4	5 1	2
u_4	4	1 2	35
u_5	4	5	1 2 3

Example 5.8.6. We show the array B associated with $K_{12,15}$.

B =

	H_1	H_2	H_3	H_4	H_5	H_6	H_7
u_1	1,2	3,4	5,6,7	8,9	10,11	12,13	14,15
u_2	2,3	4,5,6	7,8	9,10	11,12	13,14,15	1
u_3	3,4,5	6,7	8,9	10,11	12,13,14	15,1	2
u_4	5,6	7,8	9,10	11,12,13	14,15	1,2	3,4
u_5	6,7	8,9	10,11,12	13,14	15,1	2,3	4,5
u_6	7,8	9,10,11	12,13	14,15	1,2	3,4,5	6
u_7	8,9,10	11,12	13,14	15,1	2,3,4	5,6	7
u_8	10,11	12,13	14,15	1,2,3	4,5	6,7	8,9
u_9	11,12	13,14	15,1,2	3,4	5,6	7,8,9	10
u_{10}	12,13	14,15,1	2,3	4,5	6,7,8	9,10	11
u_{11}	13,14,15	1,2	3,4	5,6,7	8,9	10,11	12
u_{12}	-	-	-	-	-	-	$1 \rightarrow 15$

The subgraphs H_1, ... , H_6 each have 25 edges. A spanning tree needs 26 edges. Notice that H_7 has cycles, H_7^* has 10 components, $C = 4$, and $M = 5$. Therefore, in order to decompose $K_{12,15}$ into nearly all spanning trees, we must first move edge $u_2 1$ from H_7 to H_2 and edge $u_{10}1$ from H_2 to H_7 . We give the array D which represents a decomposition of $K_{12,15}$ into 6 spanning trees and one tree with 24 edges in Example 5.8.7.

Example 5.8.7. We show the array D associated with $K_{12,15}$. Array D represents a decomposition of $K_{12,15}$ into 7 trees, 6 of which are spanning trees.

D =

	T_1	T_2	T_3	T_4	T_5	T_6	T_7
u_1	1,2	3,4	5,6,7	8,9	10,11	12,13	14,15
u_2	2,3	4,5,6,1	7,8	9,10	11,12	13,14,15	-
u_3	3,4,5	6,7	8,9	10,11	12,13,14	15,1	2
u_4	5,6	7,8	9,10	11,12,13	14,15	1,2	3,4
u_5	6,7	8,9	10,11,12	13,14	15,1	2,3	4,5
u_6	7,8	9,10,11	12,13	14,15	1,2	3,4,5	6
u_7	8,9,10	11,12	13,14	15,1	2,3,4	5,6	7
u_8	10,11	12,13	14,15	1,2,3	4,5	6,7	8,9
u_9	11,12	13,14	15,1,2	3,4	5,6	7,8,9	10
u_{10}	12,13	14,15	2,3	4,5	6,7,8	9,10	11,1
u_{11}	13,14,15	1,2	3,4	5,6,7	8,9	10,11	12
u_{12}	14	3	5	8	11	13	1,2,4,6,7,9,10,12,15

5.9 Decomposition of $K_{m,n}$ into nearly all spanning trees for $1 \leq m \leq n$

Theorem 5.9.1. $K_{m,n}$ is decomposable into $t = \tau(K_{m,n})$ trees where at least t-1 of the trees are spanning trees.

Proof. There are several cases to consider:

I) If m=1, then $K_{1,n}$ is already a tree.

II) If $2 \leq m \leq n < (m-1)^2$ and

A) $\sum_{j=1}^{t-1} b(m-1,j) \leq n$ and

1) $2 \leq m \leq n < 2m-3$ and

a) $\sum_{j=1}^{t-1} b(m-1,j) < n$, then apply Lemma 5.8.3.

b) $\sum_{j=1}^{t-1} b(m-1,j) = n$, then apply Lemma 5.8.2.

2) $2m-3 \leq n < (m-1)^2$, then apply Lemma 5.7.6.

B) If $\sum_{j=1}^{t-1} b(m-1,j) > n$, then apply Lemma 5.6.1.

III) If $n \geq (m-1)^2$, then apply Lemma 5.1.1.

Thus, $K_{m,n}$ is decomposable into t trees where at least t-1 of the trees are spanning trees. •

Theorem 5.9.1 proves that all complete bipartite graphs $K_{m,n}$ can be decomposed into t trees where at least t-1 of them are spanning trees. In Chapter 6 we consider the minimum number of trees required to decompose the complete tripartite graphs.

Chapter 6

Tree Decomposition of the Complete Tripartite Graphs

In Chapter 6, we find an upper and lower bound for $\tau(K_{m,n,p})$ for all m,n,p. We then decompose a few families of $K_{m,n,p}$ into $\tau(K_{m,n,p})$ trees. We close by summarizing our results and stating a conjecture.

6.1 Preliminary Results

Without loss of generality, we assume that $m \le n \le p$, and we let $t = \lceil \frac{mn+mp+np}{m+n+p-1} \rceil$.

Lemma 6.1.1. The minimum number of trees, $\tau(K_{m,n,p})$, required to decompose $K_{m,n,p}$ is at least t.

Proof. $K_{m,n,p}$ contains mn+mp+np edges and m+n+p vertices. Therefore, the maximum number of edges in any tree contained in $K_{m,n,p}$ is m+n+p-1. Hence, the minimum number of trees required to decompose $K_{m,n,p}$ is at least $\frac{mn+mp+np}{m+n+p-1}$. Since $\tau(K_{m,n,p})$ is an integer, it follows that $\tau(K_{m,n,p}) \ge t$. •

Lemma 6.1.2. $m+1 \le \tau(K_{m,n,p}) \le m+n$.

Proof. To see that $m+1 \le \tau(K_{m,n,p})$, consider the subgraph H of $K_{m,n,p}$ that is isomorphic to $K_{m,m,m}$. Since H is a subgraph of $K_{m,n,p}$, $\tau(H) \le \tau(K_{m,n,p})$. By Lemma 6.1.1, $\tau(K_{m,m,m}) \ge \lceil \frac{3m^2}{3m-1} \rceil = \lceil m + \frac{m}{3m-1} \rceil$ $= m+1$. Therefore, $m+1 \le \tau(K_{m,n,p})$. To see that $\tau(K_{m,n,p}) \le m+n$, notice that we could easily decompose $K_{m,n,p}$ into m+n trees in the following way: for $1 \le i \le m$, let u_i be adjacent to all vertices in V and W. (See Figure 6.1.1) Then for $1 \le j \le n$, let v_j be adjacent to all vertices in W. (See Figure 6.1.2) In this fashion $K_{m,n,p}$ has been decomposed into m+n trees. Since $\tau(K_{m,n,p})$ is the minimum number of trees required to decompose $K_{m,n,p}$, it follows that $\tau(K_{m,n,p}) \le m+n$. •

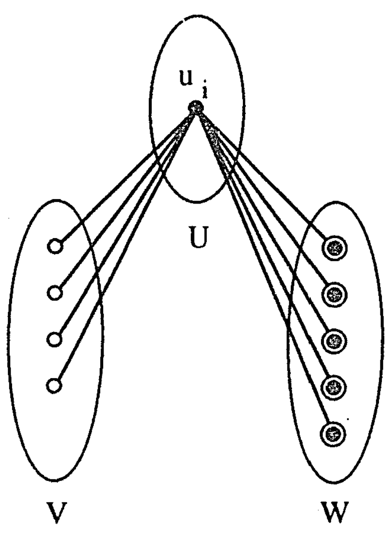

Figure 6.1.1

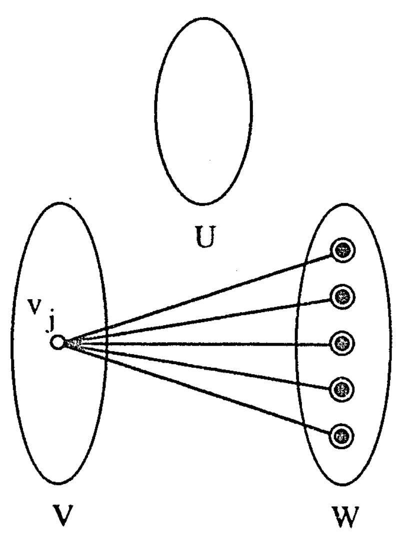

Figure 6.1.2

Proposition 6.1.3. When $p \geq (m-1)^2+(n-1)^2+mn$, then $\tau(K_{m,n,p}) = m+n$.

Proof. Since $p > (m-1)^2+(n-1)^2+mn-1 = m^2-2m+n^2-2n+mn+1$, it follows that $m^2+mn+n^2-m-n < m+n+p-1$. So, $\frac{m^2+mn+n^2-m-n}{m+n+p-1} < 1$. By definition, $t = \lceil \frac{mn+mp+np}{m+n+p-1} \rceil = \lceil m+n - \frac{m^2+mn+n^2-m-n}{m+n+p-1} \rceil = m+n$ when $\frac{m^2+mn+n^2-m-n}{m+n+p-1} < 1$. Hence $\tau(K_{m,n,p}) \geq m+n = t$ when $p \geq (m-1)^2+(n-1)^2+mn$ by Lemma 6.1.1. But, $\tau(K_{m,n,p}) \leq m+n$ by Lemma 6.1.2. Therefore $\tau(K_{m,n,p}) = m+n$ when $p \geq (m-1)^2+(n-1)^2+mn$. •

Proposition 6.1.4. $\tau(K_{m,n,p}) \leq \min \{ \tau(K_{m,n+p}) + \tau(K_{n,p}), \tau(K_{n,m+p}) + \tau(K_{m,p}), \tau(K_{p,m+n}) + \tau(K_{m,n}) \}$.

Proof. Consider $K_{m,n,p}$ with partite sets U, V, and W. Let $G_1 = \langle U \cup V \rangle \cup \langle U \cup W \rangle = K_{m,n+p}$ and $G_2 = \langle V \cup W \rangle = K_{n,p}$. Then, $G_1 \cup G_2 = K_{m,n,p}$. See Figure 6.1.3. Using Beineke's method, given in Chapter 4, we can decompose G_1 into $\tau(K_{m,n+p})$ trees and G_2 into $\tau(K_{n,p})$ trees. Using this decomposition, we can decompose $K_{m,n,p}$ into $\tau(K_{m,n+p}) + \tau(K_{n,p})$ trees. Similarly, let $H_1 = \langle V \cup U \rangle \cup \langle V \cup W \rangle = K_{n,m+p}$ and $H_2 = \langle U \cup W \rangle = K_{m,p}$. Then $H_1 \cup H_2 = K_{m,n,p}$. Again using Beineke's method, we can decompose $K_{m,n,p}$ into $\tau(K_{n,m+p}) + \tau(K_{m,p})$ trees. Likewise, we can decompose $K_{m,n,p}$ into $\tau(K_{p,m+n}) + \tau(K_{m,n})$ trees. Therefore, $\tau(K_{m,n,p}) \leq \min \{ \tau(K_{m,n+p}) + \tau(K_{n,p}), \tau(K_{n,m+p}) + \tau(K_{m,p}), \tau(K_{p,m+n}) + \tau(K_{m,n}) \}$. •

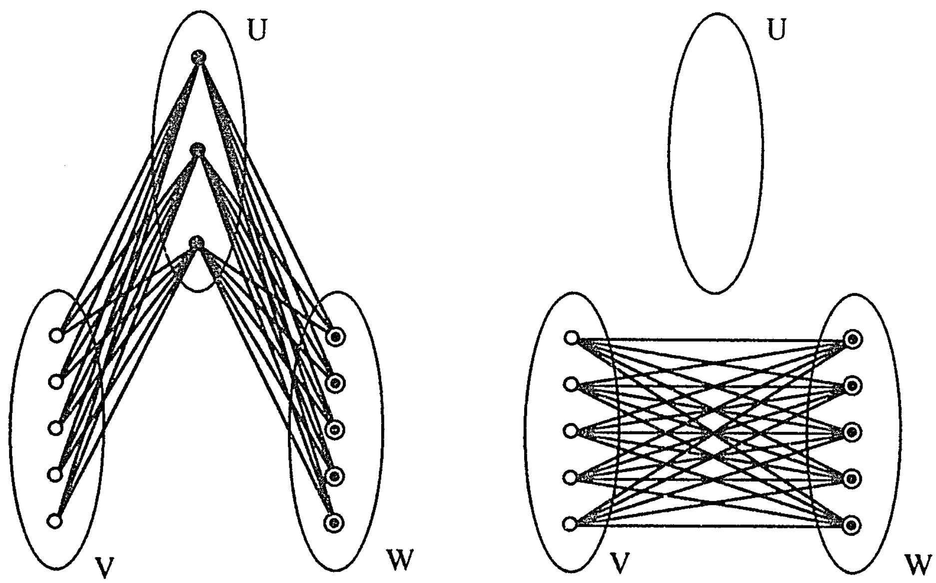

Figure 6.1.3

The following well known theorem and its corollary will be used in the subsequent sections.

Theorem 6.1.5 (König). Every regular bipartite graph of degree $r \geq 1$ is 1-factorable.

Proof. A proof is given in [6] on page 235.

Corollary 6.1.6. $K_{n,n}$ is 1-factorable.

Proof. Observe $K_{n,n}$ is n regular and apply Theorem 6.1.5. •

6.2 Tree decomposition of $K_{m,n,p}$, $1 \le n\text{-}1 \le m \le n \le p \le n+1$

We show that $K_{m,n,p}$, $1 \le n-1 \le m \le n \le p \le n+1$, can be decomposed into t trees by providing the decomposition. Then, by Lemma 6.1.1, $\tau(K_{m,n,p}) = t$ for $1 \le n-1 \le m \le n \le p \le n+1$. The possible cases are: $K_{n-1,n,n}$, $K_{n-1,n,n+1}$, $K_{n,n,n}$, and $K_{n,n,n+1}$.

Proposition 6.2.1. $\tau(K_{n-1,n,n}) = n$, for all $n \ge 2$.

Proof. By definition, $t = \lceil \frac{n^2-n+n^2-n+n^2}{3n-2} \rceil = \lceil \frac{3n^2-2n}{3n-2} \rceil = n$. Let $H = \langle V \cup W \rangle = K_{n,n}$. By Corollary 6.1.6, form n edge-disjoint 1-factors, $F_1,\dots,F_n$, of H. Next, form the following n subgraphs $T_1, \dots, T_n$ whose union will be shown to be $K_{n-1,n,n}$. (For n = 4, the decomposition is given in Figure 6.2.1.)

For $1 \le j \le n-1$, let $E(T_j) = \{u_jv_i \mid 1 \le i \le n\} \cup E(F_j)$ $\cup \{u_iw_{i-j+1(\text{mod}\,n)} \mid 1 \le i \le n-1, i \ne j\}$ and let $E(T_n) = E(F_n)$ $\cup \{w_1u_i \mid 1 \le i \le n-1\} \cup \{u_iw_{i+1} \mid 1 \le i \le n-1\}$.

Each T_j has 3n-2 edges and 3n-1 vertices and is connected and therefore each T_j, $1 \le j \le n$, is a spanning tree. We must show that each edge of $K_{n-1,n,n}$ is in one of the T_j's. Any edge joining V and W is in a 1-factor, F_j, and hence in a T_j. Each edge of the form u_jv_i is in T_j only. Consider edges of the form u_rw_s. If s = 1, then u_rw_s is in T_n. If s > 1, then $u_rw_s = u_rw_{((r+1)-(r+1-s))(\text{mod}\,n)}$ that is in $T_{r+1-s(\text{mod}\,n)}$. Thus, $T_1, \dots, T_n$ are a spanning tree decomposition of $K_{n-1,n,n}$ and $\tau(K_{n-1,n,n}) = n$. ●

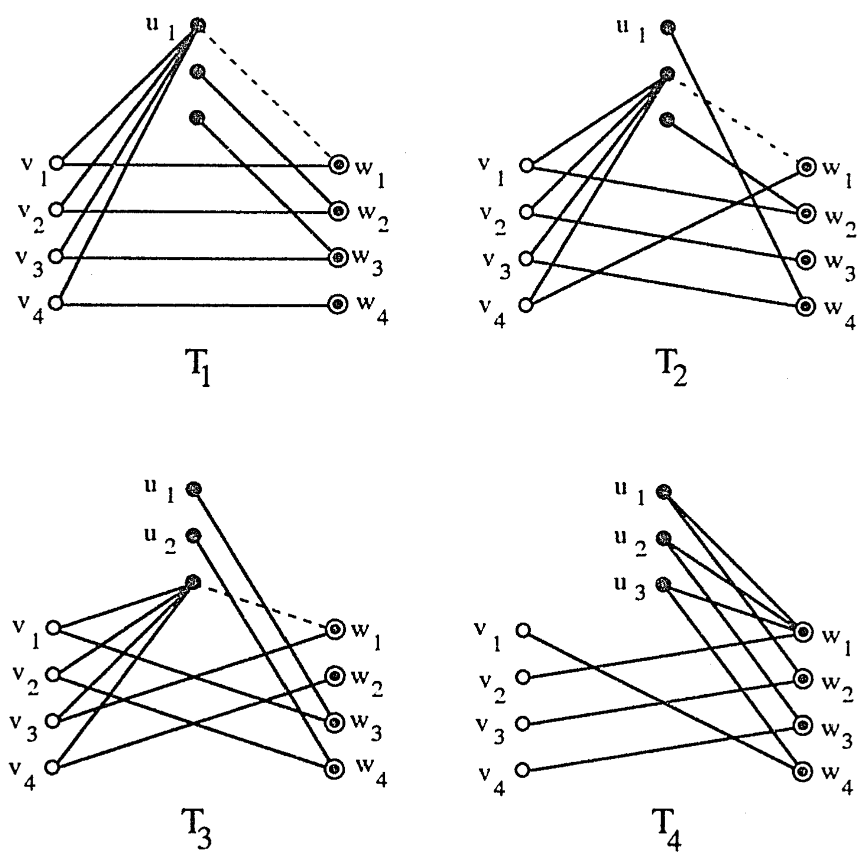

Figure 6.2.1

Proposition 6.2.2. $\tau(K_{n-1,n,n+1}) = t = n+1$, for all $n \geq 2$.

Proof. By definition, $t = \lceil \frac{3n^2-1}{3n-1} \rceil = \lceil n + \frac{n-1}{3n-1} \rceil = n+1$. In Proposition 6.2.1 we proved that $\tau(K_{n-1,n,n}) = n$. Therefore, to decompose $K_{n-1,n,n+1}$ into n+1 trees we use the n trees given in Proposition 6.2.1 and $E(T_{n+1}) = \{ w_{n+1}u_i \mid 1 \leq i \leq n-1\} \cup \{ w_{n+1}v_i \mid 1 \leq i \leq n\}$. In this fashion $K_{n-1,n,n+1}$ is decomposed into $t = n+1$ trees. •

Proposition 6.2.3. For $n \geq 1$, $\tau(K_{n,n,n}) = t = n+1$.

Proof. First notice that $t = \lceil \frac{3n^2}{3n-1} \rceil = \lceil n + \frac{n}{3n-1} \rceil = n+1$. For $n = 1$, the proposition is obvious. For $n \geq 2$, notice that in Proposition 6.2.1 we proved that $\tau(K_{n-1,n,n}) = n$. Therefore, to decompose $K_{n,n,n}$ into $n+1$ trees we use the n trees given in Proposition 6.2.1 and we let $E(T_{n+1}) = \{u_n v_i \mid 1 \leq i \leq n\} \cup \{u_n w_i \mid 1 \leq i \leq n+1\}$. In this fashion $K_{n,n,n}$ is decomposed into $t = n+1$ trees. ∘

Proposition 6.2.4. For $n = 1$, $\tau(K_{n,n,n+1}) = t = n+1$.

Proof. By definition, $t = \lceil \frac{3n^2+2n}{3n} \rceil = \lceil n + \frac{2}{3} \rceil = n+1$. For $n = 1$ the proposition is obvious. For $n \geq 2$, let $H = \langle U \cup V \rangle = K_{n,n}$. By Corollary 6.1.6, form n edge-disjoint 1-factors, $F_1, \ldots, F_n$, of H. Next, form the following $n+1$ subgraphs $T_1, \ldots, T_{n+1}$ whose union will be shown to be $K_{n,n,n+1}$. (The decomposition of $K_{3,3,4}$ into $t = 4$ trees is given in Figure 6.2.2.)

For $1 \leq j \leq n$, let $E(T_j) = \{w_j u_i \mid 1 \leq i \leq n\} \cup E(F_j)$ $\cup \{v_i w_{i+j-1(\mathrm{mod}\, n)} \mid 2 \leq i \leq n\}$ and let $E(T_{n+1}) = \{w_{n+1} u_i \mid 1 \leq i \leq n\}$ $\cup \{w_{n+1} v_i \mid 1 \leq i \leq n\} \cup \{v_1 w_i \mid 1 \leq i \leq n\}$.

T_j, $1 \leq j \leq n$, has $3n-1$ edges, $3n$ vertices and is connected. Therefore, T_j, $1 \leq j \leq n$, is a tree. T_{n+1} has $3n$ edges, $3n+1$ vertices, and is connected. Thus T_{n+1} is a spanning tree. $\sum_{j=1}^{n+1} E(T_j)$ $= (3n-1)n + 3n = 3n^2 + 2n$ as required. We must show that each edge of $K_{n,n,n+1}$ is in a T_j, $1 \leq j \leq n+1$. Any edge joining U and V is in a 1-factor, F_j, and hence in a T_j. Each edge of the form $w_j u_i$ is in T_j.

Consider edges of the form $v_r w_s$. If $r = 1$ or $s = n+1$, then $v_r w_s$ is in T_{n+1}. If $1 < r \leq n$ and $1 \leq s < n+1$, then $v_r w_s = v_r w_{((r-1)+(s+1))(\mathrm{mod}\, n)}$ is in T_{s+1}. Thus, $T_1, \ldots, T_{n+1}$ are a tree decomposition of $K_{n,n,n+1}$ and hence, $\tau(K_{n,n,n+1}) = n+1$. •

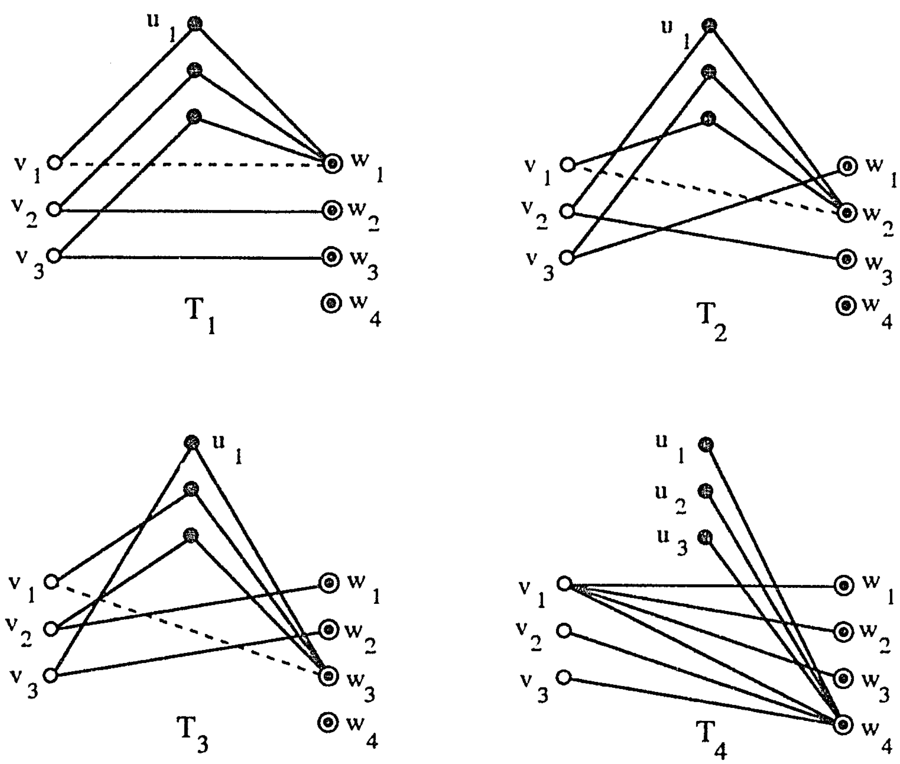

Figure 6.2.2

6.3 Tree Decomposition of $K_{n,n,n+a}$, $0 \leq a \leq 4$

We show that $K_{n,n,n+a}$, $2 \leq a \leq 4$, can be decomposed into t tree by providing the decomposition. Then, by Lemma 6.1.1, $\tau(K_{n,n,n+a}) = t$ for $0 \leq a \leq 4$. The case when $a = 0$ was done in Proposition 6.2.3, and the case when $a = 1$ was done in Proposition 6.2.4. In both of these cases $t = n+1$.

Proposition 6.3.1. $\tau(K_{n,n,n+2}) = t = n+1$.

Proof. First notice that $t = \lceil \frac{3n^2+4n}{3n+1} \rceil = \lceil n+1 - \frac{1}{3n+1} \rceil = n+1$. Next, form n edge-disjoint 1-factors, $F_1, \ldots, F_n$, between sets U and V. Then, form the following $n+1$ subgraphs $T_1, \ldots, T_{n+1}$ whose union will be shown to be $K_{n,n,n+2}$. (We give the decomposition of $K_{3,3,5}$ into $t = 4$ trees in Figure 6.3.1.)

Let $E(T_1) = \{w_1v_i \mid 1 \le i \le n\} \cup E(F_1) \cup \{u_iw_{i+1} \mid 1 \le i \le n\}$. For $2 \le j \le n$, let $E(T_j) = \{w_jv_i \mid 1 \le i \le n\} \cup \{w_{n+2}v_j\} \cup E(F_j) \cup \{u_iw_{i+j(\mathrm{mod}\, n+1)} \mid 1 \le i \le n\}$. Let $E(T_{n+1}) = \{w_{n+1}v_i \mid 1 \le i \le n\} \cup \{w_{n+2}u_i \mid 1 \le i \le n\} \cup \{w_{n+2}v_1\} \cup \{u_iw_i \mid 1 \le i \le n\}$.

T_1 has $3n$ edge, $3n+1$ vertices and is connected. Thus T_1 is a tree. T_j, $2 \le j \le n+1$, has $3n+1$ edges, $3n+2$ vertices and is connected. Thus each T_j, $2 \le j \le n+1$, is a spanning tree. $\sum_{j=1}^{n+1} E(T_j) = 3n + n(3n+1) = 3n^2 + 4n$, as required. Next, we need to check that each edge is in a T_j, $1 \le j \le n+1$. An edge joining U and V is in an F_j and hence is in a T_j. An edge of the form w_jv_i, $1 \le j \le n+1$, is in T_j. An edge of the form $w_{n+2}v_j$, $j \ne 1$, is in T_j, while $w_{n+2}v_1$ is in T_{n+1}. For $1 \le s \le n+1$, edge $u_rw_s = u_rw_{((r+(s-r))(\mathrm{mod}\, n+1)}$ is in $T_{s-r \ (\mathrm{mod}\ n+1)}$. An edge $w_{n+2}u_i$, $1 \le i \le n$, is in T_{n+1}. Thus $T_1, \ldots, T_{n+1}$ are a tree decomposition of $K_{n,n,n+2}$ and $\tau(K_{n,n,n+2}) = n+1 = t$. •

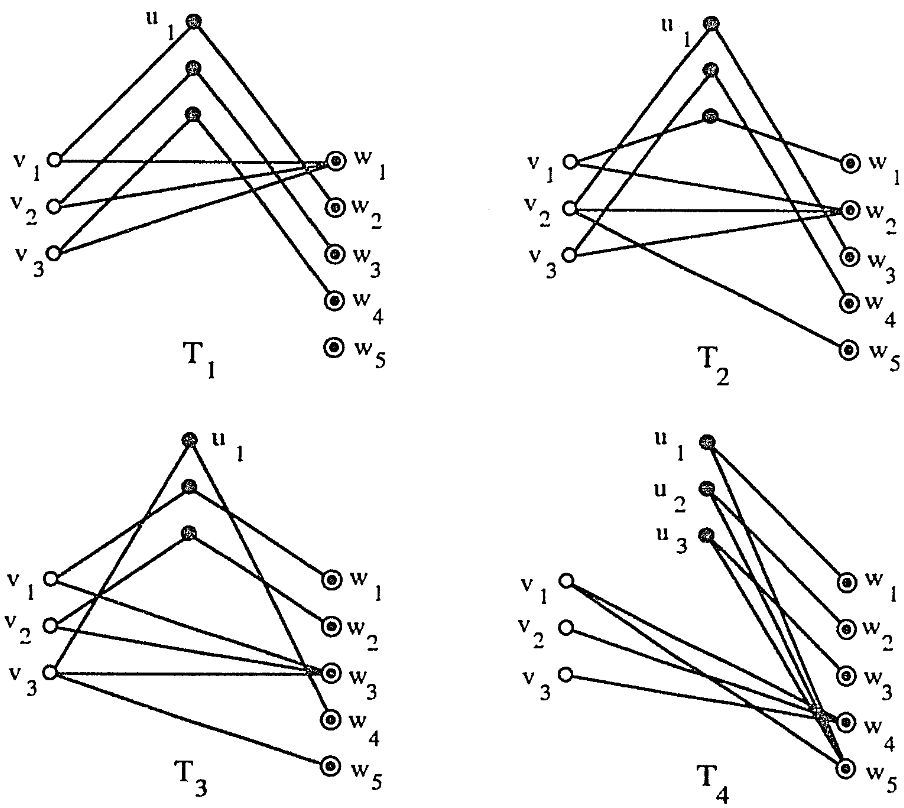

Figure 6.3.1

Proposition 6.3.2. For $n > 2$, $\tau(K_{n,n,n+3}) = t = n+2$.

Proof. First notice that $t = \lceil \frac{3n^2+6n}{3n+2} \rceil = \lceil n+2 - \frac{2n+4}{3n+1} \rceil = n+2$ when $n > 2$. In Proposition 6.3.1 we proved that $\tau(K_{n,n,n+2}) = n+1$. Therefore, to decompose $K_{n,n,n+3}$ into n+2 trees we use the n+1 trees given in Proposition 6.3.1 and we let $E(T_{n+2}) = \{w_{n+2}u_i \mid 1 \le i \le n\} \cup \{w_{n+2}v_i \mid 1 \le i \le n\}$. In this fashion $K_{n,n,n+3}$ is decomposed into $t = n+2$ trees. ●

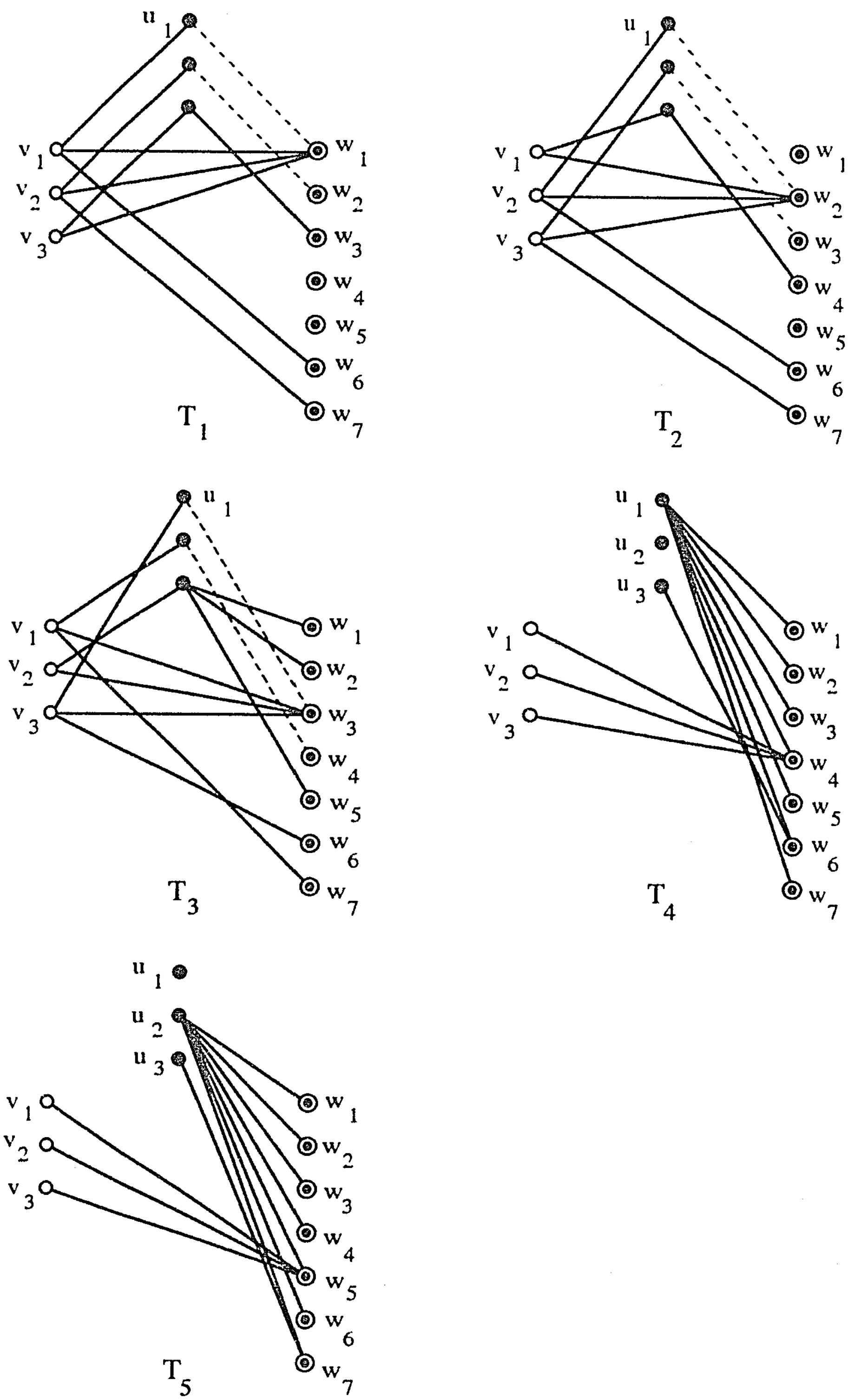

Figure 6.3.2

Proposition 6.3.3. For $n \geq 2$, $\tau(K_{n,n,n+4}) = t = n+2$.

Proof. First notice that $t = \lceil \frac{3n^2+8n}{3n+3} \rceil = \lceil n+2 - \frac{n+6}{3n+3} \rceil = n+2$ when $n \geq 2$. When $n = 2$, $t = 4$ and therefore $K_{2,2,6}$ can be decomposed into $4 = m+n$ tree by Lemma 6.1.2. For $n > 2$, first form n edge-disjoint 1-factors, $F_1, \ldots, F_n$, between sets U and V. Then, form the following n+2 subgraphs $T_1, \ldots, T_{n+2}$ whose union will be shown to be $K_{n,n,n+4}$. (We give the decomposition of $K_{3,3,7}$ into $t = 5$ trees in Figure 6.3.2.)

Let $E(T_1) = \{w_1v_i \mid 1 \leq i \leq n\} \cup \{w_{n+3}v_1, w_{n+4}v_2\} \cup E(F_1) \cup \{u_iw_i \mid 3 \leq i \leq n\}$. Let $E(T_2) = \{w_2v_i \mid 1 \leq i \leq n\} \cup \{w_{n+3}v_2, w_{n+4}v_3\} \cup E(F_2) \cup \{u_iw_{i+1} \mid 3 \leq i \leq n\}$. For $3 \leq j \leq n$, let $E(T_j) = \{w_jv_i \mid 1 \leq i \leq n\} \cup \{w_{n+3}v_j, w_{n+4}v_{j+1(\mathrm{mod}\, n)}\} \cup E(F_j) \cup \{u_jw_1, u_jw_2\} \cup \{u_iw_{i+j-1} \mid 3 \leq i \leq n-j+3\} \cup \{u_iw_{i+j-1-n} \mid n-j+4 \leq i \leq n\}$. Let $E(T_{n+1}) = \{w_{n+1}v_i \mid 1 \leq i \leq n\} \cup \{u_1w_i \mid 1 \leq i \leq n+4\} \cup \{w_{n+3}u_i \mid 3 \leq i \leq n\}$. Let $E(T_{n+2}) = \{w_{n+2}v_i \mid 1 \leq i \leq n\} \cup \{u_2w_i \mid 1 \leq i \leq n+4\} \cup \{w_{n+4}u_i \mid 3 \leq i \leq n\}$.

T_1 and T_2 each have 3n edges, 3n+1 vertices, and are connected. Hence T_1 and T_2 are trees. T_j, $3 \leq j \leq n+2$, has 3n+2 edges, 3n+3 vertices, and is connected. Hence, T_j, $3 \leq j \leq n+2$, is a tree. $\sum_{j=1}^{n+2} E(T_j) = 6n + n(3n+2) = 3n^2 + 8n$, as required. Next, we need to check that each edge is in a T_j, $1 \leq j \leq n+2$. An edge joining U and V is in an F_j and hence is in a T_j. An edge of the form w_jv_i, $1 \leq i \leq n$, $1 \leq j \leq n+2$, is in T_j. An edge of the form $w_{n+3}v_j$ is in T_j. An edge of the form $w_{n+4}v_j$ is in $T_{j-1(\mathrm{mod}\, n)}$. An edge of the form u_1w_i, $1 \leq i \leq n+4$, is in T_{n+1}. An edge of the form u_2w_i, $1 \leq i \leq n+4$, is in T_{n+2}. An edge of the from u_jw_1 or u_jw_2, $3 \leq j \leq n$, is in T_j. An edge of the form u_iw_{n+3}, $3 \leq i \leq n$, is in T_{n+1}. An edge of the form

$u_i w_{n+4}$, $3 \leq i \leq n$, is in T_{n+2}. If $s \geq r$, $3 \leq r \leq n$, $3 \leq s \leq n+2$, an edge $u_r w_s = u_r w_{(r-1)+(s-r+1)}$ is in T_{s-r+1}. If $s < r$, $3 \leq r \leq n$, $3 \leq s \leq n+2$, an edge $u_r w_s = u_r w_{(r-n-1)+(s-r+n+1)}$ is in $T_{s-r+n+1}$. Thus each edge is in a T_j, $1 \leq j \leq n+2$. Therefore, the edge-disjoint union of T_1 through $T_{n+2} = K_{n,n,n+4}$ and $\tau(K_{n,n,n+4}) = n+2$. •

6.4 Closing Remarks

In Chapters 4 and 5, we showed that it was possible to decompose $K_{m,n}$ into $\tau(K_{m,n}) = t = a(K_{m,n}) = \lceil \frac{mn}{m+n-1} \rceil$ trees. Recall that $\tau(K_{m,n})$ had to be greater than or equal to t in order to insure that no subgraph used in the decomposition contained a cycle. In Chapter 4, we described Beineke's result that $K_{m,n}$ can be decomposed into t trees where the trees are nearly equal in size. In Chapter 5, we showed that $K_{m,n}$ can be decomposed into t trees where all but perhaps one of the trees are spanning.

In Chapter 6, we considered complete tripartite graphs $K_{m,n,p}$ and found upper and lower bounds for $\tau(K_{m,n,p})$. We then decomposed a few families of $K_{m,n,p}$ into $t = \lceil \frac{mn+mp+np}{m+n+p-1} \rceil$ trees. Here also t is the minimum possible number of subgraphs such that no subgraph contains a cycle.

In addition to the families given in Chapter 6, we also considered $K_{1,n,p}$, $K_{2,n,p}$, $K_{3,n,p}$, and a few other families of $K_{m,n,p}$. We were able to decompose $K_{1,n,p}$ into $\tau(K_{n,p+1}) + 1$ trees by considering $K_{1,n,p} = K_{n,p+1} \cup K_{1,p}$ and by decomposing $K_{n,p+1}$ into $\tau(K_{n,p+1})$ trees using Beineke's method. We also found that $\tau(K_{1,n,p}) \geq \tau(K_{n,p+1})$ by direct calculation or by noticing that $K_{n,p+1}$ is a subgraph of $K_{1,n,p}$. Similarly, we were able to prove that $\tau(K_{n+1,p+1}) \leq \tau(K_{2,n,p}) \leq \tau(K_{n+1,p+1})+1$, and $\tau(K_{n+1,p+2}) + 1 \leq \tau(K_{3,n,p}) \leq \tau(K_{n+1,p+2}) + 2$.

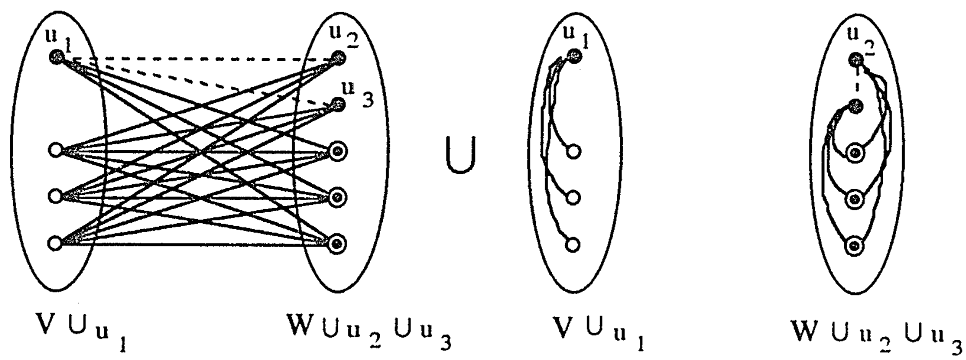

Figure 6.4.1

However, for some values of m,n,p, t equals the lower bound and for some values t equals the upper bound. When t equals the upper bound we are easily able to decompose $K_{m,n,p}$ into t trees. However, when t equals the lower bound, in order to decompose $K_{m,n,p}$ into t trees it is necessary to show that p edges of the form $u_m w_j$, $1 \leq j \leq p$, and sometimes n edges of the form $u_{m-1}v_j$, $1 \leq j \leq n$, can be added to the first t trees. For all particular values of n and p that we tried, we were always able to decompose $K_{m,n,p}$, $1 \leq m \leq 3$, into t trees, but we were unable to generalize the results. This method of partitioning the smallest partite set into the other two partite sets (See Figure 6.4.1 for an example of $K_{3,n,p}$) seems to be promising for $1 \leq m \leq 4$, but the method does not seem to extend for $m \geq 5$.

At present we feel that it is relatively easy to decompose other families of $K_{m,n,p}$ into t trees, but we are unaware of a method that will work for all m,n,p. Still, we conjecture that $K_{m,n,p}$ can be decomposed into $t = \lceil \frac{mn+mp+np}{m+n+p-1} \rceil$ trees. We leave it as an open question whether $K_{m,n,p}$ can be decomposed into t trees where at least t-1 of the trees are spanning.

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