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A Survey of polynomial invariants of knots

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A SURVEY OF POLYNOMIAL INVARIANTS OF KNOTS

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ABSTRACT

A SURVEY OF POLYNOMIAL INVARIANTS OF KNOTS

by Rebecca G. Wahl

This thesis is a survey of polynomial invariants of knots which includes the classical invariant, called the Alexander polynomial, as well as several modern knot polynomials.

The first half of this thesis is devoted to the development of the necessary preliminary concepts of knot theory and to the description and definition of the Alexander polynomial invariant of knots. This material is primarily based on *Introduction to Knot Theory* by Crowell and Fox.

The contemporary polynomial invariants of knots described in the second half of this thesis are of particular interest. Research conducted on this subject reveals a renewed excitement in the field of knot theory which was fueled by the recent discovery of a new knot invariant by V. F. R. Jones.

ACKNOWLEDGEMENTS

I wish to express sincere appreciation to professor Kubelka for his assistance and guidance in researching and writing this thesis.

In addition, I would like to thank my family for their unending support. Particularly, I extend special thanks go to my brother Steven for introducing me to the joy and beauty of mathematics.

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INTRODUCTION

The classical problem of knot theory involves determining whether or not two knots are the same. Much early progress toward this end came in 1928 when J. Alexander discovered the Alexander polynomial invariant of knots. This polynomial is a mathematical expression which depends only on the knot and does not depend on any particular picture of the knot. That is, different pictures of the same knot yield the same polynomial and knots with different polynomials actually are different knots. However, there are simple knots that the Alexander polynomial cannot distinguish.

Knot theory has been the object of a recent resurgence of interest. In 1985 V. F. R. Jones announced the discovery of a new polynomial invariant of knots now called the Jones polynomial. Discovered while Jones was working on von Neumann algebras, the Jones polynomial invariant of knots has revealed surprising connections between the apparently disparate fields of knot theory and statistical mechanics. In addition, E. Witten has found that the Jones polynomial invariant occurs in the setting of quantum field theory.

This thesis provides an introduction to the basic definitions and concepts of knot theory as well as to the definitions and descriptions of the classical Alexander polynomial invariant of knots, the Jones polynomial and several recently discovered polynomial invariants.

CHAPTER 1

BASIC KNOT THEORY

In this chapter we develop a minimal familiarity with the terminology, notation, and concepts involved in the study of knots and links. We begin with preliminary definitions of knots and their projections and then examine the various notions of equivalence of knots. Finally, we examine a presentation for the fundamental group of a knot.

1.1 KNOTS

1.1.1 Definition. A *knot* K is the homeomorphic image in $\mathbf{R}^3$ of the unit circle $\mathbf{S}^1$.

Certainly, any two knots are homeomorphic since both are homeomorphic to the unit circle $\mathbf{S}^1$. Thus, any useful notion of equivalence of knots must account for the way in which the knots are embedded in $\mathbf{R}^3$. We have the following definition of equivalence of knots:

1.1.2 Definition. Knots K_1 and K_2 are *equivalent* if there exists a homeomorphism of $\mathbf{R}^3$ onto itself which maps K_1 onto K_2.

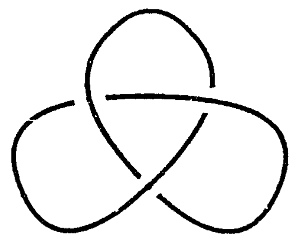
This relation is a true equivalence relation on the set of all knots, and each equivalence class of knots is referred to as a *knot type.* All knots equivalent to $\mathbf{S}^1$ with the standard embedding – called the *unknot* – are called *trivial* and constitute the *trivial type.*

We say that a homeomorphism of $\mathbf{R}^3$ is *orientation preserving* if the image of a right-hand screw is a right-hand screw. If, on the other hand, the image of a right-hand screw is a left-hand screw, we say that the homeomorphism is *orientation reversing*.

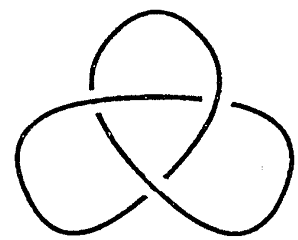
1.1.3 Definition. An *isotopic deformation* of a topological space $\mathbf{X}$ is a family of homeomorphisms h_t, $0 \le t \le 1$, of $\mathbf{X}$ onto itself such that h_0 is the identity and the function defined by $H(p,t) = h_t(p)$ is continuous in both t and p, for all $p \in \mathbf{X}$.

1.1.4 Definition. Knots K_1 and K_2 are said to be of the same *isotopy type* if there exists an isotopic deformation $\{h_t\}$ of $\mathbf{R}^3$ such that $h_1 K_1 = K_2$. Equivalently, knots K_1 and K_2 are of the same *isotopy type* if there exists an orientation preserving homeomorphism of $\mathbf{R}^3$ onto $\mathbf{R}^3$ which maps K_1 onto K_2. (Note that the equivalence of these definitions is far from obvious. See, e.g., [6].)

1.1.5 Examples. The simplest possible knot is the *overhand* or *trefoil* knot. This knot has three overcrossings and has either a left-handed or right-handed form.



left-hand trefoil



right-hand trefoil

There is exactly one knot with four crossings and it is referred to by the following names: the *figure-eight knot*, *four knot*, and *Listing's knot.*

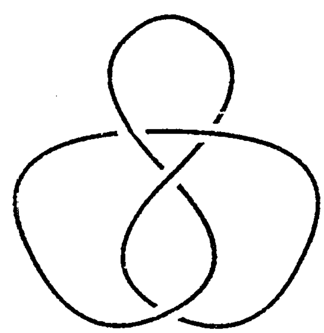


figure-eight

1.1.6 Definition. A knot K is said to be *amphicheiral* if there exists an orientation reversing homeomorphism h of $\mathbf{R}^3$ onto itself such that $hK = K$.

1.1.7 Definition. By the *mirror image* of a knot K we shall mean the image of K under the reflection defined by $(x, y, z) \to (x, y, -z)$.

Note that since the reflection map is a homeomorphism of $\mathbf{R}^3$ taking K onto the mirror image of K, every knot is equivalent to its mirror image.

1.1.8 Theorem. A knot is amphicheiral if and only if there exists an orientation preserving homeomorphism of $\mathbf{R}^3$ onto itself which maps K onto its mirror image.

If K_1 and K_2 belong to the same isotopy type, then they are equivalent. The converse, however, is false. It can be shown, although Fox [6] notes that this result is hard, that the trefoil knot is not amphicheiral. Assuming this result, we have that the trefoil knot and its mirror image are equivalent but not of the same isotopy type. Hence, isotopy is a sufficient but not necessary condition for the equivalence of knots.

1.1.9 Definition. An *oriented* knot is a knot to which a direction of travel is assigned. This preferred direction around the knot is called an *orientation.*

1.1.10 Definition. A knot K is *invertible* if there exists an orientation preserving homeomorphism h of $\mathbf{R}^3$ onto itself such that the restriction $h|_K$ is an orientation reversing homeomorphism of K onto itself.

1.1.11 Example. The trefoil and figure-eight knots are invertible. Suppose, for example, that the figure-eight knot depicted in the previous figure is embedded in $\mathbf{R}^3$ so that the positive z-axis of a right-hand system is oriented directly out of the page. Consider rotating the knot through an angle of π about the y-axis. This rotation is an orientation preserving homeomorphism of $\mathbf{R}^3$ onto itself which takes the figure-eight knot onto itself while reversing the orientation.

H. F. Trotter proves in [20] that an infinite family of non-invertible knots exist. The simplest member of this family is the knot $C_{7,3,5}$.

1.1.12 Definitions. A *polygonal knot* is a knot which is a finite union of straight line segments in $\mathbf{R}^3$. The straight line segments are called *edges* and the points where the straight line segments meet are called *vertices*. A knot is *tame* if it is equivalent to a polygonal knot; otherwise it is *wild*. We will restrict our attention to tame knots.

1.1.13 Definitions. Knots are represented by their projections PK, where $P: \mathbf{R}^3 \to \mathbf{R}^3$ is given by $P(x,y,z) = (x,y,0)$. A point p in the image of PK is said to be a *multiple point* if the inverse image $P^{-1}(p) \cap K$ consists of more than one point. The *order* of $p \in PK$ is the cardinality of $P^{-1}(p) \cap K$.

1.1.14 Definition. A polygonal knot is said to be in *regular position* if: (i) the only multiple points of K are double points, and there are only finitely many of them; (ii) no double point is the image of a vertex of K.

It is a known fact that any polygonal knot is equivalent under an arbitrarily small rotation of $\mathbf{R}^3$ to a polygonal knot in regular position. Thus, we may assume any polygonal knot to be in regular position.

Each double point of the projected image of a polygonal knot in regular position is the image of two points of the knot. The point with the larger z-coordinate is called an *overcrossing*, and the other point is called an *undercrossing*. A polygonal knot in regular position is said to be *alternating* if the overcrossings and undercrossings alternate around the knot. A closed arc segment of a projection of a polygonal knot in regular position containing exactly one overcrossing and no undercrossings is called an *overpass*. An *underpass* is similarly defined. For convenience we will refer to a closed arc segment of a projection of a polygonal knot in regular position which contains neither overcrossings nor undercrossings as a *null overpass* or a *null underpass*. This definition will allow us to divide the projection of any knot into overpasses and underpasses which alternate around the knot.

1.1.15 Definition. A *link* is the union of a finite number of disjoint knots in $\mathbf{R}^3$. The mutually disjoint knots which make up a link are called the *components* of the link. A link is called *tame* if all of its components are tame.

1.1.16 Definition. An *oriented* link is a link with a direction of travel assigned to each component. This preferred direction around a component is called an *orientation.*

1.2 THE KNOT GROUP

The *knot group* of a knot K in regular position is the fundamental group of the space $\mathbf{R}^3 - K$, which is called the *knot complement.* A brief discussion of the notion of the fundamental group of a space is given first, followed by an examination of a presentation of the knot group. A complete exposition of the fundamental group of a space can be found in [16].

1.2.1 Definition. Let $\mathbf{X}$ be a topological space and let $b \in \mathbf{X}$. A *loop* γ in $\mathbf{X}$ *based* at b is a continuous mapping $\gamma\colon [0,1] \to \mathbf{X}$ such that $\gamma(0) = \gamma(1) = b$. The point b is called the *basepoint.*

1.2.2 Definition. Two loops γ_0 and γ_1 based at b are *homotopic with basepoint fixed*, denoted $\gamma_0 \simeq \gamma_1$, if there is a continuous map F: [0,1] x [0,1] $\to$ **X** such that

$$F(s, 0) = \gamma_0(s), \quad 0 \le s \le 1,$$

$$F(s, 1) = \gamma_1(s), \quad 0 \le s \le 1,$$

$$F(0, t) = b, \quad 0 \le t \le 1,$$

$$F(1, t) = b, \quad 0 \le t \le 1.$$

The map F is referred to as a *homotopy* between γ_0 and γ_1. For each $t \in [0,1]$ the map γ_t: [0,1] $\to$ **X** defined by

$$\gamma_t(s) = F(s, t), \quad 0 \le s \le 1,$$

is a loop based at b. As the parameter t moves from 0 to 1, the loop γ_0 is continuously deformed to the loop γ_1 through the γ_t's.

1.2.3 Lemma. The relation "homotopic with basepoint fixed" is an equivalence relation on the set of all loops in **X** based at b.

The equivalence classes of loops in **X** based at b modulo this equivalence relation are called the *homotopy classes* of loops based at b. The homotopy class of the loop γ will be denoted $[\gamma]$, and $[\gamma_0] = [\gamma_1]$ means $\gamma_0 \simeq \gamma_1$.

Multiplication of loops can be defined by concatenation for any two loops α, β based at a common point b. Although the multiplication of loops is not associative, we can define the multiplication of homotopy classes by noting that the class of a product of two loops depends only on the classes of the given loops, and that this multiplication of classes is an associative operation. In addition, the homotopy class [b] of the constant map b is a multiplicative identity for the set of homotopy classes of loops based at b. Finally, given any loop γ, let γ^{-1} denote the loop defined by

$$\gamma^{-1}(t) = \gamma(1 - t), \qquad t \in [0,1];$$

that is, the loop γ traversed in the opposite direction. Then if $[\gamma]$ and $[\gamma^{-1}]$ denote the homotopy classes of γ and γ^{-1} respectively, $[\gamma][\gamma^{-1}] = [\gamma^{-1}]\,[\gamma] = [b]$ so that each homotopy class of loops based at b has an inverse. In summary, the set of all homotopy classes of loops in **X** based at b satisfies the axioms for a group.

1.2.4 Definition. The class of all loops in **X** based at a point b under the equivalence relation of homotopy yields a group, denoted by $\pi_1(\mathbf{X}, b)$, called the *fundamental group* of the space **X** based at b.

1.2.5 Definition. If K is any knot in $\mathbf{R}^3$ and p_0 is any point in $\mathbf{R}^3 - K$, then the fundamental group $\pi_1(\mathbf{R}^3 - K, p_0)$ is called *the group of the knot K.*

It can be easily shown that if **X** is path-connected, the fundamental group $\pi_1(\mathbf{X}, b)$ is independent of the choice of basepoint b. For this reason, we may omit reference to the basepoint p_0, and refer only to $\pi_1(\mathbf{R}^3 - K)$ as the group of the knot K.

1.3 THE PRESENTATION OF A KNOT GROUP

The following is a description of a method for deriving the *over presentation* of any polygonal knot K in regular position. Fox proves in [6] that the over presentation is in fact a group presentation of $\pi_1(\mathbf{R}^3 - K, p_0)$, the fundamental group of the complement space of the knot K. A similar construction exists for deriving the *under presentation* of a knot. Together, the over and under presentations constitute a pair of *dual presentations*; that is, there exists a presentation equivalence from one to the other. Thus, we shall forgo the latter and focus on the over presentation of a polygonal knot in regular position.

1.3.1. The Over Presentation. Let K be a polygonal knot in regular position with projection P. For some integer n, select a subset Q of K containing 2n points, none of which is an overcrossing or undercrossing, so that these points divide K into overpasses and underpasses which alternate around the knot. Note that this selection of the 2n points can be done in many ways. However, if K is alternating and if the projection of K contains double points, then a good choice for the number n is the number of double points in the projection. Denote the overpasses by $A_1, A_2, \ldots, A_n$ and their union $A_1 \cup A_2 \cup \ldots \cup A_n$ by **A**; denote the underpasses by $B_1, B_2, \ldots, B_n$ and their union $B_1 \cup B_2 \cup \ldots \cup B_n$ by **B**. For simplification we will assume that $Q \subset R^2$, $(K \setminus Q) \cap \mathbf{R}^2 = \varnothing$ and that all points in $\mathbf{A} \setminus Q$ have positive z-coordinate and all points in $\mathbf{B} \setminus Q$ have negative z-coordinate.

We next choose an orientation for K and a basepoint p_0 lying above the knot. For convenience, assume $p_0 = (0,0,z_0)$ for some positive z_0. Also, we choose a point $q_0 \in \mathbf{R}^2 - PK$. We require the following definition.

1.3.2 Definition. We call a path **a** in $\mathbf{R}^2$ *simple* if it satisfies the following conditions:

(i) **a** is polygonal,

(ii) neither the initial nor the terminal point of **a** belongs to PK, and

(iii) **a** intersects PK in only a finite number of points, no one of which is a vertex of **a** or a vertex of PK.

A simple path is shown in Figure 1.3.1.

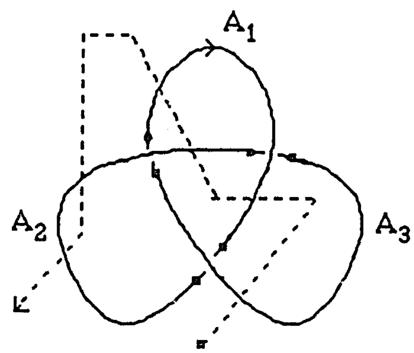


Figure 1.3.1

Let **a** in $\mathbf{R}^2 - PB$ be any simple path. The simple path **a** will intersect the projected overpasses $PA_1, \ldots, PA_n$ in a finite number of points. Let F(**x**) be the free group generated by $\mathbf{x} = \{x_1, \ldots, x_n\}$ with one generator for each projected overpass.

Since **a** is a path in $\mathbf{R}^2 - PB$, **a** is necessarily oriented; let K have the previously chosen orientation. Suppose the projected overpasses crossed by **a** are, in order, $PA_{i_1}, PA_{i_2}, \ldots, PA_{i_m}$. If **a** crosses under A_{i_k} from the left (with respect to the orientation of K), let $e_k = 1$, and if **a** crosses under A_{i_k} from the right, let $e_k = -1$. Then to each simple path $\mathbf{a} \in R^2 - PB$ we assign a word $\mathbf{a}^\# \in F(\mathbf{x})$ defined by

$$\mathbf{a}^\# = x_{i_1}^{e_1} \ldots x_{i_m}^{e_m}.$$

For example, to the path **a** shown in Figure 1.3.1, we assign the word $\mathbf{a}^\# = x_3 x_1 x_2^{-1} x_1^{-1} x_2 x_2^{-1}$. Note that the assignment $\mathbf{a} \to \mathbf{a}^\#$ is product preserving,

$$(\mathbf{a}_1 \cdot \mathbf{a}_2)^\# = \mathbf{a}_1^\# \cdot \mathbf{a}_2^\#,$$

but not necessarily onto F(**x**). For any point $p \in \mathbf{R}^2$, let $\overline{p}$ be the path which runs linearly from p_0 parallel to $\mathbf{R}^2$ to a point directly over p and then down to p. Now, if **a** is a path in $\mathbf{R}^2$, we set

$$^*\mathbf{a} = \overline{\mathbf{a}(0)} \cdot \mathbf{a} \cdot [\overline{\mathbf{a}(1)}]^{-1},$$

where $\mathbf{a}(0)$ is the initial point of $\mathbf{a}$ and $\mathbf{a}(1)$ is the terminal point of $\mathbf{a}$. In this way we produce a loop $^*\mathbf{a}$ in $\mathbf{R}^2 - K$, based at the point p_0, which includes the path $\mathbf{a}$.

We are now in a position to define a homomorphism of F(**x**), the free group of the over presentation, onto $\pi_1(\mathbf{R}^3 - K, p_0)$, the fundamental group of the knot complement based at p_0. Define the desired homomorphism $\phi: F(\mathbf{x}) \to \pi_1(\mathbf{R}^3 - K, p_0)$ as follows: Let $\mathbf{a}_j$ be a simple path in $\mathbf{R}^3 - \mathit{PB}$ such that $\mathbf{a}_j^{\#} = x_j$, j = 1, ...,n. Define for j = 1, ..., n,

$$\phi(x_j) = [^*\mathbf{a}_j],$$

the equivalence class in $\pi_1(\mathbf{R}^3 - K, p_0)$ of the p_0 - based loop $^*\mathbf{a}_j$. The homomorphism ϕ is then the unique extension of this assignment on the generators $x_1, \ldots, x_n$ to the entire group F(**x**). Thus,

$$\phi(\mathbf{a}^{\#}) = [^*\mathbf{a}],$$

for any path $\mathbf{a}$ in $\mathbf{R}^3 - \mathit{PB}$. That the $\phi(x_1), \ldots, \phi(x_n)$ generate $\pi_1(\mathbf{R}^3 - K, p_0)$ is proved in Fox [6]. However, this result seems plausible since it is geometrically evident that every p_0 - based loop in $\mathbf{R}^3 - K$ is equivalent to a product of the loops $^*\mathbf{a}_j$ and their inverses for j = 1, ..., n (see Figure 1.3.2 taken from [6]). Thus, we have generators $x_1, \ldots, x_n$ for the over presentation.

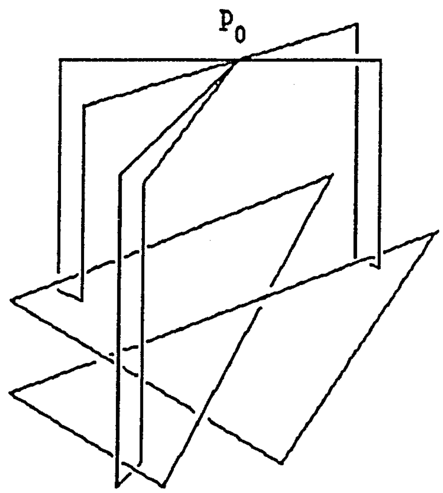


Figure 1.3.2

Now we proceed to find relators for the over presentation. The images of the underpasses PB_i, $i = 1, \ldots, n$, are disjoint arcs in $\mathbf{R}^2$. Thus, we may choose disjoint, simply-connected, open sets $V_1, \ldots, V_n$ in $\mathbf{R}^2$ such that $PB_i \subset V_i$, $i = 1, \ldots, n$, and such that their boundaries are disjoint simple loops $v_1, \ldots, v_n$ which run counterclockwise (from above) around $V_1, \ldots, V_n$ respectively. The only restriction placed on the choices of the regions V_j is that the point $q_0 \in \mathbf{R}^2 - K$ lie outside the closures of the V_j.

Select a set of simple paths $c_1, \ldots, c_n$ in $\mathbf{R}^2$ connecting the point q_0 to the initial point of each of the loops v_i. More precisely, select $c_1, \ldots, c_n$ such that each c_i has initial point $c_i(0) = q_0$ and terminal point $c_i(1) = v_i(0)$, with the provision that

$c_i(t) \in \mathbf{R}^2 - \left[\bigcup_{k=1}^{n} \mathrm{cl}\,(V_k) \right]$ for all $t \leq 1$, where $\mathrm{cl}\,(V_k)$ denotes the closure of V_k.

Next, let $r_i = (c_i \cdot v_i \cdot c_i^{-1})^{\#}$. Then the *over presentation* of $\pi_1(\mathbf{R}^3 - K, p_0)$ is

$$(x_1, \ldots, x_n \colon r_1, \ldots, r_n)_\phi .$$

To see that the assignment $\phi(r_i) = 1$ is reasonable, consider

$$\phi(r_i) = [{}^*(c_i \cdot v_i \cdot c_i^{-1})]$$

the equivalence class of the p_0 - based loop ${}^*(c_i \cdot v_i \cdot c_i^{-1})$. It is geometrically evident that the contraction to the point p_0 of a typical loop ${}^*(c_i \cdot v_i \cdot c_i^{-1})$ in this equivalence class is possible by sliding the loop below the underpass B_i. Therefore, such a loop must be homotopically trivial and, hence, can be taken to be a relator in the presentation of the group. Moreover, the collection of relators obtained in this way suffices to generate the kernel of ϕ.

We will assume the proof that the over presentation is indeed a group presentation of $\pi_1(\mathbf{R}^3 - K, p_0)$. Additionally, the proof in Fox [6] yields the following computationally important fact.

1.3.3 Proposition. In an over presentation, any one of the relators $r_1, \ldots, r_n$ is a consequence of the other $n - 1$.

We now calculate the over presentations of a few knots using the method outlined in 1.3.1.

1.3.4 Example. *Trivial knot* (Figure 1.3.3). The projection of the trivial knot below has one overpass shown as a heavy line and one underpass shown as a thin line. The path v_1 is shown as a dashed line. By using the method described in 1.3.1, we follow v_1 counterclockwise and note the overpasses which are crossed. Since no overpasses are crossed, $\pi_1(\mathbf{R}^3 - K)$, for the trivial knot K, has no relators. Thus, $\pi_1(\mathbf{R}^3 - K) = | x : |$ and so the group of the trivial knot is infinite cyclic.

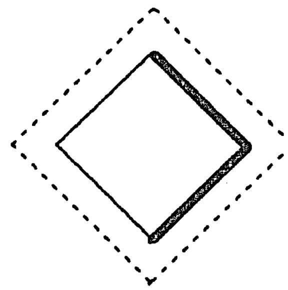


Figure 1.3.3

1.3.5 Example. *Right-hand trefoil knot* (Figure 1.3.4). Once the knot has been given an orientation, we label any three consecutive underpasses B_1, B_2 and B_3. The overpasses following B_1, B_2 and B_3 we denote by A_1, A_2 and A_3 respectively and we assign to these overpasses the generators x, y, z so that $x = a_1^{\#}$, $y = a_2^{\#}$ and $z = a_3^{\#}$. We mark x, y, z with arrows so that the generator crosses from the left with respect to the orientation of the knot. Next, we find loops v_1, v_2 and v_3 which run counterclockwise around the open sets V_i, where $PB_i \subset V_i$, i = 1, 2, 3, as shown in Figure 1.3.4.

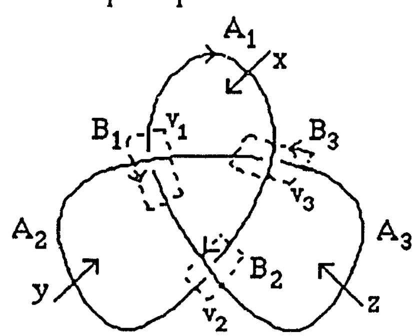


Figure 1.3.4

To simplify our computations, it is useful to note that a presentation involving only the words $v_i^{\#}$ assigned to the loops v_i,

$$(x_1, \ldots, x_n \colon v_1^{\#}, \ldots, v_n^{\#})_{\phi},$$

can be obtained from the over presentation

$$(x_1, \ldots, x_n : r_1, \ldots, r_n)_\phi .$$

This can be seen by noting that $\phi(r_i) = 1$ where $r_i = (c_i \cdot v_i \cdot c_i^{-1})^{\#}$, and since the assignment $v_i \to v_i^{\#}$ is product prescrving, any element $v_i^{\#}$ is also mapped into 1 by ϕ.

With the above preparation complete, the relators can be arrived at by reading around each loop v_i and noting which of the generators x, y or z are crossed. Thus, we have relators

$$v_1^{\#} = x^{-1} y z y^{-1}$$
$$v_2^{\#} = y^{-1} z x z^{-1}$$
$$v_3^{\#} = z^{-1} x y x^{-1}.$$

With one relator dropped, $(x, y, z{:}\ x^{-1} y z y^{-1}, z^{-1} x y x^{-1})$ is a group presentation and, if we write these relators as relations, we have for the right-hand trefoil knot,

$$\pi_1(\mathbf{R}^3 - K) = | x, y, z{:}\ x = y z y^{-1}, z = x y x^{-1} |.$$

By substituting $z = x y x^{-1}$, we get

$$\pi_1(\mathbf{R}^3 - K) = | x, y{:}\ x = y x y x^{-1} y^{-1} |$$
$$= | x, y : x y x = y x y |,$$

and we see that for the right-hand trefoil knot K, the presentation of the group $\pi_1(\mathbf{R}^3 - K)$ is $(x, y : x y x = y x y)$.

1.3.6 Example. *Figure-eight knot* (Figure 1.3.5) As in the previous example, we assign an orientation to the knot, label the four underpasses and overpasses, and assign generators x, y, z, and w to the four overpasses so that $x = a_1^{\#}$, $y = a_2^{\#}$, $z = a_3^{\#}$, and $w = a_4^{\#}$. We then mark each of these generators with an arrow so that the generator crosses from the left with respect to the orientation of the knot.. Next, we find loops v_1, v_2, v_3 and v_4 which run counterclockwise around the open sets V_i, where $PB_i \subset V_i$,

$i = 1, \ldots, 4$, as shown in Figure 1.3.5.

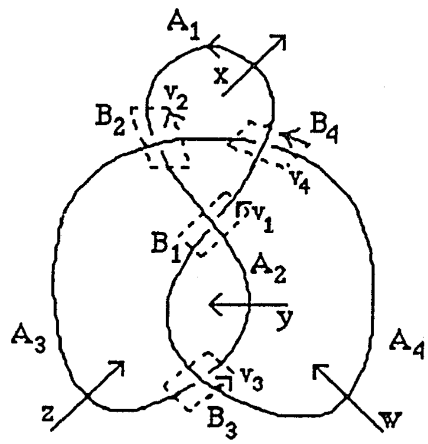


Figure 1.3.5

The relators can now be obtained by reading around each loop v_i and noting which of the generators x, y, z or w, which are crossed.

$$v_1^{\#} = x^{-1} y w y^{-1}$$
$$v_2^{\#} = z y^{-1} z^{-1} x$$
$$v_3^{\#} = w y w^{-1} z^{-1}$$
$$v_4^{\#} = x^{-1} z x w^{-1}.$$

Dropping relator $v_4^{\#}$, we get

$$\pi_1(\mathbf{R}^3 - K) = | x, y, z, w: \; x^{-1} y w y^{-1}, z y^{-1} z^{-1} x, \; w y w^{-1} z^{-1} |.$$

Writing relations instead of relators yields

$$\pi_1(\mathbf{R}^3 - K) = | x, y, z, w: \; x = y w y^{-1}, y = z^{-1} x z, z = w y w^{-1} |.$$

Substituting $z = w y w^{-1}$ in the other relations, we obtain

$$\pi_1(\mathbf{R}^3 - K) = | x, y, w: \; x = y w y^{-1}, y = w y^{-1} w^{-1} x w y w^{-1} |.$$

From the first relation we have $y^{-1} x y = w$ and substituting this gives

$$\pi_1(\mathbf{R}^3 - K) = | x, y: \; y = y^{-1} x y y^{-1} y^{-1} x^{-1} y x y^{-1} x y y y^{-1} x^{-1} y |$$

or

$$\pi_1(\mathbf{R}^3 - K) = | x, y: \; y\, x^{-1} y\, x\, y^{-1} = x^{-1} y\, x\, y^{-1} x\, |.$$

Thus, the presentation of the group $\pi_1(\mathbf{R}^3 - K)$, where K is the figure-eight knot, is

$$(x, y: \; y\, x^{-1} y\, x\, y^{-1} = x^{-1} y\, x\, y^{-1} x).$$

CHAPTER 2

THE ALEXANDER POLYNOMIAL OF A KNOT

In this chapter our aim is to describe the classical polynomial invariant of knots, the Alexander polynomial. Introduced by Alexander in 1928, the Alexander polynomial is effective in distinguishing specific knots.

Our work in Chapter 1 culminated with a presentation of the fundamental group of the knot complement . Unfortunately, there can be no generic algorithm for determining whether or not two presentations define isomorphic groups. This is the classic *isomorphism problem* for finitely presented groups, first proved unsolvable by Adian [1957] and Rabin [1958]. In fact, as noted in a paper by J. Stillwell [19] the problem which led to the first statement of the *word problem*, given by Dehn [1910], is the problem of deciding when two knots are the same. In hopes of generating useful invariants of knots we must, therefore, derive some invariants of group presentation types. To this end, we shall first consider the group ring of an arbitrary multiplicative group. Next, we define formal derivatives on this group ring and define the Alexander matrix of a group presentation. We arrive at a true invariant of group presentation type in the *elementary ideals* which are defined on the Alexander matrix. Finally, the Alexander polynomial of a knot is defined in terms of the elementary ideals.

2.1 THE GROUP RING

We shall first define what is meant by a group ring and then consider some properties of a particular group ring, **Z**[G], the group ring with respect to the integers.

2.1.1 Definition. Given a multiplicative group G and a commutative ring R with identity, the *group ring* , denoted by R[G], consists of all finite formal sums $\sum_{\sigma \in G} a_\sigma \sigma$, with addition given by

$$\sum_{\sigma \in G} a_\sigma \sigma + \sum_{\sigma \in G} b_\sigma \sigma = \sum_{\sigma \in G} (a_\sigma + b_\sigma)\sigma,$$

and multiplication given by

$$\sum_{\sigma \in G} a_\sigma \sigma \sum_{\tau \in G} b_\tau \tau = \sum_{\sigma, \tau \in G} a_\sigma b_\tau \sigma\tau,$$

where a_σ, b_σ, and $b_\tau \in R$. These operations make R[G] into a ring. Furthermore, if 1_R is the identity in R and $e \in G$ is the identity in G, then $1_R e$ is the multiplicative identity in R[G].

In the study of knot invariants, we are solely concerned with group rings **Z**[G], group rings with respect to the integers. Note that **Z**[G] is free abelian with a basis given by G.

The following theorem and its proof are due to Fox [6]:

2.1.2 Theorem. An arbitrary function $\phi : G \rightarrow A$, where G is a multiplicative group and A is an additive abelian group, has a unique extension to an additive homomorphism $\phi : \mathbf{Z}[G] \rightarrow A$. Moreover, if A is a ring and ϕ preserves products on G, the extension is a ring homomorphism.

Proof: Set $\phi(0) = 0$. Every element of **Z**[G] has a unique expression $\sum_{i=1}^{k} n_i g_i$, where $n_i \neq 0$, $i = 1, \ldots, k$, and $g_1, \ldots, g_k$ are distinct. To obtain the extension, we define

$$\phi\left(\sum_{i=1}^{k} n_i g_i\right) = \sum_{i=1}^{k} n_i \phi(g_i). \qquad (*)$$

Defined in this way, ϕ is trivially addition preserving. Since any extension of ϕ to an additive homomorphism $\mathbf{Z}[G] \to A$ must satisfy (∗), the uniqueness of the extension is assured. Finally, if A is a ring and ϕ is product preserving on G,

$$\begin{aligned}\phi\left(\sum_i n_i g_i \sum_j n_j' g_j'\right) &= \phi\left(\sum_{i,j} n_i n_j' g_i g_j'\right) \\ &= \sum_{i,j} n_i n_j' \phi(g_i)\phi(g_j') \\ &= \sum_i n_i \phi(g_i) \sum_j n_j' \phi(g_j') \\ &= \phi\left(\sum_i n_i g_i\right)\phi\left(\sum_j n_j' g_j'\right)\end{aligned}$$

q.e.d.

2.2 DERIVATIVES IN THE GROUP RING

Before defining formal derivatives in a group ring, we will need to consider two important ring homomorphisms defined on the group ring of every group: the *abelianizer* and the *trivializer*.

2.2.1 Definition. For any group G, an element of the form $aba^{-1}b^{-1}$ is called a *commutator* of the group.

2.2.2 Definition. For any group G, the *commutator subgroup* is the subgroup of G generated by all the commutators of G and is denoted [G,G].

2.2.3 Definition. The quotient group $G/[G,G]$ is called the *commutator quotient group* or the *abelianization of G*, and the canonical homomorphism

$$\mathfrak{a} : G \to G/[G,G]$$

is called the *abelianizer.* We will also denote by $\mathfrak{a}$ the unique extension of $\mathfrak{a}$ to a homomorphism between group rings.

2.2.4 Definition. For any group G, consider the mapping $\mathfrak{t}: G \to \mathbf{Z}$ defined by $\mathfrak{t}(g) = 1$, for all $g \in G$. The *trivializer* is the unique extension of $\mathfrak{t}$ to a ring homomorphism $\mathbf{Z}[G] \to \mathbf{Z}$. Clearly,

$$\mathfrak{t}\,(\sum_i n_i g_i) = \sum_i n_i.$$

Now we consider the definition of a formal derivative in a group ring $\mathbf{Z}[G]$, as given in Fox [6].

2.2.5 Definition. Given a group ring $\mathbf{Z}[G]$, any function $D: \mathbf{Z}[G] \to \mathbf{Z}[G]$ which satisfies

(i) $D(v_1 + v_2) = Dv_1 + Dv_2$

(ii) $D(v_1 v_2) = (Dv_1)\mathfrak{t}(v_2) + v_1 Dv_2$,

is called a *derivative* in $\mathbf{Z}[G]$, where $v_1, v_2 \in \mathbf{Z}[G]$ and $\mathfrak{t}$ denotes the trivializer. As Fox [6] notes, for elements of G, 2.2.5(ii) takes the simpler form

(iii) $D(g_1 g_2) = Dg_1 + g_1 Dg_2$ for all $g_1, g_2 \in G$.

By theorem 2.1.2, an arbitrary function $G \to A$, where A is abelian, has a unique extension to an additive homomorphism from the group ring $\mathbf{Z}[G]$ into A. In fact, a derivative may be defined as the unique linear extension to $\mathbf{Z}[G]$ of any function $D : G \to \mathbf{Z}[G]$ which satisfies (iii).

2.2.6 Theorem. Let D be a derivative in a group ring $\mathbf{Z}[G]$. Then

(i) $D(\sum_i n_i g_i) = \sum_i n_i D g_i$

(ii) $Dn = D(n \cdot e) = 0$, for $n \in \mathbf{Z}$ and e the identity in G

(iii) $Dg^{-1} = -g^{-1}Dg$, $g \in G$.

2.2.7 Remark. Another useful fact, a formula for the derivative of a power, follows by defining, for any $g \in G$ and $n \in \mathbf{Z}$, the group ring element

$$\frac{g^n - 1}{g - 1} = \begin{cases} 0 & \text{if } n = 0 \\ \sum_{i=0}^{n-1} g^i & \text{if } n > 0 \\ -\sum_{i=n}^{-1} g^i & \text{if } n < 0 \end{cases}$$

Using this definition we have:

2.2.8 Theorem. Let D be a derivative in a group ring $\mathbf{Z}[G]$. Then

$$Dg^n = \frac{g^n - 1}{g - 1} Dg \quad \text{for any } g \in G.$$

Derivatives in a group ring of a free group enjoy special properties:

2.2.9 Theorem. Let F be a free group with free basis $x_1, x_2, \ldots$. To each free generator x_j there corresponds a unique derivative

$$D_j = \frac{\partial}{\partial x_j}$$

in $\mathbf{Z}[F]$, called the derivative with respect to x_j, which has the property

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}, \text{ where } \delta_{ij} \text{ is the Kronecker delta function.}$$

This result and its proof can be found in Fox [6].

2.2.10 Example. *Right-hand trefoil knot.* In 1.3.5 we found that for the right-hand trefoil knot K, the presentation of the group $\pi_1(\mathbf{R}^3 - K)$ is $(x, y : xyx = yxy)$. As Fox [6] notes, any relation $r_i = s_i$ corresponds to the relator $r_i s_i^{-1}$. In fact, the derivative of such a relator is

$$D(r_i s_i^{-1}) = Dr_i - r_i s_i^{-1} Ds_i, \text{ by 2.2.5 (iii) and 2.2.6 (iii).}$$

Since $\gamma : F(x) \to \frac{F(x)}{R} = |\mathbf{x} : \mathbf{r}|$ maps every relator onto 1, the calculation of the derivative is further simplified by noting

$$\gamma\left[D(r_i s_i^{-1})\right] = \gamma\left[Dr_i - Ds_i\right],$$

and in terms of partial derivatives we have

$$\gamma\left[\frac{\partial r_i s_i^{-1}}{\partial x_j}\right] = \gamma\left[\frac{\partial r_i}{\partial x_j} - \frac{\partial s_i}{\partial x_j}\right].$$

We shall now calculate the derivative of the relation $xyx = yxy$ and each of the partial derivatives.

By the above remark and properties of derivatives we have under γ,

$$\begin{aligned} D[(xyx)(yxy)^{-1}] &= D(xyx) - D(yxy) \\ &= Dx + xDyx - Dy - yDxy \\ &= Dx + x[Dy + yDx] - Dy - y[Dx + xDy] \\ &= Dx + xDy + xyDx - Dy - yDx - yxDy. \end{aligned}$$

Using 2.2.9 we see that the partial derivatives with respect to each variable under γ are

$$\begin{aligned} \frac{\partial}{\partial x}[(xyx)(yxy)^{-1}] &= 1 + 0 + xy - 0 - y - 0 \\ &= 1 + xy - y, \text{ and} \\ \frac{\partial}{\partial y}[(xyx)(yxy)^{-1}] &= 0 + x + 0 - 1 - 0 - yx \\ &= x - 1 - yx. \end{aligned}$$

2.2.11 Example. *Figure-eight knot.* As calculated in Example 1.3.6, if K is the figure-eight knot, the presentation of the group $\pi_1(\mathbf{R}^3 - K)$ is

$(x, y: y x^{-1} y x y^{-1} = x^{-1} y x y^{-1} x)$. We now calculate the derivative of the relation $y x^{-1} y x y^{-1} = x^{-1} y x y^{-1} x$, as well as each of the partial derivatives of this relation. Under the map $\gamma : F(x) \rightarrow \frac{F(x)}{R} = |\mathbf{x} : \mathbf{r}|$, we have

$$D[(y x^{-1} y x y^{-1}) (x^{-1} y x y^{-1} x)^{-1}]$$

$$= D(y x^{-1} y x y^{-1}) - D(x^{-1} y x y^{-1} x)$$

$$= Dy + yDx^{-1}yxy^{-1} - (-x^{-1}Dx + x^{-1}Dyxy^{-1}x)$$

$$= Dy - yx^{-1}Dx + yx^{-1}Dyxy^{-1} + x^{-1}Dx - x^{-1}Dy - x^{-1}y\,Dxy^{-1}x$$

$$= Dy - yx^{-1}Dx + yx^{-1}Dy + yx^{-1}yDxy^{-1} + x^{-1}Dx - x^{-1}Dy - x^{-1}yDx - x^{-1}yxDy^{-1}x$$

$$= Dy - yx^{-1}Dx + yx^{-1}Dy + yx^{-1}yDx + yx^{-1}yxDy^{-1} + x^{-1}Dx - x^{-1}Dy - x^{-1}yDx + x^{-1}yxy^{-1}Dy - x^{-1}yxy^{-1}Dx$$

$$= Dy - yx^{-1}Dx + yx^{-1}Dy + yx^{-1}yDx - yx^{-1}yxy^{-1}Dy + x^{-1}Dx - x^{-1}Dy - x^{-1}yDx + x^{-1}yxy^{-1}Dy - x^{-1}yxy^{-1}Dx.$$

The partial derivatives with respect to each variable under γ are

$$\frac{\partial}{\partial x}[(y x^{-1} y x y^{-1}) (x^{-1} y x y^{-1} x)^{-1}] = 0 - yx^{-1} + 0 + yx^{-1}y - 0 + x^{-1} - 0 - x^{-1}y + 0 - x^{-1}yxy^{-1}$$

$$= - yx^{-1} + yx^{-1}y + x^{-1} - x^{-1}y - x^{-1}yxy^{-1}, \text{ and}$$

$$\frac{\partial}{\partial y}[(y x^{-1} y x y^{-1}) (x^{-1} y x y^{-1} x)^{-1}] = 1 - 0 + yx^{-1} + 0 - yx^{-1}yxy^{-1} + 0 - x^{-1} - 0 + x^{-1}yxy^{-1} - 0$$

$$= 1 + yx^{-1} - yx^{-1}yxy^{-1} - x^{-1} + x^{-1}yxy^{-1}.$$

2.3 THE ALEXANDER MATRIX OF A PRESENTATION

2.3.1 Definition. Let $(\mathbf{x} : \mathbf{r})$ be a finite presentation of a group G. The *Alexander matrix* associated with this presentation is the matrix $[a_{ij}]$ defined by $a_{ij} = \mathfrak{a}\gamma\left(\frac{\partial r_i}{\partial x_j}\right)$,

where $\mathfrak{a}$ denotes the abelianizer and γ is the extension of the canonical ring homomorphism induced by $\gamma\colon F(x) \to \frac{F(x)}{R} = |\mathbf{x} : \mathbf{r}|$. Note that γ takes elements of $\mathbf{Z}[F]$ into $\mathbf{Z}|\mathbf{x} : \mathbf{r}|$, where every consequence of $\mathbf{r}$ equals 1. More importantly, $\mathfrak{a}$ then carries everything into a commutative ring. That is,

$$\mathfrak{a} \circ \gamma \circ \frac{\partial}{\partial x_j} : \mathbf{Z}[F] \to \mathbf{Z}[F] \to \mathbf{Z}|\mathbf{x} : \mathbf{r}| \to \mathbf{Z}[H],$$

where H denotes the abelianized group of $|\mathbf{x} : \mathbf{r}|$.

2.3.2 Definition. If A and A' are two matrices with entries in a commutative ring R, we define A to be *equivalent* to A', denoted $A \sim A'$, if there exists a finite sequence of matrices $A = A_1, \ldots, A_n = A'$ such that A_{i+1} is obtained from A_i, or vice-versa, by one of the following operations:

(i) Permuting rows or permuting columns.

(ii) Adjoining a row of zeros, $A \to \begin{bmatrix} A \\ \mathbf{0} \end{bmatrix}$

(iii) Adding to a row a linear combination of the other rows.

(iv) Adding to a column a linear combination of the other columns.

(v) Adjoining a new row and column such that the entry in the intersection of the new row and column is 1, and the remaining entries in the new row and column are all 0, $A \to \begin{bmatrix} A & 0 \\ \mathbf{0} & 1 \end{bmatrix}$. As noted by Fox [6],

(v') $A \to \begin{bmatrix} A & 0 \\ \mathbf{a} & 1 \end{bmatrix}$ can replace (v) since $\begin{bmatrix} A & 0 \\ \mathbf{a} & 1 \end{bmatrix}$ may be obtained from A by one application of (v) followed by n applications of (iv), where n is the number of columns of A. Also, the familiar multiplication of a row or a column by a unit $e \in R$ preserves equivalence:

$$\text{(vi)} \quad \begin{bmatrix} A \\ a \end{bmatrix} \text{(ii)} \rightarrow \begin{bmatrix} A \\ a \\ 0 \end{bmatrix} \text{(iii)} \rightarrow \begin{bmatrix} A \\ a \\ ea \end{bmatrix} \text{(i)} \rightarrow \begin{bmatrix} A \\ ea \\ a \end{bmatrix} \text{(iii)} \rightarrow \begin{bmatrix} A \\ ea \\ a - e^{-1}ea \end{bmatrix} \text{(ii)} \leftarrow \begin{bmatrix} A \\ ea \end{bmatrix}$$

$$\text{(vii)} \quad [\, A \quad a \,] \text{ (v')} \rightarrow \begin{bmatrix} A & a & 0 \\ 0 & -e^{-1} & 1 \end{bmatrix} \text{(iv)} \rightarrow \begin{bmatrix} A & a & ea \\ 0 & -e^{-1} & 0 \end{bmatrix} \text{(i)} \rightarrow \begin{bmatrix} A & ea & a \\ 0 & 0 & -e^{-1} \end{bmatrix}$$

$$\text{(iv)} \rightarrow \begin{bmatrix} A & ea & 0 \\ 0 & 0 & -e^{-1} \end{bmatrix} \text{(vi)} \rightarrow \begin{bmatrix} A & ea & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (v)} \leftarrow [\, A \quad ea \,]$$

2.3.3 Example. *Right-hand trefoil knot.* In Example 2.2.10 we found that

$$\gamma \circ \frac{\partial}{\partial x}[(x\,y\,x)\,(y\,x\,y)^{-1}] = 1 + xy - y, \quad \text{and}$$

$$\gamma \circ \frac{\partial}{\partial y}[(x\,y\,x)\,(y\,x\,y)^{-1}] = x - 1 - yx.$$

By 2.3.1 we see that the 1 x 2 Alexander matrix is $A = [a_{11}\ a_{12}]$ where $a_{ij} = \mathfrak{A}\gamma\left(\frac{\partial r_i}{\partial x_j}\right)$.

So $A = [\ \mathfrak{A}(1 + xy - y) \quad \mathfrak{A}(x - 1 - yx)\ ]$.

2.3.4 Example. *Figure-eight knot.* In Example 2.2.11 we found that

$$\gamma \circ \frac{\partial}{\partial x}[(y\,x^{-1}\,y\,x\,y^{-1})\,(x^{-1}\,y\,x\,y^{-1}\,x)^{-1}] = -yx^{-1} + yx^{-1}y + x^{-1} - x^{-1}y - x^{-1}yxy^{-1}, \text{ and}$$

$$\gamma \circ \frac{\partial}{\partial y}[(y\,x^{-1}\,y\,x\,y^{-1})\,(x^{-1}\,y\,x\,y^{-1}\,x)^{-1}] = 1 + yx^{-1} - yx^{-1}yxy^{-1} - x^{-1} + x^{-1}yxy^{-1}.$$

The Alexander matrix is $A = [\ a_{11} \quad a_{12}\ ]$ where

$$a_{11} = \mathfrak{A}(-yx^{-1} + yx^{-1}y + x^{-1} - x^{-1}y - x^{-1}yxy^{-1}), \text{ and}$$

$$a_{12} = \mathfrak{A}(1 + yx^{-1} - yx^{-1}yxy^{-1} - x^{-1} + x^{-1}yxy^{-1}).$$

2.4 THE ELEMENTARY IDEALS

The invariance of the elementary ideals is the primary focus of this section. We first give the definition of the k-th elementary ideal of an arbitrary matrix with elements in any commutative ring with 1, followed by some immediate consequences of the definition. Next, we define the k-th elementary ideal of a group presentation $(\mathbf{x} : \mathbf{r})$ in terms of the Alexander matrix of $(\mathbf{x} : \mathbf{r})$. Finally, the invariance of the elementary ideals is proved by examining the invariance of the elementary ideals under Tietze operations **I** and **II**.

2.4.1 Definition. Let R be an arbitrary commutative ring with a nonzero multiplicative identity 1, and consider an $m \times n$ matrix A with entries in R. For a non-negative integer k, we define the *k-th elementary ideal* $E_k(A)$ of A as follows:

If $0 < n - k \leq m$, then $E_k(A)$ is the ideal generated by the determinants of all $(n - k) \times (n - k)$ submatrices of A.

If $n - k > m$, then $E_k(A) = 0$.

If $n - k \leq 0$, then $E_k(A) = R$.

2.4.2 Theorem. The elementary ideals of A form an ascending chain

$$E_0(A) \subset E_1(A) \subset \ldots \subset E_n(A) \subset E_{n+1}(A) = \ldots = R.$$

2.4.3 Theorem. Equivalent matrices define the same chain of elementary ideals.

2.4.4 Theorem. Let ϕ be an arbitrary ring homomorphism $\phi: R \rightarrow R'$, where R and R' are any two commutative rings with multiplicative identities, and define for any matrix $A = [a_{ij}]$, $a_{ij} \in R$, the image matrix $\phi A = [\phi(a_{ij})]$. If ϕ is onto, then $\phi E_k(A) = E_k(\phi A)$.

2.4.5 Definition. For any finite group presentation $(\mathbf{x} : \mathbf{r})$ and non-negative integer k, we define the *k-th elementary ideal of* $(\mathbf{x} : \mathbf{r})$ to be the k-th elementary ideal of the Alexander matrix of $(\mathbf{x} : \mathbf{r})$.

2.4.6 Definition. A *mapping* $f : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ *of presentations* consists of the two presentations $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ and a homomorphism $f : F(\mathbf{x}) \to F(\mathbf{y})$ which satisfies the condition that the image $f(\mathbf{r})$ of $\mathbf{r}$ under f is contained in the consequence of $\mathbf{s}$. Every presentation mapping $f : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ induces a group homomorphism $f_* : |\mathbf{x} : \mathbf{r}| \to |\mathbf{y} : \mathbf{s}|$ satisfying $f_*\gamma = \gamma f$, where the canonical homomorphisms $F(\mathbf{x}) \to |\mathbf{x} : \mathbf{r}|$ and $F(\mathbf{y}) \to |\mathbf{y} : \mathbf{s}|$ are both denoted by γ.

$$\begin{array}{ccc} F(\mathbf{x}) & \xrightarrow{f} & F(\mathbf{y}) \\ \gamma\downarrow & & \downarrow\gamma \\ |\mathbf{x} : \mathbf{r}| & \xrightarrow{f_*} & |\mathbf{y} : \mathbf{s}| \end{array}$$

2.4.7 Definition. Presentation mappings $f_1, f_2 : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ are called *homotopic*, $f_1 \simeq f_2$, if for every $x \in \mathbf{x}$, the element $f_1(x)f_2(x^{-1})$ belongs to the consequence of $\mathbf{s}$.

2.4.8 Definition. Presentations $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ are said to be of the same *type* if there exist presentation mappings $(\mathbf{x} : \mathbf{r}) \underset{\xleftarrow{g}}{\xrightarrow{f}} (\mathbf{y} : \mathbf{s})$ such that $gf \simeq 1$ and $fg \simeq 1$. The pair of mappings f, g is called a *presentation equivalence.*

The following definition of the *Tietze equivalences* **I, I', II, II'** is given in Fox [6].

2.4.9 Definition. Let $(\mathbf{x} : \mathbf{r})$ be an arbitrary presentation, and let s be in the consequence of $\mathbf{r}$. Consider the presentation $(\mathbf{y} : \mathbf{s})$ made up of $\mathbf{y} = \mathbf{x}$ and $\mathbf{s} = \mathbf{r} \cup s$. In this case the consequence of $\mathbf{r}$ equals the consequence of $\mathbf{s}$ since s is in the consequence of $\mathbf{r}$. Hence $(\mathbf{x} : \mathbf{r})$, $(\mathbf{y} : \mathbf{s})$, and the identity automorphism $\mathbf{I} : F(\mathbf{x}) \to F(\mathbf{y})$ define a presentation mapping $\mathbf{I} : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$. Similarly, $(\mathbf{y} : \mathbf{s})$, $(\mathbf{x} : \mathbf{r})$, and the identity 1 define a presentation mapping $\mathbf{I'} : (\mathbf{y} : \mathbf{s}) \to (\mathbf{x} : \mathbf{r})$.

Again, let $(\mathbf{x} : \mathbf{r})$ be an arbitrary presentation, let y be any member of the underlying set of generators that is not contained in $\mathbf{x}$, and let z be any element of $F(\mathbf{x})$. Consider the presentation $(\mathbf{y} : \mathbf{s})$ made up of $\mathbf{y} = \mathbf{x} \cup y$ and $\mathbf{s} = \mathbf{r} \cup yz^{-1}$. The homomorphism $\mathbf{II} : F(\mathbf{x}) \to F(\mathbf{y})$, defined by the rule $\mathbf{II}(x) = x$ for any $x \in \mathbf{x}$, maps $\mathbf{r}$ into the consequence of $\mathbf{s}$ so that $(\mathbf{x} : \mathbf{r})$, $(\mathbf{y} : \mathbf{s})$, and $\mathbf{II} : F(\mathbf{x}) \to F(\mathbf{y})$ define a presentation map $\mathbf{II} : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$. Also, the homomorphism $\mathbf{II'} : F(\mathbf{y}) \to F(\mathbf{x})$ defined by the rule $\mathbf{II'}(x) = x$ for any $x \in \mathbf{x}$ and $\mathbf{II'}(y) = z$ maps $\mathbf{s}$ onto $\mathbf{r} \cup 1$ and hence into the consequence of $\mathbf{r}$. It follows that $(\mathbf{y} : \mathbf{s})$, $(\mathbf{x} : \mathbf{r})$, and $\mathbf{II'} : F(\mathbf{y}) \to F(\mathbf{x})$ define a presentation map $\mathbf{II'} : (\mathbf{y} : \mathbf{s}) \to (\mathbf{x} : \mathbf{r})$. The composition $\mathbf{II'II}$ is the identity. Also, for every $x \in \mathbf{x}$, $\mathbf{II}\,\mathbf{II'}(x)x^{-1} = 1$ and $\mathbf{II}\,\mathbf{II'}(y)y^{-1} = \mathbf{II}(z)y^{-1} = zy^{-1} = (yz^{-1})^{-1}$ which belongs to the consequence of $\mathbf{s}$, so that $\mathbf{II}\,\mathbf{II'} \simeq 1$. Thus the pair $\mathbf{II}, \mathbf{II'}$ is a presentation equivalence. The presentation equivalences $\mathbf{I}, \mathbf{I'}, \mathbf{II}, \mathbf{II'}$ defined above are called the *Tietze equivalences*.

We shall require the following classic result by Tietze, a proof of which is given in Fox [6].

2.4.10 Theorem. (Tietze) Suppose that $(\mathbf{x} : \mathbf{r}) \underset{\leftarrow g}{\overset{f \rightarrow}{\rightleftarrows}} (\mathbf{y} : \mathbf{s})$ is a presentation equivalence and that the presentations $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ are both finite. Then there exists a finite sequence $T_1, T_1'; \ldots; T_k, T_k'$ of Tietze equivalences such that $f = T_1 \ldots T_k$ and $g = T_k' \ldots T_1'$.

The following lemma due to Fox [6].

2.4.11 Lemma. If the pair f, g is a presentation equivalence, then each of f_{**} and g_{**} is an isomorphism and the inverse of the other.

Proof: If $f : (\mathbf{x} : \mathbf{r}) \rightarrow (\mathbf{y} : \mathbf{s})$ is an arbitrary presentation mapping, there is induced a homomorphism $f_*: |\mathbf{x} : \mathbf{r}| \rightarrow |\mathbf{y} : \mathbf{s}|$ on the groups of the presentations. This mapping induces a homomorphism f_{**} from the abelianization of $|\mathbf{x} : \mathbf{r}|$ into that of $|\mathbf{y} : \mathbf{s}|$. If $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ are known to be of the same type, there exists a presentation equivalence $(\mathbf{x} : \mathbf{r}) \xrightarrow{f} (\mathbf{y} : \mathbf{s}) \xrightarrow{g} (\mathbf{x} : \mathbf{r})$ and

$$f_*g_* = (fg)_* = \text{identity},$$

$$f_{**}g_{**} = (f_*g_*)_* = \text{identity}.$$

Similarly, $g_{**}f_{**}$ is the identity. q.e.d.

2.4.12 Theorem. (The invariance of the elementary ideals) If $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ are finite group presentations and

$$f : (\mathbf{x} : \mathbf{r}) \rightarrow (\mathbf{y} : \mathbf{s})$$

is a presentation equivalence, then the k-th elementary ideal of $(\mathbf{x} : \mathbf{r})$ is mapped by f_{**} onto the k-th elementary ideal of $(\mathbf{y} : \mathbf{s})$, where f_{**} is the extension of the above f_{**} to the group ring.

Proof: As a result of Theorem 2.4.10, the proof reduces to checking only the invariance of the elementary ideals under the Tietze equivalences **I, I', II, II'**. Since $\mathbf{I'}\,\mathbf{I} \simeq \mathbf{I}\,\mathbf{I'} \simeq 1$ and

$\mathbf{II'II} \simeq \mathbf{II\,II'} \simeq 1$, by Lemma 2.4.11 it suffices to check only Tietze equivalences **I** and **II**.

Tietze **I**. Let $(\mathbf{x} : \mathbf{r}) = (x_1, \ldots, x_n : r_1, \ldots, r_m)$ be any group presentation. Let s be in the consequence of **r**. A Tietze equivalence of type **I**, is a mapping $\mathbf{I}: (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$, where $\mathbf{s} = \mathbf{r} \cup s$ and $\mathbf{y} = \mathbf{x}$. Since s is in the consequence of **r**, and $\mathbf{I}: F(\mathbf{x}) \to F(\mathbf{y})$ is the identity mapping, the induced homomorphisms $\mathbf{I}_*$ and $\mathbf{I}_{**}$ are also identities. Thus, by Theorem 2.4.3, we need only to check that $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ have equivalent Alexander matrices. Consider the Alexander matrix of each of these presentations. If A denotes the Alexander matrix of $(\mathbf{x} : \mathbf{r})$, and if A' denotes the Alexander matrix of $(\mathbf{y} : \mathbf{s})$ then A is the $m \times n$ matrix $A = \left[\mathfrak{a}\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right]$ and A' is the $(m + 1) \times n$ matrix $A' = \left[\mathfrak{a}\gamma\left(\frac{\partial s_i}{\partial y_j}\right)\right]$. Since $(\mathbf{y} : \mathbf{s})$ has the same generators as $(\mathbf{x} : \mathbf{r})$, A' has the same number of columns as A and A' will have identical rows with the exception of the additional row of A' corresponding to the relator s. However, since s is in the consequence of **r**, the additional row will only amount to a linear combination of the rows of A. This fact can easily be seen by writing the new relator s as a product of conjugates of powers of the relators r_i, $i = 1, \ldots, n$, and calculating the $(m + 1)$-st row of the matrix A'. For $j = 1, \ldots, n$, let

$$s = \prod_{k=1}^{p} h_k r_{\beta_k}^{\alpha_k} h_k^{-1}, \text{ where } h_k \in F(\mathbf{x}) \text{ and } \alpha_k, \beta_k \in \mathbf{Z}.$$

Then,

$$\frac{\partial s}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\prod_{k=1}^{p} h_k r_{\beta_k}^{\alpha_k} h_k^{-1}\right]$$

$$= \frac{\partial}{\partial x_j}\left[h_1 r_{\beta_1}^{\alpha_1} h_1^{-1}\right] + h_1 r_{\beta_1}^{\alpha_1} h_1^{-1} \frac{\partial}{\partial x_j}\left[h_2 r_{\beta_2}^{\alpha_2} h_2^{-1}\right]$$

$$+ \ldots + \prod_{k=1}^{p-1} h_k r_{\beta_k}^{\alpha_k} h_k^{-1} \frac{\partial}{\partial x_j}\left[h_p r_{\beta_p}^{\alpha_p} h_p^{-1}\right].$$

Since $\gamma(r_i) = 1$, we have

$$\gamma\left[\frac{\partial s}{\partial x_j}\right] = \sum_{k=1}^{p} \gamma\left[\frac{\partial}{\partial x_j}\left(h_k r_{\beta_k}^{\alpha_k} h_k^{-1}\right)\right].$$

Now, by 2.2.5(iii), 2.2.6(iii), and 2.2.8,

$$\frac{\partial}{\partial x_j}\left[h_k r_{\beta_k}^{\alpha_k} h_k^{-1}\right] = \frac{\partial h_k}{\partial x_j} + h_k \frac{\partial}{\partial x_j}\left[r_{\beta_k}^{\alpha_k} h_k^{-1}\right]$$

$$= \frac{\partial h_k}{\partial x_j} + h_k \frac{\partial}{\partial x_j}\left[r_{\beta_k}^{\alpha_k}\right] - h_k r_{\beta_k}^{\alpha_k} h_k^{-1} \frac{\partial h_k}{\partial x_j}$$

$$= \frac{\partial h_k}{\partial x_j} + h_k \frac{r_{\beta_k}^{\alpha_k} - 1}{r_{\beta_k} - 1} \frac{\partial}{\partial x_j}\left[r_{\beta_k}\right] - h_k r_{\beta_k}^{\alpha_k} h_k^{-1} \frac{\partial h_k}{\partial x_j}.$$

By Definition 2.2.7,

$$\gamma\left[\frac{r_{\beta_k}^{\alpha_k} - 1}{r_{\beta_k} - 1}\right] = \gamma\left[\sum_{i=0}^{\alpha_k - 1} r_{\beta_k}^{i}\right] = \sum_{i=0}^{\alpha_k - 1} \gamma(r_{\beta_k}^{i}) = \sum_{i=0}^{\alpha_k - 1} 1^i = \alpha_k.$$

Hence,

$$\gamma\left[\frac{\partial}{\partial x_j}\left(h_k r_{\beta_k}^{\alpha_k} h_k^{-1}\right)\right] = \gamma\left[\frac{\partial h_k}{\partial x_j}\right] + \gamma(h_k)\,\alpha_k\,\gamma\left[\frac{\partial r_{\beta_k}}{\partial x_j}\right] - \gamma(h_k)\,\gamma(r_{\beta_k}^{\alpha_k})\,\gamma(h_k^{-1})\,\gamma\left[\frac{\partial h_k}{\partial x_j}\right]$$

$$= \alpha_k\,\gamma(h_k)\,\gamma\left[\frac{\partial r_{\beta_k}}{\partial x_j}\right].$$

Setting $\alpha_k \gamma(h_k) = c_k$, we obtain

$$\mathfrak{a}\gamma\left[\frac{\partial s}{\partial x_j}\right] = \sum_{k=1}^{p} c_k \mathfrak{a}\gamma\left[\frac{\partial r_{\beta_k}}{\partial x_j}\right].$$

Thus, the Alexander matrix of $(\mathbf{y} : \mathbf{s})$ is the same as the Alexander matrix of $(\mathbf{x} : \mathbf{r})$ except for having an additional row which is a linear combination of the other rows, and so the two matrices are equivalent.

Tietze II. Consider the presentation $(\mathbf{y} : \mathbf{s})$ where $\mathbf{y} = \mathbf{x} \cup y$ and $\mathbf{s} = \mathbf{r} \cup y z^{-1}$ for any $z \in F(\mathbf{x})$ and where y is any member of the underlying set of generators that is not contained in $\mathbf{x}$. By Tietze, $(\mathbf{x} : \mathbf{r})$, $(\mathbf{y} : \mathbf{s})$ and the homomorphism $\mathbf{\Pi}$: $F(\mathbf{x}) \to F(\mathbf{y})$ given by $\Pi(x) = x$ for every $x \in \mathbf{x}$, define a presentation map $\mathbf{\Pi}$: $(\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ where Π is the inclusion homomorphism. Setting $G = |\mathbf{x} : \mathbf{r}|$ and $G' = |\mathbf{y} : \mathbf{s}|$ and H and H' equal to the abelianizations of G and G' respectively, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{Z}F(\mathbf{x}) & \xrightarrow{\Pi} & \mathbf{Z}F(\mathbf{y}) \\
\gamma\downarrow & & \downarrow\gamma' \\
\mathbf{Z}G & \xrightarrow{\Pi_*} & \mathbf{Z}G' \\
\mathfrak{a}\downarrow & & \downarrow\mathfrak{a}' \\
\mathbf{Z}H & \xrightarrow{\Pi_{**}} & \mathbf{Z}H'
\end{array}$$

where $\gamma'\mathbf{\Pi} = \mathbf{\Pi}_*\gamma$, and $\mathfrak{a}'\mathbf{\Pi}_* = \mathbf{\Pi}_{**}\mathfrak{a}$. Let A denote the Alexander matrix of $(\mathbf{x} : \mathbf{r})$ and let A' denote the Alexander matrix of $(\mathbf{y} : \mathbf{s})$. Then

$$A = \left[\mathfrak{a}\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right], \qquad i = 1, \ldots, m, j = 1, \ldots, n.$$

The group presentation $(\mathbf{y} : \mathbf{s})$ contains a new generator y and relator $y z^{-1}$ for some $z \in F(x)$, so that

$$\frac{\partial r_i}{\partial y} = 0, \qquad i = 1, \ldots, m,$$

and

$$\frac{\partial (y z^{-1})}{\partial y} = \frac{\partial y}{\partial y} + y \frac{\partial z^{-1}}{\partial y} = 1.$$

Hence,

$$A' = \begin{bmatrix} \mathbf{II}_{**} A & 0 \\ * & 1 \end{bmatrix},$$

where $*$ denotes the row of elements $\mathfrak{a}'\gamma'\left(\frac{\partial}{\partial x_j} yz^{-1}\right)$, $j = 1, \ldots, n$. By definition 2.3.2(v'), A' is equivalent to $\mathbf{II}_{**}A$. Thus, by theorems 2.4.3 and 2.4.4 we have

$$E_k(A') = E_k(\mathbf{II}_{**} A) = \mathbf{II}_{**} E_k(A). \qquad \text{q.e.d.}$$

2.5 THE ALEXANDER POLYNOMIAL

In this section we define a sequence of knot polynomials, one of which is the *Alexander polynomial* of a knot.

Recall that for a finite presentation $(\mathbf{x} : \mathbf{r})$ of a group G, the Alexander matrix associated with this presentation is the matrix $[a_{ij}]$ defined by $a_{ij} = \mathfrak{a}\gamma\left(\frac{\partial r_i}{\partial x_j}\right)$. We will see that the *Alexander polynomial* of a knot is defined in terms of the Alexander matrix of the presentation $(\mathbf{x} : \mathbf{r})$ of the knot group and consequently, the knot polynomial depends on the abelianization of the knot group. Our next result, due to Fox [6], examines the structure of the abelianized group of the knot group.

We require the following preliminary algebraic lemma.

2.5.1 Lemma. Any homomorphism of a group into an abelian group can be factored through the commutator subgroup.

2.5.2 Theorem. The abelianization of any knot group is infinite cyclic.
Proof: Let G be a knot group and $(x_1, \ldots, x_n : r_1, \ldots, r_n)_\phi$ an over presentation of G. Consider a typical region of the knot as shown in Figure 2.5.1. This region of the knot shows an underpass followed by a null overpass followed by an underpass.

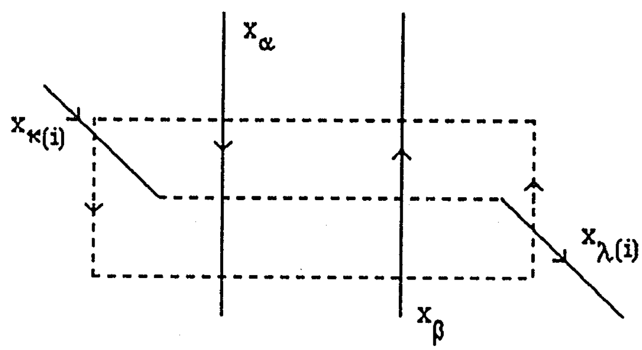


Figure 2.5.1

Let $x_{\kappa(i)}$ and $x_{\lambda(i)}$ be the generators corresponding to the two overpasses adjacent to the underpass and assume that, with respect to the orientation of K, the generator $x_{\kappa(i)}$ precedes the generator $x_{\lambda(i)}$. Calculating the relator r_i by reading around the underpass, we find that $r_i = x_{\kappa(i)} x_\alpha^{-1} x_\beta x_{\lambda(i)}^{-1} x_\beta^{-1} x_\alpha$. If θ is any homomorphism of G into an abelian group,

$$\begin{aligned} 1 = \theta(\phi r_i) &= \theta[\phi x_{\kappa(i)}\, \phi x_\alpha^{-1}\, \phi x_\beta\, \phi x_{\lambda(i)}^{-1}\, \phi x_\beta^{-1}\, \phi x_\alpha] \\ &= \theta(\phi x_{\kappa(i)})\, \theta(\phi x_\alpha^{-1})\, \theta(\phi x_\beta)\, \theta(\phi x_{\lambda(i)}^{-1})\, \theta(\phi x_\beta^{-1})\, \theta(\phi x_\alpha) \\ &= \theta(\phi x_{\kappa(i)})\, \theta(\phi x_{\lambda(i)})^{-1}. \end{aligned}$$

Since the projection of any knot is connected, for every pair of generators x_i, x_j we then have

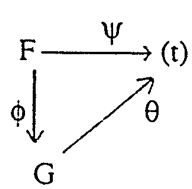
$$\theta(\phi x_i) = \theta(\phi x_j).$$

Thus any element of the image group $\theta(G)$ is a power of the single element $t = \theta(\phi x_j)$, $j = 1, \ldots, n$. In particular, since the abelianizer $\mathfrak{a} : G \to G/[G,G]$ takes G onto an abelian group, $G/[G,G]$ is cyclic.

To see that $G/[G,G]$ is infinite, let (t) denote the infinite cyclic group generated by t and let $\phi: F(x) \to \pi_1(R^3 - K, p_0)$ given by $\phi x_j = [{}^*a_j]$ be an over presentation of G. Since $F(x)$ is a free group, the assignment $\psi x_j = t, j = 1, \ldots, n$, can be extended to a homomorphism of F onto (t). Define $\theta: G \to (t)$ by $\theta(\phi u) = \psi u$, for all $u \in F(x)$. Notice that $\psi r_i = t^{s_i}$, $i = 1, \ldots, n$, where $s_i = \sum_{j=1}^{n} t\left(\frac{\partial r_i}{\partial x_j}\right) = \sum_{j=1}^{n}(\delta_{j,\kappa(i)} - \delta_{j,\lambda(i)})$. Thus, $\psi r_i = t^0 = 1$ for $i = 1, \ldots, n$, and the consequence of $r_1, \ldots, r_n$, which is in the kernel of ϕ, is contained in the kernel of ψ. Since $\ker\phi \subset \ker\psi$, θ is well-defined. For if for some $x \in G$, $x = Ru = Ru'$, where $u, u' \in F(x)$ and R denotes the consequence of $r_1, \ldots, r_n$, then $u = ru'$ for some $r \in R$. Hence,

$$\psi u = \psi ru' = \psi r \psi u' = 1\, \psi u' = \psi u'.$$

Also, since ψ is onto, so is θ, and the following diagram commutes:



Next, consider the abelianizer $\mathfrak{a} : G \to G/[G,G]$. By Lemma 2.5.1, there exists a homomorphism θ': $G/[G,G] \to (t)$ such the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\theta} & (t) \\ \mathfrak{a}\downarrow & \nearrow \theta' & \\ G/[G,G] & & \end{array}$$

Since θ is onto so is θ'. Since a function whose image is infinite cannot have a finite domain, $G/[G,G]$ must be infinite and the result is proved. q.e.d.

In view of Theorem 2.5.2, the image of any generator x_j, $j = 1, \ldots, n$, under the composition of maps $\mathfrak{a} \circ \gamma\colon Z[F] \to Z|x : r| \to Z[H]$, where H denotes the abelianization of $|x : r|$, can be set equal to t. That is, $\mathfrak{a}\gamma x_j = t$ for every generator x_j, $j = 1, \ldots, n$. Hence, the elements of the Alexander matrix are elements of the group ring of an infinite cyclic group with generator t. The elements of such a group ring are finite formal sums with possible negative exponents and thus are Laurent polynomials in t.

2.5.3 Definition. An element d of a commutative ring R is called a *greatest common divisor*, abbreviated g.c.d., of a finite set of elements $a_1, \ldots, a_n \in R$ if $d|a_i$, $i = 1, \ldots, n$, and, for any $e \in R$, if $e|a_i$, $i = 1, \ldots, n$, then $e|d$.

2.5.4 Definition. A ring R is called a *g.c.d. domain* if it is an integral domain and every finite set of elements has a g.c.d.

2.5.5 Theorem. The group ring of an infinite cyclic group is a g.c.d. domain.

2.5.6 Theorem. The group ring of an infinite cyclic group has only trivial units, i.e., the powers of a generator t and their negatives.

2.5.7 Definition. Let R be a commutative ring with multiplicative identity 1. The *ideal* E *generated by a subset* S is the set of all finite sums $\sum_i a_i b_i$ where $a_i \in S$, $b_i \in R$. An ideal is called a *principal ideal* if it is generated by a single element.

2.5.8 Theorem. In a g.c.d. domain with identity, the g.c.d. of any finite set of elements is the generator of the smallest principal ideal that contains them.

With the above preparation complete, we are finally in a position to define a sequence of knot polynomials of the knot group, one of which is the *Alexander polynomial.*

2.5.9 Definition. For any integer $k \geq 0$, the *k-th knot polynomial* Δ_k of a finite presentation $(\mathbf{x} : \mathbf{r}) = (x_1, \ldots, x_n : r_1, \ldots, r_m)$ of a knot group is the g.c.d. of the determinants of all $(n-k) \times (n-k)$ submatrices of the Alexander matrix of $(\mathbf{x} : \mathbf{r})$. Moreover,

$$\Delta_k = 0 \quad \text{if} \quad n - k > m,$$

$$\Delta_k = 1 \quad \text{if} \quad n - k \leq 0.$$

The first knot polynomial Δ_1 is called the *Alexander polynomial* of the knot group and is usually written without the subscript.

2.5.10 Definition. We shall say that a knot polynomial Δ_k has been *normalized* if Δ_k has no negative powers of t and has a positive constant term.

By their definitions, the knot polynomials are Laurent polynomials in t. Note that a representation of an element of the group ring of an infinite cyclic group has two representations as a Laurent polynomial, depending on which of the two generators of the group is set equal to t.

2.5.11 Example. *Right-hand trefoil knot.* In 2.3.3 we found the Alexander matrix for the right-hand trefoil knot to be

$$A = [\ \mathfrak{a}(1 + xy - y) \quad \mathfrak{a}(x - 1 - yx)\ ].$$

Now, setting $\mathfrak{a}\gamma x = \mathfrak{a}\gamma y = t$ we have $A = [\ t^2 - t + 1 \quad -t^2 + t - 1\ ]$. By Definition 2.5.9, the Alexander polynomial is the g.c.d. of the determinants of all 1 x 1 submatrices of the matrix A, so we have

$$\Delta_1 = t^2 - t + 1, \text{ and}$$

$$\Delta_k = 1, \text{ for } k \geq 2.$$

2.5.12 Example. *Figure-eight knot.* In Example 2.3.4, we found that the Alexander matrix for the figure-eight knot is $A = [\ a_{11} \quad a_{12}\ ]$ where

$$a_{11} = \mathfrak{a}(-yx^{-1} + yx^{-1}y + x^{-1} - x^{-1}y - x^{-1}yxy^{-1}), \text{ and}$$

$$a_{12} = \mathfrak{a}(1 + yx^{-1} - yx^{-1}yxy^{-1} - x^{-1} + x^{-1}yxy^{-1}).$$

Setting $\mathfrak{a}\gamma x = \mathfrak{a}\gamma y = t$, we have

$$A = [\ -3 + t + t^{-1} \quad 3 - t - t^{-1}\ ].$$

Again, the Alexander polynomial is the g.c.d. of all 1 x 1 submatrices of A, or

$$\Delta_1 = -3 + t + t^{-1} \text{ and}$$

$$\Delta_k = 1, \text{ for } k \geq 2.$$

Upon normalizing we have

$$\Delta_1 = t^2 - 3t + 1.$$

2.5.13 Theorem. The knot polynomials are unique to within $\pm t^n$, where $n \in \mathbf{Z}$ and t is the generator of the infinite cyclic abelianization of the presentation $(\mathbf{x} : \mathbf{r})$ of the knot group.

Together with the definitions of the knot polynomials Δ_k and the elementary ideals E_k, Theorem 2.5.8 gives us the following theorem.

2.5.14 Theorem. Each knot polynomial Δ_k is the generator of the smallest principal ideal containing the elementary ideal E_k.

Since equivalent matrices have the same elementary ideals, it is a corollary of Theorem 2.5.14 that the knot polynomials of a presentation can be calculated from any matrix equivalent to the Alexander matrix.

2.5.15 Theorem. $\Delta_{k+1} | \Delta_k$.

Proof: Let (Δ_k) and (Δ_{k+1}) denote the principal ideals generated by Δ_k and Δ_{k+1} respectively. Then since the elementary ideals form an ascending chain,

$$E_k \subset E_{k+1} \subset (\Delta_{k+1}).$$

By Theorem 2.5.14, (Δ_k) is the smallest principal ideal containing E_k,

$$(\Delta_k) \subset (\Delta_{k+1}).$$

Thus, $\Delta_k = a\Delta_{k+1}$ for $a \in Z[G]$, where G is the abelianization of $(\mathbf{x} : \mathbf{r})$, and so $\Delta_{k+1} | \Delta_k$. q.e.d.

The next result which is of fundamental importance implies that the knot polynomials are invariants of knot type.

2.5.16 Theorem. (The invariance of the knot polynomials.)
Let f_{**} be the induced linear extension to the group ring of an isomorphism of the abelianization of $|\mathbf{x} : \mathbf{r}|$ onto that of $|\mathbf{y} : \mathbf{s}|$. If $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ are finite presentations of knot groups and $f: (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ is a presentation equivalence, then, to within units, the k-th knot polynomial Δ_k of $(\mathbf{x} : \mathbf{r})$ is mapped by f_{**} onto the k-th knot polynomial Δ_k' of $(\mathbf{y} : \mathbf{s})$.

Proof: Let (Δ_k) and (Δ_k') denote the principal ideals generated by Δ_k and Δ_k', respectively, and let E_k and E_k' denote the elementary ideals of $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$, respectively. Then

$$E_k \subset (\Delta_k),\ E_k' \subset (\Delta_k'), \text{ and } f_{**}(E_k) = E_k'$$

by 2.5.14 and the invariance of the elementary ideals. It is a known algebraic fact that the isomorphic image of a principal ideal is principal, and

$$E_k' = f_{**}(E_k) \subset (f_{**}(\Delta_k)).$$

Since (Δ_k') is the smallest principal ideal containing E_k',

$$(\Delta_k') \subset (f_{**}(\Delta_k)).$$

By the same argument,

$$E_k = f_{**}^{-1}(E_k') \subset (f_{**}^{-1}(\Delta_k')), \text{ and since } (\Delta_k) \text{ is minimal,}$$

$$(\Delta_k) \subset (f_{**}^{-1}(\Delta_k')).$$

Thus,

$$(f_{**}(\Delta_k)) = (\Delta_k').$$

Since $(f_{**}(\Delta_k))$ is generated by $f_{**}(\Delta_k)$, and $(f_{**}(\Delta_k)) = (\Delta_k')$ we see that $f_{**}(\Delta_k)$ and Δ_k' generate the same principal ideal. Thus for some unit $u \in Z[G]$, where G denotes the abelianization of $(\mathbf{x} : \mathbf{r})$,

$$f_{**}(\Delta_k) = u\Delta_k'.$$

Hence, to within units, the k-th knot polynomial Δ_k of $(\mathbf{x} : \mathbf{r})$ is mapped by f_{**} onto the k-th knot polynomial Δ_k' of $(\mathbf{y} : \mathbf{s})$. q.e.d.

A significant computational aid is given in the following theorem.

2.5.17 Theorem. If a finite presentation of a knot group satisfies

$$\mathfrak{a}\gamma x_i = \mathfrak{a}\gamma x_j, \quad i, j = 1, \ldots, n,$$

then the Alexander matrix A is equivalent to the matrix obtained by replacing any column of A with a column of zeros.

(Note that although the hypothesis of this theorem may seem strong, the condition is satisfied by any over presentation.)

CHAPTER 3
THE JONES POLYNOMIAL

In 1985, V. F. R. Jones [11] announced the discovery of a new polynomial invariant for knots, which led in part to his being awarded the Fields Medal in 1990. Jones used a representation of the braid group to the group of units of a Hecke Algebra on which he defined a trace function. This trace function resulted in a (Laurent) polynomial invariant for knots. After a brief description of braids and representations of the braid group, we shall examine in this chapter the Jones polynomial and calculate the Jones polynomial for the right-hand trefoil knot.

3.1 THE BRAID GROUP

We shall first define what is meant by an n-braid and show that, modulo a certain equivalence relation, we can define a group structure on the set of all n-braids. The resulting group is the *n-th braid group*.

3.1.1 Definition. A (tame) *n-braid* or a *braid on n strings* is any structure in $\mathbf{R}^3$ given by the following data:

(i) let $P_1, \ldots, P_n$ be the n points in $\mathbf{R}^3$ given by $P_i = (i,0,1)$;

(ii) let $Q_1, \ldots, Q_n$ be the n points in $\mathbf{R}^3$ given by $Q_i = (i,0,0)$;

(iii) for every $i = 1, \ldots, n$, let there be a polygonal path joining P_i to $Q_{\sigma(i)}$, where σ is a permutation of $\{1, \ldots, n\}$, such that along the path from P_i to $Q_{\sigma(i)}$ the z-coordinate strictly decreases and such that no two of the distinct paths intersect.

The reader is encouraged to visualize the abstract notion of an n-braid – as in Figure 3.1.1 showing the common 3-braid used to braid hair – and we shall adopt the convention that our viewpoint is from the negative y-axis.

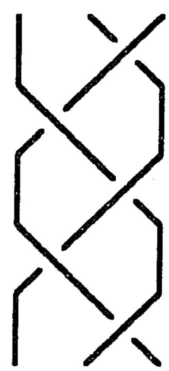


Figure 3.1.1

3.1.2 Definition. For any n-braid, the polygonal path joining P_i to $Q_{\sigma(i)}$ is called the *i-th string* and the permutation σ is called the *permutation* of the braid. The straight line segments of any string are called *edges* and the points where the straight line segments meet are called *vertices*.

3.1.3 Definition. Two n-braids are called *equivalent* or *string isotopic* if and only if there exists an isotopic deformation of $\mathbf{R}^3$ fixing the points P_i and Q_i, $i = 1, \ldots, n$, which takes one n-braid onto the other n-braid and which satisfies the condition (iii) of Definition 3.1.1 throughout the deformation.

3.1.4 Definition. For $1 \le i \le n-1$, we define the *elementary n-braid* , denoted by σ_i, to be that braid in which the i-th string crosses over the (i + 1)-th string (when viewed from the negative y - axis) exactly once and in which all other strings go directly from top to bottom. (See Figure 3.1.2.)

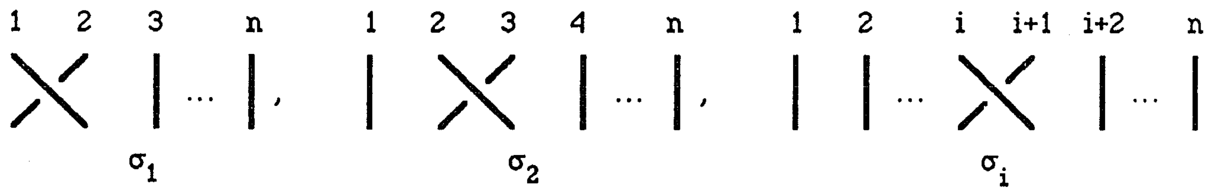


Figure 3.1.2.

Braids, like knots, are represented by their projections $P\beta$, $\beta \in B_n$, where $P: \mathbf{R}^3 \to \mathbf{R}^3$ is given by $P(x,y,z) = (x,0,z)$. A point p in the image of $P\beta$ is said to be a *multiple point* if the inverse image $P^{-1}(p) \cap \beta$ consists of more than one point. The *order* of $p \in PK$ is the cardinality of $P^{-1}(p) \cap \beta$. A multiple point of order two is called a *double point.*

3.1.5 Definition. An n-braid β is said to be in *regular position* if:

(i) the only multiple points of β are double points;

(ii) no double point is the image of a vertex of β.

Each double point of the projected image of an n-braid in regular position is the image of two points of the braid. The point with the larger y-coordinate is called an *undercrossing*, and the other is called an *overcrossing*. Standard arguments show that any n-braid is equivalent to an n-braid in regular position and thus we will assume any n-braid to be in regular position. Also, we will assume that we get transversal crossings of the strings if we project the braid orthogonally onto the xz-plane and we will indicate these over and under crossings in our diagrams. Also note that (up to equivalence) we may assume that the crossings of the strings occur at different levels with respect to the z-axis. By isolating these crossings, any n-braid may be resolved into a concatenation of elementary n-braids. For example, the common hair braid of Figure 3.1.1 can be resolved into the concatenation $\sigma_2^{-1}\,\sigma_1\,\sigma_2^{-1}\,\sigma_1\,\sigma_2^{-1}$ of elementary n-braids, as shown in Figure 3.1.3.

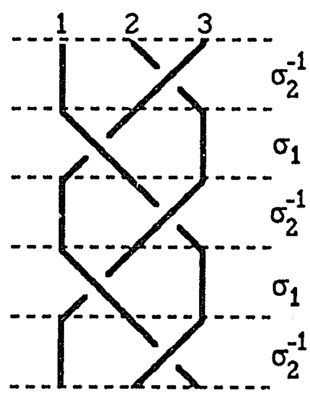


Figure 3.1.3

This leads to the following definition:

3.1.6 Definition. Let B_n denote the set of all equivalence classes of n-braids. Let β_1 and β_2 be elements of B_n. The *product* of β_1 and β_2 is defined to be the vertical concatenation of β_1 and β_2 formed by glueing the n points Q_i^1 of β_1 to the n points P_i^2 of β_2 followed by a vertical deformation so that the product lies between the planes $z = 1$ and $z = 0$.

3.1.7 Definition. The *trivial braid,* ι, is the n-braid in which all strings go straight down from the plane $z = 1$ to the plane $z = 0$.

3.1.8 Definition. The *inverse braid*, β^{-1}, of the braid β is defined as the mirror image of β with respect to the plane $z = \frac{1}{2}$. For the elementary n-braid σ_i, $1 \leq i \leq n-1$, the inverse braid σ_i^{-1} is obtained by changing the overcrossing of the (i + 1)-th string by the i-th string to an undercrossing. The projection of the common hair braid, its inverse and their product are shown in Figure 3.1.4.

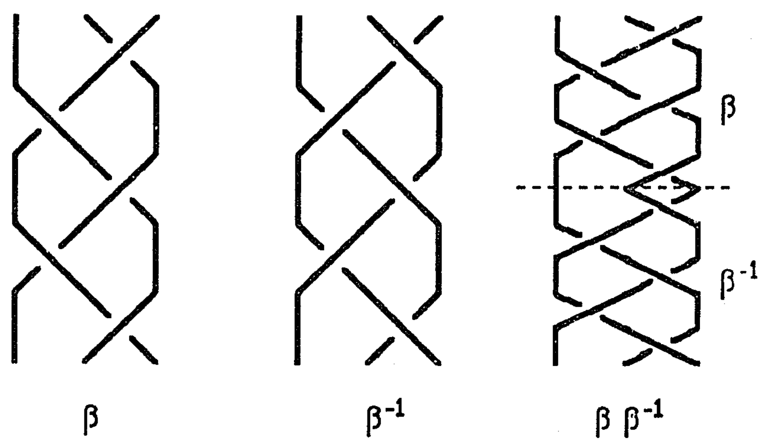


Figure 3.1.4

Figure 3.1.5 shows an isotopic deformation of the product $\beta\beta^{-1}$ to the identity braid ι.

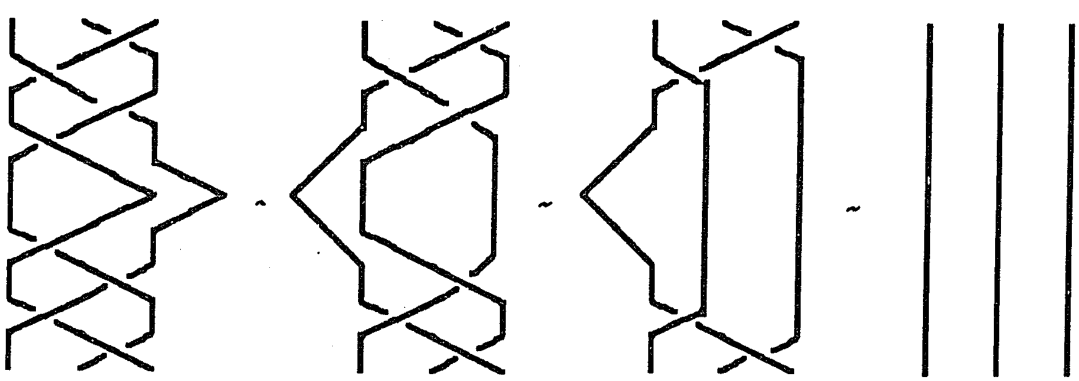


Figure 3.1.5

With the above defined product, identity element ι, and inverse elements, the set B_n of equivalence classes of geometric n-braids forms a group called the *Artin braid group* of braids on n strings. In his paper on braid groups, Artin [2] proved that the group B_n admits a presentation with generators: $\sigma_1, \ldots, \sigma_{n-1}$, and defining relations:

$$\text{(i)}\quad \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} \text{ for } 1 \leq i \leq n-2$$

$$\text{(ii)}\quad \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i \text{ for } |\, i - j \,| \geq 2,\ 1 \leq i, j \leq n-1.$$

Note that (ii) is readily visualized, since if $|\, i - j \,| \geq 2$, then the strings involved in each of the factors of the products $\sigma_i \cdot \sigma_j$ and $\sigma_j \cdot \sigma_i$, are isolated with respect to each other and so the products are easily seen to be equivalent. The reader may also note the similarity between the generators and relations given above for B_n and those for the symmetric group on n letters, S_n. This similarity can be seen by recalling that S_n is generated by the $n-1$ transpositions $\tau_1 = (12), \tau_2 = (23), \ldots, \tau_{n-1} = (n-1\ \ n)$, and that these generators satisfy (i), (ii), and the additional relation $\tau_i^2 = 1, i = 1, \ldots, n-1$. In fact, S_n is the quotient of B_n by the subgroup generated by the relators σ_i^2. Moreover, the canonical map $\pi: B_n \rightarrow S_n$ actually associates to each braid the permutation of the braid, σ, as defined in 3.1.2.

3.2 THE BRAID CORRESPONDING TO A LINK

After a few preliminary definitions, we shall examine how an n-braid can be derived from a given link and how a link can be derived from a given n-braid.

3.2.1 Definition. Given an n-braid $\beta \in B_n$, let l be an axis parallel to the x-axis placed behind β so that l has a greater y-coordinate than any point on any string of β and so that l has a z-coordinate of 1/2. The *closure* of β, denoted by $\hat{\beta}$, is the unoriented link formed when the braid β is closed around the axis l by identifying the points P_i with the points Q_i, for $i = 1, \ldots, n$. The link formed by closing an n-braid in this manner is called the *corresponding link* and is denoted by $L(\beta)$.

3.2.2 Example. In this example we examine the closure of the 3-braid β used to braid hair. To visualize the closure of a particular n-braid β, draw the plane projection of β horizontally as in Figure 3.2.1. Next, wrap the projection of β around a cylinder which has been placed behind the n-braid and join P_i to Q_i, $i = 1, \ldots, n$, by a path on the cylinder along the n-th circular section (see Figure 3.2.2). Note that this is equivalent to Definition 3.2.1. Finally, remove the cylinder and simplify the resulting link projection of $L(\beta)$ by isotopic deformation as in Figure 3.2.3.

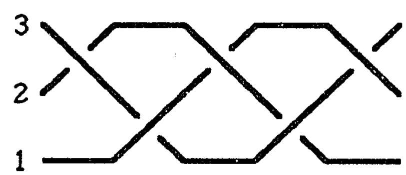


Figure 3.2.1

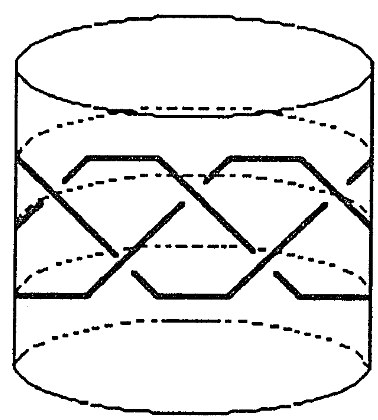


Figure 3.2.2

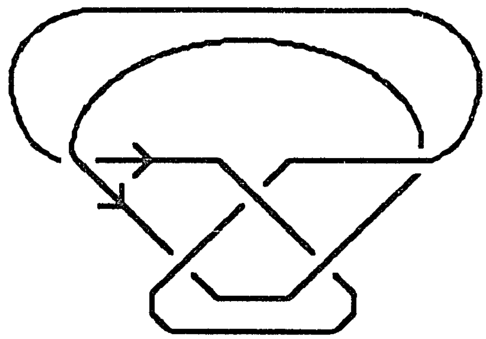


Figure 3.2.3

In the study of links, we wish to use braids to gain information about a particular link. Thus, our interest here is to describe the process that reverses the closure operation; that is, we want to obtain from a link K a corresponding n-braid β so that L(β) is string isotopic to K. The following description of this process is due to Hansen [9].

3.2.3 Definition. Let K be a link in $\mathbf{R}^3$ and let l be an arbitrary but fixed line which does not intersect the link K. The line l is called the *axis* for K.

3.2.4 Definition. A link K is said to be in *general position* with respect to the axis l if none of its edges are coplanar with l.

3.2.5 Notation. We shall use the notation [a b] to denote an edge of a link with vertices a and b.

3.2.6 Definition. Let K be a link in $\mathbf{R}^3$ which is in general position with respect to an axis l. Suppose K is oriented and fix an orientation of l. An edge [a b] on K is said to be *positive* when the half-plane P determined by l and a point on the edge [a b] turns on a right-hand screw around l, when the point on [a b] moves along the edge in the positive direction determined by the orientation of K. Similarly, an edge is said to be *negative* when the half-plane P turns on a left hand screw around l when [a b] is traversed in the positive direction.

3.2.7 Definition. A *closed braid* in $\mathbf{R}^3$ is an oriented link K which admits an oriented axis l such that all edges are positive. Less formally, the closed braids are exactly the links in $\mathbf{R}^3$ which arise by closing braids.

An important theorem due to Alexander [1] states that every link in $\mathbf{R}^3$ is isotopic to a closed n-braid. Thus, in the following discussion of a method for obtaining the braid representation of a link we will assume that every link is a closed n-braid. (The reader should note that although Alexander's theorem guarantees that an axis l can be placed so that K is a closed braid, finding the correct position for l can be difficult in practice.)

Let K be a link in $\mathbf{R}^3$ with axis l. Suppose that both K and l are oriented and that K is in general position with respect to l. Choose a plane π in $\mathbf{R}^3$ which is orthogonal to l. The orientation of l induces an orientation of the plane π, so that with respect to l the orientation of the plane π turns on a right-hand screw. Denote this orientation around the point where l intersects π by . Next, project K orthogonally onto π indicating the over and under crossings as usual.

Since we are assuming that K is a closed braid, every edge of K is positive. A braid word representing K can be read from the projection as follows. We will rotate a half-line in π, originating from the point of intersection of the axis l and the plane π, one full turn in the direction of the orientation of the plane π and in the process, read off the braid word.

First, choose an initial position of the half-line arbitrarily with the exception that it cannot intersect any double point of the projection of the link K. Once the initial position is chosen, the points of intersection between the half-line in initial position and the projection of the link K are numbered 1, ..., n. The point of intersection farthest from the line l is labeled 1, the point of intersection next farthest from the line l is labeled 2, etc.

Next, rotate the half-line in the direction of the orientation of the plane π. As the half-line rotates, after each crossing in the projection of the link K draw a half-line so that after one full rotation m half-lines have been indicated where m is the number of crossings in the projection of the link K. Once this family of m half-lines has been indicated, the projection of the link is cut open along the initial position half-line and the family of half-lines is bent into parallel position yielding a projection of a braid. Taking into account the over and under crossings of the strings, a braid representation of the link can be read from the resulting braid projection.

3.2.8 Example. *Borromean Rings.* We shall calculate the braid word representing the link known as the Borromean Rings, pictured in Figure 3.2.4. This three component link has the property that if any one of the components is removed, then the remaining two components form a link composed of two disjoint unknots.

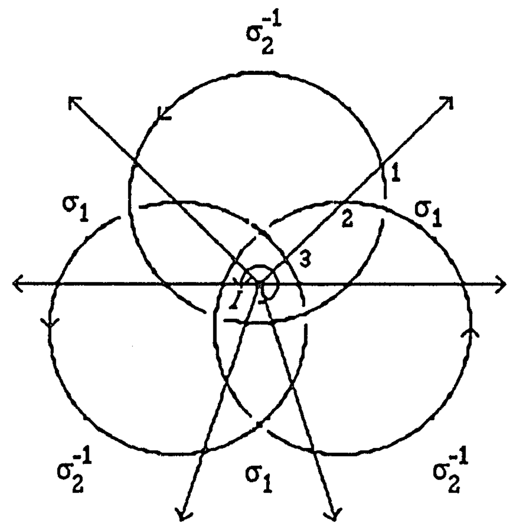


Figure 3.2.4

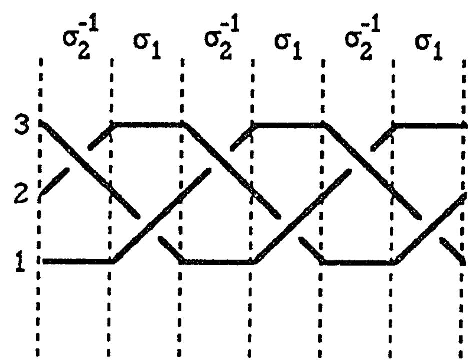


Figure 3.2.5

The braid word for the Borromean Rings can be read directly from Figure 3.2.5 and is

$$\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1 \text{ or } (\sigma_2^{-1}\sigma_1)^3.$$

It is interesting to note that this braid is a section of the common braid used to braid hair (see Figure 3.1.1). In addition, note that the closure of another section of the common hair braid in Example 3.2.2 yielded a two-component link.

3.2.9 Example. *Right-hand trefoil knot.* Next, we will calculate the braid word for the right-hand trefoil knot.

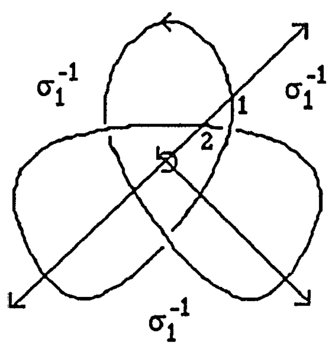


Figure 3.2.6

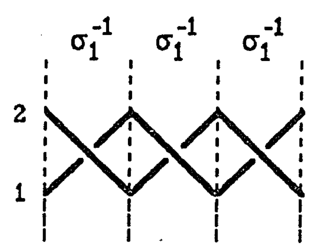


Figure 3.2.7

Finally, we are able to read the braid word for the right-hand trefoil knot as σ_1^{-3}.

As we have stated, Alexander proved that every oriented link in $\mathbf{R}^3$ is isotopic to the closure of some braid. Thus, we can represent any oriented link in $\mathbf{R}^3$ by the pair (β, n) where β is a braid in B_n. Unfortunately, the correspondence $(\beta, n) \rightarrow \hat{\beta}$ is many-to-one. For example, using this notation, the unknot can be written as $(\sigma_1\, \sigma_2 \ldots \sigma_{n-1}, n)$ or

$(\sigma_1^{-1}\sigma_2^{-1}\ldots\sigma_{n-1}^{-1}, n)$ since the closures (3.2.1) of both of these elements yield the unknot. The following theorem due to Markov provides the necessary and sufficient conditions for braids $\alpha \in B_n$ and $\beta \in B_m$ to have isotopic closures. We shall first define the two Markov moves.

3.2.10 Definition. A *Markov move of type* I is changing $\alpha \in B_n$ to $\beta\alpha\beta^{-1} \in B_n$ for any $\beta \in B_n$, and a *Markov move of type* II is changing $\alpha \in B_n$ to $\alpha\sigma_n^{\pm 1} \in B_{n+1}$, or the inverse of this operation.

3.2.11 Theorem. For $\alpha \in B_n$ and $\beta \in B_m$, α and β have isotopic closures if and only if there exists a finite sequence of Markov moves of type I and II (followed possibly by an isotopic deformation) which takes α to β.

Although the question of equivalence has been decided within each braid group [8], no algorithm has yet been found to decide when $\alpha \in B_n$ and $\beta \in B_m$ are equivalent for $n \neq m$. For this reason, attempts to use braids directly to study links are unsatisfactory. Recently, much progress in the study of links has been made by using *representations* of the braid group and we shall devote the next section to this topic.

3.3 GROUP REPRESENTATIONS

The Jones polynomial invariant for links uses a representation of the braid group described in 3.1 into the group of invertible elements of a quotient of a Hecke algebra. A trace function is then defined on this quotient algebra. The value of this trace function is then used in defining the resulting Laurent polynomial invariant for links. In this section we shall develop a minimal familiarity with group representation theory and define the Jones polynomial via a representation of the braid group.

3.3.1 Definition. Let G be a finite multiplicative group with identity element 1 and let V be an n-dimensional vector space over a field K. A *representation* π of G is a homomorphism $\pi : G \to \mathrm{Aut}(V)$ where $\mathrm{Aut}(V)$ denotes the set of all vector space isomorphisms of V onto V. (Recall that $\mathrm{Aut}(V)$ forms a group under the operation of composition of functions which is isomorphic to $GL_n(K)$, the general linear group.) The vector space V is called the *representation space* and the dimension of V over K is called the *degree* of the representation.

We have the following analogous definition for the representation of an algebra.

3.3.2 Definition. Let A be a finite dimensional algebra over a field K and M a finite-dimensional vector space over K. A *representation* of A with representation space M is an algebra homomorphism $T: A \to \mathrm{Hom}_K(M,M)$.

The Jones polynomial uses a representation of the braid group into a certain operator algebra. Thus, we will consider a representation of B_n to be the restriction to the group B_n of a representation of the group algebra KB_n.

Recall that the braid group B_n has a presentation with generators: $\sigma_1, \ldots, \sigma_{n-1}$, and defining relations:

$$\text{(i)}\quad \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} \text{ for } 1 \le i \le n-2$$

$$\text{(ii)}\quad \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i \text{ for } |i-j| \ge 2,\ 1 \le i, j \le n-1.$$

The key to seeing that the representation of B_n into the group of units of a certain Hecke algebra could be useful, is to notice the similarity between the presentation of the braid group and the following presentation of the Hecke algebra of type A_{n-1}.

3.3.3 Definition. For each $q \in \mathbb{C}$, the *Hecke algebra* $H(q,n)$ of type A_{n-1} has a presentation with generators $g_1, \ldots, g_{n-1}$, and defining relations:

$$\text{(i)}\ \ g_i^2 = (q-1)\,g_i + q,\ i = 1, \ldots, n-1$$

$$\text{(ii)}\ \ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},\ i = 1, \ldots, n-2$$

$$\text{(iii)}\ \ g_i g_j = g_j g_i,\ \ |i-j| \geq 2.$$

We will consider $H(q,n)$ as embedded in $H(q, n+1)$ by identifying the g_i's. For $q \neq 0$, we obtain a representation of B_n inside the Hecke algebra $H(q,n)$ by sending σ_i to g_i, for each $i = 1, \ldots, n-1$.

Jones [12] originally discovered $*$ - algebras A_n with generators $1, e_1, \ldots, e_n$ and relations

$$e_i^* = e_i, \quad e_i^2 = e_i$$

$$e_i e_{i+1} e_i = \tau e_i$$

$$e_i e_j = e_j e_i \quad \text{if } |i-j| \geq 2,$$

where τ is a real number.

These relations do not yield a presentation of A_n for all τ. Rather, the structure of A_n depends on the existence of a trace, denoted by tr, on A_n. For our purposes, it is enough to know that this trace is uniquely defined by the following property:

$$\operatorname{tr}(x e_{n+1}) = \tau \operatorname{tr}(x) \ \text{ if } x \in A_n \qquad (*)$$

$$\operatorname{tr}(1) = 1.$$

As noted by Jones [12], this property is similar to the Markov moves of type II, (changing $\alpha \in B_n$ to $\alpha\sigma_n^{\pm 1} \in B_{n+1}$ or the inverse of this operation). We shall call traces which exhibit this similarity *Markov traces.*

The algebra A_n is actually a quotient of the Hecke algebra given in 3.3.3.

Let $t \in \mathbf{R}$ be such that $\tau^{-1} = 2 + t^{-1} + t$. We may note that the assignment $\pi_0(\sigma_i) = t e_i - (1 - e_i)$ gives a representation of B_n into A_{n-1} and may now define the Jones polynomial $V_L(t)$ via this representation π_0.

3.3.4 Definition. Given $\beta \in B_n$, let e denote the exponent sum of β when β is written as a word in the generators σ_i, $i = 1, \ldots, n-1$. Define the *Jones polynomial* $V_L(t)$ by

$$V_L(t) = \left(-\frac{t+1}{\sqrt{t}}\right)^{n-1} (\sqrt{t})^e \operatorname{tr}\left[\pi_0(\beta)\right].$$

3.3.5 Example. In this example we shall use the techniques developed in this chapter to calculate the Jones polynomial of the right-hand trefoil knot. We have already calculated the braid word for the right-hand trefoil knot to be $\sigma_1^{-3} \in B_2$. Using the representation of B_2 into A_1 given above, we see that

$$\begin{aligned}\pi_0(\sigma_1^{-3}) &= \left[t^{-1}e_1 - (1 - e_1)\right]^3 \\ &= t^{-3}e_1^3 - 3t^{-2}e_1^2(1 - e_1) + 3t^{-1}e_1(1 - e_1)^2 - (1 - e_1)^3.\end{aligned}$$

Since e_1 is idempotent, $1 - e_1$ must also be idempotent. Moreover, $e_1(1 - e_1) = e_1 - e_1^2 = e_1 - e_1 = 0$. Thus, we have

$$\begin{aligned}\pi_0(\sigma_1^{-3}) &= t^{-3}e_1 - (1 - e_1)^3 \\ &= t^{-3}e_1 - (1 - e_1) \\ &= t^{-3}e_1 + e_1 - 1.\end{aligned}$$

Next, we calculate the trace of $\pi_0(\sigma_1^{-3}) = t^{-3}e_1 + e_1 - 1$ by using (*).

$$\operatorname{tr}\left[\pi_0(\sigma_1^{-3})\right] = \operatorname{tr}(t^{-3}e_1 + e_1 - 1)$$

$$= t^{-3}\tau + \tau - 1.$$

Since $\tau = \dfrac{t}{(t+1)^2}$, we have

$$\begin{aligned} \mathrm{tr}\left[\pi_0(\sigma_1^{-3})\right] &= t^{-3}\frac{t}{(t+1)^2} + \frac{t}{(t+1)^2} - 1 \\ &= \frac{t}{(t+1)^2}(t^{-3}+1) - 1. \end{aligned}$$

The exponent sum for σ_1^{-3} is -3, so we have for the right-hand trefoil knot 3_1

$$\begin{aligned} V_{3_1}(t) &= \left(-\frac{t+1}{\sqrt{t}}\right)^{2-1} \left(\sqrt{t}\right)^{-3} \mathrm{tr}\left[\pi_0(\sigma_1^{-3})\right] \\ &= -\frac{(t+1)}{t^2}\left[\frac{t}{(t+1)^2}(t^{-3}+1) - 1\right] \\ &= -\frac{(t+1)}{t^2}\left[\frac{t}{(t+1)^2}\left(\frac{1+t^3}{t^3}\right) - 1\right]. \end{aligned}$$

Writing $t^3 + 1$ as $(t+1)(t^2 - t + 1)$ yields

$$\begin{aligned} V_{3_1}(t) &= -\frac{1}{t^4}(t^2 - t + 1) + \frac{t+1}{t^2} \\ &= \frac{t^3 + t - 1}{t^4} \end{aligned}$$

and finally,

$$V_{3_1}(t) = -t^{-4} + t^{-3} + t^{-1}.$$

3.3.6 Remarks. We shall see in Chapter 4 that both the Alexander and the Jones polynomials are special cases of a more general polynomial invariant called the *HOMFLY P-polynomial.* However, the Jones polynomial already gives us new information; for example, as remarked by the authors of [15], the Alexander polynomial fails to distinguish the rather complicated pretzel knot $C_{-3,5,7}$ from the unknot, while the Jones polynomial for the same knot is nontrivial.

CHAPTER 4

OTHER POLYNOMIAL INVARIANTS

We now examine several polynomial invariants of knots which are calculated using the projection of a knot. Each invariant is defined inductively by considering the planar projections, K_+, K_-, and K_0, of three oriented links that are exactly the same except near one crossing, where they are as depicted in Figure 4.1.1.

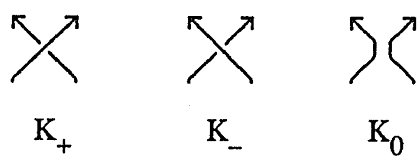


Figure 4.1.1

Conway [5] was the first to develop this combinatorial approach and he showed that the normalized Alexander polynomial ($\Delta(t) = \Delta(t^{-1})$ and $\Delta(1) = 1$) satisfies the formula

$$\Delta_{K_+}(t) - \Delta_{K_-}(t) + (t^{1/2} - t^{-1/2})\, \Delta_{K_0}(t) = 0.$$

By substituting $z = (t^{1/2} - t^{-1/2})$, Δ_K can be expressed as a one variable polynomial,

$\nabla_K(z)$, called the *Conway potential function.* When the Jones polynomial was discovered in 1985 and shown to be given by the recursive definition

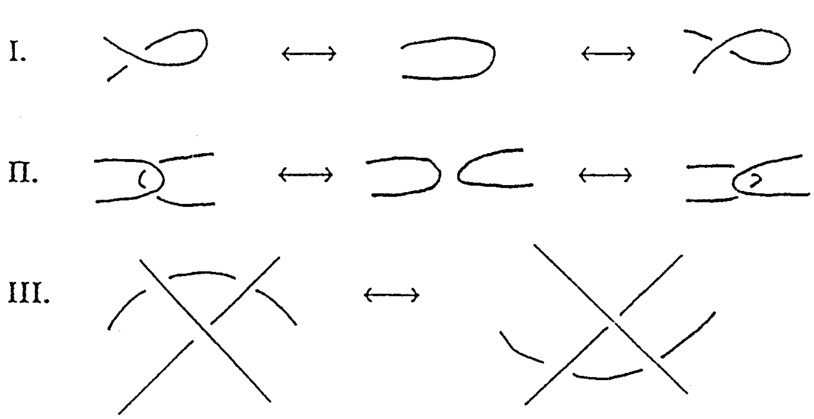
$$t^{-1}V_{K_+}(t) - t\,V_{K_-}(t) = (t^{1/2} - t^{-1/2})\, V_{K_0}(t),$$

four independent teams of six individuals, Freyd-Yetter, Hoste, Lickorish-Millett, and Ocneanu [7] realized that the Jones and Conway polynomials were both particular cases of a new polynomial invariant, called the HOMFLY polynomial after their initials. Shortly thereafter, Brandt, Lickorish, Millett [4] and Ho [10] discovered yet another new invariant for unoriented links, distinct from the HOMFLY polynomial, which was generalized to the Kauffman polynomial. In the following sections we will examine each of these invariants and give a few elementary calculations.

4.1 PRELIMINARIES

In the combinatorial approach of this chapter it is useful to take as the definition of equivalence of knots the equivalence of *isotopy* given in Definition 1.1.4.

4.1.1 Definition. A *Reidemeister move* of type **I, II** or **III** (shown below), is a deformation performed on the projection of a link where no other strands of the projection are present locally other than those depicted in the moves.



4.1.2 Definition. Two projections P and P' are *topologically equivalent* if there exists a homeomorphism of $\mathbf{R}^2$ which maps P onto P'.

In [18] Reidemeister proves the following theorem:

4.1.3 Theorem. Two links are equivalent if and only if their projections differ by a finite sequence of Reidemeister moves combined with planar topological equivalences of their projections.

4.2 THE CONWAY POLYNOMIAL ∇_K

This pioneering diagrammatic approach to finding useful invariants of links was introduced in 1970 by Conway [5]. In his paper, Conway describes a variation of the classical Alexander polynomial called the Conway potential function, denoted $\nabla_K(z)$.

4.2.1 Definition. The *Conway polynomial* is the polynomial satisfying the following three axioms:

Axiom 1: To each oriented link K there is associated a polynomial $\nabla_K(z) \in \mathbf{Z}[z]$, where $\mathbf{Z}[z]$ denotes the ring of polynomials in z with integer coefficients. Equivalent links receive identical polynomials: $K \sim K' \Rightarrow \nabla_K(z) = \nabla_{K'}(z)$.

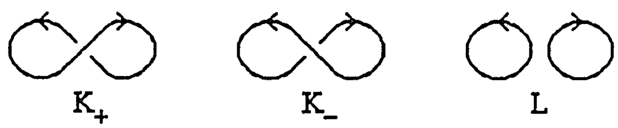
Axiom 2: If K is equivalent to the unknot, then $\nabla_K(z) = 1$.

Axiom 3: Suppose that three links K_+, K_-, K_0 differ at the site of one crossing as shown in Figure 4.1.1. Then $\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_0}(z)$. This is called the *exchange identity.*

A proof of the consistency of these axioms can be found in Kauffman [13] and it can also be shown that the polynomial is uniquely defined.

The exchange identity given in Axiom 3 is the foundation for all of the inductive calculations of the Conway polynomial. For a given link K, we will describe a triple of links which differ at the site of one crossing only, where K is one of the links in the triple and the other two links in the triple have known Conway polynomials. Then it is possible to calculate $\nabla_K(z)$ for the given link by using the exchange identity. Since all calculations of the Conway polynomial are inductive, we shall take as our starting point the following example.

4.2.2 Example. If L is the union of two disjoint unknots, then $\nabla_L = 0$. To see this, choose for K_+ and K_- the projections of the unknot as shown below.



Since are both K_+ and K_- are projections of the unknot, we have $\nabla_{K_+} = \nabla_{K_-} = 1$.

Then, using the exchange identity

$$\nabla_{K_+} - \nabla_{K_-} = z\nabla_L,$$

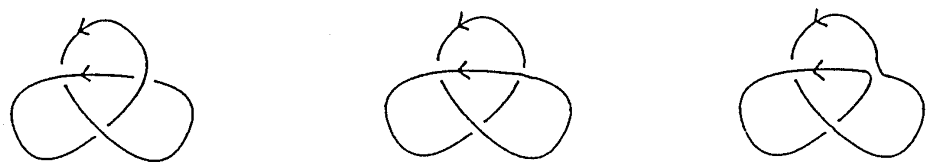
we have

$$1 - 1 = z\nabla_L$$

and so

$$\nabla_L = 0.$$

4.2.3 Example. We now calculate the Conway polynomial of the right-hand trefoil knot. Let K_+ be the right-hand trefoil knot and let K_- and K_0 be as shown on the next page.



K_+ K_- K_0

To calculate ∇_{K_+}, we need to know ∇_{K_-} and ∇_{K_0}. By inspection, K_- is a projection of the unknot and so $\nabla_{K_-} = 1$. However, ∇_{K_0} is not known and will require a separate calculation.

4.2.4 Example. Let K_+, K_- and K_0 be given as below.

K_+ K_- K_0

Clearly, K_0 is a projection of the unknot and so $\nabla_{K_0} = 1$. By Example 4.2.2, $\nabla_{K_-} = 0$.

Thus, using the exchange identity $\nabla_{K_+} - \nabla_{K_-} = z\nabla_{K_0}$,

we have

$$\nabla_{K_+} - 0 = z \cdot 1$$

and finally,

$$\nabla_{K_+} = z.$$

Returning to the calculation of the right-hand trefoil knot, we have

$$\nabla_{K_{\text{RH-trefoil}}} - \nabla_{K_-} = z\nabla_{K_0}$$

so

$$\nabla_{K_{\text{RH-trefoil}}} - 1 = z(z)$$

and

$$\nabla_{K_{\text{RH-trefoil}}} = z^2 + 1.$$

4.3 THE HOMFLY POLYNOMIAL P(K)

Shortly after Jones' first announcement, a polynomial for oriented links, called the HOMFLY P-polynomial and denoted by P(K), was discovered almost simultaneously by four independent teams of six authors, Freyd-Yetter, Hoste, Lickorish-Millett, and Ocneanu [7]. This new polynomial generalized both the Alexander-Conway and the Jones polynomial and it can be described as follows.

4.3.1 Definition. The *HOMFLY P-polynomial* is the unique Laurent polynomial in commuting variables l and m, such that equivalent links have the same polynomial and

(i) P(unknot) = 1.

(ii) If K_+, K_- and K_0 are any three oriented links that are identical except near a point where they are as in Figure 4.3.1, then

$$l\,P(K_+) + l^{-1}\,P(K_-) + m\,P(K_0) = 0.$$

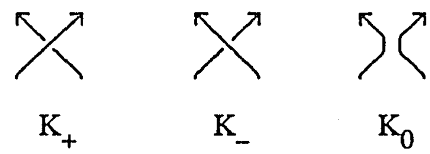


Figure 4.3.1

4.3.2 Example. We first calculate P(K) for two oriented disjoint unknots (compare with Example 4.2.2). Let K_+, K_- and K_0 be as depicted in Figure 4.3.2.

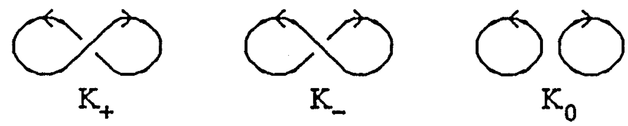


Figure 4.3.2

Since K_+ and K_- are projections of the unknot, $P(K_+) = P(K_-) = 1$. By 4.3.1.(ii) we have

$$l\,(1) + l^{-1}(1) + m\,P(K_0) = 0$$
$$m\,P(K_0) = -l - l^{-1}$$
$$P(K_0) = -m^{-1}\,(l + l^{-1}).$$

4.3.3 Example. Next, we calculate P(K) for the right-hand trefoil knot (cf. 4.2.3.). Consider K_+, K_- and K_0 depicted in Figure 4.3.3.

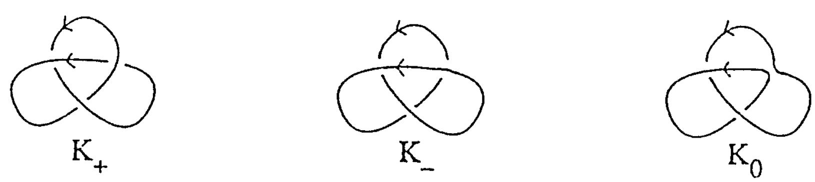


Figure 4.3.3

We wish to calculate $P(K_+)$ for the right-hand trefoil knot K_+. Clearly, K_- is a projection of the unknot so $P(K_-) = 1$. However, we do not yet know $P(K_0)$. To calculate $P(K_0)$, let K_0 be K_+ in Figure 4.3.4 and consider the trio of links in Figure 4.3.4.

K_+ K_- K_0

Figure 4.3.4

As calculated in Example 4.3.2, $P(K_-) = -m^{-1}(l + l^{-1})$ and $P(K_0) = 1$, since K_0 is a projection of the unknot. Thus,

$$l\,P(K_+) + l^{-1}\left[-m^{-1}(l + l^{-1})\right] + m(1) = 0$$
$$lP(K_+) - m^{-1}(1 + l^{-2}) + m = 0$$
$$lP(K_+) = m^{-1}(1 + l^{-2}) - m$$

so that

$$P(K_+) = m^{-1}(l^{-1} + l^{-3}) - m\,l^{-1}.$$

Returning to the calculation of P for the right-hand trefoil, we now know the polynomials of two of the three links shown in Figure 4.3.3. thus,

$$lP(K_+) + l^{-1}(1) + m\left[m^{-1}(l^{-1} + l^{-3}) - m\,l^{-1}\right] = 0$$
$$lP(K_+) = -l^{-1} - (l^{-1} + l^{-3}) + m^2 l^{-1},$$

and finally,

$$P(K_+) = -2l^{-2} - l^{-4} + m^2 l^{-2}.$$

4.3.4 Examples. Similar calculations yield $P(K) = m^{-1}(l^3 + l) - lm$. where K is the link formed by two interlocking unknots as in Figure 4.3.5 (compare with Example 4.2.4), and $P(K) = -2l^2 - l^4 + l^2 m^2$ for K the left-hand trefoil knot.

Figure 4.3.5

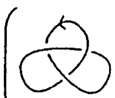
At this point we have calculated the following P-polynomials:

P() $= -m^{-1}(l + l^{-1})$

P() $= m^{-1}(l^3 + l) - lm$

P() $= m^{-1}(l^{-3} + l^{-1}) - l^{-1} m$

P() $= -2l^2 - l^4 + l^2 m^2$

P() $= -2l^{-2} - l^{-4} + l^{-2} m^2.$

These results illustrate the following properties of the P-polynomial as stated by the authors of [15]:

1. If $\bar{K}$ is the mirror image of a link K, then $P_{\bar{K}}(l, m) = P_K(l^{-1}, m)$. Thus, for a link to be of the same isotopy type as its mirror image its polynomial must be symmetric in l and l^{-1}.

2. If we consider each summand as having a factor m^k, then the lowest power of m in P(K) is equal to $1 - c$, where c is the number of components of K.

3. The powers of l and m are either all even or odd depending upon whether the number of components of K is odd or even, respectively.

4.4 THE JONES POLYNOMIAL $V_K(t)$

Recall that $V_K(t)$ is a Laurent polynomial in the variable $t^{1/2}$ satisfying

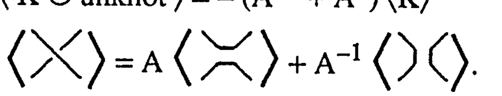
$$V(\text{unknot}) = 1, \text{ and}$$
$$t^{-1}V(K_+) - t\,V(K_-) + (t^{-1/2} - t^{1/2})\,V(K_0) = 0$$

where K_+, K_- and K_0 are oriented links as in Figure 4.1.1. In this section we outline a proof of the existence of $V_K(t)$ which is due to Kauffman [13].

4.4.1 Definition. For each link projection K define a Laurent polynomial $\langle K \rangle$ in one variable A by the following:

(i) $\langle \text{unknot} \rangle = 1$

(ii) $\langle K \cup \text{unknot} \rangle = -(A^{-2} + A^2)\langle K \rangle$

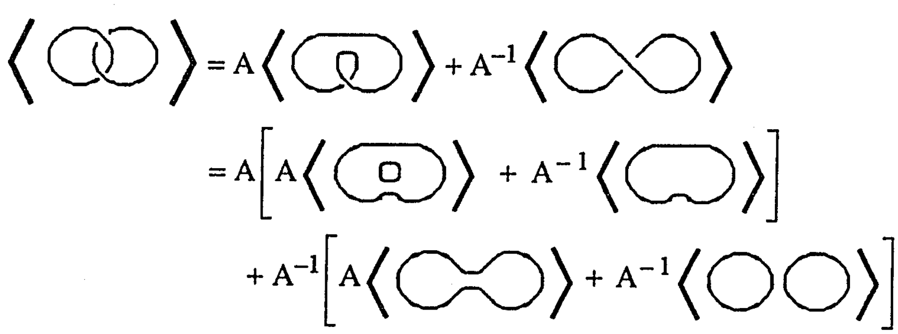
(iii) 

This $\langle K \rangle$ is called the *bracket polynomial* of K.

Note that in 4.4.1 (ii), $K \cup \text{unknot}$ denotes the projection that consists of K and another component that contains no crossing, and 4.4.1 (iii) refers to projections of three links which are exactly the same except where they are as shown. The first projection in the triple of 4.4.1 (iii) shows a crossing, the other two pictures in 4.4.1 (iii) show this crossing eliminated. Given a picture of a crossing, the other two pictures can be distinguished by the following: if, when moving along the underpass towards the crossing one swings to the right,up onto the overpass, one creates the second projection of the triple in 4.4.1 (iii). Notice that no arrows are required for this determination.

4.4.2 Example. As an example, we calculate ⟨K⟩ where K is the projection of the link formed by two interlocking unknots as in Figure 4.4.1.

Figure 4.4.1



$$= A^2\left[-(A^{-2} + A^2)\right] + 2 + A^{-2}\left[-(A^{-2} + A^2)\right]$$

$$= -(A^{-2} + A^2)^2 + 2.$$

Note that in the calculation of ⟨K⟩, each use of 4.4.1 (iii) reduces the number of crossings in the projection until there are no more crossings; then 4.4.1 (i) and (ii) finish the calculation. Also, though no proof will be given here, the choice of the order in which the crossings are eliminated is irrelevant, so 4.4.1 defines unambiguously a polynomial for each unoriented link projection. In addition, it can be checked that the bracket polynomial is invariant under Reidemeister moves II and III. However, the bracket polynomial fails to be invariant under Reidemeister move I:

$$= A\left[-(A^{-2} + A^{2})\langle \smile \rangle\right] + A^{-1}\langle \smile \rangle$$
$$= \left[-A(A^{-2} + A^{2}) + A^{-1}\right]\langle \smile \rangle$$
$$= -A^{3}\langle \smile \rangle.$$

Similarly,  $= -A^{-3}\langle \smile \rangle$. To correct the failure of invariance of the bracket polynomial under Reidemeister move I, give the link projection K an orientation and consider the following definitions.

4.4.3 Definition. For any crossing in the projection of an oriented link K, a crossing which obeys the right-hand rule convention is called *positive* and a crossing which does not obey the right-hand rule is called *negative.*

positive negative

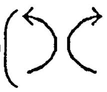
4.4.4 Definition. The *writhe* of an oriented link K, denoted by $\omega(K)$, is the algebraic sum of the crossings of K, counting $+1$ for a positive crossing and -1 for a negative crossing.

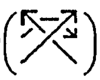
4.4.5 Example.

$$\omega(\text{left-hand trefoil}) = \omega\Big($$  $$\Big) = -3$$

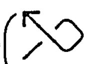
$$\omega(\text{right-hand trefoil}) = \omega\Big($$  $$\Big) = 3.$$

Now, $\omega(K)$ is invariant under Reidemeister moves II and III:

Move II: $\omega\Big($  $\Big) = -1 + 1 = 0 = \omega\Big($  $\Big)$

Move III: $\omega\Big($  $\Big) = -1 - 1 + 1 = -1 = \omega\Big($  $\Big)$

However, under Reidemeister move I, $\omega(K)$ is not invariant:

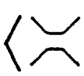
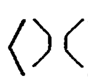
$\omega\Big($  $\Big) = -1 \neq 0 = \omega\Big($  $\Big)$.

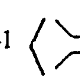
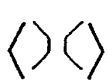
Finally, we arrive at a polynomial which is invariant under all three Reidemeister moves by defining

$$X(K) = (-A)^{-3\omega(K)}\langle K\rangle.$$

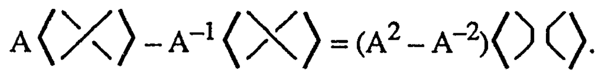
In this new polynomial the non-invariant behaviors of $\omega(K)$ and $\langle K\rangle$ 'cancel' so that $X(K)$ is invariant under Reidemeister move I. Also, any combination of $\omega(K)$ and $\langle K\rangle$ is invariant under Reidemeister moves II and III. Thus, $X(K)$ is a well-defined invariant of oriented links where $\langle\ \rangle$ ignores the orientation of K.

Next we find the normalization of $X(K)$ which yields $V_K(t)$. Using 4.4.1 (iii) twice, we have

$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$, and

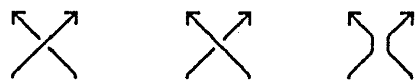
$\Big\langle$  $\Big\rangle = A^{-1} \Big\langle$  $\Big\rangle + A \Big\langle$  $\Big\rangle$.

Multiplying the first equation by A^{-1}, the second by A, and then subtracting the first from the second gives



$(*)$

Call the projections in the above equation K_+, K_- and K_0 and give them the following orientations:



K_+ K_- K_0

Then $\omega(K_+) = 1$, $\omega(K_-) = -1$ and $\omega(K_0) = 0$. Substitution of the bracket polynomials into $(*)$ gives

$$A\left[\frac{X(K_+)}{-A^{-3\omega(K_+)}}\right] - A^{-1}\left[\frac{X(K_-)}{-A^{-3\omega(K_-)}}\right] = (A^2 - A^{-2})\left[\frac{X(K_0)}{-1}\right], \text{ or}$$

$$-A^4X(K_+) + A^{-4}X(K_-) = (A^{-2} - A^2)X(K_0).$$

Letting $A = t^{-1/4}$, we have

$$-t^{-1}X(K_+) + tX(K_-) = (t^{1/2} - t^{-1/2})X(K_0),$$

and finally,

$$t^{-1}X(K_+) - tX(K_-) + (-t^{-1/2} + t^{1/2})X(K_0) = 0.$$

Thus, under the substitution $A = t^{-1/4}$, $X(K)$ is the original Jones polynomial $V_K(t)$ since they satisfy the same defining formula.

Lickorish and Millett remark in [15] that the Jones polynomial, $V_K(t)$, and the Alexander polynomial, $\Delta_K(t)$, are related to $P_K(l, m)$ by

$$V_K(t) = P_K(i\,t^{-1}, i\,(t^{-1/2} - t^{1/2})), \text{ and}$$
$$\Delta_K(t) = P_K(i, i\,(t^{1/2} - t^{-1/2})), \text{ where } i^2 = -1.$$

Since Jones used the presentation of the Artin braid group generated by the inverses of the elementary n-braids given in Chapter 3, the polynomials calculated by the method of Chapter 3 will be equivalent to the polynomials obtained via the above relation upon substituting t for t^{-1}.

4.5 THE Q-POLYNOMIAL

The Q-polynomial invariant for unoriented links is a Laurent polynomial in one variable and was discovered independently by Brandt, Lickorish and Millett [4] and by Ho[10] as an extension of the ideas of the P-polynomial.

4.5.1 Definition. For any link K there exists a unique Laurent polynomial $Q(K)$ in one variable x called the *Q-polynomial*, such that equivalent links have the same polynomial and

(i) $Q(\text{unknot}) = 1$

(ii) if K_+, K_-, K_0 and K_∞ are any four links that are identical except near a point where they are as in figure 4.5.1, then

$$Q(K_+) + Q(K_-) = x(Q(K_0) + Q(K_\infty)).$$

K_+ K_- K_0 K_∞

Figure 4.5.1

4.6 THE KAUFFMAN F-POLYNOMIAL F(K)

In 1986 Kauffman [13] discovered how to insert a second variable into the Q-polynomial, producing a new two-variable polynomial invariant for oriented links called the F-polynomial. The development of the F-polynomial is similar to that of the Jones polynomial in section 4.4 and begins with the definition of a Laurent polynomial $\Lambda(K)$ for unoriented links in variables a and x. This polynomial is invariant under Reidemeister moves II and III. Giving $\Lambda(K)$ an orientation and adjusting $\Lambda(K)$ by a power of a will result in a polynomial invariant for oriented links called the *Kauffman F-polynomial.*

4.6.1 Definition. For a projection of an unoriented link K, define a Laurent polynomial $\Lambda(K)$ in two variables a and x by

(i) $\Lambda(\text{unknot}) = 1$;

(ii) $\Lambda(\boxed{K}\!\propto) = a\,\Lambda(K)$, $\Lambda(\boxed{K}\!\propto) = a^{-1}\Lambda(K)$ (where $\boxed{K}$ denotes the portion of the projection of the link K not involved in the crossing depicted), and $\Lambda(K)$ does not change under a Reidemeister move of type II or III;

(iii) $\Lambda(K_+) + \Lambda(K_-) = x\left[\Lambda(K_0) + \Lambda(K_\infty)\right]$ where K_+, K_-, K_0 and K_∞ are projections of unoriented links that are exactly the same except near a point where they are as given in Figure 4.5.1.

Note that 4.6.1 (ii) means that if a positive kink $\propto$ is removed, the Λ-polynomial is multiplied by a (and similarly by a^{-1} for a negative kink $\propto$). Thus the Λ-polynomial is not invariant under a Reidemeister move of type I. Note also that the ambiguity involved in the labeling of the crossings in Figure 4.5.1 is irrelevant due to the symmetry of 4.6.1 (iii).

We now arrive at a true polynomial invariant of oriented links.

4.6.2 Theorem. For any oriented link K, let

$$F(K) = a^{-\omega(K)}\Lambda(K).$$

This F(K), called the *Kauffman F-polynomial*, is a well-defined invariant of oriented links in $\mathbf{R}^3$.

4.6.3 Remarks. The F-polynomial is sometimes called 'semioriented' since although K must be oriented to define F(K), changing the orientation only changes F(K) by multiplication by a power of a. Note that when $a = 1$ the resultant polynomial is the Q-polynomial.

4.7 CONCLUSION

We present here a summary of the individual characteristics of the polynomial invariants presented in this chapter.

4.7.1 The Alexander Polynomial $\Delta_K(t)$.

Although effective in distinguishing many knots, the Alexander polynomial invariant cannot distinguish a knot from its mirror image. In addition, as mentioned at the end of Chapter 3, it fails to detect knottedness in the complicated pretzel knot $C_{-3,5,7}$. The authors of [15] also note that of the 84 knots having less than 10 crossings, three pairs have the same Alexander polynomial.

4.7.2 The Conway Polynomial $\nabla_K(t)$.

Discovered in 1970, the Conway polynomial, $\nabla_K(t)$, is related to the Alexander polynomial by

$$\nabla_K(t) = \Delta_K(t^{1/2} - t^{-1/2}).$$

The Conway polynomial can distinguish many links with an even number of components from their mirror images.

4.7.3 The Jones Polynomial $V_K(t)$.

The discovery of the Jones polynomial, $V_K(t)$, in 1985 brought new excitement to the subject of knot theory. Most surprising were its connections with the fields of statistical mechanics and quantum field theory. In addition, it proved to be better at distinguishing knots than previous knot polynomials. For example, it is capable of distinguishing the pretzel knot $C_{-3,5,7}$ from the unknot. In fact, no knot has been found for which $V_K(t)$ cannot detect knottedness.

4.7.4 The HOMFLY Polynomial P(K).

The HOMFLY Polynomial, P(K), is a generalization of the Jones polynomial which specializes to both the Alexander and the Jones polynomials by the relations given at the end of Section 4.4. The authors of [15] point out that P(K) is better than both of these in detecting the difference between a knot and its mirror image, for P(K) can distinguish the mirror image of the knot 11_{388} but both $\Delta_K(t)$ and $V_K(t)$ fail to do so. Although it distinguishes the left and right trefoil knots, it cannot detect the reversal of orientation of a knot [15].

4.7.5 The Kauffman F-Polynomial F(K).

The Kauffman F-Polynomial, F(K), can detect the difference between a knot and its mirror image [14]. In addition, it is capable of distinguishing the knot 8_8 from the knot

10_{129} (the HOMFLY polynomial cannot [15]). However, the F-polynomial cannot distinguish 11_{255} from 11_{257} but both the Alexander and the P-polynomial can.

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