2000

The LU-factorization of totally positive and strictly totally positive matrices

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The LU-Factorization of Totally Positive and Strictly Totally Positive Matrices

A Thesis Presented to
The Faculty of the Department of Mathematics and Computer Science
San Jose State University

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts

by

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May, 2000

UMI Number: 1399822

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Abstract

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by Anna Cooper Strong

A real matrix is totally positive "TP" (strictly totally positive "STP") if all its minors are non-negative (strictly positive). While singular and non-singular matrices in general may or may not have an LU-factorization, it is a fact that any TP matrix does have an LU-factorization and **L** and **U** can be chosen to be TP also. Analogously, an STP matrix has an LU-factorization so that **L** and **U** are triangular STP, i.e., each non-trivial minor of **L** and **U** is positive. The converses follow easily by using the classic Cauchy-Binet Theorem. These results are due to C. W. Cryer. This thesis contains a detailed exposition of Cryer's work and the necessary background, including proofs of all supporting identities and lemmas.

Acknowledgments

The author wishes to express her deep appreciation to her thesis advisor, Dr. Jane Day, for her encouragement, counsel, and support throughout the process of research and writing this thesis.

The author also wishes to thank her committee members, Dr. Leslie Foster and Dr. Richard P. Kubelka, for their careful reading of this thesis and their helpful comments and suggestions.

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I Introduction

I.A OVERVIEW

Let $\mathbf{A}$ be an $n \times n$ matrix. Then $\mathbf{A}$ is said to admit an **LU-factorization** if there exist lower triangular $\mathbf{L}$ and upper triangular $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$. This concept can be generalized to an $m \times n$ matrix $\mathbf{A}$, but only square matrices $\mathbf{A}$ will be examined in this paper. Assume henceforth that all matrices are $n \times n$.

Not every matrix $\mathbf{A}$ has an LU-factorization. However, if row exchanges are allowed, then a permuted matrix $\mathbf{A}'$ can be formed from $\mathbf{A}$, so that $\mathbf{A}'$ will admit an LU-factorization.

This type of factorization has long been of interest for many reasons. An LU-factorization can be used to solve a linear system $\mathbf{Ax} = \mathbf{b}$, and this method is quite efficient. If row exchanges are allowed, it can be applied to any linear system. One needs approximately $n^3/3$ operations to find $\mathbf{L}$ and $\mathbf{U}$, then approximately $n^2/2$ operations each to solve $\mathbf{Ly} = \mathbf{b}$ by forward substitution and $\mathbf{Ux} = \mathbf{y}$ by back substitution, making a total of approximately $(n^3/3) + n^2$ operations. The term operation is used here to indicate a multiplication or a division. This method is more efficient than calculating $\mathbf{A}^{-1}$, a process requiring about n^3 floating point operations, then multiplying $\mathbf{A}^{-1}$ by $\mathbf{b}$ to get $\mathbf{x}$, requiring n^2 more operations for a total of $n^3 + n^2$ operations [Lay: 32].

An LU-factorization of $\mathbf{A}$ also gives insight about the properties of $\mathbf{A}$ and its determinants.

Since not every matrix has an LU-factorization, as background the conditions required to ensure the existence of such a factorization for singular and non-singular matrices in general will be examined and proved in Section (II) of this thesis.

The purpose of this thesis is to study LU-factorizations of two particular types of matrices called Totally Positive (TP) and Strictly Totally Positive (STP) matrices. The material here is based primarily upon two papers by C. W. Cryer, dated 1973 and 1976, where he proves that there exists an LU-factorization for any STP or TP matrix, whether singular or non-singular, and regardless of singularity or non-singularity of leading principal minors. The main results of the two Cryer papers will be proven in detail, along with examples and proofs of all supporting lemmas and identities, in Sections (III) and (IV).

Although it will not be detailed here, the reader may be interested to know that an important area of application of such matrices is in the theory of kernel functions, k(x,y), which comes from mechanics. A grid approximation of the integral of a kernel function over the unit square s, such as the classic $\iint_{s} e^{-x^2-y^2}\,dydx$, yields a totally positive matrix. This is discussed in detail in [GK: 1-9].

I.B NOTATION AND DEFINITIONS

$\underline{n}$ will denote the ordered set $(1,2,\dots,n)$.

$Q^{(p,n)}$: If $1 \le p \le n$ then $Q^{(p,n)}$ will denote the set of strictly increasing sequences $\alpha = (\alpha_1,\dots,\alpha_p)$ of p integers chosen from $\underline{n}$.

$\alpha \le \beta$: If $\alpha,\beta \in Q^{(p,n)}$ then $\alpha \le \beta$ will refer to the partial ordering where $\alpha_k \le \beta_k$ for $k \in \underline{p}$. $\alpha < \beta$, $\alpha \ge \beta$, and $\alpha > \beta$ are defined analogously.

$\hat{\alpha}$: If $\alpha \in Q^{(p,n)}$ then $\hat{\alpha} \in Q^{(n-p,n)}$ is defined to be the ordered complement of α in $\underline{n}$.

$\alpha\hat{\alpha}$: For the above α and $\hat{\alpha}$, $\alpha\hat{\alpha}$ is defined to be their concatenation, $(\alpha_1,\dots,\alpha_p,\hat{\alpha}_1,\dots,\hat{\alpha}_{n-p})$, which is a permutation of $\underline{n}$.

d(α): If $\alpha \in Q^{(p,n)}$, the **discrepancy** of α is defined as

$$d(\alpha) = \sum_{i=1}^{p-1} (\alpha_{i+1} - \alpha_i - 1).$$

M_n will denote the set of $n \times n$ matrices with real entries.

$A[\alpha;\beta]$ or $A[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_r]$: If $A \in M_n$ is a matrix and $\alpha \in Q^{(p,n)}$, $\beta \in Q^{(r,n)}$ then $A[\alpha;\beta]$ or $A[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_r]$ will denote the submatrix of A formed by taking the entries in rows α and columns β of A. A "$\underline{k}$" may be used in the row or column position to indicate that initial rows or columns, respectively, 1 through k are being selected.

$A(\alpha;\beta)$ will denote det $A[\alpha;\beta]$ when $\alpha,\beta \in Q^{(p,n)}$.

If $B = A[\alpha;\beta]$, then $\hat{B}[\alpha_i;\beta_j]$ or $A[\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_p;\beta_1,\ldots,\hat{\beta}_j,\ldots,\beta_r]$ will denote the submatrix of B which is formed by deleting row i and column j from B. $\hat{B}[\varnothing;\beta_j]$ or $\hat{B}[\alpha_i;\varnothing]$ indicate that no rows or columns, respectively, are omitted.

A **principal submatrix** of A is a $k \times k$ submatrix $A[\alpha;\alpha]$ where $\alpha \in Q^{(k,n)}$ for some $k \le n$. The notation $A[\alpha]$ is also used.

A **leading principal submatrix** of $\mathbf{A}$ is a $k \times k$ submatrix $\mathbf{A}[\underline{k};\underline{k}]$. The notation $\mathbf{A}[\underline{k}]$ is also used.

$\mathbf{A}(\alpha)$ and $\mathbf{A}(\underline{k})$ will denote $\det \mathbf{A}[\alpha]$ and $\det \mathbf{A}[\underline{k}]$, respectively.

diag(a,b,c) will denote the matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ where a, b, and c may be scalars or matrices.

$(\alpha_1,\ldots,\alpha_p)_N$: If $\alpha_1,\ldots,\alpha_p$ is an indexed set chosen from $\underline{n}$, where $p \le n$ and $\alpha_1,\ldots,\alpha_p$ are not necessarily in increasing order, then $(\alpha_1,\ldots,\alpha_p)_N$ is the corresponding ordered sequence in $Q^{(p,n)}$.

A **minor** of $\mathbf{A}$ is the determinant of some square submatrix $\mathbf{A}[\alpha;\beta]$ of $\mathbf{A}$.

A real matrix is **totally positive** (TP) if all its minors are non-negative.

A real matrix is **strictly totally positive** (STP) if all its minors are strictly positive.

(Note that all the entries of a TP matrix must be non-negative and all the entries of an STP matrix must be positive.)

A triangular real matrix is **triangular strictly totally positive** (ΔSTP) if all its non-trivial minors are strictly positive. Specifically, if $\mathbf{L}$ is an $n \times n$ lower triangular matrix, then the non-trivial minors of $\mathbf{L}$ are those $\mathbf{L}(\alpha;\beta)$ for which $\alpha, \beta \in Q^{(k,n)}$, and $\beta \le \alpha$ [Cry73: 84].

(The TP and STP terminology is that used by Cryer [CRY73], [CRY76] and Karlin [KAR], and is that which has been adopted here. Readers should be aware, however, that other authors such as Gantmacher and Krein [GK] use the phrases "totally non-negative" and "totally positive" to describe the same concepts as TP and STP, respectively.)

A **unit triangular matrix** is a triangular matrix whose diagonal entries are all ones.

II BACKGROUND

II.A OVERVIEW

To better understand LU-factorizations in general, a well-known algorithm for computing such factorizations, when they exist, and the standard existence theorems for general matrices will be presented first.

These theorems state conditions sufficient to ensure the existence of an LU-factorization of a singular matrix, and conditions both necessary and sufficient for an LU-factorization to exist for a non-singular matrix. A modification of the algorithm is used by Cryer in his proof of the existence of LU-factorizations for TP matrices.

II.B ALGORITHM

In order to understand why an LU-factorization can fail to exist in some cases, first the basic LU-factorization algorithm will be discussed [Lay: 130-131], [HJ: 158-163].

The strategy is to reduce a matrix $\mathbf{A}$ to an upper triangular form $\mathbf{U}$ by a sequence of row-replacement operations, if possible, and to store "multipliers" (to be described momentarily) in a unit lower triangular square matrix $\mathbf{L}$.

This algorithm may be applied to any rectangular $m \times n$ matrix $\mathbf{A}$, and if it completes it yields a square $m \times m$ $\mathbf{L}$ and rectangular $m \times n$ $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$ is true. However, only square matrices will be of interest here.

An example will be helpful.

II.B.1 Example:

Let $\mathbf{A} = \begin{bmatrix} 2 & 4 & 5 & -2 \\ -4 & -5 & -8 & 1 \\ 2 & -5 & 1 & 8 \\ -6 & 0 & -3 & 1 \end{bmatrix}$ be a 4×4 matrix. Perform a Gaussian row-reduction upon $\mathbf{A}$ to yield an upper triangular $\mathbf{U}$ as follows:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 5 & -2 \\ -4 & -5 & -8 & 1 \\ 2 & -5 & 1 & 8 \\ -6 & 0 & -3 & 1 \end{bmatrix} \sim \mathbf{A}' = \begin{bmatrix} 2 & 4 & 5 & -2 \\ 0 & 3 & 2 & -3 \\ 0 & -9 & -4 & 10 \\ 0 & 12 & 12 & -5 \end{bmatrix}$$

$$\sim \mathbf{A}'' = \begin{bmatrix} 2 & 4 & 5 & -2 \\ 0 & 3 & 2 & -3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 7 \end{bmatrix} \sim \mathbf{A}''' = \begin{bmatrix} 2 & 4 & 5 & -2 \\ 0 & 3 & 2 & -3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \mathbf{U}.$$

Define $\mathbf{L} = \begin{bmatrix} a_{11}/a_{11} & 0 & 0 & 0 \\ a_{21}/a_{11} & a'_{22}/a'_{22} & 0 & 0 \\ a_{31}/a_{11} & a'_{32}/a'_{22} & a''_{33}/a''_{33} & 0 \\ a_{41}/a_{11} & a'_{42}/a'_{22} & a''_{43}/a''_{33} & a'''_{44}/a'''_{44} \end{bmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{vmatrix}$, so $\mathbf{L}$

is unit lower triangular. The entries below the diagonal of $\mathbf{L}$ are often referred to as "multipliers". Then $\mathbf{A} = \mathbf{LU}$.

To better understand why, when the algorithm does complete, these multipliers work to create an $\mathbf{L}$ such that $\mathbf{A} = \mathbf{LU}$, consider how $\mathbf{A}$ is transformed into $\mathbf{U}$. In a standard Gaussian row-reduction, one works on the columns of $\mathbf{A}$ one at a time from left to right, working in each from the element below the pivot to the bottom of the column to "clear" entries, that is make them zero. If at some stage there is a zero in the next pivot position, but non-zero entries below it, the algorithm cannot proceed.

Specifically, in the example above, six separate entries are cleared using six separate elementary row operations. This can be thought of as the equation $\mathbf{E}_6\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U}$, where

$$\mathbf{E}_1 = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_3 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 3 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{E}_4 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 3 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_5 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & -4 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_6 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -2 & 1 \end{bmatrix}, \text{ and}$$

$\mathbf{A}' = \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}$,

$\mathbf{A}'' = \mathbf{E}_5\mathbf{E}_4\mathbf{A}'$,

$\mathbf{A}''' = \mathbf{E}_6\,\mathbf{A}''$.

It is easy to consolidate the $\mathbf{E}_i$ which clear a single column:[1]

$$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \mathbf{L}_1 = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ -1 & 0 & 1 & \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{E}_5\mathbf{E}_4 = \mathbf{L}_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 3 & 1 & \\ 0 & -4 & 0 & 1 \end{bmatrix}$$

$$\mathbf{E}_6 = \mathbf{L}_3 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -2 & 1 \end{bmatrix}, \text{ yielding } \mathbf{L}_3\mathbf{L}_2\mathbf{L}_1\mathbf{A} = \mathbf{U}.$$

Since all elementary row operation matrices are invertible, their products are also, hence $\mathbf{A} = (\mathbf{L}_3\mathbf{L}_2\mathbf{L}_1)^{-1}\mathbf{U} = \mathbf{L}_1^{-1}\mathbf{L}_2^{-1}\mathbf{L}_3^{-1}\mathbf{U}$.

But the inverse of such a product of row-replacement matrices which clear a column is simply the same matrix with sign changes in the off-

[1] Products of $\mathbf{E}_i$'s which clear entries in different columns are not so simple, e.g.: $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & b & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ a & 1 & \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ a & 1 & \\ ab & b & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & & \\ a & 1 & \\ 0 & b & 1 \end{bmatrix}$

diagonal entries of that column,[2] so:

$$\mathbf{L}_1^{-1} = \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & 0 & 1 & \\ -3 & 0 & 0 & 1 \end{bmatrix}, \mathbf{L}_2^{-1} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & -3 & 1 & \\ 0 & 4 & 0 & 1 \end{bmatrix}, \mathbf{L}_3^{-1} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

The product of $\mathbf{L}_1^{-1}\mathbf{L}_2^{-1}\mathbf{L}_3^{-1}$ is extremely simple[3]: it is

$$\mathbf{L} = \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}.$$

Thus $\mathbf{A} = \mathbf{LU}$, and the i,j entry of $\mathbf{L}$ for $i>j$ is exactly the multiplier used in the elementary row-replacement operation which pivoted on the j,j entry of $\mathbf{A}^{(k)}$ to clear the i,j entry of $\mathbf{A}^{(k)}$, $k=\{\ ,\ ',\ '',\ ''',\ldots\}$.

Notice that this algorithm creates a unit (thus non-singular) lower triangular $\mathbf{L}$. An analogous algorithm using column-replacement operations to reduce $\mathbf{A}$ to a lower triangular $\mathbf{L}_1$ would create a unit, non-singular $\mathbf{U}_1$ such that $\mathbf{A} = \mathbf{L}_1\mathbf{U}_1$. Cryer used a column-replacement strategy to prove that a totally positive matrix always has an LU-factorization. See Section (III).

Also observe that, although the algorithm described above did complete for the particular example shown, the algorithm would halt if the

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[2] The inverse of a unit lower triangular matrix with off-diagonal entries in different columns is not so simple, e.g.:

$$\begin{bmatrix} 1 & & \\ a & 1 & \\ 0 & b & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ -a & 1 & \\ ab & -b & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & & \\ -a & 1 & \\ 0 & -b & 1 \end{bmatrix}.$$

[3] The consolidation of unit lower triangular matrices with entries in columns from right to left is not so straightforward, e.g.:

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & c & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ a & 1 & \\ b & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ a & 1 & \\ ac+b & c & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & & \\ a & 1 & \\ b & c & 1 \end{bmatrix}$$

entry in the next pivot position were zero and a row-exchange type operation were needed in order to proceed.

II.C LU-FACTORIZATION OF SINGULAR MATRICES

It will be shown in this section that if $\mathbf{A}$ is an $n \times n$ matrix of rank r, and all leading principal sub-matrices of size one through r have non-zero determinants, then $\mathbf{A}$ admits an LU-factorization. If $\mathbf{A}$ is singular, this factorization is highly non-unique; in particular, either $\mathbf{L}$ or $\mathbf{U}$ may be chosen to be non-singular.

The converse is not true: When $\mathbf{A}$ is singular of rank r and some leading principal minor of size one through r is zero, there may or may not exist an LU-factorization of $\mathbf{A}$.

II.C.1 Example:

The matrix $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ has rank 1, and $\mathbf{A}(\underline{1}) = 0$. An LU-factorization is not guaranteed by the result above, but in this case one does exist. In fact, for any scalar a, $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & a \end{bmatrix}\begin{bmatrix} 1 & 2-a \\ 0 & 1 \end{bmatrix} = \mathbf{LU}$.

It is easy to see that even though LU-factorizations exist for this $\mathbf{A}$, $\mathbf{L}$ must be singular for each such factorization.

II.C.2 Example:

The matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ has rank 2 and $\mathbf{A}(\underline{1}) = \mathbf{A}(\underline{2}) = 0$, so an LU-factorization is not guaranteed, and in fact does not exist.

Proof:

Suppose there were an LU-factorization. Then

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \mathbf{A} = \mathbf{LU} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{bmatrix} = \begin{bmatrix} ag & ah & \cdot \\ bg & \cdot & \cdot \\ dg & \cdot & \cdot \end{bmatrix}$$

Since $a_{11} = ag = 0$, either a or g must be zero. But $a_{12} = ah = 1$ and $a_{31} = dg = 1$ require that neither a nor g be zero, a contradiction. Thus $\mathbf{A}$ has no LU-factorization.

Q.E.D.

II.C.3 Example:

The matrix $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$ has rank 1 and $\mathbf{A}(\underline{1}) \neq 0$, so by the result quoted above there exists an LU-factorization for $\mathbf{A}$. Here are some of them:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}, \text{ with } \mathbf{L} \text{ chosen to be non-singular}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}, \text{ with } \mathbf{U} \text{ chosen to be non-singular}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \text{ with both } \mathbf{L} \text{ and } \mathbf{U} \text{ singular}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}, \text{ for any } a$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & a \end{bmatrix}, \text{ for any } a$$

Notice that although LU-factorizations are guaranteed where one of **L** or **U** may be chosen to be non-singular, it is possible in this example to find $\mathbf{A} = \mathbf{LU}$ where neither **L** nor **U** is non-singular.

II.C.4 Theorem: When **A** is singular of rank r and $\mathbf{A}(\underline{s}) \neq 0$ for $s=1,\ldots,r$, then **A** has an LU-factorization. Furthermore, either **L** or **U** may be chosen to be non-singular.

Proof:

This proof is an adaptation of one used in [HJ: 160-161].

Suppose **A** is an $n \times n$ singular matrix, **A** has rank r, and $\mathbf{A}(\underline{s}) \neq 0$ for $s=1,\ldots,r$.

A constructive proof will be presented to show that there exist a non-singular unit lower triangular **L** and a singular upper triangular **U** such that $\mathbf{A} = \mathbf{LU}$.

Let $\mathbf{A}_{11} = \mathbf{A}[\underline{r}]$ and partition **A** as $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$. Suppose one has an LU-factorization of $\mathbf{A}_{11}$, $\mathbf{A}_{11} = \mathbf{L}_{11}\mathbf{U}_{11}$ where $\mathbf{L}_{11}$ is unit lower triangular and $\mathbf{U}_{11}$ upper triangular. Then $\mathbf{L}_{11}$ and $\mathbf{U}_{11}$ are both invertible since $\mathbf{A}_{11}$ is.

Now one needs to construct $\mathbf{L} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{0} & \mathbf{U}_{22} \end{bmatrix}$ so that $\mathbf{L}_{22}$ is unit lower triangular, $\mathbf{U}_{22}$ is upper triangular, and $\mathbf{A} = \mathbf{LU}$ is true, that is:

$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{0} & \mathbf{U}_{22} \end{bmatrix}$, which means the following must be true:

$$\mathbf{A}_{11} = \mathbf{L}_{11}\mathbf{U}_{11}$$

$$\mathbf{A}_{12} = \mathbf{L}_{11}\mathbf{U}_{12}$$

$$\mathbf{A}_{21} = \mathbf{L}_{21}\mathbf{U}_{11}$$

$$\mathbf{A}_{22} = \mathbf{L}_{21}\mathbf{U}_{12} + \mathbf{L}_{22}\mathbf{U}_{22}.$$

From these equations it is clear that $\mathbf{U}_{12}$ and $\mathbf{L}_{21}$ are now determined, since $\mathbf{U}_{12} = \mathbf{L}_{11}^{-1}\mathbf{A}_{12}$ and $\mathbf{L}_{21} = \mathbf{A}_{21}\mathbf{U}_{11}^{-1}$. As will be seen below, $\mathbf{L}_{22}$ and $\mathbf{U}_{22}$ can be chosen in many different ways.

Here is a way to construct $\mathbf{L}_{11}$ and $\mathbf{U}_{11}$.

First define $l_{ii} = 1$ for $i = 1,\ldots,r$, $l_{ij} = 0$ for $i < j$, and $u_{ij} = 0$ for $i > j$. Using the matrix equation $\mathbf{A}_{11} = \mathbf{L}_{11}\mathbf{U}_{11}$ and the hypothesis that $0 \neq \mathbf{A}(\underline{s}) = \mathbf{L}(\underline{s})\mathbf{U}(\underline{s})$ for s=1,...,*r*, entries l_{ij} and u_{ij} will be calculated one-by-one in the $r \times r$ leading principal $\mathbf{L}[\underline{r}]$ and $\mathbf{U}[\underline{r}]$ blocks.

Now l_{11}=1 has been set. Then by the definition of matrix multiplication,

$$a_{1j} = \sum_{k=1}^{r} l_{1k}u_{kj} = l_{11}u_{1j} \quad \text{since } l_{1k} = 0 \text{ for } k>1$$

$$= u_{1j} \quad \text{since } l_{11} = 1.$$

Thus the first row of $\mathbf{U}[\underline{r}]$ equals the first row of $\mathbf{A}[\underline{r}]$.

Also by the definition of matrix multiplication,

$$a_{i1} = \sum_{k=1}^{r} l_{ik}u_{k1} = l_{i1}u_{11}, \text{ since } u_{k1} = 0 \text{ for } k>1.$$

Since $u_{11} = a_{11} = \mathbf{A}(\underline{1}) \neq 0$, this gives $l_{i1} = a_{i1} / u_{11}$.

Thus the first column of $\mathbf{L}[\underline{r}]$ has been determined.

Next consider row 2 of $\mathbf{U}[\underline{r}]$:

$$a_{2j} = \sum_{k=1}^{r} l_{2k}u_{kj} = l_{21}u_{1j} + l_{22}u_{2j} \quad \text{because } l_{2k} = 0 \text{ when } k>2$$

$$= l_{21}u_{1j} + u_{2j} \quad \text{since } l_{22} = 1.$$

Thus $u_{2j} = a_{2j} - l_{21}u_{1j}$ for $j=2,\dots,r$ and row 2 of $\mathbf{U}[\underline{r}]$ has been found.

Also, $a_{i2} = \sum_{k=1}^{r} l_{ik}u_{k2} = l_{i1}u_{12} + l_{i2}u_{22}$ since $u_{k2} = 0$ for $k>2$.

This gives $l_{i2} = (a_{i2} - l_{i1}u_{12}) / u_{22}$, where $u_{22} \neq 0$ since block multiplication of the leading 2×2 blocks of **L** and **U** yields $\mathbf{L}[\underline{2}]\mathbf{U}[\underline{2}] = \mathbf{A}[\underline{2}]$ and the hypothesis $0 \neq \mathbf{A}(2)$ implies there can be no diagonal zero entry in $\mathbf{U}[2]$.

Thus column 2 of $\mathbf{L}[\underline{r}]$ has been determined.

Continue solving for the next row of $\mathbf{U}[\underline{r}]$, then the next column of $\mathbf{L}[\underline{r}]$, each time using one equation in one unknown. Each equation will be solvable since each $u_{ss} \neq 0$, because the non-zero determinant of $\mathbf{A}[\underline{s}]$ requires a non-zero determinant in $\mathbf{U}[\underline{s}]$, thus requiring non-zero diagonal entries u_{ss}.

This completes the factorization of submatrix $\mathbf{A}[\underline{r}]$ into $\mathbf{L}[\underline{r}]\mathbf{U}[\underline{r}]$ yielding $\mathbf{A}_{11} = \mathbf{L}_{11}\mathbf{U}_{11}$; $\mathbf{A}_{11}, \mathbf{L}_{11}, \mathbf{U}_{11} \in \mathbf{M}_r$.

If $r=n$, the construction of **L** and **U** is complete. Suppose $r<n$ and let $\mathbf{L}_{11} = \mathbf{L}[\underline{r}], \mathbf{U}_{11} = \mathbf{U}[\underline{r}]$, and $\mathbf{A}_{11} = \mathbf{A}[\underline{r}]$. Since $\mathbf{A}_{11}$ is non-singular, by hypothesis,

and $\mathbf{A}_{11} = \mathbf{L}_{11}\mathbf{U}_{11}$, then both $\mathbf{L}_{11}$ and $\mathbf{U}_{11}$ must be non-singular as well. As noted earlier, one must define $\mathbf{L}_{21} = \mathbf{A}_{21}\mathbf{U}_{11}^{-1}$ and $\mathbf{U}_{12} = \mathbf{L}_{11}^{-1}\mathbf{A}_{12}$. To see how to define $\mathbf{L}_{22}$ and $\mathbf{U}_{22}$, observe the following.

Since rank $\mathbf{A}$ equals the dimension of the row space of $\mathbf{A}$, and this equals rank $\mathbf{A}_{11}$, each row of $[\mathbf{A}_{21} \ \ \mathbf{A}_{22}]$ is a unique linear combination of the rows in $[\mathbf{A}_{11} \ \ \mathbf{A}_{12}]$. That is, for some unique $(n\text{-}r) \times r$ matrix $\mathbf{B}$

$\mathbf{A}_{21} = \mathbf{B}\mathbf{A}_{11}$ and $\mathbf{A}_{22} = \mathbf{B}\mathbf{A}_{12}$. So

$$\begin{aligned}
\mathbf{A}_{22} &= \mathbf{L}_{21}\mathbf{U}_{12} + \mathbf{L}_{22}\mathbf{U}_{22} \\
&= \mathbf{A}_{21}\mathbf{U}_{11}^{-1}\mathbf{L}_{11}^{-1}\mathbf{A}_{12} + \mathbf{L}_{22}\mathbf{U}_{22} \\
&= \mathbf{A}_{21}(\mathbf{L}_{11}\mathbf{U}_{11})^{-1}\mathbf{A}_{12} + \mathbf{L}_{22}\mathbf{U}_{22} \\
&= (\mathbf{B}\mathbf{A}_{11})\mathbf{A}_{11}^{-1}\mathbf{A}_{12} + \mathbf{L}_{22}\mathbf{U}_{22} \\
&= \mathbf{B}\mathbf{A}_{12} + \mathbf{L}_{22}\mathbf{U}_{22} \\
&= \mathbf{A}_{22} + \mathbf{L}_{22}\mathbf{U}_{22}.
\end{aligned}$$

Therefore one can now choose $\mathbf{L}_{22}$ and $\mathbf{U}_{22}$ to be any matrices of shape $(n-r) \times (n-r)$ whose product is $\mathbf{0}$. For example, to obtain a unit lower triangular $\mathbf{L}$, one could let $\mathbf{L}_{22} = \mathbf{I}_{n-r}$; then $\mathbf{U}_{22}$ must be $\mathbf{0}$, and one now has a non-singular unit lower triangular $\mathbf{L} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{0} \\ \mathbf{A}_{21}\mathbf{U}_{11}^{-1} & \mathbf{I}_{n-r} \end{bmatrix}$ and a singular upper triangular $\mathbf{U} = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{L}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ such that $\mathbf{A} = \mathbf{L}\mathbf{U}$.

Q.E.D.

To obtain a non-singular unit upper triangular $\mathbf{U}$, either apply the above process to $\mathbf{A}^T$, or carry out an analogous proof by first defining $u_{ii} = 1$ for i=1,...,r, then alternately solving for columns of $\mathbf{L}[\underline{r}]$ then rows of $\mathbf{U}[\underline{r}]$, and finally setting $\mathbf{U}_{22} = \mathbf{I}_{n-r}$ and $\mathbf{L}_{22} = \mathbf{0}$.

II.D LU-FACTORIZATION OF NON-SINGULAR MATRICES

It follows at once from (II.C.4) that if an $n \times n$ matrix $\mathbf{A}$ is non-singular, thus having rank n, and $\mathbf{A}(\underline{s}) \neq 0$ for $s=1,\dots,n$, then $\mathbf{A}$ admits an LU-factorization.

It will be shown in this section that the converse is true as well for a non-singular $\mathbf{A}$: if $\mathbf{A}$ has an LU-factorization then all leading principal submatrices of $\mathbf{A}$ have non-zero determinants. Furthermore, $\mathbf{L}$ and $\mathbf{U}$ are "essentially" unique. Specifically, this means that $\mathbf{A} = \mathbf{LU}$ can be factored further to yield $\mathbf{A} = \mathbf{L'DU'}$, where $\mathbf{L'}$ and $\mathbf{U'}$ are unit lower and upper triangular matrices, respectively, $\mathbf{D}$ is a diagonal matrix, and all are unique to $\mathbf{A}$.

II.D.1 Example:

The non-singular matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\mathbf{A}(\underline{1}) = 0$, and has no LU-factorization.

Proof:

Suppose there were one. Then $\mathbf{A} = \mathbf{LU} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Then $l_{11}u_{11} = 0$ implies either l_{11} or u_{11} is zero, which would make either $\mathbf{L}$ or $\mathbf{U}$ singular, but $\mathbf{A}$ is non-singular. Thus there does not exist an LU-factorization for $\mathbf{A}$.

Q.E.D.

II.D.2 Example:

The non-singular matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has $\mathbf{A}(\underline{2}) = 0$ and has no LU-factorization.

Proof:

Suppose there were one. Let $\mathbf{A}_{11} = \mathbf{A}[\underline{2}]$, $\mathbf{L}_{11} = \mathbf{L}[\underline{2}]$, $\mathbf{U}_{11} = \mathbf{U}[\underline{2}]$ and $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{0} & \mathbf{U}_{22} \end{bmatrix} = \mathbf{LU}$. By block multiplication $\mathbf{A}_{11} = \mathbf{L}_{11}\mathbf{U}_{11}$, thus $0 = \det \mathbf{A}_{11} = \det \mathbf{L}_{11} \det \mathbf{U}_{11}$. So either $\mathbf{L}_{11}$ or $\mathbf{U}_{11}$ must have a diagonal entry of zero since one of these must have a zero determinant. But then $\mathbf{L}$ or $\mathbf{U}$ must have determinant 0, implying that $\mathbf{A}$ is singular, a contradiction. Thus, there does not exist an LU-factorization for $\mathbf{A}$.

Q.E.D.

II.D.3 Example:

Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 7 & 2 \\ 3 & 6 & 2 \end{bmatrix}$, so $\mathbf{A}$ has $\mathbf{A}(\underline{1}), \mathbf{A}(\underline{2}), \mathbf{A}(\underline{3}) \neq 0$. Then $\mathbf{A}$ has an LU-factorization.

In fact $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

The first LU-factorization is the one which would be obtained using the algorithm presented in Section (II.B). An analogous algorithm using column operations to reduce **A** to a lower triangular **L** would produce the second LU-factorization. The third factorization is one of an infinite number of further possibilities for LU-factorization.

Observe that each of the LU-factorizations above could be further factored into $\mathbf{A} = \mathbf{L'DU'} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

In fact, in all LU-factorizations of this **A**, the non-unit diagonal entries in **L** and **U** can be factored out into diagonal matrices on the right and left, respectively, leaving $\mathbf{L}'$ and $\mathbf{U}'$. The product of the diagonal matrices is **D**, and this is why the factorization $\mathbf{A} = \mathbf{L'DU'}$ is unique.

II.D.4 **Theorem:** Suppose **A** is non-singular of order $n \times n$. Then $\mathbf{A}(\underline{s}) \neq 0$ for s=1,...,n if and only if **A** has an LU-factorization. Furthermore, when **A** has an LU-factorization,

(i) **L** and **U** must both be non-singular, and

(ii) there exist $\mathbf{L}', \mathbf{D}, \mathbf{U}'$ such that $\mathbf{L}'\mathbf{D}\mathbf{U}'$ is a unique factorization of $\mathbf{A}$, where $\mathbf{L}'$ and $\mathbf{U}'$ are unit lower and upper triangular matrices, respectively, and $\mathbf{D}$ is a non-singular diagonal matrix with $\mathbf{D}(\underline{s}) = \mathbf{A}(\underline{s})$ for $s=1,\ldots,n$.

Proof:

Suppose $\mathbf{A}$ is a non-singular $n \times n$ matrix with $\mathbf{A}(\underline{s}) \neq 0$ for $s=1,\ldots,n$. It follows immediately from results of Section (II.C) that $\mathbf{A}$ has an LU-factorization, showing *sufficiency* of the hypothesis.

Now suppose a non-singular $n \times n$ matrix $\mathbf{A}$ admits an LU-factorization $\mathbf{A} = \mathbf{LU}$. Then because $\mathbf{A}$ is non-singular, its determinant is non-zero. Hence the determinants consisting of the products of the diagonal entries of **L** and **U** are non-zero. Partition $\mathbf{A}$, $\mathbf{L}$, and $\mathbf{U}$ so that $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{0} & \mathbf{U}_{22} \end{bmatrix} = \mathbf{LU}$, where $\mathbf{A}_{11}$, $\mathbf{L}_{11}$, and $\mathbf{U}_{11}$ are square $s \times s$ blocks. Then $\mathbf{A}_{11} = \mathbf{L}_{11}\mathbf{U}_{11}$, and $\mathbf{A}(\underline{s}) = \det \mathbf{A}_{11} = \det \mathbf{L}_{11} \det \mathbf{U}_{11} = \mathbf{L}(\underline{s})\mathbf{U}(\underline{s}) \neq 0$ for $s=1,\ldots,n$. This shows *necessity* of the hypothesis and proves (i).

To show (ii), suppose $\mathbf{A}$ is non-singular and $\mathbf{A} = \mathbf{LU}$ is an LU-factorization of $\mathbf{A}$. Let $\mathbf{L}' = [\mathbf{L}_1 / l_{11} \quad \mathbf{L}_2 / l_{22} \quad \ldots \quad \mathbf{L}_n / l_{nn}]$, where the $\mathbf{L}_i$ are the columns of **L**. Let $\mathbf{D}_\mathbf{L} = \operatorname{diag}(l_{11}\ l_{22} \ldots l_{nn})$ and $\mathbf{D}_\mathbf{U} = \operatorname{diag}(u_{11}\ u_{22} \ldots u_{nn})$.

Finally, let $\mathbf{U}' = [\mathbf{U}_1/u_{11} \quad \mathbf{U}_2/u_{22} \quad \dots \quad \mathbf{U}_n/u_{nn}]^T$ where the $\mathbf{U}_i$ are the rows of $\mathbf{U}$. Then $\mathbf{L} = \mathbf{L}'\mathbf{D}_L$ and $\mathbf{U} = \mathbf{D}_U\mathbf{U}'$. All entries in $\mathbf{L}'$ and $\mathbf{U}'$ are well-defined because the non-singularity of $\mathbf{L}$ and $\mathbf{U}$ guarantees that column and row divisors l_{ii} and u_{ii} are non-zero. Let $\mathbf{D} = \mathbf{D}_L\mathbf{D}_U$. Then $\mathbf{A} = \mathbf{LU} = \mathbf{L}'\mathbf{D}_L\mathbf{D}_U\mathbf{U}' = \mathbf{L}'\mathbf{D}\mathbf{U}'$. Partition these matrices so that $\mathbf{A}_{11}$, $\mathbf{L}'_{11}$, $\mathbf{D}_{11}$, and $\mathbf{U}'_{11}$ are the leading principal $s \times s$ blocks for s=1,...,n. Then $\mathbf{A}_{11} = \mathbf{L}'_{11}\mathbf{D}_{11}\mathbf{U}'_{11}$, hence $\det \mathbf{A}_{11} = \det(\mathbf{L}'_{11}\mathbf{D}_{11}\mathbf{U}'_{11})$. Since $\mathbf{L}'$ and $\mathbf{U}'$ are unit triangular matrices, both $\det \mathbf{L}'_{11}$ and $\det \mathbf{U}'_{11}$ equal one, so $\det \mathbf{A}_{11} = \det \mathbf{D}_{11}$, or $\mathbf{A}(\underline{s}) = \mathbf{D}(\underline{s})$ for s=1,...n.

To see that $\mathbf{L}', \mathbf{D}$, and $\mathbf{U}'$ are unique, suppose there were two such $\mathbf{L}'\mathbf{D}\mathbf{U}'$ factorizations for $\mathbf{A}$: $\mathbf{A} = \mathbf{L}'_1\mathbf{D}_1\mathbf{U}'_1$, and $\mathbf{A} = \mathbf{L}'_2\mathbf{D}_2\mathbf{U}'_2$. Let $\mathbf{D}_1\mathbf{U}'_1 = \mathbf{U}_1$ and $\mathbf{D}_2\mathbf{U}'_2 = \mathbf{U}_2$, so the factorizations may be restated $\mathbf{A} = \mathbf{L}'_1\mathbf{U}_1$ and $\mathbf{A} = \mathbf{L}'_2\mathbf{U}_2$ where $\mathbf{L}'_1$ and $\mathbf{L}'_2$ are unit lower triangular and $\mathbf{U}_1$ and $\mathbf{U}_2$ are upper triangular. Then $\mathbf{L}'_1\mathbf{U}_1 = \mathbf{L}'_2\mathbf{U}_2$, hence $\mathbf{L}'^{-1}_2\mathbf{L}'_1 = \mathbf{U}_2\mathbf{U}_1^{-1}$.

Because both inverses of and products of unit lower triangular matrices remain unit lower triangular, and the product of upper triangular matrices is upper triangular, the above equation in matrix form looks like this: $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix} = \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix}$. Therefore $\mathbf{L}'^{-1}_2\mathbf{L}'_1 = \mathbf{U}_2\mathbf{U}_1^{-1} = \mathbf{I}$, i.e., $\mathbf{L}'_1 = \mathbf{L}'_2$ and $\mathbf{U}_1 = \mathbf{U}_2$, so the factorization $\mathbf{A} = \mathbf{L}'\mathbf{D}\mathbf{U}'$ is unique.

Q.E.D.

III LU-Factorization of Totally Positive Matrices

III.A OVERVIEW

It will be shown in Section (III.H) below that for a square real matrix **A**, **A** is totally positive (TP) if and only if **A** has an LU-factorization such that **L** and **U** are TP (main theorem). Several substantial theorems about determinants will be used in the proof of this result and these are described and proven in Sections (III.B) through (III.F).

A lemma by Cryer, also used in the proof of the main theorem, is presented in Section (III.G). The main theorem was first presented in a paper by C. W. Cryer, "Some Properties of Totally Positive Matrices" [Cry76]. Among other things, the theorem guarantees that there exists an LU-factorization for every TP matrix, singular or non-singular, whether or not its leading principal minors are non-zero.

His constructive proof of the theorem provides an algorithm for computing TP matrices **L** and **U** that give an LU-factorization of a TP matrix **A**; and if **A** is not TP, the algorithm will detect that at some point and halt. Thus the algorithm provides an efficient means for testing the total positivity of **A**.

Both of these features of the algorithm are significant. The well-known algorithm for finding **L** and **U** presented in Section (II.B) is guaranteed to work only when all leading principal minors of size one through rank A are non-zero. Although some singular matrices have LU-factorizations when the condition on leading principal minors is not satisfied, there is no general algorithm guaranteed to find such a factorization in such cases. Cryer has provided such an algorithm for singular TP matrices.

To determine if a matrix is TP, Cryer's algorithm is considerably more efficient than testing all minors. It is also more efficient than earlier special tests which allow some proper subset of all the minors of a matrix to be examined for positivity in order to determine the total positivity of that matrix. Thus this algorithm vastly reduces the amount of work required to test whether a matrix is TP.

III.B ZERO LEMMA

Recall from Section (I.B) the definition of a TP matrix: "A real matrix is **totally positive** (TP) if all its minors are non-negative."

III.B.1 Examples:

The following are TP matrices:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}, \begin{bmatrix}3 & 1\\0 & 4\end{bmatrix}, \begin{bmatrix}0 & 2\\0 & 3\end{bmatrix}$$
$$\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 1\\1 & 0 & 1\\1 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\1 & 1 & 1\end{bmatrix}$$

III.B.2 Examples:

The following matrices are not TP:

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\1 & 0 & 1\\1 & 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\1 & 0 & 1\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\0 & 0 & 1\\1 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}$$

The last example is particularly interesting: it has a negative determinant, but all its proper minors are non-negative.

III.B.3 **Zero Lemma.** Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ TP matrix with $a_{ij} = 0$ for some $1 \le i, j \le n$. Then at least one of the following is true:

(i) row i is zero

(ii) column j is zero

(iii) $\mathbf{A}[1,\dots,i;j,\dots,n]$ is zero

(iv) $\mathbf{A}[i,\dots,n;1,\dots,j]$ is zero

$$\begin{array}{cccc}
\begin{array}{c} \\ i \\ \\ \end{array}\begin{bmatrix} & j & \\ & & \\ 0 & 0 & 0 \\ & & \end{bmatrix}, &
\begin{array}{c} \\ i \\ \\ \end{array}\begin{bmatrix} & j & \\ & 0 & \\ & 0 & \\ & 0 & \end{bmatrix}, &
\begin{array}{c} \\ i \\ \\ \end{array}\begin{bmatrix} & j & \\ & 0 & 0 \\ & 0 & 0 \\ & & \end{bmatrix}, &
\begin{array}{c} \\ i \\ \\ \end{array}\begin{bmatrix} & j & \\ & & \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix} \\
\text{(i)} & \text{(ii)} & \text{(iii)} & \text{(iv)}
\end{array}$$

Proof:

Suppose row i is not zero and column j is not zero. Since $a_{ij} = 0$, only the following two cases are possible.

<u>Case 1</u>: $a_{kj} > 0$ for some $k>i$.

Consider the following diagrams:

$$\begin{array}{ccc}
\begin{bmatrix} & & \\ 0 & a_{im} & \\ a_{kj} & a_{km} & \\ & & \end{bmatrix} &
\begin{bmatrix} a_{pl} & a_{pj} & \\ a_{il} & 0 & \\ & & \\ & & \end{bmatrix} &
\begin{bmatrix} 0 & a_{pm} & \\ & & \\ a_{kj} & a_{km} & \\ & & \end{bmatrix} \\
\text{(a)} & \text{(b)} & \text{(c)}
\end{array}$$

Then for any $m>j$ (a) $\det\begin{bmatrix} 0 & a_{im} \\ a_{kj} & a_{km} \end{bmatrix} = 0 - a_{im}a_{kj} \geq 0$ and $a_{kj} > 0$, so $a_{im} = 0$.

So row i is zero from j to n. But since not all of row i is zero, there exists $a_{il} > 0$ for some $l<j$. Then for any $p<i$ (b) $\det\begin{bmatrix} a_{pl} & a_{pj} \\ a_{il} & 0 \end{bmatrix} = 0 - a_{pj}a_{il} \geq 0$ implies that $a_{pj} = 0$. Thus column j is zero from 1 to i. Then for $p<i$ and $m>j$, (c) $\det\begin{bmatrix} 0 & a_{pm} \\ a_{kj} & a_{km} \end{bmatrix} = 0 - a_{pm}a_{kj} \geq 0$ implies that $a_{pm} = 0$ because $a_{kj} > 0$. Thus

$\mathbf{A}\,[1,\ldots,i;j,\ldots,n]$ is zero: $\mathbf{A} = \begin{bmatrix} & 0 & 0 \\ & 0 & 0 \\ & & \end{bmatrix}$.

<u>Case 2</u>: $a_{kj} > 0$ for some $k<i$.

Consider the following diagrams:

$$\begin{bmatrix} a_{km} & a_{kj} & \\ a_{im} & 0 & \\ & & \end{bmatrix} \qquad \begin{bmatrix} & & \\ & 0 & a_{il} \\ & a_{pj} & a_{pl} \\ & & \end{bmatrix} \qquad \begin{bmatrix} a_{km} & a_{kj} & \\ & & \\ a_{pm} & 0 & \\ & & \end{bmatrix}$$

(a) (b) (c)

An argument analogous to Case 1 can be made to conclude that

$\mathbf{A}[i,\ldots,n;1,\ldots,j]$ is zero: $\mathbf{A} = \begin{bmatrix} & & \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix}$.

Since Cases 1 and 2 are mutually exclusive and exhaustive, the proof is complete.

Q.E.D.

Observe that, indeed, one of the four structures outlined in the Zero Lemma is present in each of the examples of TP matrices in (III.B.1). As seen in the proof, one of these structures is required to ensure that no 2×2 submatrix has a negative determinant. However, the existence of one of these structures is not sufficient to guarantee total positivity in a square matrix of order 3 or larger. The last example in (III.B.2) exhibits this fact.

A Zero Lemma also applies to an $m \times n$ TP matrix, $m \neq n$, and such a variation will be used in Section (III.G). The proof is analogous to that above.

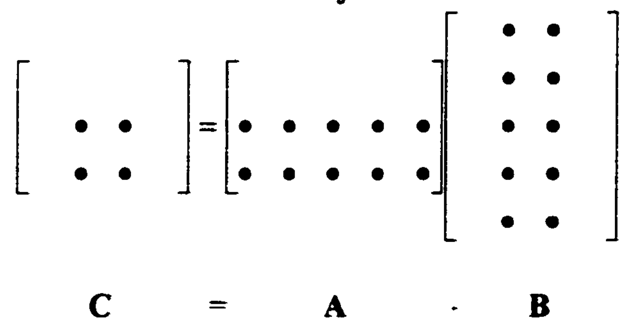
III.C CAUCHY-BINET THEOREM

Let **C** be an $n \times n$ matrix, and suppose $\mathbf{C} = \mathbf{AB}$ where **A** is $n \times p$ and **B** is $p \times m$. Suppose $1 \le r \le \min\{n, p, m\}$. The Cauchy-Binet Theorem gives a formula for calculating the determinant of any $r \times r$ submatrix $\mathbf{C}[\alpha;\beta]$ where $\alpha \in Q^{(r,n)}$ and $\beta \in Q^{(r,m)}$. This will prove especially useful in the sequel.

The Cauchy-Binet Theorem states that such a determinant $\mathbf{C}(\alpha;\beta)$ is equal to the sum of the products of all possible matched pairs of $r \times r$ minors taken from rows α of **A** and columns β of **B**, where "matched" means the columns of the minor of **A** match the rows of the minor of **B**. That is, the formula states:

$$\mathbf{C}(\alpha;\beta) = \sum_{\gamma \in Q^{(r,p)}} \mathbf{A}(\alpha;\gamma)\mathbf{B}(\gamma;\beta)$$

When α and β are sets of consecutive numbers, the following picture illustrates the $r \times r$ submatrix $\mathbf{C}[\alpha;\beta]$, and the sources of $r \times r$ minors from **A** and **B** used in the Cauchy-Binet sum:



A specific example illustrating the formula will be described after some background on determinants is presented.

III.C.1 Definitions:

If **A** is an $n \times n$ matrix, then the term **elementary product** refers to any product of n entries of **A**, no two of which are from the same row or column [AR: 68].

Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a permutation of $\underline{n}$. An **inversion** in σ is defined to be a pair (σ_i, σ_j) where $i<j$ but $\sigma_i > \sigma_j$. Let s(σ) be the total number of distinct inversions in σ. Then the sign of σ, **sgn** σ is defined to be (+1) if s(σ) is even and (-1) if s(σ) is odd [AK: 118-125].

Remark. Some authors define sgn σ differently, as follows.

A **transposition** is defined to be a pair-wise interchange of elements in an ordered set. When σ is given, count the minimum number of transpositions necessary to produce σ from $\underline{n}$ and define **sgn** σ to be (+1) if that number is even, and (-1) if that number is odd [FRA: 172], [HJ: 8].

It is true that these two definitions of sgn σ agree with one another.

III.C.2 Example:

Let σ=(6,5,1,3,2,4) be a permutation of <u>6</u>. The inversions in σ are (6,5), (6,1), (6,3), (6,2), (6,4), (5,1), (5,3), (5,2), (5,4), and (3,2). Their number, s(σ)=10, so sgn σ=+1.

Now sgn σ will be computed using transpositions. Using cycle notation, the permutation of (1,2,3,4,5,6) to (6,5,1,3,2,4) is (1643)(25). This can be re-expressed as a product of transpositions (13)(14)(16)(25) [FRA: 105], whose number is even so sgn σ=+1.

Let $\mathbf{A} \in \mathrm{M}_n$. In elementary linear algebra texts, det **A** is usually defined in one of the following ways.:

III.C.3 Definition of det A using signed elementary products:

$$\det \mathbf{A} = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \prod_{i=1}^{n} a_{i\sigma_i}$$

III.C.4 Definition of det A using the Laplace Expansion about row *i*:

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij} \quad \text{where } \mathbf{A}_{ij} = \hat{\mathbf{A}}[i;j]$$

An analogous expansion can be performed about column *j*.

There is also an axiomatic definition of the determinant function; it can be proved that the axioms are satisfied by at most one function, and the two definitions above both yield functions which do satisfy the axioms, hence they are equivalent [APOS: 71-79].

Before proving the Cauchy-Binet Theorem, here is an example and some preliminary information that will be helpful in the proof.

III.C.5 Example:

Consider the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathrm{M}_3$, with

$$\mathbf{C} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & -2 & 6 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \mathbf{AB}.$$ Let $\alpha = (1,2)$ and $\beta = (2,3)$. Then

$$\mathbf{C}[\alpha;\beta] = \begin{bmatrix} 1 & 0 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix}$$ and the determinant of $\mathbf{C}[\alpha;\beta]$ will be computed in two ways:

Directly: $\mathbf{C}(\alpha;\beta) = \mathbf{C}(1,2;2,3) = \begin{vmatrix} 1 & 0 \\ -2 & 6 \end{vmatrix} = 6$

Using Cauchy-Binet:

$$\mathbf{C}(\alpha;\beta) = \sum_{\gamma \in \mathrm{Q}^{(2,3)}} \mathbf{A}(\alpha;\gamma)\mathbf{B}(\gamma;\beta)$$

$$= \mathbf{A}(1,2;1,2)\mathbf{B}(1,2;2,3) + \mathbf{A}(1,2;1,3)\mathbf{B}(1,3;2,3) + \mathbf{A}(1,2;2,3)\mathbf{B}(2,3;2,3)$$

$$= \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix}\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= (3) \cdot (-2) + (-2) \cdot (0) + (-3) \cdot (-4)$$

$= -6 + 0 + 12$

$= 6$

The proof given below for the Cauchy-Binet Theorem is an adaptation of one presented in Broida and Williamson [BW: 208-213].

The following notation and observations will be used in the proof.

III.C.6 Notation:

Let $1 \leq p \leq n$, and recall from (I.B) that $\underline{n}=(1,\ldots,n)$ and $\underline{p}=(1,\ldots,p)$. A map, or function $\alpha:\underline{p}\to\underline{n}$ will usually be denoted by $\alpha=\{\alpha_1,\ldots,\alpha_p\}$, where $\alpha_i = \alpha(i)$.

MAP(p,n) is the set of all mappings from $\underline{p}$ to $\underline{n}$.

INC(p,n) is the set of all strictly increasing functions from $\underline{p}$ to $\underline{n}$.

INJ(p,n) is the set of all one-to-one functions in MAP(p,n), i.e., all permutations of elements of INC(p,n).

PER(p) = INJ(p,p). This set is more commonly called S_p.

Note that INC(p,n) is the same set that is called $Q^{(p,n)}$ elsewhere in this paper, but for purposes of clarity in this proof only, the Broida-Williamson INC(p,n) notation will be used.

The orders of these sets are:

$$O(\mathrm{MAP}(p,n)) = n^p,$$
$$O(\mathrm{INC}(p,n)) = \binom{n}{p},$$

$O(PER(p)) = p!$,

$O(INJ(p,n)) = n!/(n-p)! = O(INC(p,n))O(PER(p))$.

III.C.7 Observation:

For any integers $p,n \geq 1$ and scalars x_{ij}, $1 \leq i \leq p$ and $1 \leq j \leq n$,

$$\prod_{i=1}^{p}(\sum_{j=1}^{n} x_{ij}) = \sum_{\gamma \in MAP(p,n)} (\prod_{i=1}^{p} x_{i\gamma_i}).$$

Proof:

$$\prod_{i=1}^{p}(\sum_{j=1}^{n} x_{ij}) = \prod_{i=1}^{p}(x_{i1}+\ldots+x_{in})$$

$$= (x_{11}+\ldots+x_{1n})(x_{21}+\ldots+x_{2n})\cdots(x_{p1}+\ldots+x_{pn})$$

$$= \sum_{\gamma \in MAP(p,n)} x_{1\gamma_1}x_{2\gamma_2}\cdots x_{p\gamma_p}$$

$$= \sum_{\gamma \in MAP(p,n)} (\prod_{i=1}^{p} x_{i\gamma_i}).$$

Q.E.D.

III.C.8 Observation:

If $1 \leq p \leq n$, and $\mathbf{A} \in M_{n,p}$ then $\sum_{\gamma \in MAP(p,n)} \mathbf{A}(\gamma;\underline{p}) = \sum_{\gamma \in INJ(p,n)} \mathbf{A}(\gamma;\underline{p})$

Proof:

If $\gamma = \{\gamma_1, \ldots, \gamma_p\} \in$ MAP(p,n) is not an injection, then the submatrix A[γ;$\underline{p}$] has a repeated row, so $A(\gamma;\underline{p}) = \det A[\gamma;\underline{p}] = 0$. Thus $\sum_{\gamma \in MAP(p,n)} A(\gamma;\underline{p}) = \sum_{\gamma \in INJ(p,n)} A(\gamma;\underline{p})$.

Q.E.D.

III.C.9 Observation:

Let $\beta \to Q_\beta$ be a function from INJ(p,n) to the real numbers, and when $\sigma \in$ PER(p) and $\gamma \in$ INC(p,n), let $\gamma\sigma$ denote their composition. Then $\sum_{\beta \in INJ(p,n)} Q_\beta = \sum_{\gamma \in INC(p,n)} \sum_{\sigma \in PER(p)} Q_{\gamma\sigma}$.

Proof:

Let $\beta \in$ INJ(p,n). Then $\beta = \gamma\sigma$ for some $\gamma \in$ INC(p,n) and $\sigma \in$ PER(p): Let σ permute $\underline{p}$ so that $\beta\sigma$ is increasing, and define $\gamma = \beta\sigma^{-1}$. Then $\gamma\sigma = \beta$. Thus the map INC(p,n) $\times$ PER(p) $\to$ INJ(p,n) defined by $(\gamma,\sigma) \to \gamma\sigma$ is onto. The domain and range have the same order, hence this map is 1-1. That is, each β equals $\gamma\sigma$ for unique γ and σ.

Q.E.D.

III.C.10 Observation:

Let $1 \le p \le n$, $\sigma \in$ PER(p), $\gamma \in$ INC(p,n), and $\mathbf{A} \in M_{p,n}$; then $A(\gamma\sigma;\underline{p}) = (\text{sgn } \sigma) A(\gamma;\underline{p})$.

Proof:

The conclusion follows when one recalls that $\mathbf{A}[\gamma\sigma;\underline{p}]$ can be obtained from $\mathbf{A}[\gamma;\underline{p}]$ by a sequence of row-exchange operations, each of which alters the determinant by a factor of (-1), until the rows of $\mathbf{A}[\gamma;\underline{p}]$ have been rearranged to get $\mathbf{A}[\gamma\sigma;\underline{p}]$. Thus $\mathbf{A}(\gamma\sigma;\underline{p}) = (-1)^r \mathbf{A}(\gamma;\underline{p})$ where r is the number of transpositions (pair-wise interchanges) in the permutation σ (III.C.1), and $(-1)^r = \operatorname{sgn}\sigma$.

Q.E.D.

III.C.11 Theorem: Cauchy-Binet Theorem. Let $\mathbf{A} \in \mathrm{M}_{p,n}$, $\mathbf{B} \in \mathrm{M}_{n,m}$, $\mathbf{C} = \mathbf{AB}$, $1 \le r \le \min\{p,n,m\}$, $\alpha \in \mathrm{INC}(r,p)$, and $\beta \in \mathrm{INC}(r,m)$; then

$$\mathbf{C}(\alpha;\beta) = \sum_{\gamma \in \mathrm{INC}(r,n)} \mathbf{A}(\alpha;\gamma)\mathbf{B}(\gamma;\beta).$$

Proof:

First, consider a special case. Suppose $\mathbf{A} = [a_{ij}] \in \mathrm{M}_{p,n}$ and $\mathbf{B} = [b_{ij}] \in \mathrm{M}_{n,p}$, $p \le n$, so that $\mathbf{C} = \mathbf{AB} = [c_{ij}]$ is in M_p. The Cauchy-Binet Formula will be shown to hold for det $\mathbf{C}$:

$\det \mathbf{C} = \sum_{\sigma \in S_p} (\operatorname{sgn}\sigma) \prod_{i=1}^{p} c_{i\sigma_i}$, by the definition of det $\mathbf{C}$ (III.C.3)

$= \sum_{\sigma \in S_p} (\operatorname{sgn}\sigma) \prod_{i=1}^{p} \sum_{k=1}^{n} a_{ik} b_{k\sigma_i}$, by the definition of matrix multiplication,

$$= \sum_{\sigma \in S_p} (\text{sgn } \sigma) \sum_{\gamma \in MAP(p,n)} \prod_{i=1}^{p} a_{i\gamma_i} b_{\gamma_i \sigma_i} \text{ by (III.C.7)}$$

$$= \sum_{\gamma \in MAP(p,n)} \sum_{\sigma \in S_p} ((\text{sgn } \sigma) \prod_{i=1}^{p} a_{i\gamma_i} \prod_{i=1}^{p} b_{\gamma_i \sigma_i})$$

$$= \sum_{\gamma \in MAP(p,n)} ((\prod_{i=1}^{p} a_{i\gamma_i})(\sum_{\sigma \in S_p} (\text{sgn } \sigma) \prod_{i=1}^{p} b_{\gamma_i \sigma_i}))$$

$$= \sum_{\gamma \in MAP(p,n)} (\prod_{i=1}^{p} a_{i\gamma_i})(\mathbf{B}(\gamma; \underline{p})) \text{, by the definition of det } \mathbf{B}[\gamma; \underline{p}] \text{ (III.C.3)}$$

$$= \sum_{\gamma \in INJ(p,n)} (\prod_{i=1}^{p} a_{i\gamma_i})(\mathbf{B}(\gamma; \underline{p})) \text{, by (III.C.8),}$$

$$= \sum_{\gamma \in INJ(p,n)} \mathbf{B}(\gamma; \underline{p}) \prod_{i=1}^{p} a_{i\gamma_i}$$

$$= \sum_{\gamma \in INC(p,n)} \sum_{\sigma \in S_p} \mathbf{B}(\gamma\sigma; \underline{p}) \prod_{i=1}^{p} a_{i(\gamma\sigma)_i} \text{, by (III.C.9), letting } Q_\gamma = \mathbf{B}(\gamma; \underline{p}) \prod_{i=1}^{p} a_{i\gamma_i}$$

$$= \sum_{\gamma \in INC(p,n)} \mathbf{B}(\gamma; \underline{p}) \sum_{\sigma \in S_p} (\text{sgn } \sigma) \prod_{i=1}^{p} a_{i(\gamma\sigma)_i} \text{, by (III.C.10),}$$

$$= \sum_{\gamma \in INC(p,n)} \mathbf{B}(\gamma; \underline{p}) \mathbf{A}(\underline{p}; \gamma) \text{, by the definition of det } \mathbf{A}[\underline{p}; \gamma] \text{ (III.C.3).}$$

Thus, $\det \mathbf{C} = \sum_{\gamma \in INC(p,n)} \mathbf{A}(\underline{p}; \gamma) \mathbf{B}(\gamma; \underline{p})$.

Now let $\mathbf{A} \in M_{p,n}$, $\mathbf{B} \in M_{n,m}$, $\mathbf{C} = \mathbf{AB} \in M_{p,m}$, $1 \le r \le \min\{p,n,m\}$,

$\alpha \in INC(r,p)$, and $\beta \in INC(r,m)$.

Since $\mathbf{C}[\alpha; \beta] = \mathbf{A}[\alpha; \underline{n}]\mathbf{B}[\underline{n}; \beta]$, by the identity proved above $\mathbf{C}(\alpha; \beta) = \sum_{\gamma \in INC(r,n)} \mathbf{A}(\alpha; \gamma) \mathbf{B}(\gamma; \beta)$.

Q.E.D.

III.D EXPANSION OF THE DETERMINANT ACROSS ROWS 1 TO *K*.

The next theorem is less well-known than the two definitions for det $\mathbf{A}$, (III.C.3) and (III.C.4), given in the previous section. But this is also valid, and will be needed in proofs of several subsequent theorems.

Recall from (I.B) that, when $\alpha \in Q^{(k,n)}$, $\hat{\alpha}$ denotes the increasing sequence in $Q^{(n-k,n)}$ which is complementary to α, and $\alpha\hat{\alpha}$ is their concatenation, $(\alpha_1,\ldots,\alpha_k,\hat{\alpha}_1,\ldots,\hat{\alpha}_{n-k})$, which is in S_n.

III.D.1 Theorem: Expansion of the Determinant across Rows 1 to *k*.

Let $\mathbf{A}$ be an $n \times n$ matrix, and $1 \le k \le n$. Then

$$\det \mathbf{A} = \sum_{\alpha \in Q^{(k,n)}} \operatorname{sgn}(\alpha\hat{\alpha})\mathbf{A}(\underline{k};\alpha)\mathbf{A}(k+1,\ldots,n;\hat{\alpha}).$$

Proof:

$$\sum_{\alpha \in Q^{(k,n)}} \operatorname{sgn}(\alpha\hat{\alpha})\mathbf{A}(\underline{k};\alpha)\mathbf{A}(k+1,\ldots,n;\hat{\alpha})$$

$$= \sum_{\alpha \in Q^{(k,n)}} \operatorname{sgn}(\alpha\hat{\alpha})\Big(\sum_{\beta \in S_k} \operatorname{sgn}\beta \prod_{i=1}^{k} a_{i\alpha_{\beta_i}}\Big)\Big(\sum_{\gamma \in S_{n-k}} \operatorname{sgn}\gamma \prod_{i=1}^{n-k} a_{(k+i),\hat{\alpha}_{\gamma_i}}\Big), \text{ by (III.C.3)}$$

$$= \sum_{\alpha \in Q^{(k,n)}} \sum_{\beta \in S_k} \sum_{\gamma \in S_{n-k}} (\operatorname{sgn}(\alpha\hat{\alpha}))(\operatorname{sgn}\beta)(\operatorname{sgn}\gamma)\Big(\prod_{i=1}^{k} a_{i\alpha_{\beta_i}}\Big)\Big(\prod_{i=1}^{n-k} a_{(k+i),\hat{\alpha}_{\gamma_i}}\Big)$$

(*)

For each $\alpha \in Q^{(k,n)}$, $\beta \in S_k$, and $\gamma \in S_{n-k}$, let $\sigma = (\alpha_{\beta_1},\ldots,\alpha_{\beta_k},\hat{\alpha}_{\gamma_1},\ldots,\hat{\alpha}_{\gamma_{n-k}})$. When *k* is fixed, there is a one-to-one correspondence between each σ in S_n

and each possible concatenation $\alpha\hat{\alpha}$ formed with all selections of such β and γ. Thus the triple sum in (*) can be consolidated to give $\sum_{\sigma\in S_n} \operatorname{sgn}\sigma\prod_{i=1}^{n} a_{i\sigma_i}$, and this equals $\det \mathbf{A}$ by (III.C.3).

Q.E.D.

In words, the formula states that $\det \mathbf{A}$ is equivalent to the sum of the signed products of all possible $k\times k$ minors $\mathbf{A}(\underline{k};\alpha)$ multiplied by the $(n-k)\times(n-k)$ minors of their complements $\mathbf{A}(k+1,\ldots,n;\hat{\alpha})$.

This formula extends the idea of the Laplace expansion about a single row. Indeed, observe that if k=1, then expansion of the determinant across rows 1 to k becomes

$$\det \mathbf{A} = \sum_{\alpha\in Q^{(k,n)}} \operatorname{sgn}(\alpha\hat{\alpha})\mathbf{A}(1;\alpha)\mathbf{A}(2,\ldots,n;\hat{\alpha})$$

$$= \sum_{j=1}^{n} (-1)^{j-1} a_{1j}\mathbf{A}(2,\ldots,n;1,\ldots,\hat{j},\ldots,n)$$, where j-1 is the number of transpositions or inversions (the numbers match when k=1) in $\alpha\hat{\alpha}$.

This equals $\sum_{j=1}^{n} (-1)^{1+j} a_{1j}\det \mathbf{A}_{1j}$, which is the Laplace expansion of $\det \mathbf{A}$ about row 1.

III.D.2 Example:

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 4 & 5 & 0 & 6 \\ 2 & 0 & 1 & -1 \\ 7 & 8 & 0 & 9 \end{bmatrix}$. Using a standard Laplace expansion about column 3 it can be seen that $\det \mathbf{A} = 0$. To calculate $\det \mathbf{A}$ by expansion across rows 1 to 2, recall that $Q^{(2,4)} = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$, and if $\alpha \in Q^{(2,4)}$, $\alpha\hat{\alpha} \in S_4$. By Theorem (III.D.1):

$$\det \mathbf{A} = \sum_{\alpha \in Q^{(2,4)}} \operatorname{sgn}(\alpha\hat{\alpha})\mathbf{A}(\underline{2};\alpha)\mathbf{A}(3,4;\hat{\alpha})$$

$$= \operatorname{sgn}(1,2,3,4)\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + \operatorname{sgn}(1,3,2,4)\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}$$

$$+ \operatorname{sgn}(1,4,2,3)\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} + \operatorname{sgn}(2,3,1,4)\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}$$

$$+ \operatorname{sgn}(2,4,1,3)\begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \operatorname{sgn}(3,4,1,2)\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

$$= (-3)(9) - (0)(8) + (-6)(-8) + (0)(25) - (-3)(-7) + (0)(16)$$

$$= 0.$$

III.E GANTMACHER-KREIN IDENTITY

Gantmacher and Krein discovered a determinantal identity [GK: 345] which is a major tool for their and Karlin's theorems about TP and STP matrices. Cryer also makes essential use of this identity. The identity, which is for certain minors in an $n \times (n+1)$ array will be stated, then an example given before its proof..

III.E.1 Theorem: Gantmacher-Krein Identity. Let $\mathbf{C} \in \mathrm{M}_{n,n+1}$, $n \geq 2$, and fix k,i so $2 \leq k \leq n$ and $1 \leq i \leq n$. Then

$$\hat{\mathbf{C}}(i;1,k)\,\hat{\mathbf{C}}(\varnothing;n+1) - \hat{\mathbf{C}}(i;1,n+1)\,\hat{\mathbf{C}}(\varnothing;k) + \hat{\mathbf{C}}(i;k,n+1)\,\hat{\mathbf{C}}(\varnothing;1) = 0.$$

Cryer remarks that sometimes in the literature the middle term is moved to the other side of the equation and the identity is written as $d_1 d_2 = d_3 d_4 + d_5 d_6$ [CRY76: 2].

III.E.2 Example:

Let $\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix} \in \mathrm{M}_{3,4}$ and let $i = 2$, $k = 3$.

The Gantmacher-Krein Identity says:

$\hat{\mathbf{C}}(2;1,3)\hat{\mathbf{C}}(\varnothing;4)-\hat{\mathbf{C}}(2;1,4)\hat{\mathbf{C}}(\varnothing;3)+\hat{\mathbf{C}}(2;3,4)\hat{\mathbf{C}}(\varnothing;1)$ must equal zero, or

$$\begin{vmatrix} c_{12} & c_{14} \\ c_{32} & c_{34} \end{vmatrix}\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}-\begin{vmatrix} c_{12} & c_{13} \\ c_{32} & c_{33} \end{vmatrix}\begin{vmatrix} c_{11} & c_{12} & c_{14} \\ c_{21} & c_{22} & c_{24} \\ c_{31} & c_{32} & c_{34} \end{vmatrix}+\begin{vmatrix} c_{11} & c_{12} \\ c_{31} & c_{32} \end{vmatrix}\begin{vmatrix} c_{12} & c_{13} & c_{14} \\ c_{22} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \end{vmatrix}=0. \quad (*)$$

This is easily checked for $\mathbf{C}=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 0 \end{bmatrix}$ With $i=2$, $k=3$, the sum

on the left in (*) is

$$\begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix}\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 0 & 1 \end{vmatrix}-\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \\ 9 & 0 & 0 \end{vmatrix}+\begin{vmatrix} 1 & 2 \\ 9 & 0 \end{vmatrix}\begin{vmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \\ 0 & 1 & 0 \end{vmatrix}$$

$$=(0)(-40) \quad - \quad (2)(-72) \quad + \quad (-18)(8)$$

$$= \quad 0\,.$$

Proof of Theorem (III.E.1)*:*

Construct $\mathbf{X}=\begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{X}_{21} & \mathbf{X}_{22} \end{bmatrix}$

$$=\begin{bmatrix} \mathbf{C} & \hat{\mathbf{C}}[\varnothing;1,k,n+1] \\ \hat{\mathbf{C}}[i;\varnothing] & \mathbf{0} \end{bmatrix}$$

$$=\left[\begin{array}{cccccccccccc} c_{11} & \cdots & c_{1k} & \cdots & c_{1n} & c_{1,n+1} & c_{12} & \cdots & \hat{c}_{1k} & \cdots & c_{1n} \\ \vdots & & & & & \vdots & \vdots & & & & \vdots \\ c_{n1} & \cdots & c_{nk} & \cdots & c_{nn} & c_{n,n+1} & c_{n2} & \cdots & \hat{c}_{nk} & \cdots & c_{nn} \\ c_{11} & \cdots & c_{1k} & \cdots & c_{1n} & c_{1,n+1} & & & & & \\ \vdots & & & & & \vdots & & & & & \\ \hat{c}_{i1} & & \hat{c}_{ik} & & \hat{c}_{in} & \hat{c}_{i,n+1} & & & \mathbf{0} & & \\ \vdots & & & & & \vdots & & & & & \\ c_{n1} & \cdots & c_{nk} & \cdots & c_{nn} & c_{n,n+1} & & & & & \end{array}\right]$$

Because of repeated rows, the first $n+1$ columns of $\mathbf{X}$ have rank $\leq n$, hence $\det \mathbf{X} = 0$. Expand $\det \mathbf{X}$ across rows 1 to n as defined in (III.D.1). Then $0 = \sum_{\alpha \in Q^{(n,2n-1)}} \operatorname{sgn}(\alpha \hat{\alpha}) \mathbf{X}(\underline{n}; \alpha) \mathbf{X}(n+1,\ldots,2n-1; \hat{\alpha})$. Notice that if $\mathbf{X}[\underline{n}; \alpha]$ does not include all n-2 columns of $\mathbf{X}_{12}$, the submatrix formed with the complementary columns $\hat{\alpha}$ will have a zero column, and thus a zero determinant, and so that term will drop out of the expansion. If the remaining two columns of $\mathbf{X}[\underline{n}; \alpha]$ are not chosen from $\{1,k,n+1\}$ then the submatrix $\mathbf{X}[\underline{n}; \alpha]$ will have a repeated column, and thus a zero determinant, so that term will drop out of the expansion. Therefore there are only three possibly non-zero terms in the sum, those for $\alpha^1 = \{1, k, n+2, \ldots, 2n-1\}$, $\alpha^2 = \{1, n+1, n+2, \ldots, 2n-1\}$, and $\alpha^3 = \{k, n+1, n+2, \ldots, 2n-1\}$.

$$\begin{aligned} \text{Thus } 0 = {} & \operatorname{sgn}(\alpha^1 \hat{\alpha}^1) \mathbf{X}(\underline{n}; \alpha^1) \mathbf{X}(n+1,\ldots,2n-1; \hat{\alpha}^1) \\ & + \operatorname{sgn}(\alpha^2 \hat{\alpha}^2) \mathbf{X}(\underline{n}; \alpha^2) \mathbf{X}(n+1,\ldots,2n-1; \hat{\alpha}^2) \\ & + \operatorname{sgn}(\alpha^3 \hat{\alpha}^3) \mathbf{X}(\underline{n}; \alpha^3) \mathbf{X}(n+1,\ldots,2n-1; \hat{\alpha}^3). \end{aligned}$$

To determine the signs, count the number of inversions. See (III.C.1):

$$\operatorname{sgn}(\alpha^1 \hat{\alpha}^1) = \operatorname{sgn}(1, k, n+2, \ldots, 2n-1, 2, \ldots, \hat{k}, \ldots, n+1) = (-1)^{0+(k-2)+(n-2)(n-1)} = (-1)^{k-2} = (-1)^k$$

$$\operatorname{sgn}(\alpha^2 \hat{\alpha}^2) = \operatorname{sgn}(1, n+1, n+2, \ldots, 2n-1, 2, \ldots, n) = (-1)^{(n-1)(n-1)} = (-1)^{n-1}$$

$$\operatorname{sgn}(\alpha^3 \hat{\alpha}^3) = \operatorname{sgn}(k, n+1, n+2, \ldots, 2n-1, 1, \ldots, \hat{k}, \ldots, n) = (-1)^{(k-1)+(n-1)^2} = (-1)^{k+n-2} = (-1)^{n+k}$$

Now, observing that the submatrices of $\mathbf{X}$ which appear in the equation above are also submatrices of $\mathbf{C}$, and letting $\hat{i}$ refer to the complement of i in $\underline{n}$, the identity can be restated:

$$0 = (-1)^k C(\underline{n};1,k,2,\ldots,\hat{k},\ldots,n)C(\hat{i};2,\ldots,\hat{k},\ldots,n+1)$$
$$+(-1)^{n-1} C(\underline{n};1,n+1,2,\ldots,\hat{k},\ldots,n)C(\hat{i};2,\ldots,n)$$
$$+(-1)^{n+k} C(\underline{n};k,n+1,2,\ldots,\hat{k},\ldots,n)C(\hat{i};1,\ldots,\hat{k},\ldots,n).$$

The columns of the first minor in each product may be reordered to become strictly increasing by a series of column exchange operations whose number s is the same as the number of transpositions. The resulting minor is the same as the original minor multiplied by $(-1)^s$. Thus:

$$0 = (-1)^k (-1)^{k-2} C(\underline{n};\underline{n})C(\hat{i};2,\ldots,\hat{k},\ldots,n+1)$$
$$+(-1)^{n-1}(-1)^{n-2} C(\underline{n};1,\ldots,\hat{k},\ldots,n+1)C(\hat{i};2,\ldots,n)$$
$$+(-1)^{n+k}(-1)^{(k-2)+(n-2)} C(\underline{n};2,\ldots,n+1)C(\hat{i};1,\ldots,\hat{k},\ldots,n)$$
$$= \hat{C}(\varnothing;n+1)\hat{C}(i;1,k) - \hat{C}(\varnothing;k)\hat{C}(i;1,n+1) + \hat{C}(\varnothing;1)\hat{C}(i;k,n+1)$$
$$= \hat{C}(i;1,k)\hat{C}(\varnothing;n+1) - \hat{C}(i;1,n+1)\hat{C}(\varnothing;k) + \hat{C}(i;k,n+1)\hat{C}(\varnothing;1).$$

Q.E.D.

III.E.3 Corollary: Alternate version of the Gantmacher-Krein Identity.

Let $C \in M_{n+1,n}$, $n \geq 2$, and fix *k,i* so $2 \leq k \leq n$ and $1 \leq i \leq n$. Then

$$\hat{C}(1,k;i)\hat{C}(n+1;\varnothing) - \hat{C}(1,n+1;i)\hat{C}(k;\varnothing) + \hat{C}(k,n+1;i)\hat{C}(1;\varnothing) = 0.$$

Proof:

Construct $\mathbf{X} = \begin{bmatrix} \mathbf{C} & \hat{\mathbf{C}}[\varnothing; i] \\ \hat{\mathbf{C}}[1, k, n+1; \varnothing] & \mathbf{0} \end{bmatrix}$ and if columns and rows are interchanged, the proof follows in a form analogous to that of (III.E.2).

Q.E.D.

III.F SYLVESTER'S IDENTITY

In this section, Sylvester's Identity will be presented and proven in two forms: "special," and "general." The general form of this identity is used in the proof of Cryer's Lemma in the next section, III.G.

III.F.1 Theorem: Sylvester's Identity, Special Form. Let $\mathbf{A}$ be an $n \times n$ matrix and fix k so $1 \leq k < n$. Define $\mathbf{C} \in \mathrm{M}_{n-k}$ thus: for each *i,j* such that $1 \leq i, j \leq n-k$, $c_{ij} = \mathbf{A}(1,\ldots,k,k+i;1,\ldots,k,k+j)$. Then $\det \mathbf{C} = \mathbf{A}(\underline{k})^{n-k-1} \det \mathbf{A}$ (where, by convention, 0^0 denotes 1, if $\mathbf{A}(\underline{n-1})$ happens to be zero).

Each c_{ij} is called a "bordered minor" since it is the determinant of a submatrix of $\mathbf{A}$ which is a leading principal $k \times k$ submatrix bordered by row *k+i* and column *k+j*.

Proof:

Partition $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$ so that $\mathbf{A}_{11} \in \mathrm{M}_k$, hence $\mathbf{A}(\underline{k}) = \det \mathbf{A}_{11}$.

<u>Case 1</u>: $\mathbf{A}(\underline{k}) \neq 0$.

Since $\mathbf{A}(\underline{k}) \neq 0$, the k rows of $\mathbf{A}_{11}$ span R^k, so multiples of rows 1 to k in $\mathbf{A}$ can be added to each of the last *n-k* rows to obtain a zero block in the (2,1) block position. Let $\mathbf{A}^{(q)}$ be the new matrix, where q indicates the number of row-replacement operations required to obtain the zero block. Such replacement-type elementary row operations do not change the determinant, so $\det \mathbf{A}^{(q)} = \det \mathbf{A}$.

So $\mathbf{A} \sim \mathbf{A}^{(q)} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22}^{(q)} \end{bmatrix}$ where $\mathbf{A}_{22}^{(q)} = [a_{k+i,k+j}^{(q)}]$. Notice that

$c_{ij} = \mathbf{A}(1,\ldots,k,k+i;1,\ldots,k,k+j) = \det \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12}[\underline{k};k+j] \\ \mathbf{0} & a_{k+1,k+j}^{(q)} \end{bmatrix} = \mathbf{A}(\underline{k}) a_{k+i,k+j}^{(q)}$. Thus

$$\mathbf{C} = \mathbf{A}(\underline{k}) \begin{bmatrix} a_{k+1,k+1}^{(q)} & \cdots & a_{k+1,n}^{(q)} \\ \vdots & & \vdots \\ a_{n,k+1}^{(q)} & \cdots & a_{nn}^{(q)} \end{bmatrix} = \mathbf{A}(\underline{k})\mathbf{A}_{22}^{(q)}, \text{ hence}$$

$\det \mathbf{C} = \det [\mathbf{A}(\underline{k})\mathbf{A}_{22}^{(q)}] = \mathbf{A}(\underline{k})^{n-k} \det \mathbf{A}_{22}^{(q)}$, since $\mathbf{A}_{22} \in \mathbf{M}_{n-k}$. (*)

Also,

$\det \mathbf{A} = \det \mathbf{A}^{(q)} = \mathbf{A}(\underline{k}) \det \mathbf{A}_{22}^{(q)}$ by expansion across rows 1 to k (III.D.1). So $\det \mathbf{A}_{22}^{(q)} = \det \mathbf{A} / \mathbf{A}(\underline{k})$, and substituting into (*) yields

$\det \mathbf{C} = \mathbf{A}(\underline{k})^{n-k} \det \mathbf{A} / \mathbf{A}(\underline{k}) = \mathbf{A}(\underline{k})^{n-k-1} \det \mathbf{A}$.

<u>Case 2</u>: $\mathbf{A}(\underline{k}) = 0$, hence rank $\mathbf{A}_{11} \le k-1$.

<u>Case 2.a</u>: $k = n-1$.

In this case $\mathbf{C} \in \mathbf{M}_{n-k} = \mathbf{M}_1$, so $\mathbf{C} = [c_{11}]$ and $\det \mathbf{C} = c_{11} = \det \mathbf{A}$. Using the convention that $0^0 = 1$, $\det \mathbf{C} = \mathbf{A}(\underline{k})^{n-k-1} \det \mathbf{A}$ is true.

<u>Case 2.b</u>: $k < n-1$.

Here $\mathbf{C}$ has size at least 2×2.

<u>Case 2.b.i</u>: Rank $\mathbf{A}_{11} < k-1$.

Because bordering $\mathbf{A}_{11}$ with one row and one column can add at most two to the rank, then rank $\mathbf{A}[1,\dots,k,k+i;1,\dots,k,k+j] < k+1$. Therefore, $c_{ij} = \mathbf{A}(1,\dots,k,k+i;1,\dots,k,k+j) = 0$ for all i and j. Thus $\mathbf{C} = \mathbf{0}_{n-k}$ and $\det \mathbf{C} = 0$.

Also $\mathbf{A}(\underline{k})^{n-k-1} = 0$ since $\mathbf{A}(\underline{k}) = 0$ and $n-k-1 \geq 1$. Thus $\det \mathbf{C} = \mathbf{A}(\underline{k})^{n-k-1} \det \mathbf{A}$ is true.

<u>Case 2.b.ii</u>: Rank $\mathbf{A}_{11} = k-1$.

As in Case 2.b.i, $\mathbf{A}(\underline{k})^{n-k-1} = 0$. It must be shown that $\det \mathbf{C} = 0$.

Because $\mathbf{A}_{11}$ has a $(k-1)\times(k-1)$ invertible submatrix, one can use the rows of $\mathbf{A}_{11}$ to do row-replacement operations on the last *n-k* rows of $\mathbf{A}$ to reduce $\mathbf{A}_{21}$ to a matrix with at most one non-zero column *m*. Similarly, use the columns of $\mathbf{A}_{11}$ to perform column replacement operations on the last *n-k* columns of $\mathbf{A}$ to reduce $\mathbf{A}_{12}$ to a matrix with at most one non-zero row *l*. If *p* is the total number of operations in both sequences, $\mathbf{A} \sim \mathbf{A}^{(p)} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{e}_l \mathbf{v}^T \\ \mathbf{x}\mathbf{e}_m^T & \mathbf{A}_{22}^{(p)} \end{bmatrix}$ for some columns $\mathbf{x}, \mathbf{v} \in C^{n-k}$ and columns $\mathbf{e}_m, \mathbf{e}_l$ of the $k \times k$ identity matrix.

Because $\mathbf{A}$ was reduced to $\mathbf{A}^{(p)}$ with only rows and columns of $\mathbf{A}_{11}$ added to other rows and columns, the bordered minor from $\mathbf{A}$ which is c_{ij} equals that same bordered minor from $\mathbf{A}^{(p)}$. A Laplace expansion (III.C.4) across the last row of the bordered submatrix $\begin{bmatrix} \mathbf{A}_{11} & \mathbf{e}_l v_j \\ x_1 \mathbf{e}_m^T & * \end{bmatrix}$ gives

$$\begin{aligned} c_{1j} &= \pm x_1 \det \begin{bmatrix} \hat{\mathbf{A}}_{11}[\varnothing;m] & \mathbf{e}_l v_j \end{bmatrix} + (*)(\det \mathbf{A}_{11}) \\ &= \pm x_1 v_j \det \hat{\mathbf{A}}_{11}[l;m]. \end{aligned}$$

Similarly, for any row p and any j,

$$c_{pj} = \pm x_p v_j \det \hat{\mathbf{A}}_{11}[l;m] = \pm(x_p / x_1)c_{1j} .$$

Therefore, each row of $\mathbf{C}$ is a multiple of the first row of $\mathbf{C}$. Thus rank $\mathbf{C} \le 1$. Here in Case 2.b, $\mathbf{C}$ has size at least 2×2, hence $\det \mathbf{C} = 0$ as required.

Q.E.D.

The proof presented in Case 2 is constructive and was supplied in private communication by C. K. Li and Steve Pierce. It is also possible to prove Case 2 by using a limit argument and Case 1.

III.F.2 Example:

To illustrate the constructions in the proof above of the special version of Sylvester's Identity, let $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ and k=1, so n-k=3 and $1 \le i, j \le 3$.

Then $\mathbf{A}^{(q)} = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & -2 & -5 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 3/2 & 3/2 & 4 \end{bmatrix}$, so $\mathbf{C} = \mathbf{A}(\underline{1})\mathbf{A}_{22}^{(q)} = \begin{bmatrix} -4 & -10 & 4 \\ 2 & 0 & 0 \\ 3 & 3 & 8 \end{bmatrix}$. On the other hand, $\mathbf{C} = [c_{ij}] \in \mathbf{M}_3$

$$
= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}
$$

$$
= \begin{bmatrix} \mathbf{A}(1,2;1,2) & \mathbf{A}(1,2;1,3) & \mathbf{A}(1,2;1,4) \\ \mathbf{A}(1,3;1,2) & \mathbf{A}(1,3;1,3) & \mathbf{A}(1,3;1,4) \\ \mathbf{A}(1,4;1,2) & \mathbf{A}(1,4;1,3) & \mathbf{A}(1,4;1,4) \end{bmatrix}
$$

$$
= \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} \end{bmatrix}
$$

$$
= \begin{bmatrix} -4 & -10 & 4 \\ 2 & 0 & 0 \\ 3 & 3 & 8 \end{bmatrix}
$$

Direct calculation shows $\det \mathbf{A} = 46$, $\det \mathbf{C} = 184$, and also $\mathbf{A}(1)^{4-1-1} \det \mathbf{A} = 2^2 \cdot 46 = 184$, thus verifying Sylvester's Identity for this example.

III.F.3 **Theorem:** Sylvester's Identity, General Form. Let $\mathbf{A}$ be an $n \times n$ matrix, fix k so $1 \le k < n$, and fix $\alpha, \beta \in \mathrm{Q}^{(k,n)}$. Recall from (I.B) that $\hat{\alpha}$ and $\hat{\beta}$ denote the elements of $\mathrm{Q}^{(n-k,n)}$ which are complementary to α and β. Define $\mathbf{C} \in \mathbf{M}_{n-k}$ as follows: for each i,j such that $1 \le i, j \le n-k$, let $\hat{\alpha}_i, \hat{\beta}_j$ be the i^{th} and j^{th} elements of the strictly increasing ordered sets $\hat{\alpha}, \hat{\beta}$, respectively, and let

$c_{ij} = \mathbf{A}((\alpha \cup \{\hat{\alpha}_i\})_N; (\beta \cup \{\hat{\beta}_j\})_N)$. Then $\det \mathbf{C} = \mathbf{A}(\alpha;\beta)^{n-k-1} \det \mathbf{A}$.

Proof:

Let $\mathbf{P}$ and $\mathbf{Q}$ be permutation matrices so that $\tilde{\mathbf{A}} = \mathbf{PAQ}$ has $\mathbf{A}[\alpha;\beta] = \tilde{\mathbf{A}}[\underline{k}]$. Let a and b, respectively, be the total number of inversions in the concatenations $\alpha\hat{\alpha}$ and $\beta\hat{\beta}$. Then $\det \mathbf{P} = (-1)^a$ and $\det \mathbf{Q} = (-1)^b$, so $\det \tilde{\mathbf{A}} = (-1)^{a+b} \det \mathbf{A}$.

For $1 \le i, j \le n-k$, define $\tilde{c}_{ij} = \tilde{\mathbf{A}}(1,\dots,k,k+i;1,\dots,k,k+j)$. Then by the Special Form of Sylvester's Identity (III.F.1) and the above equation, $\det \tilde{\mathbf{C}} = \tilde{\mathbf{A}}(\underline{k})^{n-k-1} \det \tilde{\mathbf{A}} = \mathbf{A}(\alpha;\beta)^{n-k-1} \det \tilde{\mathbf{A}} = \mathbf{A}(\alpha;\beta)^{n-k-1}(-1)^{a+b} \det \mathbf{A}$.

Recall that $\alpha\hat{\alpha}_i$ denotes the concatenation of α and $\{\hat{\alpha}_i\}$, i.e., the element of $Q^{(k+1,n)}$ which has $\alpha_1,\dots,\alpha_k$ as its first k components and $\hat{\alpha}_i$ as its $(k+1)^{st}$ component. Let a_i denote the number of inversions in $\alpha\hat{\alpha}_i$. Since $\hat{\alpha}$ is in order, this is the same as the number of inversions associated with $\hat{\alpha}_i$ in $\alpha\hat{\alpha}_i$, that is: $a = a_1 + \dots + a_{n-k}$. Similarly, for $1 \le j \le n-k$, let b_j denote the number of inversions in $\beta\hat{\beta}_j$, and then $b = b_1 + \dots + b_{n-k}$. Also one has $\tilde{c}_{ij} = \mathbf{A}(\alpha\hat{\alpha}_i; \beta\hat{\beta}_j) = (-1)^{a_i + b_j} \mathbf{A}((\alpha \cup \{\alpha_i\})_N; (\beta \cup \{\beta_j\})_N)$.

Therefore, $\tilde{\mathbf{C}}$ looks like $\mathbf{C}$ after multiplying row i by $(-1)^{a_i}$ and column j by $(-1)^{b_j}$, for each i and j. It follows that $\det \mathbf{C} = (-1)^{a+b} \det \tilde{\mathbf{C}}$. Therefore $\det \mathbf{C} = \mathbf{A}(\alpha;\beta)^{n-k-1} \det \mathbf{A}$ as claimed.

Q.E.D.

III.F.4 Example:

Let **A** be as the example in (III.F.2) above, $k = 2, \alpha = (2,4)$, and $\beta = (3,4)$. Then $\hat{\alpha} = (1,3)$ and $\hat{\beta} = (1,2)$, so $\alpha\hat{\alpha} = (2,4,1,3)$ with a=1+2=3 and $\beta\hat{\beta} = (3,4,1,2)$ with b=2+2=4. Thus

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 3 & 4 & 1 & 2 \\ 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \mathbf{PAQ}, \text{ so}$$

$$\tilde{\mathbf{A}}^{(q)} = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 3 & 4 & 1 & 2 \\ 0 & 0 & 23 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \tilde{\mathbf{C}} = \tilde{\mathbf{A}}(\underline{2})\tilde{\mathbf{A}}^{(q)} = \begin{bmatrix} -46 & 10 \\ 0 & -2 \end{bmatrix}. \text{ Thus}$$

$\det \mathbf{C} = (-1)^{a+b} \det \tilde{\mathbf{C}} = (-1)^{3+4}(92) = -92$. On the other hand,

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{A}((2,4,1)_{\mathrm{N}};(3,4,1)_{\mathrm{N}}) & \mathbf{A}((2,4,1)_{\mathrm{N}};(3,4,2)_{\mathrm{N}}) \\ \mathbf{A}((2,4,3)_{\mathrm{N}};(3,4,1)_{\mathrm{N}}) & \mathbf{A}((2,4,3)_{\mathrm{N}};(3,4,2)_{\mathrm{N}}) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{A}(1,2,4;1,3,4) & \mathbf{A}(1,2,4;2,3,4) \\ \mathbf{A}(2,3,4;1,3,4) & \mathbf{A}(2,3,4;2,3,4) \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} 2 & 3 & 0 \\ 4 & 1 & 2 \\ 1 & 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 2 & 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 4 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 3 & 4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -46 & 10 \\ 0 & 2 \end{bmatrix}, \text{ so } \det \mathbf{C} = -92.$$

As was seen before, $\det \mathbf{A} = 46$, and $\mathbf{A}(2,4;3,4) = -2$, so $\mathbf{A}(2,4;3,4)^{4-2-1} \det \mathbf{A} = (-2)^1 \cdot 46 = -92$ also, as Sylvester's Identity asserts should be true.

III.G CRYER'S LEMMA

To prove his LU-factorization theorem for a TP matrix, Cryer created the following essential result [Cry76: Lemmas 3.1 and 3.2]. It shows that some very restrictive things are true about an $m \times (n+1)$ matrix **B** if certain conditions are met. This was undoubtedly inspired by the Gantmacher-Krein Identity and the way Gantmacher, Krein, and Karlin applied it to get results about STP matrices (see the next section here for some of that work), but Cryer's contributions are very intricate and original.

III.G.1 Lemma. Cryer's Lemma. Let **B** be an $m \times (n+1)$ matrix with $m \geq n \geq 2$, such that:

(1) the first n columns of **B** form a TP matrix, and

(2) the last n columns of **B** form a TP matrix.

Also suppose there is some $\alpha \in Q^{(n,m)}$ and some k with $2 \leq k \leq n$ such that

(3) $\mathbf{B}(\alpha; 1, \ldots, \hat{k}, \ldots, n+1) < 0$.

Then

(i) columns $2, \ldots, \hat{k}, \ldots, n$ of **B** have rank n-2,

(ii) when n=2, column k=2 is zero; when n>2, column k of **B** is linearly dependent upon columns $2, \ldots, \hat{k}, \ldots, n$, and

(iii) all minors of **B** of order less than n are non-negative.

If, in addition, when $n>2$ one also assumes

(4) $\mathbf{B}(\alpha;1,\ldots,\hat{j},\ldots,n+1) \geq 0$ if $2 \leq j \leq n$ and $j \neq k$,

then

(iv) column k of **B** is zero.

When $n=2$, all of (i) to (iv) follow from conditions (1) to (3). When $n>2$, result (iv), which is needed for Cryer's main theorem, requires all conditions (1) through (4).

III.G.2 Example:

Let $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$; then $n=k=2$. Observe $\mathbf{B}(1,2;1,2) = \mathbf{B}(1,2;2,3) = 0$ but $\mathbf{B}(1,2;1,3) = -1$. Cryer's results follow: column 2 is zero, and $\mathbf{B}(i;j) \geq 0$ for all i,j.

III.G.3 Example:

Let matrix $\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \in M_{3,3+1}$, where $\mathbf{B}[\underline{3};\underline{3}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and

$\mathbf{B}[\underline{3};2,3,4] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ are TP. However, for $\alpha = (1,2,3) \in Q^{(3,3)}$ and k=2,

$$\mathbf{B}(\alpha;1,\ldots,\hat{k},\ldots,n+1) = \mathbf{B}(\underline{3};1,3,4) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1 < 0.$$

This matrix satisfies conditions (1) to (3) of Cryer's Lemma. It can be seen that results (i) to (iii) follow:

(i) columns $2,\ldots,\grave{k},\ldots,n$, i.e., column 3, has rank 1 = n-2,

(ii) column $k = 2$ of $\mathbf{B}$ is linearly dependent on columns $2,\ldots,\grave{k},\ldots,n$, i.e., column 3, of $\mathbf{B}$, and

(iii) all minors of order 1 and order 2 are non-negative by the Zero Lemma (III.B.3), since any zero entry in $\mathbf{B}$ creates either an upper right or lower left corner of zero.

Condition (4) is not satisfied because $\mathbf{B}(\alpha;1,\ldots,\grave{j},\ldots,n+1) = -1 < 0$ if j=3. And, indeed, the fact that column 2 of $\mathbf{B}$ is not zero shows result (iv) fails.

III.G.4 Example:

Let matrix $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in M_{3,3+1}$ where $\mathbf{B}[\underline{3};\underline{3}]$ and $\mathbf{B}[\underline{3};2,3,4]$ are TP,

and for $\alpha = (1,2,3) \in Q^{(3,3)}$ and k=2,

$\mathbf{B}(\alpha;1,\ldots,\check{k},\ldots,n+1) = \mathbf{B}(\underline{3};1,3,4) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1 < 0$. Furthermore, for $2 \le j \le n$

and $j \ne k$, that is $j = 3$, $\mathbf{B}(\alpha;1,\ldots,\hat{j},\ldots,n+1) = \mathbf{B}(\underline{3};1,2,4) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \ge 0$. All

conditions (1) to (4) of Cryer's Lemma are satisfied. And indeed all results (i) to (iv) follow. In particular, (iv) column k=2 of **B** is zero.

Proof of Cryer's Lemma (III.G.1):

Let $\alpha \in Q^{(n,m)}$, and define $\mathbf{C} = \mathbf{B}[\alpha;\underline{n+1}] \in M_{n,n+1}$ so **C** consists of rows α of **B** and is $n \times (n+1)$. Then by the Gantmacher-Krein Identity (III.E.1),

$\hat{\mathbf{C}}(i;1,k)\hat{\mathbf{C}}(\varnothing;n+1) - \hat{\mathbf{C}}(i;1,n+1)\hat{\mathbf{C}}(\varnothing;k) + \hat{\mathbf{C}}(i;k,n+1)\hat{\mathbf{C}}(\varnothing;1) = 0$ for $1 \le i \le n$.

Condition (3) says that $\hat{\mathbf{C}}(\varnothing;k) < 0$. Conditions (1) and (2) imply that every other minor appearing in the identity is non-negative. Therefore $\hat{\mathbf{C}}(\alpha_i;1,n+1) = 0$. Thus columns 2,...,$n$ of **C** have rank less than n-1 if row α_i is omitted. This is true for each row α_t of $\mathbf{C}[1,\ldots,n;2,\ldots,n]$, hence the rank of columns 2,...,n of **C** is less than n-1. But $\hat{\mathbf{C}}(\varnothing;k) < 0$ implies columns $1,\ldots,\hat{k},\ldots,n$ of **C** are linearly independent, thus have rank n-1. So columns

$2,\ldots,\hat{k},\ldots,n$ of **C** have rank n-2, and column k of **C** depends linearly upon columns $2,\ldots,\hat{k},\ldots,n$ of **C**.

Conclusions (i) and (ii) have thus been shown for matrix **C** and condition (i) for **B** because the rank n-2 of columns $2,\ldots,\hat{k},\ldots,n$ of **C** cannot be decreased by adding additional rows from **B**, nor can it be increased beyond the number of columns (n-2).

Conclusions (ii), (iii), and (iv) will now be shown for matrix **B** in two cases: n=2 and n>2.

Case 1: $n = 2$.

Then $k = 2$ and α has order 2 so $\mathbf{C} = \mathbf{B}[\alpha;\underline{n+1}]$ is a 2×3 matrix with each entry non-negative and column 2 of **C** equal to zero. Since $\mathbf{B}(\alpha_1,\alpha_2;1,3) < 0$ by assumption (3), and each $b_{ij} \geq 0$ by assumptions (1) and (2), then $\mathbf{B}(\alpha_1,\alpha_2;1,3) = b_{\alpha_1,1}b_{\alpha_2,3} - b_{\alpha_2,1}b_{\alpha_1,3} < 0$ with a non-negative first summand implies $b_{\alpha_2,1}b_{\alpha_1,3}$ is strictly positive; that is $b_{\alpha_2,1} > 0$ and $b_{\alpha_1,3} > 0$. So

B has the form $\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ \vdots & \vdots & \vdots \\ b_{\alpha_1,1} & 0 & b_{\alpha_1,3} > 0 \\ \vdots & \vdots & \vdots \\ b_{\alpha_2,1} > 0 & 0 & b_{\alpha_2,3} \\ \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & b_{m3} \end{bmatrix}$

Thus, by the Zero Lemma (III.B.3) applied to the TP matrix $\mathbf{B}[\underline{m};1,2]$, column 2 is zero above $b_{\alpha_2,2}$, and applied to the TP matrix $\mathbf{B}[\underline{m};2,3]$, column 2 is zero below $b_{\alpha_1,2}$. So, when $n = 2$, column $k = 2$ of **B** is zero, showing (iv) and (ii). Notice that the rank of columns $(2,...,\hat{k},...,n)=\varnothing$ of $\mathbf{B}$ is indeed 0=n-2 showing (i). Finally, (iii) follows immediately from assumptions (1) and (2) since $n=2$ requires only that all minors of order one be non-negative.

<u>Case 2</u>: $n>2$.

Condition (i) has been proved already, that is, columns $2,...,\hat{k},...,n$ of $\mathbf{B}$ have rank n-2. Since $m \geq n$, there exist α_u and α_v with $\alpha_u < \alpha_v$ such that $\mathbf{B}(\alpha_1,...,\hat{\alpha}_u,...,\hat{\alpha}_v,...,\alpha_n;2,...,\hat{k},...,n)=g \neq 0$. Furthermore, assumption (1) shows g>0.

Let $\alpha'=(\alpha_1,...,\hat{\alpha}_u,...,\hat{\alpha}_v,...,\alpha_n)$ and $\beta'=(2,...,\hat{k},...,n)$ so $\alpha',\beta' \in Q^{(n-2,n)}$. Let $(s_1,...,s_{m-n+2})=\hat{\alpha}' \in Q^{(m-n+2,n)}$, and let $t_1=1$, $t_2=k$, $t_3=n+1$. Define an $(m-n+2)\times 3$ matrix $\mathbf{D}$ by $d_{pr}=\mathbf{B}[(\alpha' \cup \{s_p\})_N;(\beta' \cup \{t_r\})_N]$. Thus $\mathbf{D}$ is a matrix whose entries are bordered minors from $\mathbf{B}$, corresponding to the matrix C defined in Sylvester's Identity (III.F.3).

Observe that each $d_{pr} \geq 0$ by hypothesis (1) or (2), because each $\mathbf{B}[(\alpha' \cup \{s_p\})_N;(\beta' \cup \{t_r\})_N]$ is a submatrix of the first n or the last n columns of $\mathbf{B}$.

Apply Sylvester's Identity (III.F.3) to the $n \times n$ matrix $\mathbf{B}[(\alpha' \cup \{s_p,s_q\})_N,(\beta' \cup \{t_1,t_2\})_N]$ to see that, for $1 \leq p < q \leq m-n+2$ $\mathbf{D}(p,q;1,2)=g\mathbf{B}((\alpha' \cup (s_p,s_q))_N;(\beta' \cup (t_1,t_2))_N)$, again non-negative by hypothesis (1).

Therefore the first two columns of $\mathbf{D}$ are TP.

Repeating this argument using hypothesis (2) and t_2, t_3 shows that the last two columns of $\mathbf{D}$ are also TP.

Apply Sylvester's Identity once more, to $\mathbf{B}[\alpha;1,\ldots,\hat{k},\ldots,n+1]$, which has a negative determinant by hypothesis (3), to get $\mathbf{D}(u,v;1,3) = g\mathbf{B}(\alpha;1,\ldots,\hat{k},\ldots,n+1) < 0$. Since $\mathbf{D}$ is an $m' \times (n'+1)$ matrix with $n' = 2$ and $m' = m-(n-2) \geq 2$, it can be seen that $\mathbf{D}$ itself satisfies conditions (1), (2), (3) of the present lemma, and since (iv) has already been shown for the case $n = 2$, it may be concluded that column two of $\mathbf{D}$ is zero.

Now let $\mathbf{B}_0 = \hat{\mathbf{B}}[\varnothing;1,n+1]$. Since $g = \mathbf{B}(\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,a_n;2,\ldots,\hat{k},\ldots,n) \neq 0$, rows $\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,a_n$ of $\mathbf{B}_0$ are linearly independent. But since column two of $\mathbf{D}$ is zero, $d_{s2} = \mathbf{B}((\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,a_n,s)_{\mathrm{N}};2,\ldots,n) = 0$ for every $s \notin \{\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n\}$, $1 \leq s \leq m$, so rows $\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n,s$ of $\mathbf{B}_0$ are linearly dependent. Thus $\mathbf{B}_0$ has row rank, and hence column rank, of n-2, that is, columns 2 through n of $\mathbf{B}$ have rank n-2.

Columns $2,\ldots,\hat{k},\ldots,n$ of $\mathbf{B}$ are independent by (i). Thus, column k of $\mathbf{B}$ must be linearly dependent upon columns $2,\ldots,\hat{k},\ldots,n$, showing (ii).

A contradiction argument will be used to show (iii) for the case $n>2$. Suppose $\mathbf{B}(\mu;\nu) < 0$ for some $\mu \in \mathrm{Q}^{(q,m)}, \nu \in \mathrm{Q}^{(q,n+1)}$ with $1 \leq q < n$. By assumptions (1) and (2) one sees that $q \geq 2$ and $\nu_1 = 1, \nu_q = n+1$.

Let l be an index not in ν, so $2 \le l \le n$. Let $\tau = (\nu_1, ..., \nu_q, l)_N$; then $\tau_1 = 1 < l < \tau_{q+1} = n+1$. Let $\mathbf{E} = \mathbf{B}[\underline{m}; \tau] \in M_{m,q+1}$. Then the first q columns of $\mathbf{E}$ are a subset of the first n columns of $\mathbf{B}$ so form a TP matrix, by assumption (1), and similarly, by assumption (2), the last q columns of $\mathbf{E}$ form a TP matrix. Finally $\mathbf{E}(\mu; \tau_1, ..., \hat{l}, ..., \tau_{q+1}) = \mathbf{B}(\mu; \nu) < 0$. Since $\mathbf{E}$ now satisfies conditions (1), (2), (3) of the present lemma, one may conclude (ii) that column l of $\mathbf{E}$ is linearly dependent upon columns $\tau_2, ..., \hat{l}, ..., \tau_q$, implying the same is true for the same columns in matrix $\mathbf{B}$. Since this is true for each l such that $2 \le l \le n$ and $l \notin \nu$, it follows that columns 2 through n of $\mathbf{B}$ have rank at most q-2<n-2. this contradicts the previous conclusion from (i) and (ii) that columns 2,...,n of $\mathbf{B}$ have rank n-2. Thus all minors of $\mathbf{B}$ of order less than n are non-negative, hence (iii) is true.

Finally, conclusion (iv) will be shown for the case n>2, assuming now (1) through (4).

By (ii) column k of $\mathbf{B}$ is a linear combination of columns $2, ..., \hat{k}, ..., n$, or $\mathbf{b}_k = \sum_{l=2, l \ne k}^{n} u_l \mathbf{b}_l$ where each u_l is a constant and $\mathbf{b}_l$ denotes a column of $\mathbf{B}$. If all u_l are zero then column $\mathbf{b}_k$ is zero.

Assume $u_j \ne 0$ for some j such that $2 \le j \le n, j \ne k$. Then $u_j \mathbf{b}_j = \mathbf{b}_k - u_2 \mathbf{b}_2 - ... - u_n \mathbf{b}_n$, so multiples of columns other than $\mathbf{b}_j$ may be added to column $\mathbf{b}_k$ so that $u_j \mathbf{b}_j$ is the new column k. The value of the determinant is unchanged because these are replace-type elementary column operations.

The coefficient u_j may be factored out in an expansion down the column in position k when computing the determinant. Finally, placing column $\mathbf{b}_j$ in order through column exchange operations would alter the determinant by the multiple $(-1)^{(k-j)+1}$, where $(k\text{-}j)+1$ is the number of inversions (III.C.1). These facts yield the following equalities:

$$\mathbf{B}(\alpha;1,..,\hat{j},..,k,..,n+1) = u_j\mathbf{B}(\alpha;1,..,\hat{j},..,j,..,n+1) = (-1)^{(k-j)+1}u_j\mathbf{B}(\alpha;1,..,j,..,\hat{k},..,n+1).$$

Because the first minor is non-negative, while the last is strictly negative, and $u_j \neq 0$, it can be seen that $(-1)^{k-j+1}u_j < 0$.

By (i), columns $2,...,\hat{k},...,n$ of $\mathbf{B}$ have rank n-2, so there exists $\gamma \in Q^{(n-2,m)}$ such that $\mathbf{B}(\gamma;2,...,\hat{k},...,n) = g \neq 0$, and g>0, by (iii). Now $\mathbf{B}(\gamma;2,...,\hat{j},...,n) = (-1)^{(k-j)+1}u_j\mathbf{B}(\gamma;2,...,j,...,\hat{k},...,n)$, as above. But here the minor on the left is non-negative, while that on the right is positive, so $(-1)^{k-j+1}u_j > 0$, a contradiction.

Thus, the assumption that $u_j \neq 0$ for some index j in $\mathbf{b}_k = \sum_{l=2,l\neq k}^{n} u_l\mathbf{b}_l$ is false, and column k of $\mathbf{B}$ is zero, hence (iv) has been proved.

Q.E.D.

III.H CRYER'S LU-FACTORIZATION OF TOTALLY POSITIVE MATRICES

III.H.1 Theorem. Let $\mathbf{A}$ be $n \times n$. Then $\mathbf{A}$ is TP if and only if there exist TP lower triangular $\mathbf{L}$ and TP upper triangular $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$.

Proof:

First suppose such $\mathbf{L}$ and $\mathbf{U}$ exist. Let $\alpha, \beta \in Q^{(k,n)}$, $1 \le k \le n$ so $\mathbf{A}(\alpha;\beta)$ is an arbitrary minor of $\mathbf{A}$. By the hypothesis that $\mathbf{A} = \mathbf{LU}$ and the Cauchy-Binet Theorem (III.C.11), $\mathbf{A}(\alpha,\beta) = \sum_{\gamma \in Q^{(k,n)}} \mathbf{L}(\alpha;\gamma)\mathbf{U}(\gamma;\beta)$; this is non-negative since all minors of $\mathbf{L}$ and $\mathbf{U}$ are non-negative by hypothesis. Thus $\mathbf{A}$ is TP.

For the converse, suppose $\mathbf{A}$ is TP. If $\mathbf{A}$ is already lower triangular the proof is done. If not, then $\mathbf{A}$ has the following form, which will be called "form (k,m)":

$$\begin{bmatrix} a_{11} & & & & & 0 & & & & & \\ & \ddots & & & & & & & & & \\ & & a_{kk} & \cdots & a_{kl} & 0 & \cdots & 0 & a_{km} & 0 & \cdots & 0 \\ & & & * & & \vdots & & \vdots & & & * & \\ & & & & & 0 & \cdots & 0 & & & & \end{bmatrix}.$$

Precisely, form (k,m) means $\mathbf{A}$ satisfies:

1) the first k-1 rows are in lower triangular form (where k may be 1),
2) in row k, $a_{km} > 0$ for some $m>k$ and if $m<n$, $a_{k,m+1},\ldots,a_{kn} = 0$ and

3) if $a_{kl} > 0$ for some l with $k \le l < m$ and $a_{kj} = 0$ for some j between l and m, then column j is zero.

Condition 3) follows from the Zero Lemma (III.B.3), because $\mathbf{A}$ is TP.

An overview with two examples will be presented before completing the proof of the converse.

III.H.2 Overview

A Gaussian column reduction will be performed on $\mathbf{A}$, starting with the topmost row which has any nonzero entry to the right of the diagonal, and starting from the rightmost entry in that row. At each step one such entry will be eliminated by using an elementary column operation yielding a new matrix $\bar{\mathbf{A}}$ and a TP upper triangular $\mathbf{U}$ so that

(i) $\mathbf{A} = \bar{\mathbf{A}}\mathbf{U}$,

(ii) $\bar{\mathbf{A}}$ has form (p,q), where $p \ge k$ and if $p=k$, then $q<m$, and $\bar{a}_{km} = 0$,

(iii) $\bar{\mathbf{A}}$ is TP.

Cryer's Lemma is the essential tool for proving (iii).

The next step will eliminate an entry $\bar{a}_{k'm'}$ for which either m' or k' is larger than k, or both, and this guarantees the algorithm will eventually stop with a lower triangular $\bar{\mathbf{A}}$ which will be called $\mathbf{L}$, and which is TP. It will also be shown that at each step i, an upper triangular TP $\mathbf{U}_i$ can be defined so that ultimately $\mathbf{A} = \mathbf{L}\mathbf{U}_w\mathbf{U}_{w-1}\cdots\mathbf{U}_2\mathbf{U}_1$. The product $\mathbf{U}$ of the $\mathbf{U}_i$'s is upper triangular and is TP, by Cauchy-Binet (III.C.11). This will complete the proof that $\mathbf{A}$ has an LU-factorization with $\mathbf{L}$ and $\mathbf{U}$ TP.

When $\mathbf{A}$ has form (k,m), it is not too hard to find a column operation that yields $\bar{\mathbf{A}}$ and $\mathbf{U}$ so that $\mathbf{U}$ is TP and $\mathbf{A} = \bar{\mathbf{A}}\,\mathbf{U}$. This is particularly easy when the column operation is a column replacement using *adjacent* columns, because the inverse of the elementary matrix $\mathbf{E}$ which does the column replacement has a TP inverse. Then $\mathbf{AE} = \bar{\mathbf{A}}$, so letting $\mathbf{U} = \mathbf{E}^{-1}$ yields $\mathbf{A} = \bar{\mathbf{A}}\,\mathbf{U}$. For example, $\mathbf{U} = \mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is clearly TP and upper triangular.

However, to eliminate a_{km}, it can happen that one needs to do a column replacement type operation using nonadjacent columns. In this case the inverse of the associated elementary matrix $\mathbf{E}$ will be upper triangular but not TP. For example, $\mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not TP. One of Cryer's essential contributions was seeing how to get around this difficulty: instead of letting $\mathbf{U} = \mathbf{E}^{-1}$, he alters $\mathbf{E}^{-1}$ a little to get an upper triangular $\mathbf{U}$ which *is* TP and so that $\mathbf{A} = \bar{\mathbf{A}}\,\mathbf{U}$ will be true.

The new $\mathbf{U}$ includes an $r \times r$ diagonal block $\mathbf{F}_r(c) = \begin{bmatrix} 1 & & c \\ & & \\ & & 1 \end{bmatrix}$ in which r represents the order of $\mathbf{F}$, $c>0$, and all unmarked entries are zero.

Specifically, suppose $\mathbf{A}$ has form (k,m) and column l is used to eliminate a_{km}, getting $\tilde{\mathbf{A}}$. For $\mathbf{A}$ of the same size as $\mathbf{F}$, $\tilde{\mathbf{A}}\mathbf{F}_r(c)$ will always equal $\tilde{\mathbf{A}}\mathbf{E}^{-1}$ since each column of $\mathbf{A}$ between columns l and m is zero.. If $\mathbf{A}$ is larger than $\mathbf{F}_r(c)$, let $\mathbf{U} = \text{diag}(\mathbf{I}_p, \mathbf{F}_r(c), \mathbf{I}_q)$ where $0 \le p,q \le n-r$ and $p+q = n-r$.

III.H.3 Example:

Let $\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \sim \tilde{\mathbf{A}} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1/2 \\ 1 & 1 & 0 & 1/2 \\ 1 & 1 & 0 & 1/2 \end{bmatrix}$. Here $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is used to obtain $\mathbf{AE} = \tilde{\mathbf{A}}$, so $\mathbf{A} = \tilde{\mathbf{A}}\mathbf{E}^{-1}$ is true. However, $\mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is not TP. Instead of using $\mathbf{E}^{-1}$, define $\mathbf{U} = \text{diag}(\mathbf{I}_1, \mathbf{F}_3(1/2)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; then $\mathbf{A} = \tilde{\mathbf{A}}\mathbf{U}$ is also true and this $\mathbf{U}$ is upper triangular and TP.

One other difficulty can occur when $\mathbf{A}$ is singular; if $a_{kk} = 0$ and there is one $a_{km} > 0$, $m > k$, then a column exchange will be needed in order to create zero in the (k,m) position. The inverse of this type of elementary matrix is neither upper triangular nor TP. For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is its own inverse and is clearly not upper triangular or TP. Cryer overcomes this by substituting for the inverse of such an elementary matrix a TP $\mathbf{U}$ which includes a diagonal block $\mathbf{G}_r = \begin{bmatrix} & 1 \\ & \\ & 1 \end{bmatrix}$ that has zeros in every position except the upper and lower right corners. Again it is easy to see that $\mathbf{U}$ is TP, upper triangular and $\mathbf{A} = \tilde{\mathbf{A}}\,\mathbf{U}$.

III.H.4 Example:

Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 3 \\ 1 & 0 & 3 \end{bmatrix} \sim \tilde{\mathbf{A}} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix}$. Here $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ yields $\mathbf{AE} = \tilde{\mathbf{A}}$, and $\mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is neither upper triangular nor TP. However, if one lets $\mathbf{U} = \text{diag}(\mathbf{I}_1, \mathbf{G}_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then $\mathbf{A} = \tilde{\mathbf{A}}\,\mathbf{U}$, and this $\mathbf{U}$ is both upper triangular and TP.

So it is not too hard to produce $\mathbf{U}$ so that $\mathbf{A} = \tilde{\mathbf{A}}\,\mathbf{U}$ satisfies (i). If (iii) is known then it is easy so see that $\tilde{\mathbf{A}}$ satisfies (ii). Thus, the major work required is to prove (iii), that the new $\tilde{\mathbf{A}}$ at each step is TP. Once that is done, there will exist an algorithm for Gaussian column reduction of a TP matrix **A** to a lower triangular TP matrix **L**, which also produces a TP upper triangular **U** so that $\mathbf{A} = \mathbf{L}\mathbf{U}$.

End Overview

Here begins the proof that if **A** is TP and has form (*k*,*m*) then there exists a TP upper triangular matrix **U** such that

(i) $\mathbf{A} = \tilde{\mathbf{A}}\mathbf{U}$,

(ii) $\tilde{\mathbf{A}}$ has form (*p*,*q*), where $p \geq k$ and if *p*=*k*, then *q*<*m*, and $\tilde{a}_{km} = 0$,

(iii) $\tilde{\mathbf{A}}$ is TP.

Let a_{km} in matrix **A** of form (*k*,*m*) be the entry to be zeroed. Search to the left of a_{km} in the same row for the nearest non-zero entry a_{kl}. Two cases exist: (1) all entries a_{kk} through $a_{k,m-1}$ are zero, in which case columns *k* and *m* will be exchanged to create zero in the (*k*,*m*) position, (Cryer performs multiple column exchanges upon columns *m* and *m*-1 in successive iterations of the process; here a single exchange of column *k* with column *m* will be done), or (2) there exists $a_{kl} > 0, k \leq l < m$, in which case a multiple of column *l* will be added to column *k* to create zero in the (*k*,*m*) position.

<u>Case 1</u>: All entries a_{kk} through $a_{k,m-1}$ are zero.

In this case, let $\tilde{\mathbf{A}}$ be obtained from $\mathbf{A}$ by exchanging columns m and k.

If one lets $\mathbf{U} = \text{diag}(\mathbf{I}_{k-1}, \mathbf{G}_{m-k+1}, \mathbf{I}_{n-m})$, a TP upper triangular block diagonal matrix, then $\mathbf{A} = \tilde{\mathbf{A}}\mathbf{U}$.

Furthermore, since columns k to m-1 are all zero by the Zero Lemma (III.B.3) (a lower left block of zeros would also create a zero column since rows 1 to k-1 are zero above the diagonal), exchanging columns k and m in $\mathbf{A}$ to create $\tilde{\mathbf{A}}$ does not alter the non-negative value of any minor. Thus $\tilde{\mathbf{A}}$ is TP and also $\tilde{\mathbf{A}}$ now has form (p,q) with $\tilde{a}_{km} = 0$, which completes Case 1.

<u>Case 2</u>: There exists $a_{kl} > 0,\ k \le l < m$.

Let $\tilde{\mathbf{A}}$ be obtained from $\mathbf{A}$ by subtracting a_{km} / a_{kl} times column l from column m. Letting $\mathbf{U} = \text{diag}(\mathbf{I}_{l-1}, \mathbf{F}_{m-l+1}(a_{km} / a_{kl}), \mathbf{I}_{n-m})$, a TP upper triangular block diagonal matrix, then $\mathbf{A} = \tilde{\mathbf{A}}\mathbf{U}$ satisfies (i) and (ii) as well as the requirements of $\mathbf{U}$.

If l=m-1, $\mathbf{U} = \text{diag}(\mathbf{I}_{l-1}, \mathbf{F}_2(a_{km} / a_{kl}), \mathbf{I}_{n-m}) = \begin{bmatrix} \mathbf{I}_{l-1} & & \\ & \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} & \\ & & \mathbf{I}_{n-m} \end{bmatrix}$ where $c = a_{km} / a_{kl}$.

If $l<m-1$, $\mathbf{U} = \begin{bmatrix} \mathbf{I}_{l-1} & & \\ & \begin{bmatrix} 1 & c \\ & \\ 0 & 1 \end{bmatrix} & \\ & & \mathbf{I}_{n-m} \end{bmatrix}$, where $c = a_{km} / a_{kl}$ and all unmarked entries are zero.

Now, it will be shown for Case 2 that (iii) $\tilde{\mathbf{A}}$ is TP, by induction on the size of minors.

Define H(s): $\tilde{\mathbf{A}}(\alpha;\beta) \geq 0$ for all $\alpha,\beta \in \mathrm{Q}^{(r,n)}$ where $1 \leq r \leq s$, that is, all minors of size $s \times s$ and smaller are non-negative.

First, H(1) is true: that is, $\tilde{\mathbf{A}}(\alpha;\beta) \geq 0$ for all $\alpha,\beta \in \mathrm{Q}^{(1,n)}$. To prove this, observe that $\mathbf{A}$ and $\tilde{\mathbf{A}}$ differ only in column m, so $\tilde{a}_{ij} = a_{ij} \geq 0$ if $j \neq m$. Within $\mathbf{A}$, columns l and m are zero above row k, so a replacement-type operation involving these two columns still has zeros in column m above row k. By construction, $\tilde{a}_{km} = 0$. Finally, if $i>k$, then because a multiple of column l was subtracted from column m, $\tilde{a}_{im} = a_{im} - (a_{km} / a_{kl})a_{il}$

$= (1 / a_{kl})(a_{kl}a_{im} - a_{km}a_{il})$

$= (1 / a_{kl})\det\begin{bmatrix} a_{kl} & a_{km} \\ a_{il} & a_{im} \end{bmatrix} \geq 0$ since $\mathbf{A}$ is TP. Thus all minors of size 1 in $\tilde{\mathbf{A}}$ are non-negative.

Now let $t \geq 2$ and assume H(*t-1*) is true: that is, $\tilde{A}(\alpha;\beta) \geq 0$ for all $\alpha,\beta \in Q^{(r,n)}$ where $1 \leq r \leq t-1$.

Suppose H(*t*) is not true; that is, there exist $\alpha,\beta \in Q^{(t,n)}$ such that $\tilde{A}(\alpha;\beta) < 0$. Since only column *m* has been changed in $\tilde{A}$, $m \in \beta$; otherwise the submatrix $\tilde{A}[\alpha;\beta] = A[\alpha;\beta]$, hence $\tilde{A}(\alpha;\beta) = A(\alpha;\beta) \geq 0$.

Also $l \notin \beta$; otherwise the submatrix $\tilde{A}[\alpha;\beta]$ would be directly obtained from $A[\alpha;\beta]$ by a replacement-type elementary column operation, which does not change the determinant, so $\tilde{A}(\alpha;\beta) = A(\alpha;\beta) \geq 0$. Since $m \in \beta$, $l \notin \beta$, and $l<m$, either $l < \beta_1$ or $\beta_{q-1} < l < \beta_q$ for some least $q \geq 2$. If $l < \beta_1$, let *q*=1. Define $\nu = (\beta_1,\ldots,\beta_t,l)_N \in Q^{(t+1,n)}$.

Since $l < m \in \beta$, then $q \leq t$. Examine the two mutually exclusive and exhaustive cases: (a) *q*=1 (i.e.: $l < \beta_1$), and (b) $2 \leq q \leq t$.

<u>Case 2.a</u>: Let *q*=1, i.e., $\nu = (l,\beta_1,\ldots,\beta_t)$.

Then $\alpha_1 > k$, for $l < \beta_1$ means all β is to the right of *l*, so if $\alpha_1 \leq k$ then the first row of $\tilde{A}[\alpha;\beta]$ is zero since $\tilde{A}$ has form (*k,m*); but this makes $\tilde{A}(\alpha;\beta) = 0$, while it is required that $\tilde{A}(\alpha;\beta) < 0$.

Let $\mu = \{k, \alpha_1, \ldots, \alpha_t\} \in Q^{(t+1,n)}$ and expand about the first row of $\tilde{A}[\mu; \nu]$ to obtain $\tilde{A}(\mu; \nu) = a_{kl}\, \tilde{A}(\alpha; \beta)$, since all entries beyond column l in row k of $\tilde{A}$ are zero. But since $\tilde{A}[\mu; \nu]$ contains both columns l and m and was obtained from $A[\mu; \nu]$ by a replacement-type elementary column operation involving columns l and m, $\tilde{A}(\mu; \nu) = A(\mu; \nu)$. Thus, $A(\mu; \nu) = a_{kl}\, \tilde{A}(\alpha; \beta)$, hence $\tilde{A}(\alpha; \beta) = A(\mu; \nu) / a_{kl} \geq 0$ since A is TP and $a_{kl} > 0$ in the present Case 2.

This contradicts the assumption that $\tilde{A}(\alpha; \beta) < 0$; thus H($t$) is true when q=1.

Case 2.b: Let $2 \leq q \leq t$, i.e., $\nu = (\beta_1, \ldots, \beta_{q-1}, l, \beta_q, \ldots, \beta_t)$.

Let B be the $t \times (t+1)$ array $B = \tilde{A}[\alpha; \nu]$.

First it will be shown that B satisfies the four conditions in Cryer's Lemma (III.G.1):

(1) the first t columns of B form a TP matrix,

(2) the last t columns of B form a TP matrix,

(3) $B(\alpha; \nu_1, \ldots, \hat{\nu}_q, \ldots, \nu_{t+1}) < 0$ for some $\alpha \in Q^{(t,t)}$ and some q with $2 \leq q \leq t$, and

(4) $B(\alpha; \nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_{t+1}) \geq 0$ if $2 \leq j \leq t$ and $j \neq q$.

Since $\nu_q = l$, then $\hat{\mathbf{B}}(\varnothing; \nu_q) = \tilde{\mathbf{A}}(\alpha; \beta) < 0$ by assumption, so Cryer's condition (3) is true.

Furthermore, if $j \neq q$, meaning $\nu_j \neq l$, then either both columns l and m are in $\hat{\mathbf{B}}[\varnothing; \nu_j] = \tilde{\mathbf{A}}[\alpha; \nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_{t+1}]$ and the replacement-type elementary column operation required to obtain $\tilde{\mathbf{A}}[\alpha; \nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_{t+1}]$ from $\mathbf{A}[\alpha; \nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_{t+1}]$ does not change the determinant; or $\nu_j = m$ and no column of $\mathbf{A}[\alpha; \nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_{t+1}]$ gets changed, so $\hat{\mathbf{B}}[\varnothing; \nu_j] = \tilde{\mathbf{A}}[\alpha; \nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_{t+1}] = \mathbf{A}[\alpha; \nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_{t+1}]$, and the determinants are the same. In either case $\hat{\mathbf{B}}(\varnothing; \nu_j) = \tilde{\mathbf{A}}(\alpha; \nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_{t+1}) = \mathbf{A}(\alpha; \nu_1, \ldots, \hat{\nu}_j, \ldots, \nu_{t+1})$ and this is non-negative because $\mathbf{A}$ is TP. This is true for any j, $1 \leq j \leq t+1$, $j \neq q$, so in particular, Cryer's condition (4) is true.

Finally, by the inductive hypothesis H(t-1), all subdeterminants of $\mathbf{B}$ of order less than t are non-negative, and $\hat{\mathbf{B}}(\varnothing; \nu_{t+1}) \geq 0$ and $\hat{\mathbf{B}}(\varnothing; \nu_1) \geq 0$ from the preceding paragraph, hence the first and last t columns of $\mathbf{B}$ are TP, i.e., Cryer's conditions (1) and (2) are true.

Thus, it may be concluded by Cryer's Lemma, that column $v_q = l$ of $\mathbf{B}$ is zero, that is, $a_{il} = 0$ if $i \in \alpha$. Hence $\tilde{a}_{im} = a_{im} - (a_{km} / a_{kl}) a_{il} = a_{im} \geq 0$ if $i \in \alpha$. Since only column m of $\tilde{\mathbf{A}}$ could differ from the corresponding column of $\mathbf{A}$, and since in rows $\alpha_1, \ldots, \alpha_t$ column m had no changes, then $\tilde{\mathbf{A}}(\alpha;\beta) = \mathbf{A}(\alpha;\beta) \geq 0$. This contradicts the assumption that $\tilde{\mathbf{A}}(\alpha;\beta) < 0$; thus H($t$) is true when $2 \leq q \leq t$.

This completes the proof of (iii) for Case 2.

Now the Gaussian column reduction described above can be repeated until $\tilde{\mathbf{A}}$ is TP lower triangular, because at each step the new $\tilde{\mathbf{A}}$ is TP and has form (k', m') where either the row index k' is higher, or if the row index is not changed, the column index m' is lower. Thus the process is guaranteed to stop eventually with a lower triangular TP matrix. One may then write $\mathbf{A} = \tilde{\mathbf{A}} \mathbf{U}_w \mathbf{U}_{w-1} \cdots \mathbf{U}_2 \mathbf{U}_1$ where $\mathbf{U}_i$ is the TP upper triangular matrix created at step i. By the Cauchy-Binet Theorem (III.C.11), $\mathbf{U} = \mathbf{U}_w \mathbf{U}_{w-1} \cdots \mathbf{U}_2 \mathbf{U}_1$ is TP; it is also upper triangular. Renaming $\tilde{\mathbf{A}}$ to $\mathbf{L}$, this completes the proof that, if $\mathbf{A}$ is TP then there exist TP lower triangular $\mathbf{L}$ and TP upper triangular $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$.

Q.E.D.

IV LU-Factorization of Strictly Totally Positive Matrices

IV.A OVERVIEW

Recall from Section (I.B) the following definitions:

A real matrix is **strictly totally positive** (STP) if all its minors are strictly positive.

A triangular real matrix is **triangular strictly totally positive** (ΔSTP) if all its non-trivial minors are strictly positive.

Although by definition every minor of a matrix $\mathbf{A}$ must be positive for $\mathbf{A}$ to be STP, various criteria have been discovered which reduce the number of minors requiring testing. An example is a result cited in [KAR: 60] which states that for an $n \times n$ matrix $\mathbf{A}$, if all minors formed from k consecutive rows and consecutive columns are positive for $1 \le k \le n$, then $\mathbf{A}$ is STP.

It will be shown in Section (IV.C) below that for a square real matrix $\mathbf{A}$, $\mathbf{A}$ is STP if and only if $\mathbf{A}$ has an LU-factorization such that $\mathbf{L}$ and $\mathbf{U}$ are ΔSTP. This yields a more efficient test for the STP property, and is the main theorem in C. W. Cryer's paper, "The LU-Factorization of Totally Positive Matrices."

Cryer's proof of this factorization theorem employs an important test for triangular strict positivity, which will be presented first, in Section (IV.B).

IV.B TESTS FOR TRIANGULAR STRICT TOTAL POSITIVITY

The main theorem of this section (IV.B.6), which is needed for Cryer's theorem in the next section, demonstrates special criteria for determining whether a lower triangular matrix is ΔSTP, namely: if every minor formed from consecutive *initial* columns of a lower triangular matrix **A** is positive, then **A** is ΔSTP. (Analogously, if every minor formed from consecutive initial rows of an upper triangular matrix **A** is positive, then **A** is ΔSTP.)

Six lemmas which are used in the proof are presented first.

Recall from (I.B) the definition of discrepancy in indices: if $\alpha \in Q^{(p,n)}$, the **discrepancy** of α is defined as $d(\alpha) = \sum_{i=1}^{p-1}(\alpha_{i+1} - \alpha_i - 1)$. Discrepancy can be thought of as a measurement of consecutivity in a string. If $d(\alpha) = 0$ then α is a string of consecutive integers; if $d(\alpha) > 0$, there is a gap in the string.

IV.B.1 Lemma: Let **A** be an $m \times p$ matrix, $m \geq p$, such that

(1) $\mathbf{A}(\alpha;\underline{p}) > 0$ for every $\alpha \in Q^{(p,m)}$ with $d(\alpha) = 0$, and

(2) $\mathbf{A}(\alpha;\underline{p-1}) > 0$ for every $\alpha \in Q^{(p-1,m)}$.

Then $\mathbf{A}(\alpha;\underline{p}) > 0$ for every $\alpha \in Q^{(p,m)}$.

Proof:

The proof will be by induction on the discrepancy of α. Let $\alpha \in Q^{(p,m)}$. If $d(\alpha) = 0$, then $A(\alpha;\underline{p}) > 0$ by (1). Suppose the result is true for $d(\alpha) < \delta$, where $\delta \geq 1$, and let $\alpha \in Q^{(p,m)}$ have $d(\alpha) = \delta$.

Since $\delta \geq 1$, let α_0 be the first integer omitted which is greater than α_1. Define $C = A[(\alpha \cup \{\alpha_0\})_N;\underline{p}] \in M_{p+1,p}$. Then by the Gantmacher-Krein Identity (III.E.3), letting $\alpha_1, \alpha_0, \alpha_p$ be rows 1, k, p+1, respectively and p be column i $\hat{C}(1,k;p)\hat{C}(p+1;\varnothing) - \hat{C}(1,p+1;p)\hat{C}(k;\varnothing) + \hat{C}(k,p+1;p)\hat{C}(1;\varnothing) = 0$, or

$A(\alpha_2,\ldots,\alpha_p;\underline{p-1})A(\alpha_1,\ldots,\alpha_0,\ldots,\alpha_{p-1};\underline{p}) - A(\alpha_2,\ldots,\alpha_0,\ldots,\alpha_{p-1};\underline{p-1})A(\alpha;\underline{p})$
$+A(\alpha_1,\ldots,\alpha_{p-1};\underline{p-1})A(\alpha_2,\ldots,\alpha_0,\ldots,\alpha_p;\underline{p}) = 0.$

The first factor in each term is positive, by (2). The second factors in the first and third terms are positive by the inductive hypothesis, since inserting α_0 in α reduces the discrepancy among any p consecutive indices in $(\alpha \cup \{\alpha_0\})_N$. Thus, the second factor in the middle term must also be positive, i.e.: $A(\alpha;\underline{p}) > 0$.

Q.E.D.

IV.B.2 **Corollary:** Let **A** be an $m \times p$ matrix, $m \geq p$, such that

(1) $A(\alpha;\underline{p}) > 0$ for every $\alpha \in Q^{(p,m)}$ with $d(\alpha) = 0$, and

(2) $A(\alpha;2,\ldots,p) > 0$ for every $\alpha \in Q^{(p-1,m)}$.

Then $A(\alpha;\underline{p}) > 0$ for every $\alpha \in Q^{(p,m)}$.

Proof:

This corollary (IV.B.2) differs from (IV.B.1) only in that assumption (2) regards minors formed from the last p-1 columns, rather than the first. This proof lets 1 be column i in the Gantmacher-Krein Identity (III.E.4), and is otherwise identical to the proof of (IV.B.1).

Q.E.D.

IV.B.3 Corollary: Let $\mathbf{A}$ be a $p \times m$ matrix, $p \le m$, such that

(1) $\mathbf{A}(\underline{p};\beta) > 0$ for every $\beta \in \mathrm{Q}^{(p,m)}$ with $\mathrm{d}(\beta) = 0$, and

(2) $\mathbf{A}(2,\ldots,p;\beta) > 0$ for every $\beta \in \mathrm{Q}^{(p-1,m)}$.

Then $\mathbf{A}(\underline{p};\beta) > 0$ for every $\beta \in \mathrm{Q}^{(p,m)}$.

Proof:

Apply Corollary (IV.B.2) to $\mathbf{A}^{T}$.

Q.E.D.

IV.B.4 Lemma: If $\mathbf{A} \in \mathrm{M}_n$ is lower triangular and

(*) $\mathbf{A}(\alpha;\underline{p}) > 0$ for every $p \in \underline{n}$ and for every $\alpha \in \mathrm{Q}^{(p,n)}$ with $\mathrm{d}(\alpha) = 0$, then

(i) $\mathbf{A}(\alpha;\underline{p}) > 0$ for every $p \in \underline{n}$ and for every $\alpha \in \mathrm{Q}^{(p,n)}$, and

(ii) $\hat{A}[1;1]$ satisfies (i): i.e., $A(\alpha;2,...r+1)>0$ for every $r \in \underline{n-1}$ and $\alpha \in Q^{(r,n)}$ such that $\alpha_1 \geq 2$. In particular, $\hat{A}[1;1]$ satisfies (*).

Note again that $d(\alpha)=0$ means that α is a string of consecutive integers, here yielding a set of consecutive rows.

Proof:

(i) The proof will be by induction on the size of minor, p. $A(\alpha;1)>0$ by (*). Let $p \geq 2$ and $q=p-1$ and assume (a) $A(\alpha;\underline{q})>0$, where $\alpha \in Q^{(q,n)}$. Note also (b) if $\alpha \in Q^{(p,n)}$ and $d(\alpha)=0$ then $A(\alpha;\underline{p})>0$ by (*).

By (b) and (a), $A[\underline{n};\underline{p}]$ satisfies hypotheses (1) and (2) of Lemma (IV.B.1). Thus $A(\alpha;\underline{p})>0$ for every $\alpha \in Q^{(p,n)}$, showing (i).

(ii) Let $\alpha \in Q^{(r,n)}$, $r \in \underline{n-1}$, such that $2 \leq \alpha_1$. It must be shown that $A(\alpha;2,...,r+1)>0$. Let $\alpha'=\{1\}\cup\alpha$. By (i), $a_{11}>0$ and $A(\alpha';\underline{r+1})>0$ are true. A is lower triangular, so $A[\alpha';\underline{r+1}]=\begin{bmatrix} a_{11} & 0 \\ * & A[\alpha;2,...,r+1] \end{bmatrix}$, thus $A(\alpha';\underline{r+1})=a_{11}A(\alpha;2,...,r+1)$. Therefore, $A(\alpha;2,...,r+1)>0$, i.e., $\hat{A}[1;1]$ satisfies (i), showing (ii).

Q.E.D.

Recall from (I.B) that if $\alpha,\beta \in Q^{(p,n)}$, the notation $\alpha \geq \beta$ means $\alpha_i \geq \beta_i$ for each i, $1 \leq i \leq p$. The notation $\alpha > \beta$, $\alpha \leq \beta$ and $\alpha < \beta$ are defined analogously.

IV.B.5 **Lemma:** Let $\alpha, \beta \in Q^{(p,n)}$ with $\alpha > \beta$. Suppose there exists $k \geq 2$ such that $\beta = (1, \ldots, k-1, \beta_k, \ldots, \beta_p)$, $\beta_k > k$.

Define $\eta = (2, \ldots, k-1, k, \beta_k, \ldots, \beta_p) \in Q^{(p,n)}$.

Then $\alpha \geq \eta$.

Proof:

If $r = 1, \ldots, k-1$ then $\alpha_r > \beta_r = r$ so $\alpha_r \geq r + 1 = \eta_r$.

If $r = k, \ldots, p$ then $\alpha_r > \beta_r = \eta_r$.

Q.E.D.

IV.B.6 **Lemma:** Let $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ are square matrices. Then $\det \mathbf{A} = \det \mathbf{B} \cdot \det \mathbf{D}$.

Proof:

Let p be the order of $\mathbf{A}$, and k be the order of $\mathbf{B}$, for $1 \leq k < p$.

If k=1, perform a Laplace expansion (III.C.4) across row 1 to get:

$$\det \mathbf{A} = \sum_{j=1}^{p} (-1)^{1+j} a_{1j} \det \mathbf{A}_{1j}$$

$$= \sum_{j=1}^{p} (-1)^{1+j} a_{1j} \hat{\mathbf{A}}(1;j)$$

$$= (-1)^{1+1} a_{11} \hat{\mathbf{A}}(1;1)$$

$$= \det \mathbf{B} \cdot \det \mathbf{D}.$$

If $k>1$, perform an expansion of the determinant of $\mathbf{A}$ across rows 1 to k (III.D.1). Because $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, columns $k+1$ to p are zero in rows 1 to k.

Thus all terms of the expansion but one are zero, and the expansion becomes:

$$\det \mathbf{A} = \sum_{\gamma \in Q^{(k,p)}} (\operatorname{sgn} \gamma \hat{\gamma}) \mathbf{A}(\underline{k}; \gamma_1, \ldots, \gamma_k) \mathbf{A}(k+1, \ldots, p; \hat{\gamma}_1, \ldots, \hat{\gamma}_{p-k})$$

$$= +\mathbf{A}(\underline{k}; \underline{k}) \mathbf{A}(k+1, \ldots, p; k+1, \ldots, p)$$

$$= \det \mathbf{B} \cdot \det \mathbf{D}.$$

Q.E.D.

IV.B.7 **Theorem:** Consecutive Leading Columns Theorem. Let $\mathbf{A}$ be an $n \times n$ lower triangular matrix. Then $\mathbf{A}$ is $\Delta\mathrm{STP}$ if and only if

(*) $\mathbf{A}(\alpha; \underline{p}) > 0$ for all $\alpha \in Q^{(p,n)}$, $1 \le p \le n$, such that $d(\alpha) = 0$.

Proof:

The sufficiency of the hypothesis is clear. The necessity will be shown by induction on n, the size of $\mathbf{A}$. If $n = 1$ or 2 the result, $\mathbf{A}$ is $\Delta\mathrm{STP}$, follows immediately from (*). Now let $n \ge 3$ and suppose the result is true for smaller matrices. This will be referred to as "the inductive hypothesis on size."

Observe that $\hat{\mathbf{A}}[n;n]$ and $\hat{\mathbf{A}}[1;1]$ are $\Delta\mathrm{STP}$. Clearly (*) is true for $\hat{\mathbf{A}}[n;n]$. Because (*) holds for $\mathbf{A}$, minors formed from consecutive initial columns in

$\hat{A}[1;1]$ are also positive by Lemma (IV.B.4.ii). Thus, by the inductive hypothesis on size, both these lower triangular matrices are ΔSTP.

Now let $\alpha, \beta \in Q^{(p,n)}$, $1 \le p \le n$, $\alpha \ge \beta$. The arbitrary non-trivial minor $A(\alpha;\beta)$ will now be evaluated for strict positivity. Three exhaustive cases will be considered.

Case 1: $A[\alpha;\beta]$ is a submatrix of $\hat{A}[\varnothing;1]$ or of $\hat{A}[n;\varnothing]$.

If $A[\alpha;\beta]$ is in $\hat{A}[\varnothing;1]$, then $\beta_1 > 1$ and $\alpha_1 \ge \beta_1 > 1$ so $A[\alpha;\beta]$ is in $\hat{A}[1;1]$, which is ΔSTP. If $A[\alpha;\beta]$ is in $\hat{A}[n;\varnothing]$, then $\alpha_p < n$ and $\beta_p \le \alpha_p < n$ so $A[\alpha;\beta]$ is in $\hat{A}[n;n]$, which is ΔSTP. Thus for each, $A(\alpha;\beta) > 0$.

Case 2: $A[\alpha;\beta]$ has $\alpha_r = \beta_r$ for some r, $1 \le r \le p$.

Recall that $A[\alpha;\beta]$ is a submatrix of order p, and let r be the first index where $\alpha_r = \beta_r$.

If r=1, then $A[\alpha;\beta]$ is a principal submatrix of A, hence its determinant is positive because $\hat{A}[n;n]$ and $\hat{A}[1;1]$ are ΔSTP, so all diagonal entries are positive.

If $1 < r \le p$, then because A is lower triangular,

$$A[\alpha;\beta] = \begin{bmatrix} A[\alpha_1,\ldots,\alpha_{r-1};\beta_1,\ldots,\beta_{r-1}] & 0 \\ * & A[\alpha_r,\ldots,\alpha_p;\beta_r,\ldots,\beta_p] \end{bmatrix}.$$

Then $A(\alpha;\beta)$ is the product of the determinants of the diagonal blocks, by (IV.B.6), and each of those is positive because $\hat{A}[n;n]$ and $\hat{A}[1;1]$ are ΔSTP.

Therefore $A(\alpha;\beta)>0$.

Case 3: $\beta_1 = 1$, $\alpha_p = n$ and $\alpha > \beta$.

An induction on the length of α,β will be performed to show that a submatrix A $[\alpha;\beta]$ of any order with the above conditions has positive determinant.

Recall that $\alpha,\beta \in Q^{(p,n)}$. If p=1, then $A(\alpha;\beta)>0$ by the hypothesis (*). Suppose p>1 and $A(\alpha';\beta')>0$ for all $\alpha',\beta' \in Q^{(s,n)}$ where $s<p$ and $\alpha' \geq \beta'$. This will be referred to as the "inductive hypothesis on length."

Now induct on column discrepancy. If $d(\beta)=0$, then $A(\alpha;\beta)>0$ by (*). Suppose that $d(\beta)>0$ and assume that $A(\alpha;\tilde{\beta})>0$ whenever $\alpha,\tilde{\beta} \in Q^{(p,n)}$, $1 \leq p \leq n$ with $\alpha \geq \tilde{\beta}$ and $d(\tilde{\beta}) < d(\beta)$. This will be called the "inductive hypothesis on discrepancy."

Because $\beta_1 = 1$ and $d(\beta)>0$ there exists a least k such that $k < \beta_k$ and $k > 1$. Construct a $p \times (p+1)$ matrix $C = A[\alpha;(\beta \cup \{k\})_N]$. It will now be shown that C satisfies hypotheses (1) and (2) of Corollary (IV.B.3). Once this has been done, $C(\underline{p};\gamma)$ will be positive for every $\gamma \in Q^{(p,p+1)}$, in particular $C(\underline{p};\beta_1,\ldots,\hat{k},\beta_k,\ldots,\beta_p) = A(\alpha;\beta)>0$.

Let $\gamma \in Q^{(p,p+1)}$ such that $d(\gamma)=0$. Then $\gamma=(1,\ldots,p)$ or (2,...,p+1). In the first case, $C[\underline{p};\gamma] = C[\underline{p};\underline{p}] = A[\alpha;\tilde{\beta}]$ where $\tilde{\beta} = (\beta_1,\ldots,k,\beta_k,\ldots,\beta_{p-1})$. Then $\alpha \geq \tilde{\beta}$ is clear because $\alpha \geq \beta$ and, because of the insertion of column k, $d(\tilde{\beta}) < d(\beta)$, so $A(\alpha;\tilde{\beta}) = C(\underline{p};\gamma) > 0$ by the inductive hypothesis on discrepancy. Similarly, in the second case $C[\underline{p};\gamma] = C[\underline{p};2,\ldots,p+1] = A[\alpha;\tilde{\beta}]$ where $\tilde{\beta} = (\beta_2,\ldots,k,\beta_k,\ldots,\beta_p)$ and again $\alpha \geq \tilde{\beta}$ is true, by Lemma (IV.B.5) this time (since $\alpha > \beta$ here in Case 3), and $d(\tilde{\beta}) < d(\beta)$, so $A(\alpha;\tilde{\beta}) = C(\underline{p};\gamma) > 0$. Thus hypothesis (1) of (IV.B.3) is satisfied.

Now let $\gamma \in Q^{(p-1,p+1)}$, so that $C[2,\ldots,p;\gamma] = A[\alpha';\beta']$ where $\alpha' = (\alpha_2,\ldots,\alpha_p)$ and β' is some (p-1)-tuple in $(\beta \cup \{k\})_N$. Observe that, whatever β' is, $\alpha' \geq \beta'$ is true. For $\alpha > \beta$ (in Case 3) and Lemma (III.B.5) imply $\alpha \geq (\beta_2,\ldots,k,\beta_k,\ldots,\beta_p)$, hence $\alpha' \geq (\beta_3,\ldots,k,\beta_k,\ldots,\beta_p)$ (where k would not appear if $k < \beta_2$). Since $(\beta_3,\ldots,k,\beta_k,\ldots,\beta_p)$ is the largest (p-1)-tuple possible, $(\beta_3,\ldots,k,\beta_k,\ldots,\beta_p) \geq \beta'$. Therefore, by the inductive hypothesis on length, $A(\alpha';\beta') = C(2,\ldots,p;\gamma) > 0$. Thus hypothesis (2) of (IV.B.3) is also satisfied.

Therefore Corollary (IV.B.3) applies here, i.e., $C(\underline{p};\gamma) > 0$ for all $\gamma \in Q^{(p,p+1)}$. In particular $C(\underline{p};\beta_1,\ldots,\hat{k},\beta_k,\ldots,\beta_p) = A(\alpha;\beta) > 0$.

This completes the induction on discrepancy and that in turn completes the induction on length. Thus the non-trivial minors $A(\alpha;\beta)$ in all three cases have been shown to be positive. Hence, the induction on size is complete and it may be concluded that A is ΔSTP.

Q.E.D.

IV.B.8 **Corollary: Consecutive Leading Rows Theorem.** Let **A** be an $n \times n$ upper triangular matrix. Then **A** is ΔSTP if and only if $\mathbf{A}(\underline{p};\beta) > 0$ for all $\beta \in Q^{(p,n)}$, $1 \le p \le n$ such that $d(\beta) = 0$.

Proof:

Apply (III.B.7) to $\mathbf{A}^T$.

Q.E.D.

IV.C CRYER'S LU-FACTORIZATION OF STRICTLY TOTALLY POSITIVE MATRICES

IV.C.1 Theorem: Let $\mathbf{A}$ be a square matrix. Then $\mathbf{A}$ is STP if and only if there exist ΔSTP lower triangular $\mathbf{L}$ and ΔSTP upper triangular $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$.

Proof:

Let $\mathbf{A}$ be $n \times n$.

First suppose such ΔSTP $\mathbf{L}$ and $\mathbf{U}$ exist. Let $\alpha, \beta \in Q^{(p,n)}$, $1 \le p \le n$, so $\mathbf{A}(\alpha;\beta)$ is an arbitrary minor of $\mathbf{A}$. Then by the Cauchy-Binet Theorem (III.C.11), since each non-trivial minor of $\mathbf{L}$ and $\mathbf{U}$ is positive,

$$\mathbf{A}(\alpha;\beta) = \sum_{\gamma \in Q^{(p,n)}} \mathbf{L}(\alpha;\gamma)\mathbf{U}(\gamma;\beta)$$

$$\ge \mathbf{L}(\alpha;\underline{p})\mathbf{U}(\underline{p};\beta)$$

$$> 0.$$

Thus $\mathbf{A}$ is STP.

For the converse, suppose $\mathbf{A} \in M_n$ is STP. Then all minors are positive; in particular, $\mathbf{A}$ is non-singular and all leading principal minors are non-zero. Thus, by Theorem (II.D.4) $\mathbf{A}$ has an LU-factorization, and by (II.C.4) $\mathbf{L}$ may be chosen to be unit lower triangular.

To show that **U** is ΔSTP, let $\beta, \gamma \in Q^{(p,n)}$ and apply a Cauchy-Binet expansion (III.C.11) to $\mathbf{A}(\underline{p};\beta)$, considering only the terms formed with non-trivial minors of **L** and **U**, i.e., $\mathbf{L}(\underline{p};\gamma)\mathbf{U}(\gamma;\beta)$ with $\gamma \le \beta, \underline{p}$ (I.B). Because $\gamma \le \beta$ refers to the partial ordering where $\gamma_k \le \beta_k$ for $k \in (1,\ldots,p)$ (I.B), $\gamma \le \underline{p}$ thus means $\gamma = \underline{p}$. So

$0 < \mathbf{A}(\underline{p};\beta) = \sum_{\gamma \subset Q^{(p,n)}, \gamma \le \beta, \underline{p}} \mathbf{L}(\underline{p};\gamma)\mathbf{U}(\gamma;\beta) = \mathbf{L}(\underline{p};\underline{p})\mathbf{U}(\underline{p};\beta) = \mathbf{U}(\underline{p};\beta)$, because **L** has a unit diagonal. Thus an arbitrary minor formed from consecutive initial rows of **U** is positive, so by Theorem (IV.B.8), the Consecutive Leading Rows Theorem, **U** is ΔSTP.

Analogously, to show that **L** is ΔSTP, this time let $\alpha \in Q^{(p,n)}$ and apply the Cauchy-Binet expansion (III.C.11) to $\mathbf{A}(\alpha;\underline{p})$ to get

$0 < \mathbf{A}(\alpha;\underline{p}) = \sum_{\gamma \in Q^{(p,n)}, \gamma \le \beta, \underline{p}} \mathbf{L}(\alpha;\gamma)\mathbf{U}(\gamma;\underline{p}) = \mathbf{L}(\alpha;\underline{p})\mathbf{U}(\underline{p};\underline{p})$. Since $\mathbf{U}(\underline{p};\underline{p}) > 0$ by above, it must be true that $\mathbf{L}(\alpha;\underline{p}) > 0$. Thus, by Theorem (IV.B.7), **L** is ΔSTP.

Therefore if **A** is STP then **A** has an LU-factorization with **L** and **U** ΔSTP.

Q.E.D.

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