2006

The doubly special relativity approach to gravitation

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THE DOUBLY SPECIAL RELATIVITY APPROACH TO

GRAVITATION

A Thesis

Presented to

The Faculty of the Department of Physics

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

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August 2006

UMI Number: 1436918

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ABSTRACT

THE DOUBLY SPECIAL RELATIVITY APPROACH TO GRAVITATION

By Dustin M. Keller

This master's thesis presents the results acquired from a theoretical research project at San Jose State University. The results affirm that construction of gravity using a deformative case of Special Relativity is not possible without modification to the founding principles. With the use of the new additive rules developed by Judes and Visser, it is possible to reformulate conservation of energy and momentum and redefine a "degree of freedom." The proposed definition of a "degree of freedom" is then used to develop the spatially and energy dependent metric that is used in the connection coefficients. The results of the research are given explicitly for the case of a probe of zero energy, and imply how to proceed for finite energy.

Acknowledgments

I must thank first of all my thesis advisor Ken Wharton for giving me the freedom to explore and learn through creation. I also need to thank all the people who set out to share their ideas of quantum gravity to make it a progressing area of physics, and the people who write texts to summarize the movement of this ever evolving field, most importantly Lee Smolin to which some views are represented here.

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I. Introduction

General Relativity (GR) gives a description of space-time and cosmology which is supported by countless observations in astronomy. GR is the working physics when describing large scale dynamics and stellar events. However, GR is limited in its capacity to describe the universe. GR breaks down when it is confronted by the behavior of subatomic particles.

Quantum mechanics is invaluable in describing many of the observed properties of particles and radiation as well as defining the stability of atoms. Quantum theory is currently viewed as incompatible with the description of space and time that underlines Einstein's general relativity theory. Quantum mechanics is a theory of probabilities and GR is a large scale field description. Unfavorably, the theories cannot just be brought together to construct a single theory that would hold from the atom up to stellar events. A theory that attempts to unify quantum mechanics and general relativity is often called a quantum theory of gravity. This is because an essential part of the development involves extending the quantum theory to a theory that includes gravity, or extending GR to include physics of the quantum nature. Gravitation is represented in the context of general relativity, which shows us that gravity is simply a symptom of the structure of space and time. This is one aspect that makes gravitation and particle physics incompatible.

No matter the approach to Quantum Gravity there is a scale that physics as it is known today is completely undefined. This physical scale is minuscule. The energy required to attain this scale is far beyond what is available in the largest particle accelerators. The spatial scales are generally 20 orders of magnitude smaller that an atomic nucleus. This is known as the Planck length, and can be derived using Planck's constant, and other fundamental constants.

The Planck constant was introduced when Planck presented his solution of the black-body-radiation paradox. There were no implications in his presentation for this constant to play a role in the structure of space-time. However, the existence of the Planck constant made it possible to define the Planck length by combining Planck's constant with the gravitational constant and the speed of light. Because of Einstein's Special Relativity, which implied that different values should be attributed to a physical length by different inertial observers, the Planck length can not be the same in all inertial reference frames.

The Planck length can be established by applying dimensional scaling to the fundamental constants. By taking a constant from quantum theory; a constant from special relativity; and the gravitational constant, from Newton's law of gravitation, one finds the natural unit of length to be L_p, where

$$L_p = \sqrt{\frac{\hbar G}{c^3}} \qquad (1)$$

The Planck length is 10^{-33} centimeters. The Planck time differs by a factor of *c*, and is 10^{-43} seconds. However, neither the Planck time nor the Planck length is Lorentz invariant in the standard theory of relativity. By Fitzgerald-Lorentz length contraction, various inertial observers would detect different values for the same physical length. But note that all of the constants in Eq. (1) are themselves Lorentz invariant, which seems to contradict the idea that this scale could be physically meaningful.

Newtonian physics assumes that space and time are continuous. But there is the possibility that space-time is quantized. If this is true, one cannot divide time indefinitely. Eventually there must be some fundamental event. A fundamental event would be undividable and is the simplest possible physics that can occur. It is then plausible to assert that if space-time is indeed quantized, the fundamental units of such a quantization are the Planck time and the Planck length.

If the Planck length remains nothing more than a coupling constant, no problem arises for the Fitzgerald-Lorentz contraction. However if the Planck length is promoted to an intrinsic characteristic of space-time structure which is observer-independent, conflicts would be expected with the Fitzgerald-Lorentz length contraction. It is very challenging to construct non-commutative versions of Minkowski space-time which conserve ordinary Lorentz symmetry. It becomes increasingly essential for the Relativity postulates to be modified. Without such modification, it would be unfeasible to attribute to the Planck length a truly fundamental (observer-independent) inherent part in the microscopic structure of space-time.

Amelino-Camelia has recently shown in [1] that it is possible to formulate the Relativity postulates in a way that does not lead to inconsistencies in the case of a space-time with a short-distance structure governed by an observer-independent length scale. The consistency of these postulates shows that it is possible to modify the rules of kinematics involving inertial observers in a way that still lead to special relativity in the low energy limit. In particular, it is possible for every inertial observer to agree on physical laws supporting deformed dispersion relations of the type $E^2 - c^2\bar{p}^2 - c^4m^2 + f(E, p, m; L_p) = 0$, where L_p is the Planck length, and like the speed of light, this value is the same in every reference frame. This is what is known as Deformative Special Relativity, or Doubly Special Relativity (DSR) [1]. Amelino-Camelia argued that there should be some examples of special-relativistic theories with two or more observer-independent scales. Amelino-Camelia's proposal is known as DSR1. While progress developed quickly in the analysis of the first DSR example, for some time no other DSR example was identified in the literature. This changed when Magueijo and Smolin proposed another DSR theory, which is know as DSR2. More recently, other possible realizations of the DSR idea have been considered, but DSR1 and DSR2 remain the focus of the majority of what is becoming a widely studied topic [14].

Section I.B presents a description of the Doubly Special Relativity theory. Then in Chapter II, the frame work for the multiparticle system is laid down. The open issue of conservation of energy within the multiparticle system is addressed, and it is reasoned that quantum mechanics can be preserved with the interpretation of the new energy additive rules purposed by Judes and Visser [3]. Noether's Theorem is used in a plausibility argument. In Chapter III the results of the Magueijo and Smolin gravitational theory are given and a discussion of the theories inadequacy when used in high energy or probing the very small, which is the regime where DSR predicts observable differences. In Chapter IV there is a presentation of an analysis on degrees of freedom in DSR and outline of the importance of different possibilities of approach. Using these results in V, It is shown that DSR cannot be generalized into a complete theory of gravitation without making some very radical changes to the underlining postulates of DSR and GR.

B. Doubly Special Relativity

To give a basic description of Doubly Special Relativity, it is best to first review the framework of relativity with group theory. An attribute of continuous groups or Lie groups is that the parameters of a product element are analytic functions. Differentiability allows the development of the concept of generators and reduces the analysis of the whole group to the investigation of the group elements infinitesimally close to the identity element. By creating exponential representations of the elements L in group G such that $L = \exp(i\theta N)$ where $\theta \rightarrow 0$, it is possible to study elements of G close to unity. The infinitesimal transformations N are then the generators of G (see appendix A). The DSR1 Lorentz generator along the z-axis can be represented as:

$$N_z = p_z \frac{\partial}{\partial E} + (\frac{L_p}{2} \vec{p}^2 + \frac{L_p(1 - e^{-2E/\lambda})}{2}) \frac{\partial}{\partial p_z} - L_p p_z (p_j \frac{\partial}{\partial p_j}). \quad (2)$$

The units used are, $c = \hbar = 1$. L_p is assumed to be of the order of the Planck length, but not necessarily given exactly by the Planck length. One can see in these units that the reciprocal of the Planck length is the Planck energy. For comparatively low energy, Eq. (2) looks very much like the ordinary Lorentz generator. These units will be used throughout this thesis unless otherwise specified. Eq. (2) is Amelino-Camelia's derivation in DSR1.

The DSR1 Lorentz generators give the description of infinitesimal Lorentz transformations. In order to obtain finite Lorentz transformations, which are used in describing Lorentz symmetry in physics, the generators must be exponentiated. Much work done in DSR 1 is developed with the use of the invariant κ. Here κ is the limit to a particle's mass and is a constant of the theory (usually taken to be the Planck mass). κ is often a signature to DSR1 within the scope of group theory and of κ–Poincaré Hopf algebras (see Appendix A).

From a mathematical standpoint, the Lorentz generators of generic κ–Poincaré Hopf algebras have been around for some time. κ–Poincaré Hopf algebras are a well studied and well documented area of importance to many forms of deformative special relativity. It has already been shown that the exponentiation procedure does not actually lead to a group. The advantage of group structure study was only available after the generators of the specific Lorentz sector of κ–Poincaré Hopf algebras were manipulated. For these specifics see [11].

A more physically derived form of the same concept comes from DSR2. In DSR2 as well as in DSR1, the boost generators are deformed while there is no deformation of the rotation generators. The difference in the derivation of the theories is found in the differential representation of the boost generators. For DSR2 it looks like:

$$N_z = p_z \frac{\partial}{\partial E} + E \frac{\partial}{\partial p_z} - \lambda p_z (E \frac{\partial}{\partial E} + p_j \frac{\partial}{\partial p_j}) \qquad (3)$$

where λ is assumed to be the Planck Length or the reciprocal of the Planck energy. Generator (3) is developed in [2], using arguments which allowed the avoidance of explicit integration of differential equations. It was shown that the DSR2 boost transformation for the descriptive case of a transformation which is purely a boost in the z

direction for a single degree of freedom and acts on the four-momentum with components (0,0,p,m) should take the form:

$$p_z(\nu) = \frac{\gamma(\nu)[p_{z,0} - \nu E]}{1 + \lambda[\gamma(\nu) - 1]E_0 - \lambda\gamma(\nu)\nu p_{z,0}} \tag{4}$$

Here, $\gamma(\nu) \equiv 1 / \sqrt{1 - \nu^2}$, and the rest mass energy is

$$E_0 = \frac{m_0 c^2}{1 + \frac{m_0 c^2}{E_p}}, \tag{5}$$

while the parameter v is a relative velocity between the observers connected by the boost. This boost can also be described in the transformed coordinate system by

$$p_0' = \frac{\gamma[E_0 - \nu p_z]}{1 + \lambda[\gamma - 1]E_0 - \lambda\gamma\nu p_z}$$

$$p_z' = \frac{\gamma[p_z - \nu E_0]}{1 + \lambda[\gamma - 1]E_0 - \lambda\gamma\nu p_z} \tag{6}$$

$$p_x' = \frac{p_x}{1 + \lambda[\gamma - 1]E_0 - \lambda\nu p_z}$$

$$p_y' = \frac{p_y}{1 + \lambda[\gamma - 1]E_0 - \lambda\nu p_z}$$

The DSR2 dispersion relation can be obtained from the commutators of the differential operators in Appendix A and the unmodified space-rotation generators, which yields

$$m^2 = \frac{E^2 - |\vec{p}|^2}{(1 - \lambda E)^2}. \tag{7}$$

It is important to note that the Planck energy is preserved in both DSR1 and DSR2 by the modified action of the Lorentz group. For example the Lorentz transform in the z

direction with velocity v is formed by taking $(E_p, 0, 0, 0)$ into$(E_p, -vE_p, 0, 0)$. We see that in Eq. (6) the four momenta of photons with $E = E_p$ is preserved under boosts in the direction the particle is traveling. For these transformations, when the particle energy is not comparable to the Planck scales, the transformation in the *z*-direction tends towards $\gamma(p_z - vE_0)$, which looks like a standard relativistic particle.

The transformations in Eq. (6) from DSR2 allow us to see that the energy is treated similarly to the speed of light in standard Special Relativity. Now there is not just one asymptotic limit but two. In a sense, DSR2 does to energy what Einstein did with velocity. DSR1 does this same thing with mass, though the energy and length are limited by this same constraint.

We see that Eq. (7) defines a mass. The definition of mass often changes in different variations of DSR1 and DSR2. Amelino-Camelia's original interpretation is that there are three distinct guises of physical mass: the "rest-energy mass," the inertial mass, and the gravitational mass. Conflicts between the different interpretations of mass arise when it is difficult or not possible to identify the rest energy mass or completely define it. Once identified, it is feasible to expect that the rest energy will still coincide with the inertial mass and the gravitational mass. Traditionally, the rest energy, inertial mass, and gravitational mass are what should preserve a physical relevance. At this point, only the concept of rest energy is clearly identifiable in a DSR theory, and as long as there is a lacking in understanding of the concept of inertial mass in DSR, it would be meaningless to formulate a fundamental relation between the rest energy, inertial mass, and gravitational mass.

II. The Multi-Particle Sector

A. Previous Work

In DSR theories, the step from the one-particle sector to multiparticle sectors is usually nontrivial. Ordinary special relativity is a linear theory and therefore no such complications are encountered; for example, one can meaningfully attribute a total momentum to a system composed of two or more particles as the sum of the momenta of the individual particles. In DSR theories, by contrast, the concept of total momentum is complex.

A key aspect of multiparticle systems in DSR theories is the kinematic conditions (energy-momentum-conservation) for particle production in collision processes. DSR theories require deformed rules for energy-momentum conservation. For example, it would be inconsistent to enforce the condition that the total energy (and momentum) of the two particle system entering a collision is just equal to the sum of the two particles coming out, since these conditions would not provide an observer-independent law. Energy and momentum have new limits.

For example if two particles each have energy comparable to the Planck energy, then at the point of collision their total energy will exceed the Planck energy, which is not possible for a single particle within the framework of the dispersion relation for a single degree of freedom in DSR ("degrees of freedom" are further explained in Chapter IV). In ordinary special relativity this condition can be derived from the linear structure of Lorentz transformations, and it does provide an observer-independent kinematical law for a collision process (they are either satisfied for all inertial observers or not satisfied for all inertial observers). The deformed nonlinear transformation laws of DSR theories would clearly not be consistent with this traditional conservation condition.

The form of the new energy-momentum conservation-like laws will naturally depend on the structure of the specific DSR theory (they must reflect the structure of the transformation laws). Much work has been done [1,2] on these laws. For the purpose of this thesis it is not necessary to go into detail on these approaches. However it is worth mentioning that most of these conservation laws "mix" the particles (the nonlinear correction terms involve properties of pairs of particles) in DSR1. There are very few attempts to construct DSR conservation laws that apply to the multiparticle sector for more than two particles. Conservation must work for more than two particles if it is to hold any ground in describing our world. There must be a rule for adding an arbitrary number of momenta. Since energy and momentum are no longer additive quantities in DSR, finding their values for composite systems and hence finding appropriate conservation laws is an intricate matter.

There are two distinct approaches to the conservation laws of DSR. One technique is to examine the nonlinear realization of the symmetry group of the DSR transformations and use its properties as constraints on the conservation laws for composite systems. The alternative is to work directly with the transformation equations. It is possible to apply physically intuitive restrictions to deduce the conservation laws. Through a combination of these two techniques, the number of possible conservation laws for DSR1 and DSR2 has been reduced to two. Judes and Visser [3] have worked along the lines of the second method to find that it is possible to uniquely identify the conservation laws for any DSR theory by applying seemingly reasonable physical principles. They give exact results for the total energy and momentum of a composite system in both DSR1 and DSR2.

Here is a summary of Judes and Visser's approach. Because the DSR symmetry group is a nonlinear realization of the Lorentz group, it is possible to find functions [3] of the physical energy-momentum $P_4 = (E, p)$ that can transform like a Lorentz 4-vector. These are called the pseudo-energy-momentum $\wp_4 = (\varepsilon, \pi)$ and at this point assume no physical significance.

They relate as:

$$P_4 = F(\wp_4) \qquad \wp_4 = F^{-1}(P_4). \tag{8}$$

The function F and its inverse both reduce to the identity in the limit where energies and momenta are small compared to the Planck scale. The Lorentz transformation acts on the auxiliary variables as usual: $(\varepsilon' ; \pi') = N(\varepsilon, \pi)$. N is the Lorentz transformation, boosting from the unprimed coordinates to the primed coordinates [3]. The boost operator for the physical energy and momentum (E,p) will be called L, and is given by:

$$P'_4 = L(P_4) = [F \circ N \circ F^{-1}](P_4). \tag{9}$$

N and F are unique as they determine the nonlinear Lorentz transformation L; however N and L are not unique as they determine the function F. The dispersion relation is:

$$[\varepsilon(E, p)]^2 - [\pi(E, p)]^2 = \mu_0^2. \tag{10}$$

where μ_0 is the Lorentz invariant developed from ε and π, not the rest energy. In terms of the rest energy m_0, developed from going to a Lorentz frame in which the particle is at rest, $\mu_0 = \varepsilon(m_0, 0)$, the relationship between L(N) and $\mu_0(m_0)$ becomes sufficient to describe the function F[3].

In the linear representation, kinematic quantities such as total energy can be defined using the pseudo-energy momentum:

$$\wp_4^{tot} = \sum_i \wp_4^i \tag{11}$$

Calculating the total physical 4-momentum is then:

$$P_4^{tot} = F(\sum_i F^{-1}(P_4^i)). \tag{12}$$

It is the total physical 4-momentum that is the quantity that will be conserved in collisions. Calculating it is simply a matter of finding F and its inverse. Again, in this context ε and π do not necessarily have any physical significance [3].

This approach is very significant, giving a complete method for finding the momentum and energy relations of a composite system and a seemingly viable approach to finding momentum and energy of a macroscopic mass. These are all requirements if there is to be any verification experimentally of the DSR theory. This is also a requirement if DSR is to bring any light to the search for quantum gravity. For even if gravity can be quantized, there must be a method of adding the quanta up to make the effect of gravity that can be observed.

So far, there is nothing in quantum gravity research to indicate that having the Planck length or Planck energy be an invariant is physically valid. However, if DSR is correct, the search for quantum gravity would become simplified. That is to say that spatial distances could become discrete and energy could exist only in finite units. This simplifies things greatly. It restricts space-time to a discrete lattice.

To look at the Judes and Visser theory in greater detail, I elaborate from the standpoint of DSR2. (DSR1 has the same basic principles but the algebra is somewhat messier.)

$$\wp_4 \equiv (\varepsilon; \pi) = F^{-1}(P_4) = \frac{(E; p)}{1 - \lambda E}. \qquad (13)$$

As before, lambda is the reciprocal of the Planck energy. The inverse mapping is

$$P_4 = (E; p) = F(\wp_4) = \frac{(\varepsilon; \pi)}{1 - \lambda\varepsilon}. \qquad (14)$$

The total physical 4-momentum is easily calculated. First observe that for the pseudo-enegy/momenta

$$\varepsilon_{tot} = \sum_i \frac{E_i}{1 - \lambda E_i}; \qquad (15)$$

and

$$\pi_{tot} = \sum_i \frac{p_i}{1 - \lambda E_i}. \qquad (16)$$

Then

$$E_{tot} = \frac{\Sigma_i E_i /(1 - \lambda E_i)}{1 + \lambda \Sigma_i E_i /(1 - \lambda E_i)} \qquad (17)$$

and

$$p_{tot} = \frac{\Sigma_i p_i /(1 - \lambda E_i)}{1 + \lambda \Sigma_i E_i /(1 - \lambda E_i)} \qquad (18)$$

Finally, the exact dispersion relation for DSR2 is

$$\frac{E^2 - p^2}{(1 - \lambda E)^2} = \mu_0^2 = \frac{m_0^2}{(1 - \lambda m_0)^2}. \tag{19}$$

Assuming the interpretation of E remains the same, it is obvious from these results that there would be no first order differences from DSR even in a macroscopic system. From Eq. (15), the total non-canonical energy never exceeds the Planck energy. This is good. If there were a discrepancy, the method would have to be discarded or reformulated immediately because to first order SR and the quantum theory are true for all experimental tests. So it is important that DSR match them both to first order [3]. The total energy in Eq. (15) does not describe an object's total mass. The description of mass, and specifically rest mass, differs slightly in the interpretation of each theory. It is important here that no distinction or limitation has yet been given to the degrees of freedom within an energy or rest energy. How the degrees of freedom play a role will be discussed in Section IV.

B. Analysis

Judes and Visser give no physical meaning to π and ε, though at fist glance they might appear closer to our macroscopic view of energy because they can exceed the Planck limit. Most objects in everyday life have far more rest mass energy than the Planck energy. The total ε for a system of particles is close to its rest energy (assuming that the internal kinetic energy, or the temperature of the system, is small enough that the mass maintains its structure). But ε does not (in the Judes/Visser interpretation) take into account binding energy or any other internal potential. The pseudo-energy and momentum are, so far, mathematical abstractions. There is no explicit result giving reasons as to why objects could have more energy than the Planck energy from just rest mass. For example, from Eq. (15) and Eq. (16), E and p can no longer exceed the Planck scale, but π and ε have no limit, which is how energy and momentum are usually thought of. From the Judes and Visser results, E and p, though changed in the additive approach, still represent the physical energy and momentum. π and ε are just devised as tools to make the total physical 4-momentum conserved in collisions.

To determine, what, if any, observable consequences from DSR would pertain to a macroscopic system, we must first apply what we know about DSR energy and momentum conservation to quantum mechanics. There is nothing to indicate which energy and momentum description corresponds to the operators $\partial/\partial t$ and $\partial/\partial x$ and how the new conservation laws should be applied to quantum mechanics. Specifically, it is unclear which energy (ε or E) is the quantum mechanical energy associated with the energy eigenvalues of the system. Quantum mechanics, and in turn statistical mechanics and all observable dynamics, would have different formulations depending on which energy is used in the quantum mechanical description.

Assume the Lagrangian L is invariant under a 4 dimensional divergence. The results of Noether's Theorem (see Appendix B) point out that the energy-momentum tensor of a field θ can be written:

$$\mathrm{T}^{\mu}{}_{\nu} \equiv \int \frac{\partial \mathrm{L}}{\partial(\partial_{\mu}\theta)} \partial_{\nu}\theta - \mathrm{L}\delta^{\mu}{}_{\nu}$$

Leading to the physically conserved energy as:

$$E = \int \frac{\partial L}{\partial(\partial_0 \theta)} \partial_0 \theta - \delta_0^0 L d^3x . \tag{20}$$

From this we see that the physically conserved energy and its operator could plausibly remain ∂/∂t, even for very high energy. In contrast, if

$$\varepsilon = \int \frac{\partial L}{\partial(\partial_0 \theta)} \partial_0 \theta - \delta_0^0 L d^3x \tag{21}$$

then the quantum mechanical description in DSR2 would become

$$E = \frac{\varepsilon}{1 + \lambda\varepsilon} = \frac{\partial \psi}{\partial t} (1 + \lambda \frac{\partial \psi}{\partial t})^{-1} . \tag{22}$$

This, of course, for low energy looks exactly like the traditional energy operator. But for high energy (close to the Planck scale per degree of freedom), quantum mechanics would have to be completely reformulated. This is something worth avoiding for the preservation of DSR. There are lots of reasons to keep quantum mechanics, or at least the postulates of quantum mechanics, "as is," but for the most part it is worth avoiding the change because it is possible to proceed without doing so. This same argument can also be applied to the momentum operator.

Letting *E* remain the energy corresponding to the quantum mechanical energy operator relieves us from having to reformulate Quantum mechanics for a particle or a multiparticle system at energy levels comparable to the Planck scale. But, because of the new additive rules, the dynamics of multiparticle systems (statistical mechanics) look slightly different. For example, think of the ultra-relativistic case of the density of states. Traditionally momentum is written as

$$p = \hbar k = \frac{\pi \hbar n_{x,y,z}}{l}, \tag{23}$$

but the new energy-momentum relationship tells us

$$E_{n_x,n_x,n_z} = \frac{\sqrt{(1 - 2\lambda m_0)[m_0^2 + (1 - \lambda m_0)^2(\pi\hbar n / l)^2]} + \lambda^2 m_0^4 - \lambda m_0^2}{1 - 2\lambda m_0}.$$

(24)

Also, the partition function for a single particle would look like

$$Z_{\text{particle}} = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} e^{-\beta(E_{n_x,n_y,n_z})}. \tag{25}$$

This would imply that the total number of partition functions would be 3^ (the number of particles). From the partition function it is possible to build interpretations for entropy and average energy. This is where it seems most possible to find a difference between special relativity and DSR. A plasma with a Planck-scale temperature would behave very different in DSR. In fact, the greater number of particles in the plasma, the greater the observable difference. But for any number of particles in a system with a kinetic energy far below the Planck energy, physics looks again like standard thermodynamics. Again these are second order results and could only be noticeable if most of the particles in the system had energy comparable to the Planck energy.

Many aspects of the above addition rules need to be looked at with a cautious eye when considering Judes and Visser's approach. One of the most fundamental constructs of physics relies on linearly additive energies that can easily be taken for granted. The Hamiltonian requires that total energy be potential energy added to kinetic energy. The additive rules for energy must be consistent with traditional physics or at least be redefined to look consistent with Newtonian mechanics at low energy. In essence this is not a difficult process, but the choice of that redefinition has great repercussions on analysis of higher energy physics and on quantum gravity. There is a similar problem with Lagrangian mechanics. Some of these questions and their implication with respect to degree of freedom and DSR gravitation will be addressed in Chapter IV.

III. Gravitation in DSR

A. Previous Work

Before venturing into the specifics of Hamiltonian and Lagrangian mechanics in DSR, it is important to first look at the issue of gravitation in DSR. One can see that there would be some observable differences between relativity and DSR that arise from the behavior of interacting particles in a high energy gas. But gravity is energy-dependent as well. It is important that General Relativity stay intact, at least to first order. DSR is a framework for encoding properties of flat quantum space-time. There should be a formalism that gets generalized to incorporate curvature.

Joao Magueijo and Lee Smolin in [4], propose a dual position space to a non-linear realization of relativity in momentum space, and show that for such a dual the space-time invariant is an energy-dependent metric. This leads to an energy-dependent connection and curvature, and a minor change to the Einstein equations.

The main result is that there is a straightforward adaptation of the principles and equations of General Relativity that fit the DSR regime. This is characterized by the feature that the geometry of space-time becomes probe-energy dependent in addition to matter-energy dependent. Quanta of different energies witness different classical geometries. Measurements of distance and time contain a new dependence on the energy of the quanta used in the measurements.

Deformed special relativity was first proposed in momentum space. Losing linearity, the dual position space no longer emulates momentum space. A reconstruction of position space [13] leads to space-time positions with energy dependent transformation laws. The metric becomes energy dependent as well.

Joao Magueijo and Lee Smolin suggest that there is no single classical space-time geometry when effects of order of the Planck energy are taken into account. They propose that classical space-time is the leading order to the true representation, which is a one parameter family of metrics, parameterized by the ratio of a probe's energy to the Planck Energy. This means the geometry of space-time depends on what fraction of the Planck Energy the particle moving in it has. This accordingly gives space-time geometry an effective but somewhat more tedious description, now that there is no single space-time dual to momentum space. The dual is now a set of metrics that rely on the energy level of the probe.

The energy dependent metric is g(E) where *E* is "the scale at which the geometry of space-time is probed." [4]. For a freely falling observer, *E* is the total energy of the probing system of particles, as measured by that observer. The construction is assumed to make the metric co-variant with regard to the dual of the non-linear representation of the Lorentz group.

In the current popular theories of quantum gravity, such as loop quantum gravity and string theory, the classical geometry of space-time is not primary. Instead, it is a low energy display in the portrayal of a very different representation. In loop quantum gravity, this is completely unequivocal. The space-time metric then depends on the ratio of a cutoff scale, *M*, to the scale studied by the probe energy, *E* [4]. Thus the classical metric can be seen as the limit

$$\lim_{E \to 0} g_{ab}(E / M) = g_{ab}^{\text{classical}} . \qquad (26)$$

Quantum gravity seemingly requires an energy dependence of the metric g_{ab}.

One technique for constructing position space in deformed special relativity requires that free particle field theories in flat space-time have plane wave solutions while the 4-momentum they carry satisfies the deformed dispersion relation. For this to be achievable, the contraction between position and momentum has to remain linear so that the waves are still plane waves:

$$dx^a p_a = dx^0 p_0 + dx^i p_i \tag{27}$$

So if momentum transforms non-linearly, then the dx^a transformation must be energy dependent. The Planck energy can be promoted to a universal constant that is the same in all inertial observer frames, resulting in a modified energy momentum relationship that now looks like:

$$E^2 f^2(\lambda E) - p \cdot p h^2(\lambda E) = m^2 \tag{28}$$

which is realized by the action of a non-linear map from momentum space to itself

$$U \cdot (E, p_i) = (f(\lambda E)E, h(\lambda E)p_i. \tag{29}$$

It is important to note that even though DSR theory carries a 4-momentum that satisfies deformed dispersion relations, the contraction between position and momentum remains linear:

$$ds^2 = -\frac{(dx^0)^2}{f^2} + \frac{(dx^i)^2}{h^2} \tag{30}$$

So, dx^a is endowed with an energy dependent quadratic invariant, that is, an energy-dependent metric [4]. For DSR2, $f = h = 1/(1 + \lambda E)$.

With the position space dual defined to deformed relativity in momentum space, it is possible to address general relativity. The deformed equivalence and correspondence principles are stated here as defined by Magueijo and Smolin in [4]:

Assume a region of space-time in which the radius of curvature R is much larger than the Planck length. "Then freely falling observers, making measurements of particles and fields with energies E, observe the laws of physics to be, to first order in $1/R$, the same as in modified special relativity. Hence freely falling observers to first order in $1/R$ can describe themselves as being inertial observers in varying flat space-time." This can be thought of as the modified equivalence principle.

"There is a restriction $1/R << E$ taken into account; that terms in $R(\partial p \;/\; p)$ coming from the fact that the wavelength of a quanta is not much smaller than the radius of curvature. The upper limit E<<Planck Energy comes from the concept that the geometry of quantum space-time does not necessarily have a smooth, classical description for energies of Planck scales or higher." This can be thought of as the new correspondence principle.

Space-time can then be described by a one parameter family of metrics with respect to a one parameter family of orthonormal frame fields. The modified equivalence principle is the essential construction of the Magueijo and Smolin gravitational theory. I will soon argue that without an intrinsic definition of the degrees of freedom of the probing energy, the principle is ineffective. But assuming this principle, the new metric can be expressed in terms of the Minkowski metric and the tensor product of the one parameter family of orthonormal frame fields,

$$g(E) = \eta^{ab} e_a(E) \otimes e_b(E). \tag{31}$$

The energy dependence of the frame fields is:

$$e_0(E) = \frac{1}{f(\lambda E)} \tilde{e}_0, \quad e_i(E) = \frac{1}{h(\lambda E)} \tilde{e}_i \tag{32}$$

where $\tilde{e}_0$ and $\tilde{e}_i$ are the orthonormal basis vectors of GR. The correspondence principle requires

$$\lim_{E / E_{Pl} \to 0} f(\lambda E) \to 1. \tag{33}$$

It is then possible to create a one parameter family of connections $\nabla(E)_\mu$ and curvature tensors $R(E)^{\sigma}_{\mu\nu\lambda}$ formulated from the standard construction. There is also a one parameter family of energy-momentum tensors $T_{\mu\nu}(E)$ and the Einstein equations are revised to a one parameter family of equations

$$G_{\mu\nu}(E) = 8\pi G(E) T_{\mu\nu}(E) + g_{\mu\nu}\Lambda(E). \tag{34}$$

G(*E*) is the energy dependent Newton's constant, where G(*0*) is the physical Newton's constant. The energy dependence of G(*E*) gives a new realization of the gravitational coupling, which will now depend on the energy scale of probing and must satisfy a renormalization group equation. $\Lambda(E)$ is simply the energy dependent cosmological constant [4].

Of course these equations must satisfy the consistency conditions of Bianchi's identities. Bianchi's identity is the mathematical equivalent of the physical law of conservation of energy. It states that a boundary of a boundary is zero. If it is possible to reconstruct and demonstrate Bianchi's identity, we could then form local energy conservation and the geodesic equation. Magueijo and Smolin do not rigorously elucidate or suggest that this should be problematic.

Considering the complex development of general relativity, this seems like a somewhat smooth transition from DSR to Doubly Special Gravitation (DGR). With the parameterized functions f and h, this formalism can be applied for any form of DSR. Even though this proposal is written in a form that expresses DSR from a generic standpoint, the generality leaves some very important points, crucial to the connection of quantum mechanics and general relativity, unaddressed and open.

B. Analysis

Given that energy and momentum are not additive in the traditional form for high energy particles in DSR, the Einstein Eq. (34) is only applicable to the low energy region where the radius of curvature is much larger than the Planck Length. Magueijo and Smolin propose that the true representation of space-time is a one parameter family of metrics parameterized by the ratio of the energy of the probing particle to the Planck energy. Magueijo and Smolin do not suggest adding the energy of space (or any field) to the probing particle if either energy is comparable to the Planck energy. This would seem to be a problem addressed in the specific additive rules for a DSR theory. But no matter which theory is used to apply the high energy additive rules of DSR, one would assume that the energy of space-time can be treated as a single degree of freedom. All of the DSR additive rules treat the energy of particles as single degrees of freedom. However, to treat a whole region of space-time as a single degree of freedom may be oversimplifying. Magueijo and Smolin analyze only the low energy case, which is expected to give results similar to GR. It is also required that matrix multiplication involving energies as matrix terms be dealt with according to the unique additive rules as well. There is no guarantee that the specific algebra will contain an operation that is equivalent to the tensor product necessary to keep the Einstein equation intact. There is no reason to believe that for the high energy case, the Einstein equation will look at all familiar.

Magueijo and Smolin give some examples in the generalized form of DSR showing differences in the FRW solution and the Schwarzschild solution. But these differences can only be applied to versions of DSR that conserve additive forms of energy.

There is also a physical concern that arises from the revised Einstein Eq. (34) in Magueijo and Smolin's theory. The description of Doubly Special General Relativity relies on the capacity to shift into a locally flat Minkowski frame, which may be a luxury left in traditional GR.

In traditional GR, some coordinate transformation can always be found which converts the metric equation to a sum of squares. The Cartesian coordinates which result from this transformation describe a space which is tangential to the curved space at the point selected. This gives the freedom to do physics in a flat Minkowski frame and then transform back into curved space. In traditional GR, the curvature of space is energy dependant, just as it is in "Deformative" GR. The difference is the way that curvature depends on energy. Because there is now more than one asymptotic path, there is a new way of looking at the effects of energy on space-time.

Consider a particle that has energy comparable to the Planck energy passing through a region of space-time. If most of this energy is kinetic, then it is still possible to shift into a local Minkowski frame to analyze this region. If most of this energy is rest energy, then there is no reason why shifting into a locally flat Minkowski frame is possible. It is only possible to shift into a local Minkowski frame if that region of space-time is differentiable. If a region of space has energy comparable to the Planck energy, there is no guarantee that the space-time is still continuous or differentiable. A baseball would then have much of the region of space-time that it sits in completely undefined. Even though Judes and Vissers' additive rules give a method of adding energy, a theory of gravity should be able to work analyzing one massive (or Energetic) particle. This is analogous to working with a quantum wave of Planck scale wave length. Consider what happens when the wavelength of a quantum at rest increases. As mentioned before, as long as the wavelength is not smaller than the radius of curvature, there is no problem. But DSR specifically tells us that the Planck length is an asymptotic limit. In order for gravity to be complete, there must be a way to describe space-time curvature around a particle with a very small wavelength, whether the quantum is moving or not. Of course there are no observed single particles that have a rest mass energy with a wavelength comparable to the Planck length. If a Planck scale rest energy particle can be described, it should be possible to describe a region of space-time containing Planck scale energy. Let us proceed as if space-time remains differentiable.

To look at this closer from standard GR, consider an event at $\wp_0$ in space-time. A local Lorentz frame at a given event $\wp_0$ is the closest thing there is to a global Lorentz frame. There exists a coordinate system with $g_{\mu\nu}(\wp_0) = \eta_{\mu\nu}$ and as many derivatives of $g_{\mu\nu}$ as possible vanishing at $\wp_0$. So if it is always possible to transform to a locally flat frame, then

$$\begin{aligned} g_{\mu\nu}(\wp_0) &= \eta_{\mu\nu} \\ g_{\mu\nu,\rho}(\wp_0) &= 0 \end{aligned} \tag{35}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Here neither the metric coefficients nor their first derivatives have anything to say about curvature. What distinguishes curved space-time from flat space-time is that the frame in free fall at $\wp_0$ differs slightly from the frame in freefall at an event a small distance away. The metric tensor and its derivatives can be obtained by Taylor expansions around $g_{\mu\nu}(\wp_0)$ and $g_{\mu\nu,\rho}(\wp_0)$ such that

$$\begin{aligned} g_{\mu\nu}(\wp_0 + \Delta x) &= \eta_{\mu\nu} + \frac{1}{2} g_{\mu\nu,\rho\sigma}\Delta x^{\rho}\Delta x^{\sigma} \\ g_{\mu\nu,\rho}(\wp_0 + \Delta x) &= g_{\mu\nu,\rho\sigma}\Delta x^{\sigma} \end{aligned} \tag{36}$$

and the change in $g_{\mu\nu}$ depends only on the second derivatives, so these derivatives must embody the curvature information. Taking the Gaussian coordinates to be x'^{μ} at $\wp_0$, the coordinates x^{μ} for which Eq. (36) holds are given by the transformation

$$x'^{\mu} = x^{\mu} + G^{\mu}_{\nu\tau} \frac{x^{\nu}x^{\tau}}{2}, \tag{37}$$

where the $G^{\mu}_{\nu\tau}$ are simply constants in GR, but in the transition to DSR gravitation, $G^{\mu}_{\nu\tau}$ would plausibly have a dependence on energy. In the same way that the metric depends on energy in the work by Magueijo and Smolin, $G^{\mu}_{\nu\tau}$ should be a member of a one-parameter family of energy-dependent connections.

The modified equivalence principle introduced by Magueijo and Smolin implies that space-time is described by a family of metrics given in terms of a one parameter family of orthonormal frame fields as described by Eq. (31). Holding this modified equivalence principle as valid, let us see where its usefulness falls off.

If we assume that the energy dependence of $G^{\mu}_{\nu\tau}$ has no other positional or directional dependence, then differentiating Eq. (37) with respect to x^{α} and then to x^{β} gives

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \delta^{\mu}_{\alpha} + G^{\mu}_{\alpha\tau}(E)x^{\tau} \tag{38}$$

and

$$\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha}\partial x^{\beta}} = G(E)^{\mu}_{\alpha\beta}\,. \tag{39}$$

Choosing $\wp_0$ to be the origin of the x^{μ} coordinates so that

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \delta^{\mu}_{\alpha}\,, \tag{40}$$

and

$$g_{\mu\nu}(E) = \left(\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\right)\left(\frac{\partial x'^{\beta}}{\partial x^{\nu}}\right)g'_{\alpha\beta}(E) \tag{41}$$

when differentiated with respect to x^{σ} gives

$$\begin{aligned} g_{\mu\nu,\sigma}(E) &= (\partial^2 x'^{\alpha} / \partial x^{\mu}\partial x^{\sigma})(\partial x'^{\beta} / \partial x^{\nu})g'_{\alpha\beta}(E) \\ &+ (\partial x'^{\alpha} / \partial x^{\mu})(\partial^2 x'^{\beta} / \partial x^{\nu}\partial x^{\sigma})g'_{\alpha\beta}(E) \\ &+ (\partial x'^{\alpha} / \partial x^{\mu})(\partial x'^{\beta} / \partial x^{\nu})g'_{\alpha\beta,\sigma}(E) \end{aligned} \tag{42}$$

and when the values of the differentials at $\wp_0$ are substituted in this expression, we have

$$g_{\mu\nu,\sigma}(E) = G^{\alpha}_{\mu\sigma}(E)g'_{\alpha\nu}(E) + G^{\beta}_{\nu\sigma}(E)g'_{\mu\beta}(E) + g'_{\mu\nu,\sigma}(E) \tag{43}$$

If locally at $\wp_0$ the second part of Eq. (36) is to be satisfied after transforming to the unprimed coordinates, we must have

$$0 = G^{\alpha}_{\mu\sigma}(E)g'_{\alpha\nu}(E) + G^{\beta}_{\nu\sigma}(E)g'_{\mu\beta}(E) + g'_{\mu\nu,\sigma}(E). \qquad (44)$$

The required transformation can be achieved if we set $G^{\alpha}_{\mu\sigma}(E)$ equal to the value of the metric connection coefficient $-\Gamma^{\alpha}_{\mu\sigma}$ at $\wp_0$. This equality would also assume that there is a tangent space for every point of interest within each energy-dependent family. This energy dependence can be thought of as the energy of the probe. Even using the smallest probe, say a photon, the energy dependence has a multiple dependent relationship to space-time based on the particle's spin state. A photon is a wave that can oscillate and corkscrew through space-time, affecting each spatial component differently as it propagates. The curves do remain smooth at low energy, but become more non-differentiable with higher energy. Magueijo and Smolin assume Eq. (44) and proceed with forming a workable approach to DGR, but it is plausible that for particles with a wavelength slightly larger than the Planck length, the energy contained within the wave function is not reducible to a single point, but is spread over some region of space-time. The additive rules of the DSR theory must also contain boundary restrictions for adding energies. Otherwise, in DGR, it is not possible to proceed. There must be a description of space-time within a boundary parameter for adding the energy degrees of freedom. In other words, there must be a space-time orientation for the new additive rules to be applied to two or more probability distributions. If two of these wave functions were to overlap, there would be no way to describe the region they occupy because their sum energy would exceed the Planck energy, violating the DSR invariance. Now that there are new additive rules and a corresponding metric, there is still no description of how far apart degrees of freedom must be to be considered two degree of freedom rather than one. No two particles can occupy the same exact point in space time, but they can have overlapping wave functions at a set of points. This implies two particles are not a single degree of freedom unless they fall within a unique boundary of interaction. The unique boundary of interaction would be the limit on the set of points in space-time where the wavefunctions overlap and the energies can sum as a single degree of freedom using the new energy additive rule.

Because of the unique boundary of interaction, the Doubly Special Gravitation term $G^{\mu}_{\nu\tau}(E)$ should be a parameter that is not only dependent on energy, but is also dependent on the type and number of degrees of freedom associated with that energy dependence. By assuming that $G^{\mu}_{\nu\tau}(E)$ has no other positional or directional dependence, the differentiation becomes trivial and we can shift into a flat Minkowski frame, but all information about the internal degrees of freedom is lost in the transformation.

To proceed in DGR preserving all degrees of freedom, equation (35) look more like:

$$\begin{aligned} g_{\mu\nu}(\wp_0) &= A_{\mu\nu}(\wp_0) \\ g_{\mu\nu;\rho}(\wp_0) &= \frac{\partial A_{\mu\nu}(\wp_0)}{\partial x^{\rho}} - \Xi^{\tau}_{\mu\rho}A_{\tau\nu}(\wp_0) - \Xi^{\tau}_{\nu\rho}A_{\mu\tau}(\wp_0) \end{aligned} \qquad (45)$$

where $\Xi^{\tau}_{\mu\rho}$ represents the energy dependent affine connections with specific relational space-time definitions, meaning $\Xi^{\tau}_{\mu\rho}$ is sensitive to spin, orientation, and all other structural components. I leave $\Xi^{\tau}_{\mu\rho}$ as an abstract entity that we can treat as a simple connection coefficient for limiting cases. $A_{\mu\nu}$ is a unique, energy dependent metric.

I define $A_{\mu\nu}$ to be the Quantum Metric which behaves as $\lim_{E \to 0} A_{\mu\nu,\rho}(E) = 0$, where $A_{\mu\nu,\rho}(E)$ is the first derivative of the energy dependent metric in the direction orthogonal to both $\tilde{e}_\mu(E)$ and $\tilde{e}_\nu(E)$. $A_{\mu\nu,\rho}(E)$ also goes to zero as the distance from the energy source gets significantly larger than the Planck length and $A_{\mu\nu}$ goes to the Minkowski metric for flat space-time. The limiting case relationship can be written as:

$$\lim_{E \to 0} \Xi_{\mu\alpha\beta} = \frac{1}{2}(A_{\mu\alpha,\beta} + A_{\mu\beta,\alpha} - A_{\alpha\beta,\mu}). \qquad (46)$$

For this case the Riemann curvature and Einstein tensor can be built accordingly (see appendix C),

$$R^{\mu}_{\nu\alpha\beta} = \Xi^{\mu}_{\nu\beta,\alpha} - \Xi^{\mu}_{\nu\alpha,\beta} + \Xi^{\mu}_{\rho\alpha}\Xi^{\rho}_{\nu\beta} - \Xi^{\mu}_{\rho\beta}\Xi^{\rho}_{\nu\alpha} \qquad (47)$$

$$G_{\mu\nu} = R_{\mu\nu} - A_{\mu\nu}R.$$

This is, of course, only the limit where the energy is essentially zero for the probing particle. As $A_{\mu\nu}$ is adjusted to take on a greater and greater probe energy, all internal components of the probe have an increasing effect on space-time. If a particle is not a single point, but a set of internal degrees of freedom that can only be defined by the way they relate to a boundary, then a boundary must be defined that describes the limit of energy within the DSR framework for each independent degree of freedom. There is no functional physics that describes what space-time looks like at scales smaller than the Planck length. A theory of quantum gravity could not rely on the hope that there exists a tangent space at all events or every point, including the points in space-time where a wavefunction is located and plays host to the uncertainty principle. In the framework of DSR or standard relativity, it is only meaningful to probe space-time at a scale asymptotically close to the Planck length.

General relativistic effects become dominant in space-time when a star collapses to a size comparable to its Schwarzschild radius $R = 2GM / c^2$. Quantum fluctuations are similarly dominant at $R = \hbar / Mc$, giving the Compton wavelength of a mass *M*, considering a fluctuation of energy *E* in which a particle/anti-particle pair are created from the vacuum and subsequently annihilate. The time of existence of this fluctuation cannot exceed a time of $t = \hbar / E$ from the uncertainty principle, and the distance of influence of these fluctuations is smaller than $R = ct = c\hbar / E$. For a mass *M*, the uncertainty in position due to quantum fluctuations is of order $R = \hbar / Mc$. At mass or energy comparable to the Planck energy and distances smaller than the Planck scale, there in no guarantee that space-time is separable from energy. It is possible that they may be identical, a photon being a gravitational wave moving at *c* with energy $\hbar\omega$ that also has electromagnetic properties. This would imply that there are no tangent spaces to shift into locally and no differentiability. The space-time region around a particle becomes completely undefined.

One approach to this complication is non-commutative geometry. The basic idea of non-commutative geometry is that in quantum physics, one can not measure the position and velocity of a particle at the same time. But it is possible to at least determine the position precisely. However, notice that a determination of the position of a particle actually involves three different measurements, for we must measure where the particle is relative to a set of three axes, which correspond to the three components of the position vector. This concept requires an extension of the uncertainty principle in which one can measure only one of these components precisely at any one time. When it is not possible to measure two quantities simultaneously, they are said to not commute, and this idea leads to a new kind of geometry, which is labeled non-commutative. Because of the uncertainty principle, one can not define a point where something may be located exactly. But there could be a set of energy-dependent metrics with a corresponding quantum geometry. That is to say that every energy state that a quantum has can be described by a different quantum geometry.

Hopf algebras are the simplest examples of a non-commutative or quantum geometry in which the commutative algebra of functions on a space is replaced by a non-commutative one. Bi-cross products, which contain the cross products, can be interpreted as the algebras of observables of quantum systems. Classically, such algebras of observables would be functions on a phase-space, so now in the quantum case we should think of them as functions on a quantum phase-space. They provide us with concrete models in which to test out some ideas of quantum geometry in actual physical examples. It could be said that a large part of the problem of unifying quantum mechanics and gravity is a matter of mathematical language. Quantum mechanics is usually formulated algebraically, whereas gravity is usually formulated more geometrically. Quantum geometry, with Hopf algebras as a basic example, addresses exactly this problem because it allows us to formulate both structures in the same language.

Finding an energy-dependent quantum metric or set of metrics could give an approach that leaves DSR intact as a general case. One would then need to find the corresponding Hopf algebra for that energy case which could only be applied to that unique single degree of freedom.

There are lots of Planck scale energy objects around, although none of them have a Schwarzschild radius anywhere near as large as the wavelength of their matter wave. A good example of this is a baseball. A baseball has about 10^25 degrees of freedom (or particles) just at first look. We have reached a point where it is no longer possible to proceed without defining what exactly a degree of freedom is.

IV. Degree of Freedom

In the DSR theory that follows from the Judes and Visser approach, a degree of freedom is usually thought to be a single particle. This develops from the statistical mechanics approach to discretizing a system of homogenous entities. In this interpretation, any one single particle can have energy asymptotically close to the Planck energy. This approach is problematic in the development of not only gravitation and field theory, but even Hamiltonian and Lagrangian mechanics. A fundamental degree of freedom must be the simplest, non-divisible bound energy fluctuation, if there is such a thing.

The physical description of the Planck energy (another way of formulating the Planck scale) is derived from the analysis, in General Relativity, of a photon. Specifically, as the wavelength of a photon approaches its Schwarzschild radius, its energy approaches the Planck energy. If this description is held over into DSR, the photon energy (being a single degree of freedom) is what is restricted to not exceed the Planck energy. There would be no restriction on a massive object (some finite number of degrees of freedom), other than each component that makes it up must have energy that does not exceed the Planck energy independently. It must be determined if a degree of freedom is then a single particle or what makes up the particle and all that the particle contains. For example a proton is made up of two up quarks and one down quark. It is unclear whether the proton energy should be treated as a single degree of freedom or as three. It is also unclear as to how to treat potential and if it should be considered part of the energy already contained in a particle or as a separate entity. The choice to proceed affects the interpretation of energy greatly. Consider the Hamiltonian $H=T+V$. Taking the rule for additive energy from Judes and Visser, if there is potential inside of the particle (internal degrees of freedom) the Hamiltonian is:

$$\hat{H} = \frac{\sum_i (T+V)_i / [1-\lambda(T+V)_i]}{1+\lambda_T \sum_i (T+V)_i / [1-\lambda(T+V)_i]}. \tag{48}$$

Here we are summing over the number of particles (I have created a notation $\hat{\mathrm{H}}$ to represent the Hamiltonian adding its energy components as internal degrees of freedom). λ_T is the reciprocal of the total number of particles times the Planck Energy. Notice that $\hat{\mathrm{H}}$ can exceed the Planck energy, because the limit is only in each particle containing $(T+V)$ as it approaches the Planck energy. The sum of the system is limited to the number of particle times the Planck energy. This equation would be applicable for a system with every particle containing a kinetic energy term and a potential energy term combined in one degree of freedom. Consider a set of protons that contained quarks. This equation could represent the total energy of that collection, letting T be the kinetic term for the ith proton while V represents the potential including mass, charge, flavor and anything else. This could also be used to describe the total energy of an atom, a baseball, or a planet. Notice that it is the sum of the potential and the kinetic energies that asymptotically approaches the Planck energy. This assumes that the potential of each particle is not a degree of freedom that will affect the outside universe; it is contained within the boundary. But if the potential is considered a second degree of freedom, the total pseudo-energy of each particle looks a little different,

$$\varepsilon_{tot} = \sum_{energy} \frac{E}{1-\lambda E} = \frac{T}{1-\lambda T} + \frac{V}{1-\lambda V} = \widetilde{E}\,. \tag{49}$$

where we are summing over energies within the particle (I can now use the notation $\widetilde{\mathrm{E}}$ to refer to this style of addition and any energy with the tilda being the pseudo-form of that energy). This implies that the total energy for one particle is:

$$\breve{\mathrm{H}} = \frac{\frac{\mathrm{T}}{1 - \lambda\mathrm{T}} + \frac{\mathrm{V}}{1 - \lambda\mathrm{V}}}{1 + \lambda(\frac{\mathrm{T}}{1 - \lambda\mathrm{T}} + \frac{\mathrm{V}}{1 - \lambda\mathrm{V}})}. \tag{50}$$

It becomes very obvious that how a degree of freedom is defined is critical in establishing Hamiltonian and Lagrangian mechanics, and it is essential to develop this distinction before developing the theory into a field theory ($\breve{\mathrm{H}}$ represents this method of adding energy that we will consider external; now, any energy with this symbol over it refers to this method of adding its components). With this new notation we can have any arbitrary energy and distinguish how its components are added as either external or internal using $\breve{\mathrm{E}}$ or $\hat{\mathrm{E}}$. Lets exploit the physical differences using simple kinematics. Take an airplane for example. Using Eq. (50) it quickly becomes confusing as to where to draw the line. If $(T+V)$ is the total energy of the plane including kinetic, potential or gravitational, then H will not exceed the Planck energy. But, an airplane has much more rest mass energy than the Planck energy. If all particles of the airplane are taken into consideration, then we notice that for each individual particle there is only a fraction of the Planck energy and the limit for any collection of particles is the limit where every particle approaches the Planck energy. So it is plausible that internal degrees of freedom must be taken into account. A clear description of how a particle with internal energy can be treated as single degree of freedom, or a description of how all internal degrees of freedom can be dealt with the same way, is required. The former possibility would look like:

$$\breve{H} = \frac{\frac{T}{1 - \lambda T} + \sum_i \frac{V_i}{1 - \lambda V_i}}{1 + \lambda(\frac{T}{1 - \lambda T} + \sum_i \frac{V_i}{1 - \lambda V_i})}. \tag{51}$$

Here we are summing over all internal potentials and this is again the total energy for one particle. Eq. (51) could be the Hamiltonian for a single particle. This equation treats all degrees of freedom and energy on equal footing. When this total energy is used to write a Hamiltonian for a system of particles, it becomes a highly "relationary" equation, meaning it depends a great deal on the possible potentials and the way the particle relates to the rest of the universe. For a classical calculation, all these minuscule potential terms are insignificant. But on the Planck scale they are not. Keep in mind again that our motivation behind DSR is an investigation of Quantum Gravity. If the quantum operators are applied to a single degree of freedom, there will be a limit to the frequency (or energy) that the degree of freedom can have.

Conventionally in DSR, the phrase "degrees of freedom" is used loosely when discussing a number of particles. If this interpretation is pressed in the framework of DSR, it is the massless particle which must be a single degree of freedom, or at least a particle containing the least degrees of freedom. An electron is a single particle but it has mass and charge as well, which manifest as gravitational and electrical potential, an energy acting on the outside universe. Though there is an electromagnetic potential associated with the photon, it can be considered local and contained within the wave packet compared to the electrical potential of an electron. And though the photon has energy and a gravitational potential associated with that energy, it does not have a structural mass. I am not suggesting that a photon is a one degree of freedom energy fluctuation—a photon actually has a few—but I am insinuating that mass is related to additional degrees of freedom even within the single particle regime. For a multiparticle system, this concept is somewhat obvious. In consideration of Eq. (19), the dispersion relation for a massive object should look like:

$$\frac{E^2 - p^2}{(1 - \lambda E)^2} = \sum_i \frac{(m_0)_i^2}{(1 - \lambda (m_0)_i)^2}. \qquad (52)$$

where m_0 is the rest energy of a single particle. But, there is no universal specification as to where the divisions are. If this mass is a cluster of electrons, perhaps we can neglect other degrees of freedom. If it is a baseball, divisions can be made down to electrons and quarks. Eq. (52) assumes that the four-vector stays intact and the three spatial degrees of freedom in the momentum three-vector all serve as one degree of freedom. It also assumes that the vector product stays intact.

This is a hopeful position. This vector product requires the sum of spatial degrees of freedom and that energy can be added to the momentum three-vector despite the fact E can have most of the particle's energy (if the particle is slow-moving or at rest). The particle's energy can also come solely from its momentum, as in the case of the photon. This could imply that energy from a quanta frequency and energy from mass are independent degrees of freedom as well. In the interpretation that a photon is a single degree of freedom, it would also appear that mass is an additional degree of freedom. The mass written in Eq. (52) is m_0. This is the rest mass of a single particle. As mentioned earlier, the description of mass changes in DSR from theory to theory. It might be time to consider a more universal definition. Rest mass is a sum of all of the energy contained within the structural form of the matter. It is essentially a collection of all internal units and the way each of the units respond to one another. Within the rest mass is contained information about the way the energy of the system is unique, ranging from atomic structure to the specific heat of the substance to the binding energy to the orientation and arrangement of the pieces that make it up. Photons, of course, do not have this complexity. In fact photons are the most fundamental particle in the known physical universe. Everything reduces to photons after annihilation. So an accurate definition of the rest mass of any object could be one half of the energy released after annihilating it with its anti-self (identical object made of anti-particles). So if we can understand and describe the number of degrees of freedom in a photon of some arbitrary energy, then we might be able to use this description to describe other particles and matter.

The field theory construction of the Hamiltonian can not be built using the present DSR framework. But if some modifications are made using the DSR assumption, we can at least loosely provide a starting point. For a discrete system one can define a conjugate momentum $\breve{p} \equiv \partial\breve{L} / \partial\dot{\breve{q}}$. Here $\breve{L}$ is the Planck energy limiting Lagrangian density, which is a function of kinetic and all potential energies from various fields that must be consider external, so they are added using Eq. (50); hence, the external adding symbol.

The Hamiltonian is then $\Sigma\breve{p}\dot{\breve{q}} - \breve{L}$. The generalization to a continuous system can be represented by the spatial points x as discretely spaced units no smaller than the Planck length. So momentum becomes:

$$\breve{p}(x) \cong \frac{\partial}{\partial\dot{\breve{\phi}}(x)} \sum_{y} \breve{L}(\phi(y), \dot{\breve{\phi}}(y)d^3y \qquad (53)$$

Thus, the Hamiltonian is then

$$\breve{H} = \sum_{x} \breve{p}(x)\dot{\breve{\phi}}(x) - \breve{L}. \qquad (54)$$

Notice that Eq. (54) is not concerned with what is going on inside of the particle. The particle is assumed to be a complete unit that does not have any internal components to act on the rest of the universe except for what is considered in the field. For high energy particles in several high energy fields, Eq. (54) would be safe until scattering occurs. During scattering, internal energy has a measurable affect on the rest of the universe. At some point during the scattering process, the internal energy of a particle must be treated as an external energy and the addition rules change. There is an obvious difference between the field equations with mass and without. The particle described by these different methods would also scatter differently. A massless particle would have less internal degrees of freedom and less of an addition complexity. Consider a theory of a single field $\phi(x)$, with the Lagrangian

$$\breve{L} = (1 + \lambda(\frac{1}{2}(\breve{\partial}_{\mu}\phi) - \frac{1}{2}(\breve{\sigma}_{\mu}\phi)))^{-1}\{\frac{1}{2(1-\lambda(\breve{\partial}_{\mu}\phi)^2)}(\breve{\partial}_{\mu}\phi)^2 - \frac{1}{2(1-\lambda(\breve{\sigma}_{\mu})^2)}(\breve{\sigma}_{\mu}\phi)^2\}. \qquad (55)$$

where σ_{μ} can be thought of as the rest mass of the particle. The rest mass in this case is a structural parameter which has its own internal degrees of freedom. Equation

(55) is difficult to do physics with, but it is the most encompassing form of the Lagrangian. If we assume one degree of freedom for each term, the external additive symbol is redundant. The Lagrangian, before mapping back from the pseudo-form, is much more familiar looking:

$$\widetilde{L} = \frac{1}{2}(\widetilde{\partial}_\mu \phi) - \frac{1}{2}(\widetilde{\sigma}_\mu \phi).$$

This Lagrangian does not necessarily have any physical meaning because it is still in the pseudo-form. This mean that $\widetilde{L}$ could add up to more than the Planck energy if needs be, but notice that could never happen if each term is limited to one degree of freedom. If we assume the liberty to proceed, from this Lagrangian the usual procedure gives the equation of motion

$$[(\widetilde{\partial}^\mu)(\widetilde{\partial}_\mu) + (\widetilde{\sigma}^\mu)(\widetilde{\sigma}_\mu)]\phi = [\eta^{ab} \frac{\partial_a}{1 - i\lambda\partial_0} \frac{\partial_b}{1 + i\lambda\partial_0} + (\widetilde{\sigma}^\mu)(\widetilde{\sigma}_\mu)]\phi = 0. \quad (56)$$

This can be thought of as a DSR friendly Klein-Gordon equation. Equation (56) has plane wave solutions that would satisfy the DSR dispersion relation. Because Hamilton's Principle requires the integral of the difference between kinetic and potential to be a constant, it is for physical reasons that we must apply any needed modification to make this procedure physically applicable. Naturally the calculus of variations will lead to an equation of motion equal to zero, which does not need to be transformed back from the pseudo-form.

Equation (56), though consistent with the DSR rules, is essentially meaningless. It assumes that for any mathematical operation, it is permissible to stay in the pseudo-form until the equations of motion are developed. It is true that the variational requirements of

Hamiltonian principle rely on the difference in pseudo-kinetic and pseudo-potential such that $\widetilde{L}$ be an extremum. Equation (56) is true for that extremum, but is not associated with any physical field.

Equation (55) has the tilda on both the kinetic and potential terms. For the case of one field this should be removed because there is only one component of energy acting on the particle. The particle in a single field can either gain energy from the field or lose it to the field. This is also true for kinetic energy, not just for this example, but in general. Kinetic energy should have no external component because it is the energy associated with the motion of an independent body. So the external additive symbols in Eq. (55) are unnecessary. However, we can now see what a degree of freedom in the DSR frame work when concerning additive energy is: *a bound entity (particle, wave, field, object) that can undergo a change resulting in a gain or loss of energy within that boundary*.

The σ_μ is indeed mass dependent and mass has a gravitational field associated with it, so even for the simplest case, it would be incorrect to include this as an independent term in the potential. Mass can also have internal components such as particles and structure, which should not be considered before the physics of the fundamental particle (the photon) is worked out.

For our analysis it is most important to consider the photon. Let us assume a divergenceless current so that $\partial_\mu J^\mu = \widetilde{\partial}_\mu J^\mu = 0$. This expresses local conservation of charge and with the help of Bianchi's identity, it sets a boundary that allows us to take the divergence in the standard manner. Using the wave equations A^μ, we can now write the

Klein-Gordon equation for a massless particle:

$$-\frac{1}{c}\frac{\partial^2 A^\mu(x)}{\partial t^2} + \nabla^2 A^\mu(x) = 0,$$

and for DSR

$$\left(-\frac{f}{h}\frac{\partial^2}{\partial t^2} + \nabla^2\right)A^\mu(x) = 0. \tag{57}$$

For DSR 2, *f* and *h* are equivalent so the speed of light does not change, but in DSR1 *h/f* gives the energy dependent speed of light which can be seen from Eq. (30). We can look at DSR plane wave solutions with momentum

$$\wp_4 \equiv (\varepsilon; \pi) = F^{-1}(P_4) = \frac{(E; p)}{1 - \lambda E}:$$

$$A^\mu(x) = ae^{-i\wp_4 \cdot x}\delta^\mu(\wp_4), \tag{58}$$

where δ^μ is the polarization vector which characterizes the spin of the photon. Here $\wp^\mu\wp_\mu = 0$. δ^μ has four components, but they are not all independent as the Lorentz condition requires that $\wp^\mu\delta_\mu = 0$. The polarization four-vector is perpendicular to the direction of propagation. There are two linearly independent four-vectors that are perpendicular to the momentum four-vector just as they would be for *p*. This leaves two independent solutions for a given momentum, one for each spin state. It is worthwhile bringing this into focus. By our new definition of degree of freedom, a spin state would seemingly not qualify because there is no change or fluctuation once the particle is created. Equation (58) treats spin as inherently built-in. Remember this is only the equation for a single massless particle. An arbitrary point in space-time would be affected quite differently from high enough energy photons of different spin. For a photon close to the Planck energy, Eq. (45) is greatly affected. Polarization affects space-time because the energy dependence of $A_{\mu\nu}$ is directly formed from A^{μ}. If an equation like (45) can be written for the energy of the probe including the basis components of all internal degrees of freedom, the form of Eqs. (46) and (47) can follow. An electron in an atom has specific bound states; when the electron is given an excessive amount of energy for that state, it breaks away to a new energy state. There is this relationship between energy and boundary throughout physics. Any one entity can be given a finite amount of energy before it makes a transition to a higher state. For a free photon in DSR, this limit is obviously the Planck energy, which is unattainable. But photons interact with the rest of the universe; they are absorbed, affected by gravity, scattered, emitted, and so on. There must be a way to determine when Bianchi's identity for the single photon is no longer applicable and the additive rules change to incorporate the new external term of energy.

V. Bianchi's Identity

Nature conveys its order to matter by means of the minimization of the world line in proper time. The Einstein equation gives the method of coupling between gravitation and the source (the stress-energy tensor) that will guarantee the automatic conservation of the source term. The mathematical reason that this conservation can work is through the concept of a boundary and Bianchi's identity: A boundary of a boundary is zero, $\partial\partial = 0$.

The pressing issues are, of course, the laws of conservation (conservation of charge; conservation of momentum-energy), $\nabla \cdot T = 0$. These conditions need not be circumstantial, which requires the source not to be a mediator, free to vary subjectively from place to place and instant to instant. The source needs a connection to something that, while having degrees of freedom of its own, will minimize the otherwise ineffectual degrees of freedom of the source sufficiently enough to guarantee that the source in question will fulfill the conservation law. Conservation demands that there is no creation or destruction of the source inside the boundary of the finite dimensional space in question. The integral of the divergence of the energy-momentum over this finite dimensional space is then required to be zero.

The source (stress-energy tensor) is indeed primary. This makes conservation of the source the number one priority of physics. When a photon is described in a way that leaves the field unchanged, it can not directly scatter or interact with anything else. The boundary requires that everything contained in $\widehat{E}$ not have an effect on the outside universe. If this were true, a photon would exist without ever having any influence. Just like the electron, there must be a region or potential associated with the photon, or any particle in a free or bound state of the applied field.

VI. The Construction of Gravity

For a description of gravity that encompasses quantum particles and probes of high energy, a sure starting point is a description of degrees of freedom that coincides with Bianchi's identity. Because DSR in general lacks this description, this gives us the opportunity to change our definitions to encompass the broad scope of gravity. Let us take the simplest case applicable to DGR. Suppose we are probing the space around a photon that has energy comparable to the Planck energy with a photon of Planck scale energy. If we are probing closer and closer to the point of scattering, there must be a limit to how close we can get before the new additive energy rule takes effect. Otherwise, there would be several points in space-time that would seem to contain more than the Planck energy as their wave functions overlap more and more. Because there is other matter and energy that can act on a particle, essentially all the energy and matter in the universe would have to be added up using the new additive rules of DSR acting within a single boundary in space-time.

As the photon is scattered closer and closer to its high energy twin, it is eventually in the region of the Planck area around it, at which point the uncertainty in momentum goes to infinity within this region. But the volume swept out from the wavefunction in position space is greater than the volume created by cubing the Planck length at the center of the Schwarzschild diameter. At the scale we are probing, there is no exact point of scattering but a region in space-time related to the scattering amplitude. There is no distinction between the region of space-time to which the new energy-additive rule is applied and not applied. For example, within the stress-energy tensor there exist terms for all of the energy in the region we are probing. Using the new additive rules for energy, the terms in the tensor look the same at each and every point where the wave functions overlap. If this were true, there would be no actual scattering. In order for this additive rule to be useful, there must be a spatial orientation related to each energy degree of freedom.

A plausible method is that the photon energy is added together with respect to the localized boundary in space-time from the probing particle and any other energy source. The metric then responds not only to energy but also to the location of any other potential in the universe according to how far away the interacting source is. The original energy dependent metric first proposed in [19] states that the empty space metric is

$$g_{ab}(E) = \begin{bmatrix} -\frac{1}{f^2(E)} & 0 & 0 & 0 \\ 0 & \frac{1}{h^2(E)} & 0 & 0 \\ 0 & 0 & \frac{1}{h^2(E)} & 0 \\ 0 & 0 & 0 & \frac{1}{h^2(E)} \end{bmatrix}. \quad (59)$$

Here E is the energy of the photon, which can not by itself exceed the Planck energy. Because space-time becomes intrinsically mixed with energy-momentum, to proceed with Doubly Special Gravitation it is a reasonable approach to allow any interaction source from outside the unique boundary of interaction to fall off with distance from the probe. As an example, consider the additive energy rules suggested in Eq. (50) without the kinetic term:

$$H = \frac{\sum_i \frac{V_i}{1 - \lambda V_i}}{1 + \lambda \sum_i \frac{V_i}{1 - \lambda V_i}} \tag{60}$$

To set a boundary in which the energy of the probing photon has a dependence on the source of interaction (using the same additive rule for internal and external degrees of freedom), Eq. (60) can be written as:

$$\xi = \frac{\sum_i \left(\frac{L_p}{S_i}\right) e^{\sqrt{1-L_p^2/S_i^2}} \left(\frac{V_i}{1 - \lambda V_i}\right)}{1 + \lambda \sum_i \left(\frac{L_p}{S_i}\right) e^{\frac{L_p}{S_i}\sqrt{1-L_p^2/S_i^2}} \left(\frac{V_i}{1 - \lambda V_i}\right)}. \tag{61}$$

Here, S is a function that gives the distance between each independent degree of freedom (or source of interaction) from the one used for probing. The parameter L_p is the Planck length, but it is essentially an arbitrary boundary limit. I choose the Planck length for reasons based on the uncertainty principle in relation to scattering of photons. It could have just as easily been any parameter relating the scattering diameter and the Schwarzschild radius to wavefunction overlap so that a unique description of energy can relate distinctively to space-time. We see that Eq. (61) goes to zero as S goes to infinity, so the interaction source becomes unaffected. With (61) it is only interaction sources that are significantly close to the Planck distance away from the probing source that show up. This gives a spatial definition to "internal" degree of freedom. All other sources can be added with the external energy equation. One can see that as S goes to the Planck length, Eq. (61) has the same form as Eq. (60). Then as S becomes smaller than the Planck energy, the resulting ξ energy (defined in Eq. 61) becomes imaginary. Then the metric looks more like

$$g_{ab}(\xi) = \begin{bmatrix} -\frac{1}{f^2(E+\xi^0)} & 0 & 0 & 0 \\ 0 & \frac{1}{h^2(E+\xi^1)} & 0 & 0 \\ 0 & 0 & \frac{1}{h^2(E+\xi^2)} & 0 \\ 0 & 0 & 0 & \frac{1}{h^2(E+\xi^3)} \end{bmatrix} \tag{62}$$

It is possible for the ξ energy to vary from one orthogonal plane to the next, so there may be a different ξ energy for each axis.

These equations give every degree of freedom as only a response to the source interaction locally. Now, each degree of freedom is a fraction of what it would be at exactly the Planck distance away form the probing energy. It also gives the capacity to take into account all rest mass and other potentials. Eq. (61) can contain interaction terms for every single particle in the universe acting on the energy of our probing photon, as long as the interaction is defined. The further away the interaction, the less the source affects the energy dependent metric. Here, the energy from a single degree of freedom as defined previously has the greatest effect when it is exactly one Planck distance away from the center of interaction. It can be said that an internal degree of freedom as applicable to Eq. (50) is defined to be exactly one Planck distance away from the interacting potential. For any source that is further away, it is only a partial degree of freedom and only responds partially to that source term. For most other source terms in the universe, S is so large that the effect is unseen.

ξ is described here just for the photon. For other particles, Eq. (61) would look radically different. There would be a dependence on the source term that involved the unique field interaction for that particle. Essentially this method gives a description to only the most fundamental case.

VII. Conclusion

It is not deduced that DSR gives any new benefit to the study of Quantum Gravity or makes any approach to Quantum Gravity more serviceable. DSR does, however, show that there are viable reformulations that can be made to create deformative cases of special relativity. The generalization to gravity is not a straightforward process. To create an energy dependent metric, by my results, implies an intrinsic dependence on the space-time location of the probing energy sources in relation to all other energy sources. This gives each energy source a fractional degree of freedom (as defined earlier) and defines a full internal degree of freedom of an energy source as a source of interaction exactly one Planck distance away. Doubly Special General Relativity can be constructed using the metric developed in Eq. (62) for the case of a photon, but there are other modifications to General Relativity and physical interpretations of these modifications are required. These results are not meant to be explicit but only meant to be a plausible example of the importance of space-time oriented degrees of freedom and a way to proceed in the development of DSR gravitation. The noticeable effects from DSR are of the same energy scale that standing theories of quantum gravity begin to diverge from the observations of Quantum Mechanics and General Relativity. In effect, there are no low energy experiments or applications of DSR. The usefulness of DSR may be nothing more than a tool to study deformative relativity with a motivation of quantizing space-time.

It seems very likely that there is some type of relational parameter, like λ, that varies with respect to local fluctuation within the boundary defined by the structure of the energy that would create an energy and or momentum invariance. There is no reason to think or not to think that that energy invariance would be the Planck energy. But, in the DSR framework this energy scale is not a parameter and does not vary; it is a constant. In order for the metric to be energy-dependent, there must be a corresponding quantum geometry based on that dependence. This in itself is a radical concept. This would imply that after discretizing energy and space-time, there would be an infinite set of alternate conformal geometries. It would be far simpler if the invariant energy parameter could change. To study Quantum Gravity under these assumptions also brings up the possibility that the symmetries observed are also energy dependent.

Most examples used have probed space-time with the photon. This is because the photon is the simplest particle and because it is used in the physical description of the Planck energy. There is an additional convenience of zero rest mass energy. In DSR the concept of rest mass is undecided. The additive rules for energy make this issue even more complex. The rest mass used in Eq. (52) is the rest mass of a single particle in DSR2. If we add up the particles of a system, we get a rest mass in terms of ε. However ε does not take into consideration binding energy and other potential energy stored within the structure of mass. The rest mass should be one-half of the energy produced by annihilating the mass with its anti-particle system. A system of particles would lose any structure that we are familiar with as each particle in the system acquired energy comparable to the Planck energy, which is also very significant. To develop a theory consistent with the interpretation of degrees of freedom and Bianchi's identity demonstrated here, ξ would need to encompass all of this information and conservation requirements when concerned with mass.

If the DSR postulates are valid, then a photon should not ever form a black hole. This does not seem like that big of a leap. It would be a very unique type of event that would emit radiation that had the Planck energy. No one quanta could have anywhere close to the Planck energy in typical physics. An extraordinary physical situation would have to be going on around that quanta in order for that physical situation to arise. There is very little research that deals with these questions.

If any degrees of freedom are somehow dependent on each other, there is the possibility that there is not only an asymptotic approach to an invariant energy, but there is also a minimum energy that the quantum could have as well. For example, it is thought that a photon with a wave length the size of the universe's diameter would be the lowest possible energy for a quanta. But if that quanta had a degree of freedom that was coupled to its energy the way mass is generally thought of, then the dilatation (in the generator) would depend on the two coupled degrees of freedom. So the maximum energy that one would have would be the invariant energy minus the minimum that the other degree of freedom would have. This would give a relational account for a minimum energy that a quanta could have. This could then be used to calculate the curvature of space-time from that quantized minimum and then quantize gravity accordingly.

Much of what seems to be lacking in a great deal of the DSR research is practical application. DSR was developed in an attempt to explain high-energy cosmic ray anomalies and to assist with the construction of quantum gravity. But a great deal of work has been pursued with attempts to make DSR a complete reformulation of Special Relativity which develops more questions than answers about the nature of space-time at the microscopic level. It does not appear that DSR can be a complete theory. At best it will teach us how to clearly define aspects needed to drive forward in the quest for quantum gravity. Issues about degrees of freedom, observer-dependent frames and relational space-time dependence are principles that quantum gravity will be based on. It would be nice if we could agree on what these principles were and what they meant.

Equation (61) comes from my developments in DSR. In this work there was an attempt to construct a case of Doubly Special Gravitation in order to find what major assumptions in the framework are necessary in order to proceed. The modified equivalence principle is inadequate to give a description of any measurable physics and insufficient to study higher energy physics. By redefining a degree if freedom and how energy relates to the universe locally, It was shown that it is plausible to proceed with DGR in the case of a photon. There are many untouched topics and open issues in my approach, such as the implication of imaginary terms in the energy at probing distances closer than the Planck length. There was also no attempt to address the physical differences between my description of total energy and energy that is measure as rest mass energy. There was no rigorously discuss the new description of mass and its physical interpretation, or the differences in rest mass and dispersion relation mass as well as inertial mass.

It is implied in the previous discussion that it would be possible to build a description for DGR using this description for a photon. In truth, this would require a method that could differentiate between species but treat each as a photon with extended degrees of freedom or energy source terms responding to a unique boundary type. Of course, this is not a problem contained only in DGR, but quantum gravity in general. Rest mass is a very expansive concept. The structural arrangement of matter has a great deal to do with the way space-time responds to it, and vice-versa. For example, a piece of matter whose components were instantaneously arranged in an isotropic fashion would attract a photon probe or passing matter less than that same piece of matter whose components were arranged more sporadically. The difference seen in the Stress-Energy tensor would be related to the specific heat of the material.

We see that all types of structure can and do affect space-time. Incorporating DSR brings new possibilities to the approach. A good example of this is the energy dependent speed of light of DSR1. Having an energy dependent invariant seems contradictory, but having loose definitions gives the freedom and leverage to manipulate old ideas.

References

[1] G. Amelino-Camelia, "Relativity in Space-Time with Short-Distance Structure Governed by an Observer Independent (Planckian) Length Scale," Int. J. Mod. D 11 (2002) 35 [arXiv:gr-qc/0012051]

[2] J. Magueijo and L. Smolin, "Lorentz Invariance With an Invariant Energy Scale," Phys. Rev Lett. 88 (2002) 190403 [arXiv:hep-th/9405107]

[3] S. Judes and M. Visser, "Conservation Laws in Double Special Relativity," Phys.Rev. D68 (2003) 045001 [arXiv:gr-qc/0205067]

[4] J. Magueijo and L. Smolin, "Gravity's Rainbow," Class.Quant.Grav. 21 (2004) 1725-1736 [arXiv:gr-qc /0305055]

[5] R. Geroch "Quantum theory of gravitation" RPP 37 (1974) 1211-56

[6] A. Magnon "Quantum fields in curved space-times" PRS A346 (1974) 375-94

[7] R. O. Hansen "A unified treatment of null and spatial infinity in general relativity" Journal of Mathematical Physics 19 (1978) 1542-66

[8] M. Streubel "On angular momentum of stationary gravitating systems" Journal of Mathematical Physics 20 (1979) 1362-5

[9] A. Magnon-Ashtekar "A geometrical approach to external potential problems in quantum field theory" GRG 12 (1980) 205-23

[10] A. Ashtekar and A. Sen, On the Role of Space-time Topology in Quantum Phenomena: Emergence of Non-trivial Vacua and Superselection of Charge, Journal of Mathematical Physics, 21, 862-867 (1980)

[11] S. Majid, Foundations of Quantum Group Theory, (Cambridge University Press, Cambridge 1995)

[12] M. Green, J. H. Schwarz , E. Witten, Superstring theory, (Cambridge University Press, Cambridge 1987)

[13] D. Kimberly, J. Magueijo and J. Medeiros, Phys.Rev. D70 (2004) 084007 [gr-qc/0303067]

[14] G. Amelino-Camelia, D. Benedetti, F. D'andrea and A. Procaccini "Comparison of relativity theories with observer-independent scales of the velocity and length/mass" Class.Quant.Grav. 20 (2003) 5353-5370 [arXiv:hep-th/0201245]

[15] S. Majid and H. Ruegg, Phys. Lett. B334 (1994) 348

[16] G. Amelino-Camelia, L. Smolin and A. Strodubtsev, Class.Quant.Grav. 21 (2004) 3095-3110 [arXiv:hep-th/0306134]

[17] M. Peskin, D. Schroeder, *An Introduction to Quantum Field Theory.* Addison-Wesley Publishing Company 1995

[18] S. Weinberg, The Quantum Theory of Fields. Cambridge press 1995

[19] L. Smolin, *Three Roads to Quantum Gravity.* Basic Books 2001

Appendix A. Generators

In the investigation of the region close to unity, one expands the group elements in a Taylor series to second order in the small group parameter θ_i because the linear terms and several quadratic terms all cancel in the product.

$$
\begin{aligned}
L_i &= \exp(i\theta_i N_i) = 1 + i\theta_i N_i - \frac{1}{2}\theta_i^2 N_i^2 + \ldots, \\
L_i^{-1} &= \exp(-i\theta_i N_i) = 1 - i\theta_i N_i - \frac{1}{2}\theta_i^2 N_i^2 + \ldots, \\
L_i^{-1} L_j^{-1} L_i L_j &= 1 + \theta_i \theta_j [N_j, N_i] + \ldots, \\
L_i^{-1} L_j^{-1} L_i L_j &= 1 + \theta_i \theta_j \sum_k c_{ji}^k N_k + \ldots,
\end{aligned}
$$

The final line holds because the product here is a group element close to unity in the group *G*. This means that its exponent is a linear combination of the generators N_k and its infinitesimal group parameter has to be proportional to the product $\theta_i \theta_j$. The closure relation of the generators of the Lie groups *G* is then

$$[N_i, N_j] = \sum_k c_{ij}^k N_k ,$$

and c_{ij}^k are the structure constants of the group *G*. For a more detailed description and applications to field theory see [18].

To generalize this approach to the vectors of special relativity it is required that translations are covariant and so space time is homogenous. Covariance under rotations is an assertion of the isotropy of space. The requirement of Lorentz covariance follows from special relativity. All three of these transformations (translations, rotations, and boosts) together form the inhomogeneous Lorentz group or the Poincaré group. When the translations are excluded, the space formations with the Lorentz transformations together form the homogenous Lorentz group. A subgroup is generated, the Lorentz transformation in which the relative velocity is along the *x* axis. The generator may be determined by considering space-time reference frames moving with a relative infinitesimal velocity. Lorentz transformations are linear in the space coordinates as well as time.

For the rotation group, the generators are the angular momentum operators j^i, which satisfy the commutation relations $[j^i, j^j] = i\varepsilon^{ijk}J^k$. For the generators of the group of Lorentz transformations, consider the expression of the four-dimensional Lorentz transformation,

$$j^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{1}$$

which contains the six operators that generate the three boosts and three rotations of the Lorentz group. To determine the commutation rules for the Lorentz algebra, just compute the commutators of the differential operators from Equation (1).

$$[j^{\mu\nu}, j^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

$g^{\mu\nu}$ is of course the metric tensor. All matrices that are to represent this algebra must obey these commutation rules.

The construction of the Lorentz group in Doubly Special Relativity is essentially the same, except the Lorentz transformations are now nonlinear, adding an extra complexity. Amelino-Camelia [1] presented DSR1 to show that Galileo's Relativity Principle can coexist with postulates introducing an extra observer-independent scale. The length/momentum invariant allows the rotations to stay the same as they are presented in special relativity.

$$[M_i, M_j] = i\varepsilon_{ijk}M_k \quad [M_i, N_j] = i\varepsilon_{ijk}N_k \quad [N_i, N_j] = i\varepsilon_{ijk}M_k$$

where *M* is the standard action of rotations on momenta and *N* is the ordinary Lorentz generator. Defining *M* as

$$[M_i, p_j] = i\varepsilon_{ijk}p_k \text{ and } [M_i, p_0] = 0.$$

From Equation (1) it is possible to isolate the operator and look at the way the ordinary Lorentz generator acts as in momentum space,

$$N = p_a \frac{\partial}{\partial p^b} - p_b \frac{\partial}{\partial p^a} = p_z \frac{\partial}{\partial E} - p_0 \frac{\partial}{\partial p_z}.$$

Where in DSR1 involves the following differential representation of the boost generator along the *z* axis,

$$N_z = p_z \frac{\partial}{\partial E} + (\frac{L_p}{2} \vec{p}^2 + \frac{L_p(1 - e^{-2E/\lambda})}{2}) \frac{\partial}{\partial p_z} - L_p p_z (p_j \frac{\partial}{\partial p_j}). \qquad (2)$$

Here all units used are, $c = \hbar = 1$. L_p is assumed to be of the order of the Planck length, but not necessarily given exactly by the Planck length. One can see in these units that the reciprocal of the Planck length is the Planck energy. For low energy, (2) looks very much like the ordinary Lorentz generator. These units will be consistent in this thesis unless otherwise specified. Equation (2) is Amelino-Camelia's derivation in DSR1.

The construct of the Lorentz generator forms elements of a group of deformed Lorentz transformations. The generators in (2) had originally emerged in mathematical studies of κ–Poincare Hopf algebras where the deformed action of boosts on momenta looks like this:

$$[N_i, p_i] = -i \frac{1}{\kappa} p_i p_j + i\delta_{ij}\left(\frac{\kappa}{2}(1 - e^{-2p_0/\kappa}) + \frac{1}{2\kappa}\vec{p}^2\right).$$

Appendix B. Noether's Theorem

The relationship between symmetries and conservation laws in classical field theory are made visible with the use of *Noether's Theorem.* Assume a continuous transformation on a field θ in infinitesimal form:

$$\theta(x) \rightarrow \theta'(x) = \theta(x) + \alpha\Delta\theta(x)$$

Here α is an infinitesimal parameter and $\Delta\theta$ is the deformation of the field configuration. The transformation is called a symmetry if it leaves the equations of motion invariant, which insures the action to be invariant under the transformation. Allowing the action to change by a surface term does not affect the derivation of the Euler-Lagrange equations of motion,

$$\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\theta)}\right) - \frac{\partial L}{\partial\theta} = 0.$$

The Lagrangian is invariant under this transformation up to a 4-divergence:

$$L(x) \rightarrow L(x) + \alpha\partial_\mu\theta(x)J^\mu(x).$$

Using the transformation, compare this expectation for ΔL to the result obtained by varying the fields:

$$\alpha\Delta L = \frac{\partial L}{\partial\theta}(\alpha\Delta\theta) + \left(\frac{\partial L}{\partial(\partial_\mu\theta)}\right)\partial_\mu(\alpha\Delta\theta)$$
$$= \alpha\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\theta)}\Delta\theta\right) + \alpha\left[\frac{\partial L}{\partial\theta} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\theta)}\right)\right]\Delta\theta$$

Of course, the term in the brackets vanishes by the Euler-Lagrange equation, then by setting the remaining term equal to $\alpha\partial_\mu J^\mu$, one gets

$$\partial_\mu j^\mu = 0, \text{ for } j^\mu(x) = \frac{\partial L}{\partial(\partial_\mu \theta)} \Delta\theta - J^\mu .$$

This states that the current $j^\mu(x)$ is conserved. For each continuous symmetry of L, there is such a conservation law.

Appendix C. Einstein Tensor

Development of the Einstein Tensor is done through the use of either the metric or the affine connections (metric connections).

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right)$$

$$\Gamma^{\mu}_{\alpha\beta} = g^{\mu\nu}\Gamma_{\nu\alpha\beta}.$$

The Riemann curvature tensor can then be expressed in terms of the affine connections:

$$R^{\alpha}_{\mu\alpha\nu} = \frac{\partial\Gamma^{\mu}_{\nu\beta}}{\partial x^{\beta}} - \frac{\partial\Gamma^{\mu}_{\nu\alpha}}{\partial x^{\alpha}} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\nu\alpha}).$$

The Riemann curvature tensor can be made into second rank:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}.$$

The Ricci scalar can then be made from the metric and second rank Riemann tensor:

$$R = g^{\mu\nu}R_{\mu\nu}$$

The Einstein tensor is then

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$$