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G-ideals in ring extensions

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G- IDEALS IN RING EXTENSIONS

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ABSTRACT

What follows is a comparative look at G-ideals in a ring R and in a ring T where T is considered first as an integral extension of R and second as a polynomial extension of R.

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TABLE OF CONTENTS

0.1
Preliminary

There are several properties that may or may not hold for the pair of rings $R \subset T$. We will continually be looking at two situations: T as an integral extension of R and T as a polynomial extension, R[x]. The ring R will always be a commutative ring with unity and x will be an indeterminate. We are concerned in Chapter 1 with finding relations between prime ideals in R and those in T, in Chapter 2 with defining G-ideals, and in Chapter 3 with finding relations between G-ideals in R and those in T. Throughout, the symbol $<$ will denote strict inclusion and the term quotient field will be abbreviated q.f. For convenience, we now state the properties of most interest to us in their general form. The definitions are from [1].

For the following, P_0 and P are ideals in R, and Q_0 and Q are ideals in T. In Chapter 1 we consider them as prime ideals and in Chapter 3 as G-ideals.

Lying Over Property: Given $R \subset T$, $P \subset R$ there exists $Q \subset T$ such that $Q \cap R = P$. We also say in this case that Q *contracts* to P.

Going Up Property: Given $R \subset T$, $P_0 \subset P \subset R$, and $Q_0 \subset T$ with $Q_0 \cap R = P_0$ there exists $Q \supset Q_0$ such that $Q \cap R = P$.

Going Down Property: Given $R \subset T$, $P_0 \subset P \subset R$, and $Q \subset T$ with $Q \cap R = P$ there exists $Q_0 \subset Q$ such that $Q_0 \cap R = P_0$.

Incomparable Property: Given $R \subset T$ there do not exist $Q_0 < Q \subset T$ such that $Q_0 \cap R = Q \cap R$.

1.1
Prime Ideals

The study of prime ideals is very extensive. In fact, most of what is in this chapter can be found in [6]. The main reason for restating much of this on prime ideals (besides the fact that some of the proofs are so lovely) is that many of the theorems from this chapter will be used in the proofs of theorems in Chapter 3. We start by looking at relations between prime ideals in a ring R and those in a ring T where T is an integral extension of R. We then move on to prime ideals in a polynomial extension ring, R[x].

1.2
Prime Ideals and Integral Extensions

In this section we will be taking a closer look at the properties that were mentioned in 0.1, where we will let T be an integral extension of the ring R, and P_0, P, Q_0 and Q will be prime ideals. This will help us examine the relations between prime ideals in a ring R and those in an integral extension of R.

Before we begin, let us note that if T is a ring containing R then a prime ideal in T will always contract to a prime ideal in R. We are now ready to look at the Lying Over Property for prime ideals.

Theorem 1.2.1 *Given rings $R \subset T$, T integral over R and P a prime ideal in R, there exists a prime ideal $Q \subset T$ with $Q \cap R = P$.*

Proof: Let S be the complement of P in R and Q an ideal in T maximal with respect to the exclusion of S. Notice, Q is necessarily prime. Also, it is clear that $Q \cap R \subset P$. Suppose there exists an element $p \in P$, $p \notin Q \cap R$. This would mean that $p \notin Q$. Consider the ideal $(Q,p) \subset T$ properly containing Q. Notice that $(Q,p) \cap S$ is not the empty set. Let $s = q + ap \in (Q,p) \cap S$ where $q \in Q$ and $a \in T$. T is integral over R so

$a^n + c_1a^{n-1} + \ldots + c_n = 0$ for some $c_1, \ldots, c_n \in R$ and some positive integer n. Multiply by p^n,

$$(ap)^n + c_1p(ap)^{n-1} + \ldots + c_np^n = 0.$$

Replace ap with $s - q$,

$$(s - q)^n + c_1p(s - q)^{n-1} + \ldots + c_np^n = 0.$$

Notice, $q \in Q$, therefore,

$$s^n + c_1 p s^{n-1} + \ldots + c_n p^n \in Q.$$

The left side of the equation is also in R so it is in $Q \cap R$ which is in P. Recall, $p \in P$ which implies, from the equation, that s^n is in P. But this would mean s is in P, a contradiction. Thus, Q is a prime ideal in T which contracts to P.

Theorem 1.2.1 states that if T is an integral extension of a ring R then there exists a prime ideal in T which lies over a given prime ideal in R. To see that if T is not integral over R this theorem may not be true, consider the following example where T is not an integral extension of R.

Example 1: Consider the ring of integers, Z, contained in the field of rational numbers, Q, and the prime ideal $3Z \subset Z$. Q is a field so the only prime ideal in Q is the zero ideal, which surely does not contract to 3Z.

Let us look now at the Going Up Property for prime ideals when T is an integral extension of R.

Theorem 1.2.2 *Given rings $R \subset T$, T integral over R, prime ideals $P_0 \subset P \subset R$ and prime ideal $Q_0 \subset T$ with $Q_0 \cap R = P_0$, there exists in T a prime ideal $Q \supset Q_0$ with $Q \cap R = P$.*

Proof: Consider S, the complement of P in R. Notice, Q_0 is disjoint from S. Expand Q_0 to a prime ideal, Q, maximal with respect to the exclusion of S. The proof of Theorem 1.2.1 proved that $Q \cap R = P$. So, we have the Going Up Property for prime ideals.

Does the Going Up Property hold when T is not integral over R? It turns out that we can use an example similar to the previous one to demonstrate that the answer to this question is no.

Example 2: Consider $Z \subset Q$, the prime ideals $\langle 0\rangle \subset 3Z \subset Z$, and $\langle 0\rangle \subset Q$. Notice $\langle 0\rangle \cap Z = \langle 0\rangle$ but since Q is a field, the only prime ideal in Q is the zero ideal so there cannot exist a prime ideal in Q containing $\langle 0\rangle$ lying over the prime ideal 3Z in Z.

We will define the *length* of a chain of prime ideals to be the number of inclusions in the chain, where the phrase '*a chain of prime ideals*' will denote 'a chain of *distinct* prime ideals.'

For an example of a ring with a chain of prime ideals of any finite length, say n, consider the ring $F[x_1, \ldots, x_n]$, where F is a field and $x_1, \ldots, x_n$ are indeterminates. $\langle 0\rangle \subset \langle x_1\rangle \subset \langle x_1,x_2\rangle \subset \ldots \subset \langle x_1, \ldots, x_n\rangle$ is a chain of prime ideals of length n.

For T integral over R, note that by using the Lying Over Property and the Going Up Property we can create a chain of prime ideals in T equal in length to any finite chain of prime ideals given in R. To see this, consider a finite chain of prime ideals in R,

$$P_0 > P_1 > \ldots > P_n.$$

Using Theorem 1.2.1, construct a prime ideal Q_n in T such that $Q_n \cap R = P_n$ and build up a chain

$$Q_n < Q_{n-1} < \ldots < Q_0$$

with Q_i contracting to P_i, $i = 0, \ldots, n-1$, by iterated use of Theorem 1.2.2. Therefore, there does exist a chain of prime ideals in T equal in length to a given finite chain in R.

Is it also true that given a finite chain of prime ideals in T we can create one of equal length in R? It turns out that by checking the Incomparable Property of prime ideals we see that this is true. We begin by looking at the following lemma.

Lemma 1.2.3 *Given rings $R \subset T$, the Incomparable Property of prime ideals holds if the following statement holds: If P is a prime ideal in R and Q is a prime ideal in T contracting to P then Q is maximal in T with respect to the exclusion of S, the complement of P in R.*

Proof: Consider prime ideals $Q_0 < Q$ in T. We must show that $Q_0 \cap R \neq Q \cap R$. Suppose $Q \cap R = P$. Q, by hypothesis, is maximal in T with respect to the exclusion of S. Clearly Q_0 is not maximal with respect to the exclusion of S, for it is strictly contained in Q. Hence, by hypothesis, $Q_0 \cap R \neq P$ and the Incomparable Property holds.

It is interesting to note that the converse of this lemma is also true. The proof can be found in [6].

Theorem 1.2.4 *Given rings $R \subset T$, T integral over R, there cannot exist two prime ideals $Q_0 < Q$ in T with $Q_0 \cap R = Q \cap R$.*

Proof: Let $Q \cap R = P$. By Lemma 1.2.3 we see that to show the Incomparable Property holds it is sufficient to show that Q is maximal with respect to the exclusion of S, the complement of P in R. Suppose it is not.

Consider J, an ideal in T, with $J \cap S = \emptyset$ and $J > Q$. Let $u \in J$, $u \notin Q$. Since T is integral over R, u is a root of a monic polynomial over R, say,

$$u^n + a_{n-1} u^{n-1} + \ldots + a_0 = 0 \qquad a_0, \ldots, a_{n-1} \in R$$

This implies that the set of all monic polynomials, f, with coefficients in R, such that $f(u) \in Q$, is non-empty. Consider one of least degree, say,

$$u^m + b_{m-1} u^{m-1} + \ldots + b_0 \in Q \subset J \qquad b_0, \ldots, b_{m-1} \in R$$

Recall $u \in J$, so $b_0 \in J$, hence $b_0 \in J \cap R \subset P \subset Q$. Finally we see,

$$u(u^{m-1} + b_{m-1} u^{m-2} + \ldots + b_1) \in Q,$$

but neither factor is in Q. Q is a prime ideal so this cannot be. So, Q must be maximal with respect to the exclusion of S, which in turn means that the Incomparable Property holds for prime ideals.

Before we answer the question regarding the length of chains of ideals in R and T, note that if T is not integral over R, the Incomparable Property for prime ideals does not necessarily hold, as is clear from the following example.

Example 3: Consider the ring of integers Z, and its polynomial extension Z[x]. Also consider prime ideals $3Z[x] \subset 3Z[x] + \langle x \rangle \subset Z[x]$ where 3Z[x] is the ideal of polynomials in Z[x] with coefficients in 3Z. Notice,
$3Z[x] \cap Z = (3Z[x] + \langle x \rangle) \cap Z = 3Z$.

Using Theorem 1.2.4 and a point we mentioned earlier about how a prime ideal in T necessarily contracts to a prime ideal in R, we can show that for every finite chain of prime ideals in T we can create a chain of equal length in R. Consider a finite chain of prime ideals in T,

$Q_0 > Q_1 > \ldots > Q_m$.

Since, by Theorem 1.2.4, the contractions $Q_i \cap R = P_i$, $i = 0, \ldots, m$, are all distinct prime ideals, the chain

$P_0 > P_1 > \ldots > P_m$

is a finite chain of prime ideals in R equal in length to that given in T.

At this point it would be appropriate to define the *dimension* of a ring R, *dim(R)*, as the supremum of the lengths of all chains of prime ideals in R. The following theorem can be easily proved as a result of our previous observations.

Theorem 1.2.5 *Given rings* $R \subset T$, *T integral over R, dim(T) = dim(R).*

Proof: If the two dimensions are finite, we can see that dim(R) ≥ dim(T) by noting that we can construct a chain of prime ideals in R equal in length to any finite chain of prime ideals in T. Also, as was stated above, given any finite chain of prime ideals in R we can construct a chain of equal length in T so dim(T) ≥ dim(R). Thus, for finite dimensions, dim(T) = dim(R).

If dim(T) = ∞, there exist arbitrarily long finite chains of prime ideals in T. Each of these contracts to a finite chain of prime ideals in R of equal length. Therefore, there exists arbitrarily long finite chains of prime ideals in R. Similarly, if dim(R) = ∞, we get that there exist arbitrarily long finite chains of prime ideals in T.

If T is not integral over R this conclusion is not always true as we will see in Section 1.3.

1.3

Prime Ideals and Polynomial Extensions

We have found that there are many correspondences between prime ideals in a ring R and those in T where T is an integral extension of R. Now we will look at relations between prime ideals in R and those in a polynomial extension, R[x]. Do similar relations exist?

Let us again begin by looking at the Lying Over Property for prime ideals. Does there exist a prime ideal in R[x] which contracts to a given prime ideal in R?

Theorem 1.3.1 *Given rings $R \subset R[x]$ and P a prime ideal in R, there exists a prime ideal Q in R[x] such that $Q \cap R = P$.*

Proof: Consider $P[x] \subset R[x]$, where P[x] is the set of all polynomials in R[x] with coefficients in P. It is easy to see that $P[x] \cap R = P$ and also that P[x] is an ideal. Now, it is only necessary to show that P[x] is prime. Consider $f = a_n x^n + \ldots + a_0$ and $g = b_m x^m + \ldots + b_0 \in R[x]$ such that $fg \in P[x]$. We must show $f \in P[x]$ or $g \in P[x]$. Suppose neither is in P[x]. There exists at least one term in each polynomial which has a coefficient not in P. In each polynomial, consider the term with the lowest degree among those that have coefficients not in P. Say for f it is the i^{th} term and for g it is the j^{th} term. Now look at the $(i + j)^{th}$ coefficient of fg,

$$a_0 b_{(j+i)} + a_1 b_{(j+i-1)} + \ldots + a_i b_j + \ldots + a_{(i+j)} b_0.$$

Notice all the terms are in P except for a_ib_j so this sum cannot be in P, which means that fg is not in P[x], a contradiction. So, P[x] is in fact prime and we have found our Q.

We see that the Lying Over Property for prime ideals holds for rings $R \subset R[x]$ as well as for $R \subset T$, T integral over R. Also note that if Q is a prime ideal in R[x] then $Q \cap R$ is a prime ideal in R. At this point it looks as though the same properties are true for $R \subset R[x]$ as were true for $R \subset T$, T integral over R. However, the following example will demonstrate that for the Going Up Property, this similarity fails.

Example 4: Consider integral domains $R \subset R[x]$ and a non-zero prime ideal $P \subset R$. Let $P_0 = \langle 0 \rangle < P \subset R$ and $Q_0 = \langle 1 + px \rangle \subset R[x]$, $0 \neq p \in P$. Q_0 is prime and $Q_0 \cap R = P_0$. Furthermore, Q, if it exists, must contain the element px. Also, $Q > Q_0 = \langle 1 + px \rangle$, thus $1 + px \in Q$. From this, we see that $1 = 1 + px - px \in Q$. Hence, Q = R[x] which is not prime. So, for this example, there is no $Q > Q_0$ such that $Q \cap R = P$ which means the Going Up Property for prime ideals does not hold for $R \subset R[x]$.

Let us now look at the Going Down Property for prime ideals.

Theorem 1.3.2 *Given rings $R \subset R[x]$, prime ideals $P_0 \subset P$ in R and prime ideal Q in R[x] such that $Q \cap R = P$, there exists in R[x] a prime ideal $Q_0 \subset Q$ such that $Q_0 \cap R = P_0$.*

Proof: First, let us show that P[x] is the smallest ideal in R[x] containing P. Suppose M is an ideal in R[x] containing P. M must contain px^n for all

$p \in P$ and $n \in Z^+$ where Z^+ is the set of all non-negative integers. Therefore, M must contain the set of all finite sums, $\{\Sigma px^n \mid p \in P, n \in Z^+\} = P[x]$. Thus, $P[x] \subset Q$. Since $P_0[x] \subset P[x] \subset Q$, $P_0[x] \cap R = P_0$ and $P_0[x]$ is prime as was shown in the proof of Theorem 1.3.1, $P_0[x]$ may be chosen as our Q_0.

It is an interesting observation that by using Theorem 1.3.1 and iterated use of Theorem 1.3.2, we can create a chain of prime ideals in R[x] equal in length to that of any finite chain of prime ideals in R. The question which immediately follows is: For every finite chain of prime ideals in R[x], can we necessarily find one of equal length in R? Recall, in Section 1.2 we went about answering this question by showing that the Incomparable Property for prime ideals holds for $R \subset T$, T integral over R. Notice, Example 3 showed that the Incomparable Property for prime ideals does not necessarily hold for $R \subset R[x]$.

So, the question still remains whether it is possible to create a chain of prime ideals in R equal in length to any given chain of prime ideals in R[x]. Example 5 will show us that at least in some instances the answer is no.

Example 5: Consider any field F and its polynomial extension ring F[x]. $\dim(F[x]) \geq 1$ and $\dim(F) = 0$ therefore, $\dim(F[x]) \neq \dim(F)$.

In fact, if $\dim(R) = n$, then $n + 1 \leq \dim(R[x]) \leq 2n + 1$, where it is possible for R[x] to have any of these dimensions. $\text{Dim}(R[x]) = n + 1$ if R is Noetherian, 0-dimensional, or if R is a Prufer domain. This fact can be found in [5].

2.1
G-Domains and G-Ideals

Chapter 2 is an accumulation of definitions, examples and equivalences that will prepare us for Chapter 3. We begin with the following definition.

G-domain: Let R be an integral domain with quotient field K. R is a G-domain if K can be generated as a ring over R by one element.

Theorem 2.1.1 *Suppose R is an integral domain. The following are equivalent:*

a) *R is a G-domain.*

b) *There exists a non-zero element contained in each non-zero prime ideal in R.*

c) *If $\cap_\alpha P_\alpha = <0>$ where P_α is prime for all α, then $P_\alpha = <0>$ for some α.*

Proof: *a) → b)* If R is a G-domain then certainly q.f.(R) = $R[u^{-1}]$ for some non-zero u in R. Suppose P is a non-zero prime ideal in R and consider $0 \neq p \in P$. Since q.f.(R) = $R[u^{-1}]$, $p^{-1} = ru^{-n}$ for some $r \in R$ and some positive integer n. Therefore $u^n = pr \in P$ and so $u \in P$.

b) → c) Consider $\{P_\alpha \mid P_\alpha \text{ is prime}, P_\alpha > <0>\}$. By hypothesis, there exists $0 \neq u \in \cap_\alpha P_\alpha$ so if $\cap_\beta P_\beta = <0>$ then $P_\beta = <0>$ for some β.

c) → a) Consider $\{P_\alpha\}$, the set of all non-zero prime ideals in R. By hypothesis, $\cap_\alpha P_\alpha \neq <0>$ so there exists some $0 \neq u \in \cap_\alpha P_\alpha$. Also, any non-zero ideal contains some u^n for some $n \geq 1$, for if one did not it could be expanded to an ideal, maximal with respect to the exclusion of $\{u^n \mid n \geq 1\}$, which would necessarily be prime. This contradicts that every non-zero prime ideal contains u. Now, consider $0 \neq a \in R$. Since $u^n \in \langle a \rangle$, $u^n = ra$ for some $r \in R$. Therefore, $a^{-1} = ru^{-n}$ and we have $a^{-1} \in R[u^{-1}]$. Thus, q.f.(R) = $R[u^{-1}]$ and R is a G-domain.

Clearly, any field F is a G-domain, for its quotient field is just F which can be written as F[1]. To see other examples of G-domains let us first look at an equivalent definition of them.

Theorem 2.1.2 *Let R be an integral domain with quotient field K. R is a G-domain if and only if K is finitely generated as a ring over R.*

Proof: If K is a G-domain then K is, by definition, generated over R by one element so is in fact finitely generated as a ring over R. On the other hand if K is finitely generated as a ring over R then $K = R[a_1/b_1, \ldots, a_n/b_n]$ for some $a_i, b_i \in R, b_i \neq 0, i = 1, \ldots, n$. With $c = b_1 \cdots b_n$ we have $K = R[1/c]$.

Using this equivalence, we immediately get the following theorem.

Theorem 2.1.3 *A principal ideal domain R is a G-domain if and only if it has only a finite number of prime ideals.*

Proof: Let R be a G-domain. Let K = q.f.(R) = $R[u^{-1}]$ for some non-zero u in R. From Theorem 2.1.1 we see that $u \in \cap_\alpha P_\alpha$ where P_α are the non-zero prime ideals in R. Suppose $u = r_1 \cdots r_n$ is a unique factorization of u into irreducibles. Let P be a non-zero prime ideal. $u \in P = \langle r \rangle$ so, $r_1 \cdots r_n \in \langle r \rangle$. Thus, $r_i \in P$ for at least one i. Therefore $r_i = rs$ for some $s \in R$. Note, s must be a unit and so $\langle r_i \rangle = \langle r \rangle$ and $\langle r_1 \rangle, \ldots, \langle r_n \rangle$ are the only prime ideals.

Therefore, if R is a G-domain it must only have a finite number of prime ideals. Conversely, suppose R has only a finite number of prime ideals, $\langle a_1 \rangle, \ldots, \langle a_n \rangle$. Since R is a P.I.D. every element of R can be written as the product of a unit and powers of a_i's. Clearly then, $K = R[1/a_1, \ldots, 1/a_n]$ is the quotient field of R and by Theorem 2.1.2, R is a G-domain.

From this theorem we can see instantly that the ring of integers, Z, with q.f.(Z) = Q, the ring of rational numbers, is clearly not a G-domain. The following theorem gives another example of a ring which is not a G-domain.

Theorem 2.1.4 *If R is an integral domain and x is an indeterminate over R, then R[x] is never a G-domain.*

Proof: Let K be the quotient field of R. If R[x] is a G-domain then so is K[x]. But K[x] is a principal ideal domain, so according to Theorem 2.1.3 we need only show that K[x] has an infinite number of prime ideals. Suppose that $p_1, \ldots, p_n$ are all the irreducible monic polynomials. Form the polynomial $1 + p_1 \cdots p_n = q$. Clearly, q is not divisible by any of the p_i's, a contradiction. So K[x] has an infinite number of prime ideals and is therefore not a G-domain and hence neither is R[x].

We have seen several cases where an integral domain is not a G-domain; let us look at a situation where we have a G-domain. It is very difficult to determine which rings are and which are not G-domains. The following theorem takes a look at a case when an integral domain is a G-domain.

Theorem 2.1.5 *A noetherian domain R is a G-domain if and only if $dim(R) \leq 1$ and R has only a finite number of maximal ideals.*

The proof of this can be found in [6].

Recall in Chapter 1 we looked at relations between prime ideals in R and T where $R \subset T$. In Chapter 3 we will be looking at relations between a different type of ideals in R and T. We now define that type of ideals.

G-ideal: A prime ideal P in a ring R is a G-ideal if R/P is a G-domain.

Clearly, any maximal ideal M is a G-ideal, for R/M is a field and a field is a G-domain.

Theorem 2.1.6 *Suppose R is a ring and P is a prime ideal in R. The following are equivalent:*

a) *P is a G-ideal.*

b) *If $\cap_\alpha P_\alpha = P$ where P_α is prime for all α, then $P_\alpha = P$ for some α.*

Proof: Apply Theorem 2.1.1 to R/P.

Valuation rings, defined below, provide other examples of G-ideals.

Valuation ring: An integral domain R is a *valuation ring* if for any *a*, *b* in R, *a* divides *b* or *b* divides *a*.

Theorem 2.1.7 gives an equivalent definition of valuation rings.

Theorem 2.1.7 *An integral domain R is a valuation ring if and only if for any $u \neq 0$ in q.f.(R), either u or u^{-1} lies in R.*

Proof: Suppose R is a valuation ring and $0 \neq u \in$ q.f.(R). If $u \in$ R we are done. If $u \notin$ R then $u = (r_1)(r_2^{-1})$, $r_1, r_2 \in R, r_2 \neq 0$. Note that r_2 does not divide r_1 or u would be in R. Thus r_1 must divide r_2, so $r_2 = (r_1)(r')$ for some $0 \neq r' \in R$. It follows that $u = (r_1)(r_1r')^{-1} = (r_1)(r_1^{-1})(r'^{-1}) = r'^{-1}$ and so $u^{-1} = r' \in R$. Conversely, suppose for any $u \in$ q.f.(R) that either $u \in$ R or $u^{-1} \in$ R. Consider $a, b \in$ R and show that a divides b or b divides a. Suppose $a,b \neq 0$, otherwise the result follows trivially. Let $u = ab^{-1} \in$ q.f.(R). Suppose $u \in$ R. Since $a = bu$, we have that b divides a in R. Suppose, on the other hand, that $u^{-1} \in$ R. Since $b = au^{-1}$, we have that a divides b in R.

And now Theorem 2.1.8 provides us with many examples of G-ideals.

Theorem 2.1.8 *Every valuation ring with finite dimension has the property that every prime ideal is a G-ideal.*

Proof: A valuation ring has the property that all prime ideals are comparable. Therefore, if R is a valuation ring with finite dimension say $\dim(R) = n$, then there exist only $n + 1$ prime ideals. It follows that if $\cap_\alpha P_\alpha = P$ where P_α and P are primes then $P_\alpha = P$ for some α. Thus, by Theorem 2.1.6, P must be a G-ideal.

Example 6 is an example of a valuation ring with dimension 1. It is followed by Example 7 which demonstrates that we can have a valuation ring R, such that $\dim(R) = n$ for any finite number n.

Example 6: Let $<p>$ be a prime ideal in Z and consider the set

$Z_{<p>} = \{^a/_b \mid a, b \in Z$ and p does not divide $b\}$. $Z_{<p>}$ is certainly a valuation ring and since $<p>Z_{<p>} = \{p(^a/_b) \mid a, b \in Z$ and p does not divide $b\}$ is the only non-zero prime ideal in $Z_{<p>}$, by the above theorem, $Z_{<p>}$ must be a G-domain. Clearly this is true since $Z_{<p>}[^1/_p]$ is a field.

Before looking at Example 7, note that there exists an equivalent definition of valuation domains using valuations and value groups. [5]

Example 7: In [3] it is shown that there exists a valuation domain with any arbitrarily prescribed value group. Consider the lexicographically ordered group Z^n. The only convex subgroups of Z^n are $\{(0, 0, \ldots, 0)\}$ and $H_i = \{(0, 0, \ldots, 0, a_i, \ldots, a_n) \mid a_i, \ldots, a_n \in Z\}, i = 1, \ldots, n$. Using the correspondence between convex subgroups and prime ideals, we see that the valuation domain with Z^n as its value group has $n + 1$ prime ideals. Therefore, its dimension is n.

From this example, it is clear how we can construct a valuation domain R such that dim(R) = n for any positive integer n. The following definitions are similar to definitions from Chapter 1.

Length: The *length* of a chain of G-ideals is the number of inclusions in the chain, where the phrase '*a chain of G-ideals*' means 'a chain of *distinct* G-ideals'.

G-dimension: The *G-dimension* of a ring R, G-dim(R), is the supremum of the lengths of all chains of G-ideals in R.

An interesting fact about the G-dimension of a noetherian ring follows.

Theorem 2.1.7 *A noetherian ring R has G-dim ≤ 1.*

Proof: Suppose G-dim(R) = n. Let $P_n \supset \ldots \supset P_0$ be a maximal chain of G-ideals in R. R/P_0 is a noetherian G-domain where $P_n/P_0 \supset \ldots \supset P_0/P_0$ is a chain of prime ideals of length n. By Theorem 2.1.5, $\dim(R/P_0) \leq 1$. Therefore, $n \leq 1$ and we have G-dim(R) = $n \leq 1$.

In fact, it follows from Theorem 2.1.5 that for a noetherian ring R, G-dim(R) = 1 if and only if there exists a prime ideal P such that P is properly contained in only a finite number of maximal ideals and there are no prime ideals strictly between P and M where M is any maximal ideal.

3.1
G-ideals and Integral Extensions

This chapter follows much of the same pattern as Chapter 1 and we will refer to Chapter 1 quite often. This first section takes another look at the properties given in 0.1 where we will let T be an integral extension of R, and P_0, P, Q_0 and Q will be G-ideals instead of prime ideals. As in Chapter 1, we will begin by looking at the Lying Over Property but first we must work our way through a few lemmas.

Lemma 3.1.1 *Suppose $R \subset T$ are rings with T integral over R, $P \subset R$ is a prime ideal and $Q \subset T$ is a prime ideal with $Q \cap R = P$. T/Q is integral over R/P, where R/P is naturally imbedded in T/Q.*

Proof: Let $t + Q \in T/Q$, and $t^n + r_{n-1}t^{n-1} + \dots + r_0 = 0$, $r_0, \dots, r_{n-1} \in R$, be an equation showing that t is an integral element over R. Clearly then we would have $(t + Q)^n + (r_{n-1} + Q)(t + Q)^{n-1} + \dots + (r_0 + Q) = 0$ in T/Q.

Lemma 3.1.2 *Let $R \subset T \subset S$ be rings and $u \in S$. If T is integral over R then the ring T[u] is integral over the ring R[u].*

Proof: Let $f(u) = t_n u^n + t_{n-1}u^{n-1} + \dots + t_0 \in T[u]$, $t_0, \dots, t_n \in T$. Since t_i is integral over R for all $i = 0, \dots, n$, it is certainly integral over $R[u]$. Since $u \in R[u]$, any power of u is integral over $R[u]$. Finite products and finite sums of integral elements are integral, so in fact $f(u)$ is integral over $R[u]$. Since $f(u)$ was arbitrary in $T[u]$, $T[u]$ is integral over $R[u]$.

Lemma 3.1.3 *Let $R \subset T$ be integral domains and suppose that T is integral over R. R is a field if and only if T is a field.*

Proof: First suppose R is a field. Let $a \in T, a \neq 0$. We must show that a has an inverse in T.

Let

$$a^n + r_{n-1}a^{n-1} + \ldots + r_0 = 0 \in T \qquad r_0, \ldots, r_{n-1} \in R,\ r_0 \neq 0$$

be an equation showing that a is integral over R. It follows that,

$$a[(a^{n-1} + r_{n-1}a^{n-2} + \ldots + r_1)(-r_0)^{-1}] = 1$$

so,

$$a^{-1} = (a^{n-1} + r_{n-1}a^{n-2} + \ldots + r_1)(-r_0)^{-1}$$

where $(a^{n-1} + r_{n-1}a^{n-2} + \ldots + r_1)(-r_0)^{-1}$ is clearly in T. Therefore, there exists $a^{-1} \in T$, for any $a \in T, a \neq 0$, thus T is in fact a field. Next, suppose T is a field. Consider $b \in R, b \neq 0$. $R \subset T$ so $b \in T$. T is a field so there exists $b^{-1} \in T$. T is integral over R so b^{-1} is a root of a monic polynomial over R say

$$(b^{-1})^n + r_{n-1}(b^{-1})^{n-1} + \ldots + r_1(b^{-1}) + r_0 = 0,\ r_0, \ldots, r_{n-1} \in R.$$

Multiply both sides by b^{n-1} to get

$$b^{-1} + r_{n-1} + r_{n-2}b + \ldots + r_1b^{n-2} + r_0b^{n-1} = 0.$$

Then,

$$b^{-1} = -(r_{n-1} + r_{n-2}b + \ldots + r_1b^{n-2} + r_0b^{n-1}).$$

Note that

$$r_{n-1} + r_{n-2}b + \ldots + r_1b^{n-2} + r_0b^{n-1} \in R.$$

Therefore, $b^{-1} \in R$ for all $0 \neq b \in R$, so R is in fact a field.

Now we are ready to look at the Lying Over Property for G-ideals where T is an integral extension of R.

Theorem 3.1.4 *Suppose $R \subset T$, T is integral over R and P is a G-ideal in R. Let Q be a prime ideal in T such that $Q \cap R = P$. Q is a G-ideal in T.*

Proof: From Lemma 3.1.1 we see that T/Q is integral over R/P, where R/P is naturally imbedded in T/Q. Let u be an element in the quotient field of R/P such that $(R/P)[u]$ is the quotient field of R/P. Lemma 3.1.2 tells us that $(T/Q)[u]$ is integral over $(R/P)[u]$ and finally from Lemma 3.1.3 we see that $(T/Q)[u]$ must also be a field. Recall, R/P is naturally imbedded in T/Q so any field containing T/Q must contain q.f.$(R/P) = (R/P)[u]$, and therefore must contain $(T/Q)[u]$. Hence, T/Q is a G-domain and Q is a G-ideal.

Corollary 3.1.5 *Suppose a ring T is integral over a ring R and P is a G-ideal in R. There exists a G-ideal $Q \subset T$ such that $Q \cap R = P$.*

Proof: Theorem 1.2.1 states that there exists a prime ideal $Q \subset T$ such that $Q \cap R = P$. This corollary follows immediately.

Corollary 3.1.6 *Suppose a ring T is integral over a ring R, $P_0 \subset P \subset R$ are G-ideals and $Q_0 \subset T$ is a G ideal with $Q_0 \cap R = P_0$. There exists a G-ideal, $Q \subset T$ such that $Q \supset Q_0$ and $Q \cap R = P$.*

Proof: Theorem 1.2.2 states that there exists a prime ideal $Q \subset T$ such that $Q \supset Q_0$ and $Q \cap R = P$ and Theorem 3.1.4 states that Q is in fact a G-ideal.

So, for T integral over R, both the Lying Over Property and the Going Up Property hold for G-ideals as well as for prime ideals. We can use a similar argument used in Chapter 1 to show that for any finite chain of G-ideals in R we can create a chain of G-ideals in T with equal length.

If T is not integral over R we can use Example 1 to show that the Lying Over Property for G-ideals does not necessarily hold. Section 3.2 will give us an example of when the Going Up Property for G-ideals fails if T is not integral over R.

Suppose again that T is integral over R. In Chapter 1 it was stated that a prime ideal in T necessarily contracts to a prime ideal in R. The next question follows immediately. Does a G-ideal in T necessarily contract to a G-ideal in R? The following lemma and theorem will answer that question.

Lemma 3.1.7 *If R[u] is a field then u is algebraic over R. Also, there exists an element, $0 \neq r \in R$, such that u is integral over $R[r^{-1}]$.*

Proof: If $u = 0$, then u is in fact integral over R so choose r to be 1. For $u \neq 0$, recall R[u] is a field so there exists $u^{-1} \in R[u]$, such that

$$u^{-1} = r_n u^n + r_{n-1} u^{n-1} + \ldots + r_0 \qquad r_0, \ldots, r_n \in R, r_n \neq 0.$$

It follows that,

$$r_n u^{n+1} + r_{n-1} u^n + \ldots + r_0 u - 1 = 0$$

and this equation shows that u is algebraic over R.

For the second part, multiply the equation by r_n^{-1} to get,

$$u^{n+1} + (r_n^{-1}) r_{n-1} u^n + \ldots + (r_n^{-1}) r_0 u - (r_n^{-1}) = 0$$

we can see that u is a root of a monic polynomial over $R[r_n^{-1}]$, so is in fact integral over $R[r_n^{-1}]$. Finally, let $r = r_n$.

Theorem 3.1.8 *Let $R \subset T$ be rings with T integral over R. A G-ideal in T contracts to a G-ideal in R.*

Proof: Let Q be a G-ideal in T, $P = Q \cap R$. P is a prime ideal in R. Let u be an element in the quotient field of T/Q such that $(T/Q)[u]$ is the quotient field of T/Q. $(T/Q)[u]$ is integral over $(R/P)[u]$, so by Lemma 3.1.3, $(R/P)[u]$ is a field. By Lemma 3.1.7, we see that there exists a non-zero element of R/P, say r, such that u is integral over $(R/P)[r^{-1}]$. So, $(R/P)[u]$ is integral over $(R/P)[r^{-1}]$. $(R/P)[u]$ is a field so, by Lemma 3.1.3, $(R/P)[r^{-1}]$ must also be a field, necessarily the quotient field of R/P because any field containing R/P would have to contain $(R/P)[r^{-1}]$. Therefore, P is a G-ideal of R.

We would like to know, whether it is true that if we start with a finite chain of G-ideals in T we can create one of equal length in R. Theorem 3.1.8 states that a G-ideal in T contracts to a G-ideal in R. If we can show the Incomparable Property is true for G-ideals, we can use an argument from Chapter 1 to show that we can create a chain of G-ideals in R equal in length to that of any given finite chain of G-ideals in T. Let us look then at the Incomparable Property for G-ideals.

Theorem 3.1.9 *Suppose a ring T is integral over a ring R and Q and Q_0 are G-ideals in T such that $Q \cap R = Q_0 \cap R$. Q and Q_0 cannot be comparable.*

Proof: Since a G-ideal is necessarily a prime ideal, Q and Q_0 are prime ideals. Theorem 1.2.4 states that the Incomparable Property holds for prime ideals and therefore Q and Q_0 cannot be comparable.

So, the Incomparable Property for G-ideals does hold and by using this theorem and Theorem 3.1.8, we can create a chain of G-ideals in R equal in length to that of any given finite chain of G-ideals in T.

Theorem 3.1.10 follows immediately.

Theorem 3.1.10 Suppose $R \subset T$ with T integral over R. Then G-dim(R) = G-dim(T).

The proof of this is similar to the proof given for Theorem 1.2.5.

3.2
G-Ideals and Polynomial Extensions

The order of this section will follow a slightly different pattern than that of previous sections. One of the reasons is that some of the proofs are not as straightforward as those we have seen. Still we are concerned with the properties given in 0.1. The first theorem in this section looks as though it has little to do with those properties but, as you will see, it does deal with the relationship between G-ideals in R and those in R[x], where x will be considered as an indeterminate, and will lead directly to situations we are familiar with.

Lemma 3.2.1 *Let h: R $\rightarrow$ S be a surjective ring homomorphism. There exists an order preserving, one-to-one correspondence between the ideals in R containing ker(h) and the ideals in S.*

Proof: It is shown in [2] that $h^{-1}(h(r)) = r + \ker(h)$ where $r \in$ R. That there exists an order preserving, one-to-one correspondence between the ideals in R containing ker(h) and the ideals in S follows.

Lemma 3.2.2 *Let h: R[x] $\rightarrow$ $R[x]/_{P[x]}$ be the homomorphism defined by h(r(x)) = r(x) + P[x] where r(x) $\in$ R[x]. The inverse image of a maximal ideal in $R[x]/_{P[x]}$ is a maximal ideal in R[x].*

Proof: It follows from Lemma 3.2.1 that there is an order-preserving, one-to-one correspondence between the ideals in R[x] containing P[x] and the ideals in $R[x]/_{P[x]}$. So certainly the inverse image of a maximal ideal in $R[x]/_{P[x]}$ is maximal in R[x].

Theorem 3.2.3 *Consider $R \subset R[x]$, with P a G-ideal in R. There exists a maximal ideal Q in $R[x]$ such that $Q \cap R = P$.*

Proof: Let u be an element of the quotient field of R/P such that $(R/P)[u]$ is the quotient field of R/P. Consider the homomorphism $g: (R/P)[x] \to (R/P)[u]$, defined by $g(x) = u$ and g = the identity on R/P. $(R/P)[u]$ is a field so $\{0\}$ is maximal in $(R/P)[u]$. Since g is onto, ker(g) is maximal in $(R/P)[x]$. Next, consider the isomorphism $f: (R/P)[x] \to R[x]/P[x]$, defined by $f((r_0 + P) + (r_1 + P)(x) + \ldots + (r_n + P)(x)^n) = (r_0 + r_1x + \ldots + r_nx^n) + P[x]$ where $r_0, \ldots, r_n \in R$. Recall, ker(g) is maximal in $(R/P)[x]$ so, f(ker(g)) = Q' is maximal in $R[x]/P[x]$. Now, consider the homomorphism $h: R[x] \to R[x]/P[x]$, defined by $h(r(x)) = r(x) + P[x]$ where $r(x) \in R[x]$. Recall Q' is a maximal ideal in $R[x]/P[x]$, and by Lemma 3.2.2, $h^{-1}(Q') = Q$ is a maximal ideal in R[x]. To see $P = Q \cap R$, consider containment in both directions.

$P \subset Q \cap R$:

$Q = h^{-1}(Q')$ contains ker(h) = P[x]. Clearly then, it contains P and it is given that R contains P, so $P \subset Q \cap R$.

$Q \cap R \subset P$:

Consider $r \in Q \cap R$. Since $r \in Q$, $g(f^{-1}(h(r))) = \{0\}$ in $(R/P)[u]$. Also, $g(f^{-1}(h(r))) = r + P$. This is only $\{0\}$ in $(R/P)[u]$ if $r \in P$, so $Q \cap R \subset P$.

Therefore, Q is a maximal ideal in R[x] with $Q \cap R = P$.

From this theorem we get the following corollaries.

Corollary 3.2.4 *Consider $R \subset R[x]$ and P a G-ideal in R. There exists a G-ideal, Q, in $R[x]$ such that $Q \cap R = P$.*

Proof: Consider the maximal ideal Q given in Theorem 3.2.3

Corollary 3.2.5 *Consider $R \subset R[x]$, G-ideals $P_0 < P$ in R, and G-ideal Q_0 in $R[x]$ with $Q_0 \cap R = P_0$. There does not necessarily exist a G-ideal, Q, with $Q_0 \subset Q \subset R[x]$, such that $Q \cap R = P$.*

Proof: Theorem 3.2.3 states that Q_0 may be a maximal ideal in which case there is no G-ideal in R[x] strictly containing Q_0 much less one that contracts to P.

Corollary 3.2.4 is a statement that the Lying Over Property for G-ideals holds for $R \subset R[x]$. Corollary 3.2.5 states that, in general, the Going Up Property does not hold. Previously we used the fact that both properties held to show that for every finite chain of G-ideals in R one of equal length could be created in R[x]. Since we cannot use that method in this situation, we proceed to prove a theorem that will enable us to show this property regardless of the fact that the Going Up Property does not hold.

Theorem 3.2.6 *Consider $R \subset R[x]$ and P a G-ideal in R. $P[x] + \langle x \rangle$ is a G-ideal in R[x].*

Proof: We will prove this theorem by showing that there is an isomorphism between $R[x]/(P[x] + \langle x \rangle)$ and the G-domain R/P. Consider the surjective ring homomorphism f: $R[x] \rightarrow R$, where $f(x) = 0$, $f(r) = r$ for $r \in R$, and the canonical homomorphism g: $R \rightarrow R/P$ defined by $g(r) = r + P$. Notice, the function gf: $R[x] \rightarrow R/P$ defined by $gf(r_0 + r_1x + \ldots + r_nx^n) = r_0 + P$ is a surjective ring homomorphism with $\ker(gf) = P[x] + x$. Therefore, $R[x]/(P[x] + \langle x \rangle)$ is isomorphic to R/P.

This shows that for every finite chain of G-ideals in R we can create one of equal length in R[x]. Is it also true that for every finite chain of G-ideals in R[x] we can create a chain of equal length in R? First let us begin by checking whether or not any G-ideal in R[x] contracts to a G-ideal in R.

Lemma 3.2.7 *Let R be a ring, R[x] a polynomial extension, Q a prime ideal in R[x] and P = Q ∩ R.* $R[x]/_Q$ *is equal to R'[u] where R' = {a + Q | a ∈ R} is isomorphic to* $R/_P$ *and u = x + Q is an element of* $R[x]/_Q$ *.*

Proof: Consider the homomorphisms f: R → R[x] defined by f(r) = r and g: R[x] → $R[x]/_Q$ defined by g(r(x)) = r(x) + Q. Notice gf: R → $R[x]/_Q$ is a homomorphism with image R' and kernel Q ∩ R = P. Thus, $R/_P$ is isomorphic to R'. Certainly R'[u] = $R[x]/_Q$.

Lemma 3.2.8 *Let R be a domain, u an element of a larger domain. If R[u] is a G-domain, then u is algebraic over R and R is a G-domain.*

Proof: If u were not algebraic over R, Theorem 2.1.4 shows that R[u] could not be a G-domain. So, u is algebraic over R which means that u is algebraic over q.f.(R). Since q.f.(R[u]) = R[u][v] for some v ∈ q.f.(R[u]) and also, q.f.(R[u]) is algebraic over q.f.(R), v is algebraic over q.f.(R). So, u and v satisfy equations with coefficients in R, say,

$$a_m v^m + \ldots + a_1 v + a_0 = 0 \qquad a_0, \ldots, a_m \in R,\ a_m \neq 0$$

$$b_n u^n + \ldots + b_1 u + b_0 = 0 \qquad b_0, \ldots, b_n \in R,\ b_n \neq 0$$

Let R' = R[a_m^{-1}, b_n^{-1}]. Notice R ⊂ R' ⊂ q.f.(R) ⊂ q.f.(R[u]). Recall, q.f.(R[u]) is generated as a ring over R by u and v so certainly q.f.(R[u]) is generated as a ring over R' by u and v. The above equations imply that u and v are integral over R'; therefore, we have that q.f.(R[u]) is integral over R'. So, by Lemma 3.1.3, R' must be a field, necessarily the quotient field of R.

Theorem 3.2.9 *Consider rings $R \subset R[x]$ and Q a G-ideal in R[x]. $Q \cap R$ is a G-ideal in R.*

Proof: Suppose $Q \cap R = P$. From Lemma 3.2.7 we know $R[x]/_Q$ is equal to R'[u], where R' and u are defined as in Lemma 3.2.7. Since $R[x]/_Q$ is a G-domain, R'[u] is also a G-domain. So by Lemma 3.2.8, R' is a G-domain. Recall also from Lemma 3.2.7 that R' is isomorphic to $R/_P$ so $R/_P$ is a G-domain, hence, P is a G-ideal.

Now that we have shown that a G-ideal in R[x] contracts to a G-ideal in R, we are interested in knowing whether or not the Incomparable Property holds for G-ideals in $R \subset R[x]$. Recall, in Section 1.3, we were able to show that this property did not hold for prime ideals in $R \subset R[x]$.

Theorem 3.2.10 *Consider rings $R \subset R[x]$. There do not exist G-ideals $Q_0 < Q \subset R[x]$ such that $Q_0 \cap R = Q \cap R$.*

Proof: Let $Q_0 < Q$ and $Q_0 \cap R = P$. In the proof of Theorem 3.2.7, we saw that $R/_P$ is isomorphic to the subset $R' = \{r + Q_0 \mid r \in R\}$ of $R[x]/_{Q_0}$. Note, finding a non-zero element in $R' \cap Q/_{Q_0}$ is equivalent to showing that $Q_0 \cap R \neq Q \cap R$. To see this, let $r \in R$, consider $0 \neq r + Q_0 \in R' \cap Q/_{Q_0}$. Clearly this would mean that $r \in Q$ but $r \notin Q_0$. Since $r \in R$, we would have $r \in Q \cap R$ but $r \notin Q_0 \cap R$ so, $Q_0 \cap R \neq Q \cap R$. On the other hand, suppose that $r \in Q \cap R$, $r \notin Q_0 \cap R$ then $r + Q_0$ is non-zero in $R[x]/_{Q_0}$. Clearly $r + Q_0 \in R' \cap Q/_{Q_0}$. Now let us show that we can find such an element.

From Theorem 3.2.9, we know that $P = Q_0 \cap R$ is a G-ideal in R. In the proof of Lemma 3.2.7 we see that $R[x]/Q_0$ is equal to $R'[u]$ where $u \in R[x]/Q_0$. According to Lemma 3.2.8, u is algebraic over R'. Consider the ring $L = (q.f.(R'))[u]$. Certainly any element in L is algebraic over q.f.(R').
Let $0 \neq v \in Q/Q_0 \subset R[x]/Q_0 = R'[u] \subset L$. Let

$$(a_n/b_n)v^n + \ldots + (a_1/b_1)v + (a_0/b_0) = 0, \text{ where } a_0, \ldots, a_n, b_0, \ldots, b_n \in R',$$

$a_0, b_0, \ldots, b_n \neq 0$, be an equation showing that v is algebraic over q.f.(R'). It follows that

$$a_0 b_1 \cdots b_n = -b_0 v(a_n b_1 \cdots b_{n-1} v^{n-1} + \ldots + a_1 b_2 \cdots b_n).$$

The left side of the equation is a non-zero element of R'. The right side is an element in Q/Q_0, an ideal in $R[x]/Q_0$. So we have found a non-zero element in $R' \cap Q/Q_0$. Thus, $Q_0 \cap R \neq Q \cap R$.

The Incomparable Property for G-ideals does hold for $R \subset R[x]$. Using this fact and Theorem 3.2.9, we can show that for every finite chain of G-ideals in $R[x]$, we can create a chain of equal length in R. Now we can state our final theorem.

Theorem 3.2.11 *Given rings $R \subset R[x]$, G-dim(R) = G-dim(R[x]).*

The proof of this is similar to the proof given for Theorem 1.2.5

An interesting fact about Hilbert rings follows. Note that a ring is a Hilbert ring if every G-ideal in the ring is maximal.

Corollary 3.2.12 *R is a Hilbert ring if and only if R[x] is a Hilbert ring.*

Proof: R is a Hilbert ring if and only if G-dim(R) = 0. Thus, by Theorem 3.2.11, G-dim(R[x]) = 0. So, every G-ideal in R[x] must be maximal and hence, R[x] is a Hilbert ring. The other direction is similar.

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