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Group representation theory with an application to P.I. algebras

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Group Representation Theory
with an Application to P.I. Algebras

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Gregory T. Daubenmire

Abstract

Group Representation Theory
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This thesis begins with a general discussion of the theory of representations of finite groups. We then look at a particular method for finding representations of the finite symmetric group S_n. This method was developed between 1900 and 1903 and involves the use of Young diagrams and Young tables. We then follow with a brief discussion of polynomial identity algebras and look at a rather suprising application of the theory of representations of the symmetric group. The theory is applied to the problem of finding an explicit identity for a polynomial identity algebra A. Finally, this procedure is used to determine an explicit identity for the tensor product of two P.I. algebras $A \otimes_F B$ over a field F of characteristic zero.

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Chapter 1 Group Representations

Introduction

In chapter one the theory of finite group representations is presented, for the most part, in general terms. Though most of the discussion assumes that the representation space is defined over the field of complex numbers, the bulk of the theorems hold for a field of characteristic zero or a field of characteristic p, where p is relatively prime to the order of group G.

The major focus of section one is the complete reducibility of a representation of a finite group. We shall see that a representation of a finite group over a finite dimensional vector space is completely and uniquely decomposable into a direct sum of irreducible representations. A discussion of normal representations completes section one. While this discussion may, at first, seem out of place we shall find it useful in later sections.

We introduce, in section two, an actual representation for a finite group called the regular representation. We also note an interesting relationship between representations of a finite group G and the normal subgroups of G. Finally we look at some representations of the finite symmetric group.

Two of the most important theorems of finite group representations, Schur's lemma and the homothety theorem, are presented in section three. These theorems describe the homomorphisms which commute with equivalent irreducible representations of a finite group. In particular they give us a tool for distinguishing between equivalent and non-equivalent irreducible representations.

In section four we define an inner product on the space of complex valued functions on a finite group G. This inner product allows us to identify the following: irreducible representations, the equivalence of two irreducible representations, and the number of equivalent irreducible representations in a given representation. It will be shown that each irreducible representation of a group G occurs in the regular representation with multiplicity equal to its degree.

We finish off chapter one with a discussion of the conjugacy classes of a group G. We then show that the number of non-equivalent irreducible representations of a finite group G is equal to the number of distinct conjugacy classes of G.

Section 1.1 Basic Definitions and Theorems

Let G be a group with identity 1_G and composition $(s,t) \to st$, $s,t \in G$. Let V be a vector space over the field of complex numbers $\mathbb{C}$, and GL(V) the group of non-singular linear transformations from V onto V. A <u>linear representation</u> of G in V is a homomorphism h from G into GL(V), that is

$$h: G \to GL(V) \quad \text{where}$$
$$h(st) = h(s)h(t) \quad \text{for each } s,t \in G.$$

Since h is a homomorphism, it follows that $h(1_G)$ is the identity transformation and $h(s^{-1}) = (h(s))^{-1}$ for each s in G. If the mapping h is also 1-1, the representation is said to be <u>faithful</u>. The vector space V is called the <u>representation space</u> of h, and the dimension of V is the <u>degree</u> of the representation. For simplicity each linear transformation h(s) shall, henceforth, be denoted h_s.

The definition makes no restriction on the order of G or the dimension of V. However, our main focus will be linear representations of the symmetric groups of finite order. It will be shown that vector spaces of finite dimension are sufficient in case the group has finite order. Thus it is to be assumed, unless stated otherwise, that groups have finite order and vector spaces have finite dimension.

When V has dimension n, each linear transformation $a:V \to V$ is defined by a square marix $A = (a_{ij})$ of order n. The coefficients a_{ij} are complex numbers dependent on a given basis (e_i) of V, and are obtained by expressing the images $a(e_i)$ in terms of the basis, so $a(e_j) = \sum_{i=1}^{n} a_{ij}e_i$. Recall that if $a \in GL(V)$ and matrix A represents a with respect to some basis (e_i), then $A \in GL(n,\mathbb{C})$, the group under multiplication, of invertible $n \times n$ matrices over the field $\mathbb{C}$. Let H_s be the matrix defining h_s, with respect to a basis (e_i), and let $h_{ij}(s)$ denote the coefficients of H_s, then for each $s,t \in G$

(1) $\det H_s \neq 0$

(2) $H_{st} = H_s \cdot H_t = (h_{ij}(st)) = \left(\sum_{k=1}^{n} h_{ik}(s) \cdot h_{kj}(t) \right)$.

Since $GL(V)$ is isomorphic with $GL(n,\mathbb{C})$, no confusion should arise if one considers the homomorphism h as a mapping of G into $GL(n,\mathbb{C})$. Though not strictly accurate, the group $h(G)$ of linear transformations or, given a basis for V, the group of matrices defining these transformations, is often referred to as the representation of G.

Linear representations h and h' of the group G in V and V', both vector spaces over $\mathbb{C}$, are said to be equivalent (similar, isomorphic) if there exists a linear isomorphism $f:V \to V'$ such that $fh_s = h'_s f$ for each $s \in G$. Or when h_s and h'_s are given in matrix form H_s and H'_s respectively, there exists invertible matrix F such that $FH_s = H'_s F$ for each

$s \in G$, commonly written $H'_s = FH_sF^{-1}$. Note V and V' must have the same dimension; therefore, h and h' have the same degree.

A subspace W of V is said to be <u>invariant</u> <u>(stable)</u> under the action of G if $h_s(W) \subseteq W$ for each $s \in G$. If also $0 < \dim W < \dim V$, W is called a <u>proper</u> <u>invariant</u> <u>subspace</u> of V. We actually have $h_s(W) = W$ since h_s is an isomorphism for each $s \in G$ and W is finite dimensional. The restriction $R^W(h_s)$ of h_s to W is then an isomorphism of W onto W with $R^W(h_{st}) = R^W(h_s)\cdot R^W(h_t)$ for each $s,t \in G$. Thus the restriction $R^W(h)\colon G \to GL(W)$ is a representation of G in W, and W is called a <u>subrepresentation</u> <u>space</u> of V.

If the representation space $V \neq \{0\}$ contains a proper invariant subspace, the representaion is said to be <u>reducible</u>. Otherwise the representation is called <u>irreducible</u>.

Let $W \neq \{0\}$ be a proper subspace of V with dimension $m < n$ and choose a basis for V, say $\beta = \{w_1,\cdots,w_m,v_1,\cdots,v_{n-m}\}$ such that the first m elements lie in W. To say that W is invariant under the action of G is equivalent to saying that each matrix H_s defining the linear transformation $h_s \in h(G)$ with respect to β has the form $H_s = \begin{pmatrix} H_s^1 & \eta \\ 0 & H_s^2 \end{pmatrix}$

where H_s^1 and H_s^2 are square matrices of order m and $n-m$ respectively, η is an $m \times n-m$ matrix, and 0 is an $n-m \times m$

zero matrix. If for any representation h, we have that each matrix H_s defining $h_s \in h(G)$ has this form, or if h is equvialent to a representation h' such that each H'_s has this form, then h is reducible; otherwise, h is irreducible. Note that if h^1 is the restriction of h to W, then H^1_s is the matrix defining h^1_s with respect to the basis $\{w_1,\cdots,w_m\}$ for each $s \in G$.

The subspaces W and W^o are said to be complementary in V if V is the direct sum of W and W^o denoted $W \oplus W^o$. Let us now suppose that in addition to a proper invariant subspace W there exists an invariant subspace W^o complementary to W. Then, as above, if we let W have dimension m, then W^o has dimension n-m. Choose a basis $\{w_1,\cdots,w_m,w^o_1,\cdots,w^o_{n-m}\}$ where the first m elements lie in W and the last (n-m) elements lie in W^o. With respect to this basis, each H_s defining $h_s \in h(G)$ has the form

$$H_s = \begin{pmatrix} H^1_s & 0 \\ 0 & H^2_s \end{pmatrix}$$

where the m by n-m matrix η is here replaced by the zero matrix of equal size. Let h^1 be the restriction of h to W and h^2 be the restriction to W^o. Then one says that h is a direct sum of h^1 and h^2, where H^1_s defines h^1_s with respect to the basis $\{w_1,\cdots,w_m\}$, and H^2_s defines h^2_s with respect to the basis $\{w^o_1,\cdots,w^o_{n-m}\}$.

A finite sum of vector subspaces W^i, denoted $W^1 + W^2 +\cdots+ W^k$, is said to be direct if for each i we have

$W^i \cap (\sum_{j \neq i} W^j) = \{0\}$. When a finite sum of subspaces W^i is direct, it shall be denoted by $W^1 \oplus \cdots \oplus W^k$.

A representation h is said to be <u>completely reducible</u> <u>analysable</u>, <u>decomposable)</u> if the representation space V can be expressed as a direct sum of irreducible invariant subspaces V^i denoted $V = V^1 \oplus \cdots \oplus V^k$. Though in general reducibility does not imply complete reducibility, in the case of finite groups this implication holds, as we shall see below.

Theorem 1.1 <u>(Maschke's Theorem)</u>. Let V be a finite dimensional representation space of finite group G. Then for each invariant subspace W of V there exists an invariant subspace W^o, such that $V = W \oplus W^o$. The proof follows two lemmas.

Lemma 1: Let h be a representation of degree n. The symmetrization of a linear map f given by $f^o = \frac{1}{|G|}\sum_{s \in G} h_{s^{-1}} f h_s$ commutes with each h_s in h(G).

Proof: $f^o h_t = \frac{1}{|G|} \sum_s h_{s^{-1}} f h_s h_t$

$$= \frac{1}{|G|} \sum_s h_t h_{t^{-1}} h_{s^{-1}} f h_s h_t$$

$$= h_t \frac{1}{|G|} \sum_s h_{(st)^{-1}} f h_{st} = h_t f^o$$

since st runs over G as s runs over G.□

Lemma 2: If a linear map $f: V \to V$ is a projection, that is $f^2 = f$, and if the image space $f(V)$ is invariant under the action of the group G, then $(f^\circ)^2 = f^\circ$ and $f^\circ(V) = f(V)$, where f° is the symmetrization of f.

Proof: Recall, if a linear map f is a projection then f acts like the identity on f(V). Also we have that the image space f(V) is invariant under G, thus each isomorphism h_s maps f(V) onto f(V). Now since V is finite dimensional, it is clear that for any finite composition of h_s's and f's that all but the right most f may be omitted. We then have

$$f^\circ f = \frac{1}{|G|} \sum_s h_{s^{-1}} f h_s f = \frac{1}{|G|} \sum_s h_{s^{-1}} h_s f = f \quad \text{and}$$

$$f f^\circ = \frac{1}{|G|} \sum_s f h_{s^{-1}} f h_s = \frac{1}{|G|} \sum_s h_{s^{-1}} f h_s = f^\circ$$

thus $(f^\circ)^2 = f^\circ(ff^\circ) = (f^\circ f)f^\circ = ff^\circ = f^\circ$.

Now, $f^\circ f = f$ and $ff^\circ = f^\circ$ implies that f and f° have equal rank, since rank $f^\circ f$ = rank ff°. Also since we have $f^\circ(V) = ff^\circ(V) \subset f(V)$ and $f(V) = f^\circ f(V) \subset f^\circ(V)$ we may conclude $f^\circ(V) = f(V)$.□

Proof of Maschke's Theorem: Given an invariant subspace W we must find an invariant subspace W° complementary to W in V. Let W' be any subspace complementary to W, and let p be the projection of V onto W along W'. Let p° be the projection of V onto W formed by the symmetrization of p, as was defined in lemma 1. Now denote by W°, the kernel of p°. Since p° is a projection of

V into itself, we have by an elementary theorem of linear algebra that V is the direct sum of W and W°, the image and kernel of p° respectively. Thus W° is complementary to W in the representation space V. According to lemma 1 we have, for each $s \in G$, $p^{\circ}h_s(W^{\circ}) = h_s p^{\circ}(W^{\circ}) = h_s(0) = 0$; thus $h_s(W^{\circ}) \subset W^{\circ}$ and W° is invariant under the action of G.∎

Theorem 1.2 (Theorem of Complete Reducibility). Every representation h of finite group G in a representation space V of finite dimension decomposes completely into a direct sum of irreducible representations.

Proof by finite induction: If representation h is irreducible we are done, otherwise by Maschke's Theorem h decomposes into two representations h^1 and h^2 with representation spaces V^1 and V^2 where $V = V^1 \oplus V^2$. Now if both h^1 and h^2 are irreducible, we are done; otherwise, apply Maschke's Theorem to h^1 or h^2 or both. This procedure can be repeated until each h^i is irreducible in representation space V^i. Clearly this is attained after a finite number of steps, since dim V is finite.∎

We may, given finite order and degree, have defined complete reducibility as follows. Representation h is completely reducible if and only if there exists a non-singular matrix F with coefficients in C such that

$$FH_sF^{-1} = \begin{pmatrix} H_s^1 & & 0 \\ & \ddots & \\ 0 & & H_s^k \end{pmatrix} \quad \text{for each } s \in G$$

where each H^i is irreducible.

Since every representation of a finite group G in a finite dimensional vector space V is decomposable into a direct sum of irreducible representations, a study of group representations simplifies to a study of irreducible representations. In particular, since equivalence divides the irreducible representations into classes, it is sufficient to determine a representation for each class. It will be shown that the number of equivalence classes is finite and in fact has group order $|G|$ as an upper bound. Furthermore, $\sqrt{|G|}$ will be shown to be an upper bound for the degree of an irreducible representation. It still remains for us to show that a decomposition of a representation h is unique.

Theorem 1.3 Theorem of Uniqueness: The decomposition of a finite dimensional representation space V of a finite group G is unique except for order and equivalence.

The proof follows two lemmas.

Lemma 1: If V is a direct sum of irreducible invariant subspaces V^i, $V = V^1 \oplus \cdots \oplus V^k$, and if W^o is an invariant subspace, then V is a direct sum of W^o and some of the V^i.

Proof: Let W^1 be the invariant subspace of V spanned by W^o and V^1. Since V^1 is irreducible and $W^o \cap V^1$ is invariant, we have either $W^o \cap V^1 = 0$ or $W^o \cap V^1 = V^1$, thus $W^1 = W^o \oplus V^1$ or $W^1 = W^o$. Now let W^2 be the invariant subspace spanned by W^1 and V^2. A similar argument yields either $W^2 = W^1 \oplus V^2$ or $W^2 = W^1$. After k-steps, we have $W^k = V$ and we are done. V is the direct sum of W^o and the V^i for which $W^{i-1} \cap V^i = 0$.□

Lemma 2: Given a representation h in a vector space V with invariant subspaces V^1, V^2 and V^3, such that $V = V^1 \oplus V^2$ and $V = V^1 \oplus V^3$, then V^2 and V^3 are equivalent subspaces.

Proof: Let h^i be a representation on V^i for $i \in \{1,2,3\}$ such that h can be expressed as either the direct sum of h^1 and h^2 or h^1 and h^3. Then we must show there exists a linear isomorphism $f\colon V^2 \to V^3$ such that $fh_s^2(v_2) = h_s^3 f(v_2)$ for all $s \in G$ and $v_2 \in V^2$. For each $v_2 \in V^2 \subset V$ there exist unique $v_1 \in V^1$ and $v_3 \in V^3$ such that $v_2 = v_1 + v_3$. Now define the map $f\colon V^2 \to V^3$ as follows $f(v_2) = f(v_1 + v_3) = v_3$. Clearly f is well defined and 1-1, f is also onto since for each $w_3 \in V^3 \subset V$ there exists $w_1 \in V^1$ and $w_2 \in V^2$ such that $w_2 = -w_1 + w_3$. Now we have

$$\begin{aligned} f(av_2 + bw_2) &= f((av_1 + bw_1) + (av_3 + bw_3)) \\ &= av_3 + bw_3 = af(v_2) + bf(w_2), \end{aligned}$$

thus f is linear. It remains only to show that $fh_s^2(v_2) = h_s^3 f(v_2)$. Let $v_2 = v_1 + v_3$ then $h_s(v_2) = h_s(v_1) + h_s(v_3)$ since each h_s is linear. Also $h_s(v_i) \subset V^i$ for $i \in \{1,2,3\}$ since each V^i is invariant in V. Thus we have $fh_s(v_2) = h_s(v_3)$. Taking h_s to be the direct sum of h^1 and h^2 on the left of the previous equality, and taking h_s to be the direct sum of h^1 and h^3 on the right gives $fh_s^2(v_2) = h_s^3(v_3)$. Finally, since $f(v_2) = v_3$, we have $fh_s^2(v_2) = h_s^3(v_3) = h_s^3 f(v_2)$ as was to be shown.□

Proof of the theorem of Uniqueness: Let the representation space V have two decompositions given by $V = V^1 \oplus \cdots \oplus V^m$ and $V = W^1 \oplus \cdots \oplus W^k$ where each V^i and W^i are irreducible invariant subspaces. One needs to show $k = m$ and that there exists a suitable renumbering of the V^i such that V^j is equivalent to W^j for $j \in \{1,\ldots,k\}$. Suppose $k \le m$. We claim $V = V^1 \oplus \cdots \oplus V^{j-1} \oplus W^j \oplus \cdots \oplus W^k$ for some renumbering. This expression is true for $j = 1$. Now assume the expression is true for any j, $1 < j \le k$. That is, there exists a suitable renumbering of the V^i for $i \in \{1,\ldots,j-1\}$ so that the relation holds. Let $U \subset V$ be given by $U = V^1 \oplus \cdots \oplus V^{j-1} \oplus W^{j+1} \oplus \cdots \oplus W^k$. Then, by lemma 1, V is a direct sum of U and some V^i with $i \ge j$. Denote this sum $V = V^1 \oplus \cdots \oplus V^{j-1} \oplus \circ\sum_{i \ge j} V^i \oplus W^{j+1} \oplus \cdots \oplus W^k$. Comparing this expression with the previous expression for V gives, by lemma 2, that W^j is equivalent to $\circ\sum_{i \ge j} V^i$. This implies $\circ\sum_{i \ge j} V^i$ is irreducible and consists of a single summand which may be assigned the number j. Thus we have

$$V = V^1 \oplus \cdots \oplus V^j \oplus W^{j+1} \oplus \cdots \oplus W^k$$

and the stated assertion follows by induction.■

The uniqueness of the decomposition assures that any method that completely reduces representation h is acceptable. Thus one may choose a method tailored to the group being represented. For example, a method which decomposes representations of a cyclic group may not decompose representations of a symmetric group. In chapter two we shall look at a method for decomposing representations of the finite symmetric group, S_n.

Let a representation space V be endowed with an inner product. This means there is a positive definite Hermitian form (inner product) in V mapping $V \times V$ into $\mathbb{C}$. Denote this inner product by $\langle u,v \rangle$ where $u,v \in V$. A linear transformation (linear operator) $h: V \to V$ is said to be <u>unitary</u> if it preserves this inner product, that is, $\langle h(u),h(v) \rangle = \langle u,v \rangle$ for all $u,v \in V$. A representation of G is called <u>normal</u> if each $h_s \in h(G)$ is equivalent to a unitary operator. That is, there exists non-singular matrix P such that $P^{-1}H_sP$ is unitary for each $s \in G$.

Now recall from linear algebra that each linear operator h has a unique adjoint operator h^* on V such that $\langle h(u),v \rangle = \langle u,h^*(v) \rangle$ for all $u,v \in V$. Moreover if H represents h with respect to an orthonormal basis, then the conjugate transpose H^* of H represents h^* in this basis. Note that if matrix H defines a linear operator h with respect to an orthonormal basis with $H^*H = I_n$, then h is unitary. This is clear, since for each $u,v \in V$ we have $\langle h(u),h(v)\rangle = \langle u,h^*h(v)\rangle = \langle u,v\rangle$. Thus the condition $H^*H = I_n$ is a sufficient condition for an operator h to be unitary.

A matrix M is called positive definite Hermitian if $M^* = M$ and $\langle M(v),v\rangle > 0$ for all non-zero v. Note that for any non-singular operator M, M^*M is positive definite Hermitian, since $(M^*M)^* = M^*M^{**} = M^*M$ and

$$\langle M^*M(v),v\rangle = \langle M(v),M(v)\rangle > 0 \text{ for all } v \neq 0.$$

Theorem 1.4 <u>Normal Representation Theorem</u>: Every representation of a finite group G in a finite vector space V over $\mathbb{C}$ is normal.

The proof follows two lemmas.

Lemma 1: If H is positive definite Hermitian, then there exists non-singular P such that $P^*HP = I$.

Proof: H is Hermitian implies, by a theorem of linear algebra, that H is unitary and similar to a diagonal matrix. That is, there exists a non-singular unitary matrix Q such that $H' = Q^*HQ$ is diagonal. Since H positive definite implies that H and therefore H' is non-singular, H' has no zero's on the diagonal. Let R be diagonal with each diagonal element r_{ii} equal to $1/\sqrt{d_i}$ where d_i is h'_{ii}, the corresponding diagonal element of H. Then, setting $P = QR$ gives $P^*HP = R^*Q^*HQR = R^*H'R = I$.□

Lemma 2: $\sum_{s\in G} H_s^* H_s$ is positive definite Hermitian.

Proof: The note prior to Theorem 1.4 gives for each $s \in G$, that $H_s^* H_s$ is positive definite Hermitian. Thus it suffices to show that a sum of two positive definite Hermitian is also positive definite Hermitian, since the conclusion would then clearly follow by finite induction on elements of G. We have $(A + B)^* = A^* + B^* = A + B$ and $\langle (A+B)(v),v\rangle = \langle A(v),v\rangle + \langle B(v),v\rangle > 0$ for $v \neq 0$.□

Proof of the normal representation theorem: One must find non-singular P such that $P^{-1}H_sP$ is unitary for each $s \in G$. To show $P^{-1}H_sP$ is unitary it is sufficient to show that $(P^{-1}H_sP)^*(P^{-1}H_sP) = I$. We have $\sum_{s\in G} H_s^* H_s$ is positive definite Hermitian by lemma 2. By lemma 1, there exists non-singular P such that $P^*(\sum_{s\in G} H_s^* H_s)P = I$. This last equality implies that $\sum_{s\in G} H_s^* H_s = (P^*)^{-1}P^{-1}$. Thus we have, for each $t \in G$

$$(P^{-1}H_tP)^*(P^{-1}H_tP) = P^*H_t^*(P^{-1})^*P^{-1}H_tP = P^*H_t^* \sum_s H_s^*H_s\, H_tP$$
$$= P^*(\sum_{s\in G} H_{st}^*H_{st})P = I \blacksquare$$

This result yields yet another proof of Maschke's theorem. For if we let h be a unitary representation with invariant subspace W in V, one can show $W^{\perp}$, the orthogonal complement of W in V, is invariant. Now each H_s maps W onto W, thus for $w \in W$ there is $w' \in W$ such that $H_s(w') = w$. Let $v \in W^{\perp}$, then $\langle H_s(v), w\rangle = \langle H_s(v), H_s(w')\rangle = \langle v, w'\rangle = 0$. Thus for each $v \in W^{\perp}$ we have $H_s(v) \in W^{\perp}$ and so $W^{\perp}$ is invariant.

If a representation h is unitary, and H_s defines h_s with respect to an orthonormal basis, then for each $s \in G$ the map $h_{s^{-1}}$ is represented by $H_{s^{-1}} = H_s^{-1} = H_s^{*}$. Also, since $H_s^{*}H_s = I = H_sH_s^{*}$ implies that

$$\sum_{k=1}^{n} h_{ik}(s)\bar{h}_{jk}(s) = \delta_{ik} = \sum_{k=1}^{n} \bar{h}_{jk}(s)h_{ik}(s),$$

we have for each H_s mutually orthonormal columns and mutually orthonormal rows.

Recall for H unitary there exists matrix P unitary such that $H' = P^{*}HP = P^{-1}HP$ is diagonal, thus H is equivalent to a diagonal matrix. Furthermore, given P and H unitary then $(PH)^{*}PH = H^{*}P^{*}PH = I$ implies that a product of unitary matrices is unitary Thus H' as defined above is unitary and $(H')^{*}H' = I$. This then implies that each diagonal element d_i of H' has absolute value one since $\bar{d}_i d_i = 1$. Thus H unitary implies H is equivalent to a diagonal matrix where each nonzero element has absolute value one.

In particular, let h be a unitary representation and let H_s define h_s with respect to an orthonormal basis. Then each H_s is equivalent to a diagonal matrix where each diagonal element has absolute value one. These diagonal elements are eigenvalues of h_s. Thus, if λ is an eigenvalue of $h_s \in h(G)$, for some representation h, then $|\lambda| = \bar{\lambda}\lambda = 1$.

This does not imply that a representation h is equivalent to a representation in which each H_s is diagonal, for this would only occur if the unitary matrix P which diagonalized H_s also diagonalized all $H_t \in h(G)$. Notice however, that given G a cyclic group and h a unitary representation of G, we can choose $s \in G$ such that s generates G, $\langle s \rangle = G$. Then H_s defining h_s with respect to an orthonormal basis generates a group of unitary matrices representing G. One need only diagonalize H_s to diagonalize each matrix H_t, representing the group element t. This is due to the identity $(BAB^{-1})^n = BA^nB^{-1}$. Thus for a cyclic group G and an irreducible representation h of G, we see that h has degree one. It will be shown later that this statement holds also for any commutative group G.

Section 1.2 Regular Representations of Finite Groups

Let V be a vector space over $\mathbb{C}$ with a basis (e_t) indexed by the elements t of the finite group G. For each $s \in G$, let h_s be the linear map of V onto V which sends e_t to e_{st}. This defines a linear representation of G, which is called the <u>regular</u> representation; its degree is equal to the order of G.

Since we have indexed the basis by elements of G, then for each matrix H_s defining h_s with respect to this basis it is reasonable to speak of a u-row and t-column where $u,t \in G$. Then for $H_s = (h_{ut}(s))$ we have $h_{ut}(s) = 1$ if $st = u$, 0 otherwise. In particular $h_{ut}(1_G) = \delta_{ut}$ and H_{1_G} is the identity matrix of order $|G|$. Also for $s \neq 1_G$ we have $h_{tt}(s) = 0$, for all $t \in G$. More will be said of this in section 1.4. For now, it is sufficient to point out that 1_G is the only element in the group G mapped to the identity element in $GL(V)$, thus the regular representation is an isomorphism from G into $GL(V)$.

Let h be a homomorphism from G into $GL(V)$ (note V need not be a vector space of dimension $|G|$) and let N be the kernel of h. Then by the 1st isomorphism theorem, N is a normal subgroup of G and $h(G)$ is isomorphic to G/N. If N consists of 1_G only, then the representation h is faithful.

Conversely, if N is a normal subgroup of G, then there is a homomorphism h from G into $GL(V)$ for some V such that the kernel of h is equal to **N**. The dimension of **V** is to be determined along with the homomorphism h. First let h' be an isomorphism from G/**N** into GL(V) for appropriate **V**. We know one exists. For example, the regular representation of G/**N** where **V** has dimension $|G/N|$. Now define h by $h(s) = h'(sN)$ for each $s \in G$. Then each member of a coset is assigned the same element of GL(V). One may say that the representation h belongs to **N**.

While we have, for each normal subgroup **N**, a representation of G which belongs to **N**, there need not exist an irreducible representation of G which belongs to **N**. For example, a group need not have a faithful representation which is irreducible. There are, however, normal subgroups which are assured the existence of one or more irreducible representations which belong to them. In particular one can demonstrate that any normal subgroup **N** for which G/**N** is isomorphic to a cyclic group will have irreducible representations with kernel **N**.

In the previous section it was noted that any irreducible representation of a cyclic group is necessarily of degree one. Thus, if h is an irreducible representation of the cyclic group C_n of order n, then h maps C_n into $\mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of nonzero complex numbers. Thus, the representation h must assign one of the nth roots of unity to a generating member of the group. If this root has order n, then the representation h is a monomorphism of C_n into C^*.

Since there exists $\phi(n)$ primitive nth roots of unity, where $\phi(n)$ is the Euler phi function, we have $\phi(n)$ faithful irreducible representaions of C_n. Recall, $\phi(n)$ equals the number of positive integers, not exceeding n, which are relatively prime to n. Thus for N a normal subgroup of G with $G/N \cong C_n$ there exists $\phi(n)$ irreducible representations of G which belong to N.

Subgroups of particular interest are N = G and N whose indices in G are two. In either case N is a normal subgroup of G. The factor group G/N, where G = N, consists of a single coset and each element of G is assigned the value 1. This results in an irreducible representation called the <u>unit (trivial)</u> representation of G.

The factor group G/N where [G:N] = 2 is isomorphic with C_2, thus there exists one, since $\phi(2) = 1$, irreducible representation of G which belongs to N. This representation assigns the value of 1 to each element of N and -1 to each element of sN where $s \in G$ and $s \notin N$.

The symmetric group on n objects, denoted by S_n, and its subgroup denoted by A_n, consisting of even permutations, are of particular interest, since subgroup A_n, called the alternating group on n objects, has index two in S_n. The irreducible representation which belongs to A_n is called the <u>alternating</u> representation of S_n.

A subgroup N is called a proper subgroup of G if $N \neq G$

and $N \neq \{1_G\}$. We have, by a classical result in algebra, that A_n is simple for $n > 4$, that is A_n has no proper normal subgroups. Also since $|A_1| = |A_2| = 1$, and $|A_3| = 3$, we have that A_n is simple for $n < 4$. Thus, we will show that if N is a proper normal subgroup of S_n with $n \neq 4$, then $N = A_n$.

Since the intersection of normal subgroups is normal, we have either $N \cap A_n = A_n$ or $N \cap A_n = \{1_G\}$. In the first case $N = A_n$, since the only proper subgroup of S_n containing A_n is A_n itself. The second case, where $N \cap A_n = \{1_G\}$, implies that each element in N, with the exception of the identity, is an odd permutation. If we let $s,t \in N$ with $s \neq 1_{S_n} \neq t$, then $st = s^2 = t^2 = 1_{S_n}$, since the product of two odd permutations is even, and the identity is the only even permutation in N. The product $st = s^2$ implies $s = t$ so N contains at most two elements. Let $N = \{s,1_{S_n}\}$ be normal in S_n. Note for $n = 2$ we have $N = S_n$, thus assume $n > 2$. Now N normal implies that for each $t \in S_n$ we have $t^{-1}st = s$. Since only the identity commutes with each element in S_n, we have $s = 1_{S_n}$ and $N = \{1_{S_n}\}$. Thus the alternating group A_n is the only proper normal subgroup of S_n for $n \neq 4$.

Each irreducible representation of a group G belongs to a normal subgroup of G. For $n > 1$ the group S_n has two irreducible representations of degree one, the unit representation belonging to S_n, and the alternating representation belonging to A_n. When $n \neq 4$, the group S_n has only one other normal subgroup, the subgroup consisting of the identity alone. Thus any other irreducible representation of S_n, $n \neq 4$, must be faithful.

When $n = 4$, the group S_n has one other proper normal subgroup. This subgroup consists of the four elements 1_{S_4}, (12)(34), (13)(24), (14)(23) and is isomorphic to the Klein 4-group, denoted by V_4. The factor group generated by this subgroup is isomorphic to the group S_3, which will be shown to have a faithful representation of degree two. Thus the symmetric group S_4 has three unfaithful irreducible representations; all other irreducible representations, as above, must be faithful.

Note that since products of complex numbers commute, any group G with a faithful representation of degree one is necessarily abelian. In fact, the group G must be cyclic since it is isomorphic to a finite subgroup of $\mathbb{C}^*$. Thus each faithful representation of S_n, for $n > 2$, has degree greater than one.

Section 1.3 The Schur Relations

Recall that linear representations h and h' of the finite group G in V and V', both vector spaces over $\mathbb{C}$, are equivalent if there exists a linear isomorphism $f:V \to V'$ such that $fh_s = h'_s f$ for each $s \in G$. We are now prepared to show, by way of Schur's lemma, that given irreducible representations h and h' and a linear map f from V to V' such that $fh_s = h'_s f$ for all $s \in G$, then either f is non-singular and h and h' are equivalent, or $f = 0$ and h and h' are non-equivalent.

Theorem 1.5: Let h and h' be linear representations of a group G in vector spaces V and V' respectively, and let f be a linear map from V' into V. Assume group G has finite order and each vector space, V and V', has finite dimension. Then, $fh_s = h'_s f$ for each $s \in G$, if and only if there exists a map $g: V\to V'$ such that $f = \frac{1}{|G|} \sum_s h'_{s^{-1}} gh_s$.

Proof: If $f: V \to V'$ is such that $fh_s = h'_s f$ for each $s \in G$, and we let $g = f$, then

$$\frac{1}{|G|}\sum_{s\in G} h'_{s^{-1}} fh_s = \frac{1}{|G|}\sum_{s\in G} h'_{s^{-1}} h'_s f = \frac{1}{|G|}\sum_{s\in G} f = f.$$

Conversely, for a map g with the above property, we have

$$\begin{aligned} fh_t = \frac{1}{|G|}\sum_{s\in G} h'_{s^{-1}} gh_s h_t &= \frac{1}{|G|}\sum_{s\in G} h'_t h'_{t^{-1}} h'_{s^{-1}} gh_s h_t \\ &= \frac{1}{|G|}\sum_{st\in G} h'_t h'_{(st)^{-1}} gh_{st} \\ &= h'_t f \quad \text{for each } t \in G. \blacksquare \end{aligned}$$

The remaining theorems in this section are concerned with irreducible representations of G.

Theorem 1.6 Schur's Lemma: For h and h' irreducible representations of G, let the homomorphism $f: V \to V'$ be such that $fh_s = h'_s f$ for each $s \in G$. If h and h' are not equivalent then $f = 0$.

Proof: There is nothing to prove when $f = 0$, so assume $f \neq 0$. Let $K(f) \subset V$ denote the kernel of f and let $Im(f) \subset V'$ denote the image of f. Now $v \in K(f)$ gives $fh_s(v) = h'_s f(v) = 0$ thus $K(f)$ is invariant under each h_s. Since V is irreducible we have either $K(f) = \{0\}$ or $K(f) = V$. Since $f \neq 0$, we have $K(f) = \{0\}$ and f is one-to-one. If we now let v be any element of V then $h'_s f(v) = fh_s(v)$ belongs to $Im(f)$, thus $Im(f)$ is invariant under each h'_s. An argument, similar to the one above, gives $Im(f) = V'$ and so the map f is onto. Therefore f is an isomorphism from V onto V' and h and h' are equivalent.∎

Notice that, in proving Schur's lemma, we have shown that any map f satisfying the condition $fh_s = h'_s f$ for each $s \in G$, must either be the zero map or be an isomorphism.

Theorem 1.7 Homothety Theorem: Given an irreducible representation of h in a finite dimensional vector space V, let the map $f: V \to V$ be such that $fh_s = h_s f$ for each $s \in G$. Then f is a homothety, that is f is a scalar multiple of the identity in GL(V).

Proof: Let λ be an eigenvalue of f. One exists since V is a vector space over the field of complex numbers. We then have $h_s(f-\lambda) = (f-\lambda)h_s$ for each $s \in G$. By the remark preceding this theorem we have $f-\lambda = 0$ since $f-\lambda$ is singular. Thus $f = \lambda$ is a homothety.∎

Let $H_s = (h_{ij}(s))$ and $H_s' = (h_{ij}'(s))$ be matrices defining linear operators h_s and h_s'. An immediate consequence of theorem 1.7 is that any irreducible representation h of an abelian group G must be of degree one. For each $t \in G$, h_t is a linear map from V into V, also for each $s \in G$ we have $h_s h_t = h_t h_s$ since G is abelian. Now if h is an irreducible representation we have, by theorem 1.7, that h_t is a homothety for each $t \in G$. That is, each matrix H_t defining h_t is a scalar multiple of the identity matrix. Let $\lambda_t \in \mathbb{C}$ be the scalar multiple associated with the martix H_t, then for each $t \in G$ we have $H_t = \lambda_t I$. Thus each $h_t \in h(G)$ is mapped to $\lambda_t \in \mathbb{C}$ and h has degree one.

Theorem 1.8 <u>Schur's</u> <u>Relations</u>: Let h and h' be irreducible representations of a finite group G in a finite dimensional vector space V over $\mathbb{C}$. If the irreducible representations h and h' are not equivalent, and n is the degree of h, then

(*i*) $\sum_s h_{ij}'(s^{-1})h_{km}(s) = 0$ for all i,j,k,m (*ii*) $\sum h_{ij}(s^{-1})h_{km}(s) = \frac{|G|}{n}\delta_{im}\delta_{jk}$.

Proof: For any matrix E of the right size let $F = \frac{1}{|G|}\sum_{s} H'_{s^{-1}} E H_s$, then by theorem 1.5 we have $FH_s = H'_s F$ for each $s \in G$. Since h and h' are not equivalent, Schur's lemma implies that F is the zero matrix. Let matrix E_{jk} be the matrix with $e_{jk} = 1$ and 0 elsewhere, then we have

$$f_{im} = \frac{1}{|G|}\sum_{\substack{s\\ j,k}} h'_{ij}(s^{-1})e_{jk}h_{km}(s) = \frac{1}{|G|}\sum_{s\in G} h'_{ij}(s^{-1})h_{km}(s) = 0.$$

Since j,k were arbitrary in defining matrix E, we have

$$\sum_{s\in G} h'_{ij}(s^{-1})h_{km}(s) = 0 \text{ for all } i,j,k,m.$$

If $h = h'$ then $F = \frac{1}{|G|}\sum_{s\in G} H'_{s^{-1}} E H_s$ is a scalar matrix by theorem 1.7. If we choose matrix E_{jk} as above then

$$f_{im} = \frac{1}{|G|}\sum_{s\in G} h_{ij}(s^{-1})h_{km}(s) = \lambda_{jk}\delta_{im},$$

where the subscript jk on λ indicates its dependence on E_{jk}. Now, summing over G by s^{-1} is equivalent to summing by s, thus we have

$$\lambda_{jk}\delta_{im} = \frac{1}{|G|}\sum_{s\in G} h_{ij}(s^{-1})h_{km}(s) = \frac{1}{|G|}\sum_{s^{-1}\in G} h_{km}(s^{-1})h_{ij}(s) = \lambda_{mi}\delta_{kj}.$$

From $\lambda_{jk}\delta_{im} = \lambda_{mi}\delta_{kj}$ for arbitrary i,j,k, and m. We obtain $\lambda_{jj} = \lambda_{ii}$, and since i and j are arbitrary we have $\lambda_{jj} = \lambda_{ii} = \lambda$ and $\frac{1}{|G|}\sum_{s\in G} h_{ij}(s^{-1})h_{km}(s) = \lambda\delta_{jk}\delta_{km}$. In order to solve for λ, set $j = k$ and $i = m$, thus $\lambda = \frac{1}{|G|}\sum_{s\in G} h_{mk}(s^{-1})h_{km}(s)$ and summing both sides over k gives $n\lambda = \frac{1}{|G|}\sum_{s\in G} h_{mm}(1_G) = \frac{1}{|G|}|G| = 1$ and so $\lambda = \frac{1}{n}$.

Thus $\sum_{s\in G} h_{ij}(s^{-1})h_{km}(s) = \frac{|G|}{n}\,\delta_{jk}\delta_{im}$ as required.■

The Schur relations can be combined if we adopt the following notation: irreducible representations h^p and h^q are equal if $p = q$, and nonequivalent if $p \neq q$. It is important to note that when $p = q$, we have h^p and h^q are actually equal; it is not sufficient to require that h^p and h^q be equivalent. If we now let n_p and n_q denote the degree of h^p and h^q respectively, the Schur relations may be written $\sum_{s\in G} h^p_{ij}(s^{-1})\ h^q_{km}(s) = \frac{|G|}{n_p}\,\delta_{im}\delta_{jk}\delta_{pq}$.

Section 1.4 Group Characters

Let V be a vector space over C with basis (e_i) of n elements and let $H = (h_{ij})$ be the matrix defining the linear operator h with respect to this basis. The trace of the operator h is defined by $Tr(h) = \sum_{i=1}^{n} h_{ii}$. For all linear operators f and h we have $Tr(hf) = \sum_i \sum_j h_{ij}f_{ji} = \sum_i \sum_j f_{ji}h_{ij} = Tr(fh)$, thus it follows that equivalent operators have the same trace. That is, given equivalent operators h and h', we have $Tr(h') = Tr(fhf^{-1}) = Tr(ff^{-1}h) = Tr(h)$. This scalar valued function is then a class function on the space of linear operators.

Recall from linear algebra that for each complex matrix H defining a linear operator h, there exists a unitary matrix U such that $H' = U^{-1}HU$ is in superdiagonal form. That is every element below the main diagonal is zero. The characteristic polynomial of the matrix H', and therefore of the matrix H, is $(h_{11} - \chi)\cdots(h_{nn} - \chi)$. Thus the trace of operator h is the sum of its eigenvalues, counted with their multiplicities.

Let h: $G \rightarrow GL(V)$ be a linear representation of a finite group G in a finite vector space V. For each $s \in G$ let $\chi_h(s) = Tr(h_s)$. This complex valued function χ_h is called the <u>character</u> of G afforded by the representation h. If the representation h is irreducible, then χ_h is said to be an irreducible character of G.

Theorem 1.9: Let χ_h be the character of a finite group G for the representation h. Let h have degree n and let $s,t \in G$. Then:

(*i*) $\chi_h(1_G) = n$

(*ii*) $\chi_h(s)$ is a sum of complex roots of unity

(*iii*) $\chi_h(s^{-1}) = \overline{\chi_h(s)}$

(*iv*) $\chi_h(t^{-1}st) = \chi_h(s)$

Proof: Part (*i*) follows since $\chi_h(1_G) = Tr(h(1_G)) = Tr(1_{GL(V)}) = n$. Now $\chi_h(s) = Tr(h_s) = \sum_{i=1}^{n} \lambda_i$ where λ_i are eigenvalues of h_s. In section one it was noted that each eigenvalue of h_s is a complex root of unity, and so we have part (*ii*). Part (*iii*) follows from the observation that for $|\lambda_i| = 1$ we have $\lambda_i^{-1} = \overline{\lambda_i}$, thus for each λ_i an eigenvalue of h_s we have λ_i^{-1} is an eigenvalue of $h_{s^{-1}}$. Therefore, we have $\chi_h(s^{-1}) = Tr(h_{s^{-1}}) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{Tr(h_s)} = \overline{\chi_h(s)}$. Finally, since $\chi_h(t^{-1}st) = Tr(h_{t^{-1}st}) = Tr(h_{t^{-1}}h_s h_t) = Tr(h_s) = \chi_h(s)$ we have part (*iv*).■

Theorem 1.10: Let G be a finite group. Let the representation $h: G \to GL(V)$ be the direct sum of linear representations $h^1,\ldots,h^k$, that is $V = V^1 \oplus \cdots \oplus V^k$ and each h^i maps group G into $GL(V^i)$. Let χ_h be the character of G in h and χ_h^i the character of G in h^i, then $\chi_h = \sum_{i=1}^{k} \chi_h^i$.

Proof: We have that for each matrix H_s defining h_s, there exists a non-singular matrix F such that

$$FH_sF^{-1} = \begin{pmatrix} H_s^1 & & 0 \\ & \ddots & \\ 0 & & H_s^k \end{pmatrix} \quad \text{for each } s \in G.$$

Thus $\chi_h(s) = Tr(H_s) = Tr(FH_sF^{-1}) = Tr(H_s^1) + \cdots + Tr(H_s^k)$ and we are done.∎

Let θ, χ be complex valued functions on group G and let $<\theta,\chi>_G = \frac{1}{|G|}\sum_{s\in G}\theta(s)\overline{\chi(s)}$, then $\overline{<\theta,\chi>_G} = \overline{\frac{1}{|G|}\sum_{s\in G}\theta(s)\overline{\chi(s)}} = \frac{1}{|G|}\sum_{s\in G}\chi(s)\overline{\theta(s)} = <\chi,\theta>_G$. Also, one can easily verify that $<a_1\theta_1 + a_2\theta_2,\chi>_G = a_1<\theta_1,\chi>_G + a_2<\theta_2,\chi>_G$, thus $<\theta,\chi>_G$ defines a Hermitian form on the vector space of complex valued functions on G. This form is positive definite since $<\theta,\theta>_G = \frac{1}{|G|}\sum_{s\in G}\theta(s)\overline{\theta(s)} > 0$, for all $\theta \neq 0$, and so $<\theta,\chi>_G$ is an inner product on the space of complex valued functions on G. If χ is a character of G then, by theorem 1.9, we have $\overline{\chi(s)} = \chi(s^{-1})$ and $<\theta,\chi>_G = \frac{1}{|G|}\sum_{s\in G}\theta(s)\chi(s^{-1}) = \frac{1}{|G|}\sum_{s\in G}\theta(s^{-1})\chi(s)$.

Theorem 1.11: Let χ_h and χ_h' be irreducible characters of G in h and h', where h is not equivalent to h'. Then

(*i*) $<\chi_h',\chi_h>_G = 0$

(*ii*) $<\chi_h,\chi_h>_G = 1$

Proof: Let $(h_{ij}(s))$ define h_s and $(h'_{ij}(s))$ define h'_s, then setting i = j and m = k in the Schur relations gives $\sum_{s\in G} h'_{jj}(s^{-1})h_{kk}(s) = 0$, and $\sum_{s\in G} h_{jj}(s^{-1})h_{kk}(s) = \frac{|G|}{n}\delta_{jk}$.

Summing by j,k on both sides of each equation then gives $\sum_{s\in G}\chi_h'(s^{-1})\chi_h(s) = 0$ and $\sum_{s\in G}\chi_h(s^{-1})\chi_h(s) = \frac{|G|}{n}\cdot n = |G|$. Part (*i*) follows since $\langle\chi_h',\chi_h\rangle_G = \frac{1}{|G|}\sum_{s\in G}\chi_h'(s^{-1})\chi_h(s) = 0$,

similarly for part (*ii*) we have

$$\langle\chi_h,\chi_h\rangle_G = \frac{1}{|G|}\sum_{s\in G}\chi_h(s^{-1})\chi_h(s) = \frac{1}{|G|}\cdot|G| = 1. \blacksquare$$

In the Schur relations $\sum_{s\in G}h^p_{ij}(s^{-1})h^q_{km}(s) = \frac{|G|}{n_p}\delta_{im}\delta_{jk}\delta_{pq}$, it was necessary that $h^p = h^q$ when $p = q$, that is, it was not sufficient that h^p be equivalent to h^q when $p = q$. This is not the case when we restrict our attention to characters since equivalent operators have the same trace. Thus we may allow that h^p be equivalent to h^q when $p = q$ and theorem 1.11 becomes $\langle\chi^p_h,\chi^q_h\rangle_G = \delta_{pq}$ where h^p is equivalent to h^q if and only if $p = q$. The irreducible characters of group G then form a orthonormal set.

Theorem 1.12: Let the representation $h: G \to GL(V)$ be the direct sum of irreducible representations $h^1,\ldots,h^k$ and let χ_h be the character of G in h. If h^o is an irreducible representation of G with character χ^o_h then the number of h^i equivalent to h^o is given by $\langle\chi_h,\chi^o_h\rangle_G$.

Proof: Let χ^i_h be the character of G in h^i, then, by theorem 1.10, $\chi_h = \sum_{i=1}^{k}\chi^i_h$ and so we have

$$\langle\chi_h,\chi^o_h\rangle_G = \langle\sum_{i=1}^{k}\chi^i_h,\chi^o_h\rangle_G = \sum_{i=1}^{k}\langle\chi^i_h,\chi^o_h\rangle_G.$$

Since $\langle\chi^i_h,\chi^o_h\rangle_G$ equals one when h^i is equivalent to h^o and zero otherwise we have our result.∎

The preceding theorem implies that the decomposition of a given representation h is completely determined by its character χ_h. This means that for any two representations with the same character, their respective decompositions contain any given irreducible representation, up to equivalence, an equal number of times. Thus two representations with the same character are equivalent. This then reduces the study of representations to that of their characters.

Let $\chi_h^1,\ldots,\chi_h^k$ be the irreducible characters of group G, and let χ_h^1 be the character afforded by the unit representation. Note $\chi_h^1(s) = 1$ for each $s \in G$. Let n_i be the degree of representation h^i which affords χ_h^i. Then, by theorem 1.9, we have $\chi_h^i(1_G) = n_i$. If θ is the character afforded by a representation of G we have, by theorem 1.12 and the remarks following it, that θ is a linear combination of irreducible characters. That is $\theta = \sum_{i=1}^{k} a_i\chi_h^i$ where each a_i represents the number of times h^i occurs in θ. Thus $<\theta,\chi_h^i>_G = a_i$ for each i and we have

$$<\theta,\theta>_G = < \theta, \sum_{i=1}^{k} a_i\chi_h^i >_G = \sum_{i=1}^{k} a_i<\theta,\chi_h^i>_G = \sum_{i=1}^{k} a^2{}_i.$$

Theorem 1.13: Let θ be the character afforded by a representation of G, then θ is irreducible if and only if $<\theta,\theta>_G = 1$.

Proof: Let $\theta = \sum_{i=1}^{k} a_i \chi_h^i$, then $<\theta,\theta>_G = \sum_{i=1}^{k} a_i^2$. If θ is irreducible then $\theta = \chi_h^j$, for some j, and $a_i = 1$ when i = j and 0 otherwise, thus $<\theta,\theta>_G = 1$. Conversely if $<\theta,\theta>_G = 1$, then $a_i = 1$ for some j and 0 otherwise, thus $\theta = \chi_h^j$.■

Let h^R be the regular representation of group G then each matrix $H_s^R = (h_{ut}^R(s))$ is defined $h_{ut}^R(s) = 1$ if st = u, zero otherwise. If $s \neq 1_G$, then for each $t \in G$ we have $st \neq t$ and so $h_{tt}^R(s) = \delta_{s1_G}$, thus we have, for $s \in G$, $Tr(H_s^R) = \sum_{t \in G} h_{tt}^R(s) = \delta_{s1_G} \cdot |G|$.

Theorem 1.14: Each irreducible representation h^i of group G occurs in the regular representation with multiplicity equal to its degree n_i.

Proof: Let χ_h^R be the character afforded by h^R, then, by the remark preceding this theorem, we have $\chi_h^R(s) = |G| \ \delta_{s1_G}$. Thus $<\chi_h^R, \chi_h^i>_G = \frac{1}{|G|}\sum_{s\in G} \chi_h^R(s)\chi_h^i(s^{-1}) = \frac{1}{|G|} \ \chi_h^R(1_G)\chi_h^i(1_G) = \frac{1}{|G|}|G| \cdot n_i = n_i$.■

As an immediate consequence of this theorem we have that $\sum_{i=1}^{k} n_i^2 = |G|$ since $|G| = \chi_h^R(1_G) = \sum_{i=1}^{k} n_i \chi_h^i(1_G) = \sum_{i=1}^{k} n_i^2$. Note that this implies that the degree n_i of each irreducible representation h^i is bounded by the square root of the order of G. Also, since $\chi_h^R(s) = 0$ for $s \neq 1_G$, we have $\chi_h^R(s) = \sum_{i=1}^{k} n_i \chi_h^i(s) = 0$ for each $s \in G$, $s \neq 1_G$.

Section 1.5 Orthogonality Relations for Characters

The group elements s and t are said to be conjugate in G if there exists $r \in G$ such that $s = r^{-1}tr$. This defines an equivalence relation on G and thus partitions the group into classes. For $s \in G$, let C_s denote the conjugacy class of s. Then $C_s = C_t$ if and only if s is conjugate to t. Also we have $C_{1_G} = \{1_G\}$ since $r^{-1}1_G r = 1_G$ for each $r \in G$. Let $h:G \to GL(V)$ be a linear representation of G with character χ_h. Then by theorem 1.9 part *iv*, we have that each element in C_s has the same character. That is $\chi_h(s) = \chi_h(r^{-1}tr) = \chi_h(t)$ for each $s,t \in C_s$. Thus χ_h is a "class function" on the group G.

The space of complex valued <u>class</u> functions on group G consists of all complex valued functions θ which assign the same value to each member of a conjugacy class. Thus θ is completely determined by its assignment of a complex value to each distinct conjugacy class of G. Since each value assigned to a conjugacy class is arbitrary the dimension of this space must equal the number of distinct conjugacy classes in G. In the last section the set of irreducible characters of G were shown to be orthonormal in the space of complex valued functions over G. Thus they must form an orthonormal set in the space of complex valued class functions. In fact we will show that they form an orthonormal basis for this space.

Theorem 1.15: Let θ be a class function on G and let $h:G \to GL(V)$ be a linear representation of G and let $f: V \to V$ be the linear operator on V defined by $f = \sum_{s\in G} \theta(s)h_s$. If h is irreducible of degree n with character χ_h, then f is a homothety of ratio λ, where $\lambda = \frac{|G|}{n} \langle\theta,\overline{\chi_h}\rangle_G$.

Proof: Let $t \in G$ be arbitrary then $h_{t^{-1}}\, fh_t = \sum_{s\in G} h_{t^{-1}}\, \theta(s)h_s h_t = \sum_{s\in G} \theta(s)h_{t^{-1}st} = \sum_{u\in G} \theta(tut^{-1})h_u$ where $u = t^{-1}st$. Since θ is a class function on G, $\theta(tut^{-1}) = \theta(u)$ and $h_{t^{-1}}\, fh_t = \sum_{u\in G} \theta(u)h_u = f$. Thus $fh_t = h_t f$ for each $t \in G$ and f is a homothety by theorem 1.7. Now, since f has degree n, we have $n\cdot\lambda = Tr(f) = Tr(\sum_{s\in G} \theta(s)h_s) = \sum_{s\in G} \theta(s)Tr(h_s)$ $= \sum_{s\in G} \theta(s)\chi_h(s) = |G|\langle\theta,\overline{\chi_h}\rangle_G$ and we have our result.■

Theorem 1.16: The set of irreducible characters $\chi_h^1,\ldots,\chi_h^k$ of a finite group G form an orthonormal basis of the space of complex valued class functions on G.

Proof: By theorem 1.11 and the remarks preceding theorem 1.15 we have that the irreducible characters form an orthonormal set in this space. We will now show that this set spans the space, or equivalently that any class function θ which is mutually orthogonal to each irreducible character χ_h^i must be the zero function. Let $h: G \to GL(V)$ be a linear representation of G. For each $s \in G$, let $h_s = \oplus\sum_{i=1}^{k} a_i h_s^i$ denote the direct sum decomposition of h into irreducible representations h^i each with degree n_i and occuring a_i times in h. Let θ be a class function such that $<\theta,\chi_h^i>_G = 0$ for each $i \in \{1,\ldots,k\}$, then also $<\overline{\theta},\overline{\chi_h^i}>_G = 0$ for each i, since $<\overline{\theta},\overline{\chi_h^i}>_G = <\chi_h^i,\theta>_G = \overline{<\theta,\chi_h^i>_G} = 0$. Now let $f\colon V \to V$ be the linear operator defined by $f = \sum\limits_{s\in G} \overline{\theta(s)}\ h_s$ then $f = \sum\limits_{s\in G} \overline{\theta(s)}\ \bigoplus\limits_{i=1}^{k} a_i h_s^i = \bigoplus\limits_{i=1}^{k} a_i \cdot \sum\limits_{s\in G} \overline{\theta(s)}\ h_s^i$. Now applying theorem 1.15 gives that each $\sum\limits_{s\in G} \overline{\theta(s)}\ h_s^i = \frac{|G|}{n_i} <\overline{\theta},\overline{\chi_h^i}>_G$ thus $f = \bigoplus\limits_{i=1}^{k} a_i \frac{|G|}{n_i} <\overline{\theta},\overline{\chi_h^i}>_G = \bigoplus\limits_{i=1}^{k} a_i \frac{|G|}{n_i}\cdot 0$ and f is the zero map for each representation h.

In particular let $h^R\colon G \to GL(V)$ be the regular representation and let the set $\{e_t\}$, $t \in G$, form the basis of V. Recall, that for each $s \in G$, h_s^R maps e_t to e_{st}, thus $h_s^R(e_{1_G}) = e_s$ for each $s \in G$. Let f be the zero map with $f = \sum\limits_{s\in G} \overline{\theta(s)} h_s^R$, then $f(e_{1_G}) = \sum\limits_{s\in G} \overline{\theta(s)}\ h_s^R(e_{1_G}) = \sum\limits_{s\in G} \overline{\theta(s)}\ e_s = \vec{0}$ and so $\theta(s) = 0$ for each $s \in G$. Thus θ is the zero function and we have the desired result.■

Since the dimension of the space of complex valued class functions on G is equal to the number of distinct conjugacy classes in G we have as a consequence of the previous theorem that this number also equals the number of inequivalent irreducible representations of G.

Theorem 1.17: (<u>Orthogonality relations</u>) Let $\chi_h^1,\ldots,\chi_h^k$ be the irreducible characters of G and let $C_1,\ldots,C_k$ be the conjugacy classes of G. Let s_i denote an element of G which belongs to the conjugacy class C_i. Then

(*i*) $<\chi_h^i,\chi_h^j>_G = \delta_{ij}$

(*ii*) $\sum_{m=1}^{k} \chi_h^m(s_i)\ \overline{\chi_h^m(s_j)} = \left|\frac{G}{C_i}\right|\ \delta_{ij}$.

Proof: Part (*i*) is a result of theorem 1.11. Let θ_j be the function equal to 1 on the class C_i and equal to 0 elsewhere. Since it is a class function on G, by the previous theorem it can be expressed as a linear combination of irreducible characters. That is $\theta_j = \sum_{m=1}^{k} a_m\chi_h^m$ where $a_m = <\theta_j,\chi_h^m>_G = \frac{1}{|G|}\sum_{t\in G}\theta_i(t)\ \overline{\chi_h^m(t)} = \left|\frac{G}{C_i}\right|\ \overline{\chi_h^m(s_i)}$. Then for each $s_i \in G$ we have

$$\delta_{ij} = \theta_j(s_i) = \sum_{m=1}^{k} a_m\chi_h^m(s_i) = \left|\frac{C_i}{G}\right| \sum_{m=1}\chi_h^m(s_i)\overline{\chi_h^m(s_j)} ,$$

and part (*ii*) follows.■

Chapter 2 Representation Theory of Group Rings over $\mathbb{C}$.

We are now prepared to look at a method for determining irreducible representations of the finite symmetric group. In the first section we define the concept of group rings and linear representations of group rings. At first this approach seems unwieldy, but it reduces the problem of finding irreducible representations to finding minimal left ideals of the group ring. Also, we shall see that the regular representation introduced in chapter one is, in this context, simply right multiplication by a fixed element of the ring. We then restate the main three theorems of section 1.1 in terms of the group ring structure.

In section two we introduce elements of the group ring called primitive idempotents which generate minimal left ideals. We then show that two minimal left ideals are equivalent if and only if there exist special non-zero ring elements that multiply minimal left ideals on the right to generate equivlalent minimal left ideals. We end the section by showing that the regular representation of the group ring is decomposable into a direct sum of simple two-sided ideals. Each of these simple two-sided ideals is, in turn, decomposable into a direct sum of equivalent minimal left ideals.

The final section of chapter two is devoted to a method for determining irreducible representations of a finite symmetric group. This method involves the introduction of Young diagrams and Young tables. These devices allow us to determine primitive idempotents, and therefore, irreducible representations of the symmetric group. We will also introduce a formula for computing the degree of an irreducible representation.

Section 2.1 Basic Definitions and Theorems

The group ring denoted R_G is the additive abelian group $\sum_{s \in G} R$ where R is a ring and G is a finite multiplicative group. Denote $a \in R_G$ by $a = \sum_{s \in G} \alpha_s \cdot s = \sum_{s \in G} s \cdot \alpha_s$ with each $\alpha_s \in R$. Addition and multiplication are defined in R_G as follows:

(*i*) $a + b = \sum_{s \in G} \alpha_s \cdot s + \sum_{s \in G} \beta_s \cdot s = \sum_{s \in G} (\alpha_s + \beta_s) \cdot s$ and

$$a \cdot b = \sum_{s \in G} \alpha_s \cdot s \cdot \sum_{t \in G} \beta_t \cdot t = \sum_{s} \sum_{t} \alpha_s \beta_t \cdot st = \sum_{u \in G} \gamma_u \cdot u$$

where $\gamma_u = \sum_{st=u} \alpha_s \beta_t = \sum_{t \in G} \alpha_{ut^{-1}} \beta_t$. Thus we have

(*ii*) $a \cdot b = \sum_{u} \sum_{t} \alpha_{ut^{-1}} \beta_t \cdot u.$

It can be easily verified that R_G, as defined, is indeed a ring. For our purpose we will take the ring R to be the field of complex numbers. Thus $R_G = \mathbb{C}_G$ is in fact a vector space over $\mathbb{C}$. Note that it is not necessary for us to choose the field of complex numbers. For what is to follow it is sufficient to choose any field of characteristic zero. The group elements {s} then form a basis for $\mathbb{C}_G$, with $s = \sum_{t \in G} \sigma_t \cdot t$ where $\sigma_t = 1$ if $t = s$, and 0 otherwise. The dimension of $\mathbb{C}_G$ as a vector space is then equal to the order of group G. Let us now define a linear representation of the group ring $\mathbb{C}_G$.

A linear representation h of a finite group G in a finite dimensional vector space V over the field of complex numbers $\mathbb{C}$ was defined in part one section one as a mapping h from G into the set of linear transformations from V into V, denoted L(V,V). We then noted that any such mapping h is a homomorphism from the group G into the group of invertible linear transformations from V onto V, denoted GL(V). We now extend this definition to define a linear representation h of the group ring $\mathbb{C}_G$ in V. The representation h need not in general map an element $a \in \mathbb{C}_G$ into GL(V). It is, however, necessary that h map each group element s into the group GL(V).

Given a finite group G and a finite vector space V over $\mathbb{C}$ a linear representation of $\mathbb{C}_G$ in V is a mapping h from $\mathbb{C}_G$ into L(V,V) such that

$$h(st) = h(s)h(t) \quad \text{for each } s,t \in G \text{ and}$$

$$h(\lambda a + \mu b) = \lambda h(a) + \mu h(b) \quad \text{for each } a,b \in \mathbb{C}_G \text{ and } \lambda,\mu \in \mathbb{C}.$$

This definition is sufficient to ensure that for any ring elements $a,b \in \mathbb{C}_G$ that $h(ab) = h(a)h(b)$. Thus h is a ring homomorphism, and h is also a vector space homomorphism over $\mathbb{C}$. It is clear from the above definition that each linear representation h of $\mathbb{C}_G$ is in fact a representation of G when one restricts h to the basis elements of $\mathbb{C}_G$.

If we begin with a representation h of a finite group G in finite vector space V over the complex field $\mathbb{C}$, there is in fact exactly one representation of $\mathbb{C}_G$, say h^*, which restricts to h. The group ring $\mathbb{C}_G$ is constructed by taking the set of all possible linear combinations of the group elements s in G with coefficients from $\mathbb{C}$. Thus if h^* restricts to h then h^* must be the mapping of $\mathbb{C}_G$ into L(V,V) derived from h by setting $h^*(a) = \sum_{s\in G} \alpha_s h(s)$ where $a = \sum_{s\in G} \alpha_s \cdot s$. Since h^* is derived from h by requiring linearity over $\mathbb{C}$, h^* is a linear representation of $\mathbb{C}_G$.

Recall two linear representations h and h' of G in V and V' are equivalent if there exists a linear isomorphism f from V onto V' such that $h'_s = fh_s f^{-1}$ for each $s \in G$. The two linear representations $(h')^*$ and h^* of $\mathbb{C}_G$ derived from h' and h, as above, are also equivalent since $(h')^*(a) = \sum_{t\in G} \alpha_t h'(t) = \sum_{t\in G} \alpha_t fh(t)f^{-1} = \sum_{t\in G} f\alpha_t h(t)f^{-1} = f(\sum_{t\in G} \alpha_t h(t))f^{-1} = fh^*(a)f^{-1}$ for each $a \in \mathbb{C}_G$. Thus the equivalence of two representations is preserved under the extension of h to h^*. Similarly if h^* and $(h')^*$are equivalent linear representations of $\mathbb{C}_G$, that is there exists an isomorphism f from V onto V' such that $(h')^*(a) = fh^*(a)f^{-1}$ for each $a \in \mathbb{C}_G$, then clearly h^* and $(h')^*$ restricted to the group elements are equivalent.

Note that if a subspace W of V is invariant under the action of $\mathbb{C}_G$ then it is necessarily invariant under the action of G. The converse also holds since W is itself a vector space over $\mathbb{C}$. It is thus closed under vector addition and scalar multiplication, and a linear representation h^* preserves these operations. Since irreducibility is defined in terms of invariant subspaces, if a representation h^* of $\mathbb{C}_G$ in V is irreducible or reducible then the restricted representation h of G in V has the same property.

A similar comment may be made regarding a representation h of G in V and the derived representation h^* of $\mathbb{C}_G$ in V. Thus the three theorems of section one, Maschke's Theorem, Theorem of Complete Reducibility, and Theorem of Uniqueness also hold for linear representations of the group ring $\mathbb{C}_G$. A restatement of these theorems in terms of the group ring structure will follow a brief discussion about equivalent subspaces and the extension of the regular representation h^R to the group ring $\mathbb{C}_G$.

Let us now turn our attention to the regular representation h^R of a finite group G. We shall show that the regular representation when extended to the group ring $\mathbb{C}_G$ is equivalent to mapping each element a of $\mathbb{C}_G$ to the linear transformation ℓ_a from $\mathbb{C}_G$ into $\mathbb{C}_G$ defined by left multiplication with element a. Recall that the representation space V has a basis (e_t) indexed by the elements t of G. Thus the dimension of V is equal to the order of group G. And as was pointed out above, the order of the group ring $\mathbb{C}_G$ viewed as a vector space is also equal to the order of group G.

We must show that there exists an isomorphism f from the vector space V into the group ring $\mathbb{C}_G$ such that $f(h_a^R)^* f^{-1} = \ell_a$ for each $a \in \mathbb{C}_G$. A natural choice is for f to map each basis element e_t of V to the group element t in $\mathbb{C}_G$. Now $h^R: G \to GL(V)$ takes each group element s to a homomorphism $h_s^R : V \to V$ defined by $h_s^R(e_t) = e_{st}$. If we now extend h^R to $(h^R)^*: \mathbb{C}_G \to L(V,V)$, then $(h^R)^*(a) = (h_a^R)^* = \sum_{s\in G} \alpha_s h^R(s) = \sum_{s\in G} \alpha_s h_s^R$ with $a = \sum_{s\in G} \alpha_s \cdot s$. We note that $(h_a^R)^*$ is a linear transformation from V into V.

Let us now define the linear mapping $\ell: \mathbb{C}_G \to L(\mathbb{C}_G, \mathbb{C}_G)$ as follows: for each $a \in \mathbb{C}_G$, $\ell(a) = \ell_a$ where $\ell_a(b) = ab = \sum_{u,t\in G}\sum \alpha_{ut^{-1}}\beta_t u$ for each $b \in \mathbb{C}_G$ with $b = \sum_{t\in G} \beta_t \cdot t$. One can easily verify that ℓ, as defined, is a linear representation of the group ring $\mathbb{C}_G$ in $\mathbb{C}_G$ viewed as a vector space over $\mathbb{C}$. We need only show that $f(h_a^R)^* f^{-1} = \ell_a$ for each a in $\mathbb{C}_G$ where f is as defined above. Let $a,b \in \mathbb{C}_G$ be defined as above, then $f(h_a^R)^* f^{-1}(b) = f(h_a^R)^*(\sum_{t\in G} \beta_t e_t)$

$$= f(\sum_{u,t\in G}\sum \alpha_{ut^{-1}}\beta_t(e_u)) = \sum_{u,t\in G}\sum \alpha_{ut^{-1}}\beta_t \cdot u = \ell_a(b).$$

Thus, left multiplication of elements of $\mathbb{C}_G$ by a fixed element a of $\mathbb{C}_G$ is equivalent to the mapping $(h_a^R)^*$ belonging to the regular representation of the group ring $\mathbb{C}_G$ in vector space V. For simplicity then, we may refer to the mapping $\ell: \mathbb{C}_G \to L(\mathbb{C}_G, \mathbb{C}_G)$ defined above as a regular representation of the group ring $\mathbb{C}_G$.

In section 1.4 theorem 1.14 we found that each irreducible representation of a finite group G occurs in the regular representation with multiplicity equal to its degree. Thus, from the above comments regarding irreducible and invariant subspaces, the problem of finding irreducible representations of a finite group G reduces to the problem of finding irreducible invariant subspaces of the vector space $\mathbb{C}_G$ under the action of left multiplication by elements of the group ring $\mathbb{C}_G$.

The invariant subspaces of a vector space $\mathbb{C}_G$ are clearly just the left ideals of the group ring $\mathbb{C}_G$. And the irreducible invariant subspaces are just the minimal left ideals.

Let N and N' be two left ideals in $\mathbb{C}_G$. Then viewed as representation spaces, N and N' are equivalent if there exists an isomorphism f from N onto N' that commutes with left multiplication. More to the point, the linear representation ℓ when restricted to the left ideal N is equivalent to ℓ restricted to N'. An example of a linear transformation which commutes with left multiplication is right multiplication by a fixed element b of $\mathbb{C}_G$. Let us denote this transformation by r_b.

Let the left ideal $N' = Nb$ and denote by $R^N(\ell)$ and $R^{N'}(\ell)$ the restrictions of the representation ℓ to the left ideals N and N' respectively, then we have $r_b R^N(\ell_a) = R^{N'}(\ell_a) r_b$ for each $a \in \mathbb{C}_G$. This is so, since by assumption each element in N' is of the form nb where $n \in N$ and b is fixed in $\mathbb{C}_G$, thus for each $n \in N$ we have $r_b R^N(\ell_a)(n) = r_b(an) = (an)b = a(nb) = R^{N'}(\ell_a)(nb) = R^{N'}(\ell_a) r_b(n)$. It will be shown later that in fact every equivalence mapping f is a right multiplication.

We are now prepared to restate, without proof, the first three theorems of chapter one: Maschke's Theorem, Theorem of Complete Reducibility and Theorem of Uniqueness. The following three theorems: 2.1, 2.2 and 2.3, then, are simply restatements of these theorems in terms of the above group ring structure and the regular representation.

We may assume for each theorem that G is a finite multiplicative group and $\mathbb{C}_G$ is the corresponding group ring constructed over the field of complex numbers $\mathbb{C}$. Also the regular representation of $\mathbb{C}_G$ is denoted by the linear mapping $\ell:\mathbb{C}_G \to L(\mathbb{C}_G,\mathbb{C}_G)$, where for each $a \in \mathbb{C}_G$, $\ell(a) = \ell_a$ is the linear transformation of $\mathbb{C}_G$ which multiplies each element of $\mathbb{C}_G$ on the left by the element a.

In the regular representation, it is worth noting that $\mathbb{C}_G$, as a group ring, is the object of the representation, and that $\mathbb{C}_G$, as a vector space over the field $\mathbb{C}$, is the representation space. Thus $\mathbb{C}_G$ plays a dual role here. It is this that gives us a rather unique point of view, that of the left ideals of the group ring $\mathbb{C}_G$ as the invariant subspaces of vector space $\mathbb{C}_G$.

Theorem 2.1. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the finite dimensional representation space of the group ring $\mathbb{C}_G$ determined by the group G. Then for each left ideal $N \leq \mathbb{C}_G$ there exists a left ideal $N^0 \leq \mathbb{C}_G$ such that $\mathbb{C}_G = N \oplus N^0$.

Theorem 2.2. The regular representation ℓ of a group ring $\mathbb{C}_G$ in the representation space $\mathbb{C}_G$ decomposes completely into irreducible representations.

This is equivalent to saying that the representation space $\mathbb{C}_G$ decomposes completely into a direct sum of minimal left ideals. In particular $\mathbb{C}_G = N^1 \oplus \cdots \oplus N^k$ where each N^i is a minimal left ideal.

Theorem 2.3. The decomposition of a finite representation space $\mathbb{C}_G$ is unique except for order and equivalence.

Section 2.2 Primitive Idempotents and Minimal Left Ideals

The left ideals of $\mathbb{C}_G$ are easily determined or generated. For example, if a is a ring element of any ring R, then the set of products ra with arbitrary $r \in R$ is a left ideal in R, which we may denote by Ra, and the set of solutions $r \in R$ of the equation $ra = 0$ is a left ideal in R. It will be shown that, in fact, every left ideal of $\mathbb{C}_G$ can be generated in both of these ways, and that we may restrict our attention to elements with particularly convenient properties, namely the idempotents.

An element e is called <u>idempotent</u> or an <u>idempotent</u> if $e^2 = e$.

Theorem 2.4. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring, over the complex field $\mathbb{C}$, determined by the group G. The following two statements are equivalent:

(1) In a left ideal $N \leq \mathbb{C}_G$ there exists at least one idempotent $e \in N$, called a <u>generating</u> <u>unit</u> of N, which generates N: $N = \mathbb{C}_G e$

(2) If $N \leq \mathbb{C}_G$ is a left ideal, then there exists an element $e \in \mathbb{C}_G$ with the following properties:

(*i*) for any $a \in \mathbb{C}_G$ we have $ae \in N$

(*ii*) for any $n \in N$ we have $n = ne$.

Proof: It is clear that statement(1) implies statement(2). Conversely we will show that statement(2) implies statement(1). Let e be as in statement(2). By part(i) we have that $1_G e \in N$, thus e belongs to N. Then we have $e = e^2$ by part(ii). Now let $n \in N$, then $n = ne$ implies that $n \in \mathbb{C}_G e$, therefore e generates N. Thus statement(1) and statement(2) are equivalent, and we are justified in using the same notation in both statements.■

It is of interest to note that theorem 2.4 implies that multiplication on the right by an idempotent e is equivalent to the projection of the group ring $\mathbb{C}_G$ onto a left ideal N. One may, therefore, consider an idempotent e as either a generating unit for N or as a projection of $\mathbb{C}_G$ onto N.

We say that a left ideal $N \leq \mathbb{C}_G$ is <u>annihilated</u> on the right by an element a if $na = 0$ for all $n \in N$. We may denote this by $Na = 0$. It is not obvious that such elements necessarily exist for each left ideal N, but in fact this is the case.

Given a generating unit e for N it is easy to show that $(1_{\mathbb{C}_G} - e)$ is also idempotent. We have, by the previous theorem, that $ne = n$ for each element n belonging to N, thus $n - ne = 0$ or $n(1_{\mathbb{C}_G} - e) = 0$ for all $n \in N$. Therefore the idempotent $(1_{\mathbb{C}_G} - e)$ annihilates the left ideal N on the right. Now, according to theorem 2.1, we have for each left ideal $N \leq \mathbb{C}_G$ there exists a left ideal N^0 such that $\mathbb{C}_G$ is the direct sum of N and N^0. In the theorem following these comments we shall see that a generating unit of N annihilates N^0 on the right and vice versa. We then see that every such left ideal is the set of solutions to an equation $ae_0 = 0$ with idempotent e_0.

Theorem 2.5. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. Let N and N^0 be left ideals such that $\mathbb{C}_G = N \oplus N^0$. Then N has a generating unit e and N^0 has a generating unit e_0, such that N is annihilated on the right by e_0, and N^0 is annihilated on the right by e. In particular we have $ee_0 = e_0e = 0$.

Proof: The method of this proof is often called Peirce's decomposition. The group ring $\mathbb{C}_G$ is a direct sum of the left ideals N and N^0. Therefore each element $a \in \mathbb{C}_G$ can be expressed uniquely as $a = n + n_0$ where $n \in N$ and $n_0 \in N^0$. Let the decomposition of the identity element in the group ring $\mathbb{C}_G$ be given by $1_{\mathbb{C}_G} = e + e_0$. Due to the identity $(1_{\mathbb{C}_G})^2 = 1_{\mathbb{C}_G}$ we have $e^2 = e$, $e_0^2 = e_0$, and $e_0e = ee_0 = 0$, since the sum is direct and the components are left ideals. Then for $a \in \mathbb{C}_G$ we have $n + n_0 = a = a \cdot 1_{\mathbb{C}_G} = ae + ae_0$. Since N and N^0 are left ideals we have that ae lies in N and ae_0 lies in N^0. Because of the uniqueness of the decomposition of element a we must have $n = ae$ and $n_0 = ae_0$. Now, if a lies in N then $a = ae$ by the idempotency of e and so $ae_0 = 0$. Thus e_0 annihilates the left ideal N on the right. Similarly, we have if a lies in N^0, then $ae = 0$ and e annihilates the left ideal N^0 on the right.∎

The previous theorem also holds if the group ring $\mathbb{C}_G$ is replaced by a left ideal N which is a direct sum of two left ideals N^1 and N^2. In this case the identity element $1_{\mathbb{C}_G}$ is replaced by a generating element e of N. We know such an element exists by theorem 2.4. With these changes the proof is very much the same as above. With this remark and the use of finite induction one can easily prove the following theorem, which is stated without proof.

Theorem 2.6. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. If a decomposition of $\mathbb{C}_G$ into left ideals is given, say $\mathbb{C}_G = N^1 \oplus \cdots \oplus N^k$, then one obtains a set $e_1, \ldots, e_k$ of generating units for these left ideals by the decomposition of the identity $1_{\mathbb{C}_G}$. Each left ideal N^i is generated by e_i and annihilated on the right by e_j when $i \neq j$, in particular we have $e_i e_j = 0$ for $i \neq j$.

An idempotent e is called <u>primitive</u> if there exists no decomposition $e = e_1 + e_2$ with $e_1 e_2 = e_2 e_1 = 0$, and $e_i^2 = e_i$, $e_i \neq 0$ for $i \in \{1,2\}$.

Theorem 2.7. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. If an idempotent e is primitive, then the left ideal $\mathbb{C}_G e$ generated by e is minimal. If a left ideal N is minimal, then every generating unit e of N is primitive.

Proof: The remark following theorem 2.5 asserts the existence of a decomposition of the generating units of non-minimal left ideals of a kind forbidden for primitive idempotents. Thus if an idempotent e is primitive then the left ideal generated by e is necessarily minimal. We now assume a left ideal N to be minimal and show indirectly that the generating units of N are primitive. Let us suppose that there exists a decomposition for a generating unit e of the left ideal N. That is $e = e_1 + e_2$ with $e_1 e_2 = e_2 e_1 = 0$ and $e_i^2 = e_i$, $e_i \neq 0$ for $i \in \{1,2\}$. These conditions imply that $e_1 e = e_1$ since $e_1 e = e_1(e_1 + e_2) = e_1^2 + e_1 e_2 = e_1 + 0$. Thus $e_1 = e_1 e$ must belong to the left ideal N generated by the idempotent e, and therefore $\mathbb{C}_G e_1 \subseteq N$. Now $\mathbb{C}_G e_1$ does not equal N since e_2, like e_1, is in N, but e_2 does not belong to $\mathbb{C}_G e_1$. Hence $\mathbb{C}_G e_1$ is a proper subspace of $N = \mathbb{C}_G e$, and we conclude that N is not minimal.∎

We are now prepared to return to the question of equivalence. Recall that two left ideals N and N' are equivalent if there exists an isomorphism f from N onto N' that commutes with left multiplication.

Theorem 2.8. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. If the left ideals N and N' are equivalent, then every equivalence mapping f from N onto N' is given by a right multiplication with some fixed element b, denoted $r_b(n) = nb$.

Proof: Let e be any generating unit of N and b its image under the equivalence mapping f. Let $R^N(\ell)$ and $R^{N'}(\ell)$ be the restrictions of the regular representation ℓ to the left ideals N and N' respectively. Then for each $a \in \mathbb{C}_G$ and for each $n \in N$ we have $f(R^N(\ell_a)(n)) = R^{N'}(\ell_a)(f(n))$. This implies that $f(an) = af(n)$, since $f(an) = f(R^N(\ell_a)(n)) = R^{N'}(\ell_a)(f(n)) = af(n)$. Now $n \in N$ implies that $n = ne$, thus $f(an) = af(n) = af(ne) = anf(e) = anb$. Therefore we have $f(an) = anb = r_b(an)$ as was to be shown.∎

The element b introduced above has several interesting properties. For one we have $N' = Nb$; this is clear since $N' = f(N) = f(Ne) = Nf(e) = Nb$. Also since $e^2 = e$ we have $b = f(e) = f(e^2) = ef(e) = e\,b$. On the other hand, since the element b is the image of idempotent e in N' we have that b is idempotent and lies in the left ideal N'. Also the element b is reproduced on the right by any generating unit e' of N'. Thus the element b has the property that $b = e\,be'$, since $b = be' = e\,be'$. All elements of the form $y = e\,xe'$ for any $x \in \mathbb{C}_G$ has this property, since $e\,ye' = e^2xe'^2 = e\,xe' = y$. Elements of the form $e\,xe'$ play an important role in the next theorem.

Theorem 2.9. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. Let N and N′ be minimal left ideals with generating units e and e'. Then right multiplication with any element $exe' \neq 0$ defines an equivalence mapping of N onto N'.

Proof: Recall that minimal left ideals N and N' are irreducible representation spaces of the group ring $\mathbb{C}_G$. Now clearly right multiplication with exe' is a linear mapping of N into N' which commutes with left multiplication ℓ_a for each $a \in \mathbb{C}_G$. We have thus met all of the conditions of Schur's lemma, therefore right multiplication by exe' is either the zero map or an isomorphism from N onto N'. Now since the idempotent e is mapped by $r_{exe'}$ to $exe' \neq 0$, we must have that the map $r_{exe'}$ defines an equivalence mapping from N onto N'.∎

We are, in fact, able to make a much stonger claim about elements of the form exe'.

Theorem 2.10. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. Two minimal left ideals N and N' with generating units e and e' are equivalent if and only if there exist non-zero elements of the form exe'. The equivalence mappings from N onto N' are given by right multiplication with these elements.

Proof: If there exist non-zero elements of the form exe', then by theorem 2.9 we have an equivalence mapping from N onto N'. Thus the minimal left ideals N and N' are equivalent. Conversely, let us assume that N and N' are equivalent minimal left ideals and show the existence of a non-zero element of the form exe'. By theorem 2.8 we have that the equivalence mapping from N onto N' is given by a right multiplication. Let the element x be such, that r_x defines this equivalence mapping, then for each $n \in N$ we have $r_x(n) = nx \in N'$. While the element x need not belong to N' the element ex must, since $ex = r_x(e) \in N'$. Also ex is the image of $e \neq 0$ under the equivalence map r_x thus $ex \neq 0$. We then have $exe' = ex \neq 0$. Now for each $n \in N$ we have $n = ne$, therefore $r_x(n) = r_x(ne) = nex = r_{ex}(n) = r_{exe'}(n)$ for all $n \in N$.∎

Let us now prove the most important property of primitive idempotents.

Theorem 2.11. Let G be a finite multiplicative group and let C_G be the group ring determined by G. If e is a primitive idempotent, then all elements exe are numerical multiples ξe of e. Conversely, if e is idempotent and $exe = \xi e$ for all $x \in C_G$, then e is primitive.

Proof: Let e be a primitive idempotent, then $C_G e = N$ is a minimal left ideal by theorem 2.7. Now according to theorem 2.10 the non-zero elements exe determine a collection M of equivalence mappings from the left ideal N onto itself. Note that different elements exe define different linear transformations, since the idempotent e is transformed into exe. Let $R^N(\ell)$ be the restriction of the regular representation ℓ to the left ideal N. The minimal left ideal N is an irreducible representation space and each equivalence mapping r_{exe} of N onto N commutes with left multiplication $R^N(\ell_a)$ for all $a \in \mathbf{C}_G$. All the conditions are met to apply the homothety theorem following Schur's lemma, thus each linear transformation in M is a scalar multiple of the identity element in GL(N). Since right multiplication by the idempotent e acts as the identity on the minimal left ideal N, we have $exe = \xi e$.

Conversely, Let $e = e_1 + e_2$ with $e_1 e_2 = e_2 e_1 = 0$ and $e_i^2 = e_i$ for $i \in \{1,2\}$. Then $ee_1 = e_1 e = e_1$, thus also $ee_1 e = e_1$. Since all exe are numerical multiples of e, we have $e_1 = \lambda e$. This is idempotent only for $\lambda = 0$ or $\lambda = 1$, thus there does not exist a proper decomposition of idempotent e.∎

The following theorems refer to two-sided ideals, that is, linear subspaces of $\mathbf{C}_G$ which are simultaneously left and right ideals. We shall show that a two-sided ideal A is decomposable into a direct sum of equivalent minimal left ideals. Moreover, we shall show that if a minimal left ideal N is contained in the two-sided ideal A, then every left ideal equivalent to N is in A.

Theorem 2.12. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. If $\mathbb{C}_G$ is the direct sum of the two-sided ideals A and A^0, then A and A^0 annihilate each other, that is $aa_0 = a_0a = 0$ for arbitrary $a \in A$ and $a_0 \in A^0$. Also the generating units e and e_0 are uniquely determined and commute with all ring elements.

Proof: If $a \in A$ and $a_0 \in A^0$ then $aa_0 \in A$, since A is a right ideal; and $aa_0 \in A^0$, since A^0 is a left ideal. Thus aa_0 is zero. Similarly, we have a_0a equal to zero. Therefore the two-sided ideals A and A^0 annihilate each other.

Let $\alpha,\beta \in \mathbb{C}_G$, then $\alpha = a + a_0$ and $\beta = b + b_0$ for some $a,b \in A$ and $a_0,b_0 \in A^0$, since $\mathbb{C}_G = A \oplus A^0$. As a result of the two-sided ideals A and A^0 annihilating each other we have $\alpha\beta = ab + a_0b_0$. Now as a result of this multiplication property, we can show that idempotents commute with group ring elements. Recall that, according to theorem 2.5, one finds certain generating units e and e_0 by decomposing the identity element $1_{\mathbb{C}_G}$. This theorem, then, holds for A and A^0 in their property as a left ideal. Let $1_{\mathbb{C}_G} = e + e_0$ and let a be any element in A, then $ae = ea$ for all $a \in A$, that is $ae = a(e + e_0) = a\cdot 1_{\mathbb{C}_G} = 1_{\mathbb{C}_G}\cdot a = (e + e_0)a = ea$. If we now let $\alpha \in \mathbb{C}_G$ be arbitrary with $\alpha = a + a_0$, then $\alpha e = ae = ea = e\alpha$. Similarly, we have $\alpha e_0 = e_0\alpha$, thus the idempotents e and e_0 commute with all group ring elements.

Finally, if e' is another generating unit of A as a left ideal, we have $e' = e'e = ee' = e$. Thus idempotent e is unique, similarly idempotent e_0 is unique.■

Theorem 2.13. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. If A is a two sided ideal and N is a minimal left ideal with generating unit e, then either N is contained in A or A is annihilated on the right by e: $ae = 0$ for all $a \in A$

Proof: We consider the set Ae of elements ae. This set is first of all a left ideal, since A is one. Secondly, all ae lie in A, since A is a right ideal. Thirdly, they all lie in N, since e is a generating unit for N. The left ideal N is minimal, therefore it follows that the set Ae is either N or $\{0\}$. In the first case $N \leq A$, in the second all $ae = 0$.■

We see, from theorem 2.13, that if the group ring $\mathbb{C}_G$ is a direct sum of two-sided ideals A and A^0, then every left ideal N lies either in A or in A^0. Otherwise, we would have $\alpha e = ae + a_0e = 0$ for each $\alpha \in \mathbb{C}_G$, which is certainly not the case, for example choose $\alpha = 1_{\mathbb{C}_G}$ or $\alpha = e$.

Theorem 2.14. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. If A is a two sided ideal, and if the minimal left ideal N lies in A, then also every left ideal N' equivalent to N lies in A.

Proof: The left ideals N and N' are equivalent thus according to theorem 2.10 there exists a right multiplication by a non-zero element exe' which transforms N into N'. Right multiplication does not lead out of A.■

In chapter one we showed that each irreducible representation h^i of finite group G occurs in the regular representation with multiplicity equal to its degree n_i. With regards to the regular representation of the group ring C_G this implies that vector space C_G is decomposable into a direct sum of minimal left ideals N^i and each N^i occurs with multiplicity equal to its dimension. We know, by theorem 2.14 and the remark preceding the theorem, that all equivalent minimal left ideals belong to the same two-sided ideal. We shall see, in fact, that the direct sum of equivalent minimal left ideals forms a two-sided ideal.

A two-sided ideal A is simple if it contains no two-sided ideal other than itself or the zero ideal. We will show that C_G is uniquely decomposable into a direct sum of simple two-sided ideals A^i, with each A^i the direct sum of equivalent minimal left ideals.

Theorem 2.15. Let G be a finite multiplicative group and let C_G be the group ring determined by G. If there exists a decomposition of C_G into a direct sum of simple two-sided ideals, say $C_G = A^1 \oplus \cdots \oplus A^k$, then there is at most one such decomposition.

Proof: Let $\mathbb{C}_G = B^1 \oplus \cdots \oplus B^j$ be a second decomposition, and let $e_1, \ldots, e_k$ be the generating units of $A^1, \ldots, A^k$, then $1_{\mathbb{C}_G} = e_1 + \cdots + e_k$. Let $B^0 = \{ b_1 e_i \mid b_1 e_i = e_i b_1 , b_1 \in B_1 \}$, then B^0 constitutes a two-sided ideal contained in A^i and in B^1. Thus either $B^0 = \{0\}$ or $B^1 = B^0 = A^i$. The first case cannot occur for all i, for this implies that $b_1 e_i = 0$ for all $b_1 \in B^1$ and all i. But this is not possible, since we have $b_1 = b_1 1_{\mathbb{C}_G} = b_1 e_1 + \cdots + b_1 e_k$ and not all $b_1 = 0$. Thus the simple two-sided ideal B^1 must be one of the A^i, and similarly for the rest.∎

It has not yet been determined whether such a decomposition exists, but if so, it must be unique. We know, however, that the group ring $\mathbb{C}_G$ is completely decomposable into a direct sum of minimal left ideals. We may then order the minimal ideals so that equivalent ideals are together. If we introduce the notation N_j^i for each minimal left ideal so that ideals with the same upper index are equivalent, and those with different upper indices are not equivalent, then $\mathbb{C}_G = N_1^1 \oplus \cdots \oplus N_s^1 \oplus \cdots \oplus N_1^k \oplus \cdots \oplus N_t^k$. Now, let $A^i = N_1^i \oplus \cdots \oplus N_r^i$ then $\mathbb{C}_G = A^1 \oplus \cdots \oplus A^k$ where each of the subspaces A^i are left ideals since they are direct sums of left ideals. In fact the following theorem shows that each A^i is a simple two-sided ideal.

Theorem 2.16. Let G be a finite multiplicative group and let $\mathbb{C}_G$ be the group ring determined by G. Let $\mathbb{C}_G$ have the following decomposition: $\mathbb{C}_G = A^1 \oplus\cdots\oplus A^k$ where each A^i is decomposed into a direct sum of equivalent minimal left ideals. And each minimal left ideal in A^i is not equivalent to any minimal left ideals in A^j when $i \neq j$. Then $A^1,\ldots,A^k$ are simple two sided ideals.

Proof: By the remarks preceding this theorem, we have that each A^i is a left ideal, thus it remains for us to show that each A^i is a right ideal and that each is simple. The decomposition of $1_{\mathbb{C}_G}$ gives us generating units e_{ij} with

$$1_{\mathbb{C}_G} = e_1+\cdots+ e_k = e_{11}+\cdots+ e_{1s}+\cdots+ e_{k1}+\cdots+ e_{kt}.$$

Then, for $i \neq j$, we have $e_i x e_j = e_j x e_i = 0$ for all $x \in \mathbb{C}_G$, since in the sum $e_i x e_j = \sum_{m,n} e_{im} x e_{jn}$ all of the summands are zero by theorem 2.10. Moreover, if $a_i \in A^i$ and $a_j \in A^j$ then $a_i a_j = a_j a_i = 0$, since $a_i = a_i e_i$ and $a_j = a_j e_j$. If now x is arbitrary with decomposition $x = x_1 +\cdots+ x_k$ and $a_i \in A^i$, then $a_i x = a_i x_i$ and therefore $a_i x \in A^i$. Thus each A^i is a right ideal.

Now let $A^0 \neq \{0\}$ be a two-sided ideal contained in A^i. Consider one of the left ideals N^i_j , according to theorem 2.13, either N^i_j is contained in A^0 or $a_0 e_{ij} = 0$ for all $a_0 \in A^0$. But $a_0 = a_0 e_i = a_0 e_{i1}+\cdots+ a_0 e_{ir}$ for all a_0 in A^0 and A^0 is not the zero ideal, thus not all summands can vanish for all j. Therefore, at least one of the N^i_j is contained in A^0. Then, by theorem 2.14, A^0 must contain all equivalent left ideals of N^i_j, and so all of A^i. Thus each A^i is simple.■

The group ring $\mathbb{C}_G$ is then uniquely decomposable into a direct sum of simple two-sided ideals A^i, each of which is a direct sum of equivalent minimal left ideals. In the regular representation each distinct irreducible representation occurs with multiplicity equal to its degree. We may then assume that the number of eqivalent left ideals N^i_j in the simple two-sided ideal A^i is equal to the dimension of N^i_j.

Also recall, by the remark following theorem 1.16, that a finite group G has as many distinct irreducible representations as there are classes of conjugate group elements in G. Therefore, the number of simple two-sided ideals A^i must equal the number of conjugacy classes in G.

Section 2.3 Young Tables and Semiidempotents

We will now restrict our attention to finding primitive idempotents for the group ring determined by the symmetric group S_n. It will prove convenient not to demand idempotence in the strong sense, but to be satisfied with "idempotence except for a numerical factor."

An element e is said to be <u>semiidempotent</u> if there exists a number $\kappa \neq 0$ such that $e^2 = \kappa e$.

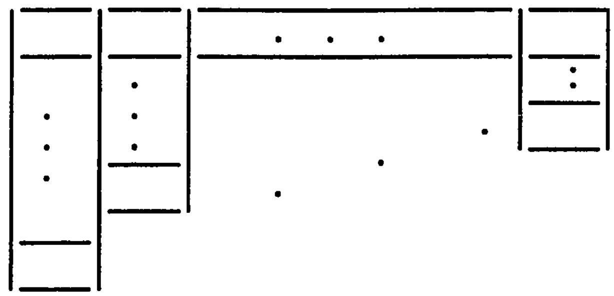
While e is not idempotent, e/κ is since $(e/\kappa)^2 = e^2/\kappa^2 = \kappa e/\kappa^2 = e/\kappa$. Rather than attach the factor $1/\kappa$ each time, we will, instead, search for semiidempotents which may be expressed more simply as e.

Let $P = \sum_{\sigma \in S_n} \sigma$, the sum of all permutations of $\{1,\ldots,n\}$, then P is semiidempotent. This is clear, since for any permutation τ we have $\tau P = P\tau = P$, thus $P^2 = \sum_{\tau \in S_n} \tau P = n!P$. The same also holds for $Q = \sum_{\sigma \in S_n} \varepsilon_\sigma \cdot \sigma$ where $\varepsilon_\sigma = \pm 1$ according to whether σ is an even or odd permutation. Here we have $\tau Q = Q\tau = \varepsilon_\tau Q$ and again $Q^2 = n!Q$. We shall see that the representations which correspond to these semiidempotents are already familiar to us.

Let $x \in \mathbb{C}_{S_n}$ with $x = \sum_{s \in S_n} \xi_s \cdot s$ then $xP = \sum_s \xi_s \cdot sP = \sum_s \xi_s \cdot P$, thus the left ideal $\mathbb{C}_{S_n} \cdot P$ is a one dimensional subspace consisting of numerical multiples of P. Similarly, we have that $\mathbb{C}_{S_n}\cdot Q$ is one dimensional; therefore both left ideals generated by P and Q are minimal, and their corresponding representations are thus irreducible.

The regular representation ℓ_a, of the group ring element $a = \sum_{s \in S_n} \alpha_s \cdot s$, when restricted to $\mathbb{C}_{S_n}\cdot P$, simply multiplies each element λP in $\mathbb{C}_{S_n}\cdot P$ by the numerical factor $\sum_s \alpha_s$. In the case of symmetric group elements the numerical factor is 1, thus the corresponding representation is the unit representation. Similarly, we see that the representation corresponding to Q is the alternating representation, which assigns 1 to every even permutation and -1 to every odd permutation.

We will now generalize the procedure, for which the two examples above are the simplest special cases. First we draw a diagram, which is called a <u>Young diagram</u>. The diagram contains r rows of squares with each row containing m_i squares, where subscript i refers to the row number. We require that $r \le n$, $m_i \ge m_{i+1} > 0$ for each i, and $\sum_{i=1}^{r} m_i = n$. The diagram is arranged so that the leftmost squares of each row form a column.



Let n_j be the number of squares in each column, where the subscript j refers to the column number, and let s be the number of columns in the diagram. Then we have $s \le n$, $n_j \ge n_{j+1} > 0$ for all $j < s$, and $\sum_{j=1}^{s} n_j = n$.

A Young table (or Young tableau) is formed by placing the numbers 1 to n in any order into the n squares of the Young diagram. To each diagram, then, corresponds n! tables.

For each table we consider two special kinds of permutations, the p and the q. Any permutation which permutes only the numbers in each row will be denoted by p. The rows, then, are invariant under the action of p. Similarly, the columns are invariant under q. Now let T be a Young table, and define the following quantities $P = \sum p$ and $Q = \sum \varepsilon_q \cdot q$ where $\varepsilon_q = \pm 1$ according to whether q is even or odd. The sums P and Q are taken over all p and q which, respectively, leave the rows and columns of T invariant. The quantities P and Q play an essential role in determining the primitive semiidempotents.

We intend to show the following: The product $PQ = e$ is semiidempotent and the left ideal $\mathbf{C}_{S_n}e$ generated by e yields an irreducible representation of S_n. It will be shown later that different diagrams yield nonequivalent representations and that representations belonging to different tables with the same diagram are equivalent.

Since two permutations of n objects are conjugate if and only if they have the same cycle structure, the number of conjugacy classes in the group S_n equals the number of partitions $\{ m_1,\ldots,m_r\}$ of n. But this is equal to the number of distinct Young diagrams. Thus, to each distinct Young diagram corresponds an irreducible representation of the group S_n.

The Young diagrams which correspond to the two representations already given turn out as follows. To the unit representation belongs the diagram with a single row of length n. While to the alternating representation belongs the diagram with a single column of length n.

Let T be a Young table and σ a permutation, then denote by σT the table, with the same diagram, which results from T after applying the permutation σ to its numbers. The transition from T to $T' = \sigma T$ is simply a renumbering of the table T. Now, let the permutation τ be arbitrary, then for the transition $T' = \sigma T$, we have $(\tau T)' = \sigma(\tau T) = \sigma\tau\sigma^{-1}\sigma T = \sigma\tau\sigma^{-1}T'$. If we let $\tau' = \sigma\tau\sigma^{-1}$ then $(\tau T)' = \tau' T'$. It is clear that τ' acts on T' in exactly the same way as τ acts on T. In particular the element in the i-th row and j-th column of table T is moved by τ to the same position that τ' moves the i-th row and j-th column element of table T'. We shall refer to $\tau' = \sigma\tau\sigma^{-1}$ as the permutation corresponding to τ for table $T' = \sigma T$. Thus, if the permutation $p' = \sigma p \sigma^{-1}$ corresponds to a permutation p which leaves the rows of T invariant then p' does the same for T'. Similarly, if q' corresponds to a permutation q which leaves the columns of T invariant then q' does the same for T'.

For the Young table T, let $\mathfrak{P} = \{p \in S_n |$ p leaves the rows of T invariant$\}$ and let $\mathfrak{Q} = \{q \in S_n |$ q leaves the columns of T invariant$\}$, then clearly $\mathfrak{P}$ and $\mathfrak{Q}$ are subgroups of S_n. Now, for $T' = \sigma T$, the corresponding groups $\mathfrak{P}'$ and $\mathfrak{Q}'$ are given by $\mathfrak{P}' = \sigma\mathfrak{P}\sigma^{-1}$ and $\mathfrak{Q}' = \sigma\mathfrak{Q}\sigma^{-1}$. Furthermore, the quantities corresponding to $P = \sum_{p \in \mathfrak{P}} p$ and $Q = \sum_{q \in \mathfrak{Q}} \varepsilon_q q$ are $P' = \sigma P \sigma^{-1}$ and $Q' = \sigma Q \sigma^{-1}$. Regarding Q' we note that σ^{-1} and σ are either both even or both odd, the same holds for $\sigma q \sigma^{-1}$ and q, thus $\varepsilon_{q'} = \varepsilon_q$. Finally, for the element $e = PQ$ we have $e' = \sigma e \sigma^{-1}$, since $e' = P'Q' = \sigma P \sigma^{-1} \sigma Q \sigma^{-1} = \sigma P Q \sigma^{-1}$.

Let us now take a closer look at the element e, and the terms pq which make up the double sum $e = PQ = \sum_{p \in \mathfrak{P}} \sum_{q \in \mathfrak{Q}} \varepsilon_q pq$. We actually have $e = \sum_{pq \in S_n} \varepsilon_q pq$, since every element in S_n can be written in the form pq in at most one way. For, if we let $pq = p_0 q_0$, then we have $p_0^{-1} p = q_0 q^{-1}$. Now, p and p_0 belong to the group $\mathfrak{P}$ implies that $p_0^{-1} p \in \mathfrak{P}$, similarly, we have $q_0q^{-1} \in \mathfrak{Q}$. The only permutation which belongs to both $\mathfrak{P}$ and $\mathfrak{Q}$ is 1_{S_n}, therefore $p = P_0$ and $q = q_0$.

We observe by simple enumeration that the pq's do not account for all permutations in S_n. Consider for example this simple diagram for n = 3.

Here $\mathfrak{P}$ and $\mathfrak{Q}$ each consists of the identity and one transposition, thus there are four elements of the form pq. Since S_3 has six elements this leaves two permutations which are not expressible in the form pq.

Each permutation of the form pq transforms the table T as follows: Permutation p acts first leaving the rows of T invariant. Next the permutation pqp^{-1} acts on pT. Note that pqp^{-1} is the permutation corresponding to q for pT, and that $(pqp^{-1})p = pq$. Thus we have $T' = pqT = pqp^{-1}pT$. It is useful to note that pqp^{-1} leaves the columns of pT invariant, thus $pqp^{-1} \in \mathfrak{Q}'$ for $T' = pqT$.

Theorem 2.17. Let T be a Young table, and let $\mathfrak{P}$ and $\mathfrak{Q}$ be the corresponding groups of permutations which, respectively, leave the rows and columns of T invariant. Two numbers which occur in the same row in the table T can never occur in the same column in the table $T' = pqT$, where $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$.

Proof: This follows immediately since $pq = (pqp^{-1})p$ transforms T by permutation p first which leaves each row invariant, thus the two numbers still occur in the same row and so in different columns. The permutation pqp^{-1} leaves each column of pT invariant, thus the two numbers cannot occur in the same column of $T' = pqT$.■

Theorem 2.18 Let T be a Young table, and let $\mathfrak{P}$ and $\mathfrak{Q}$ be the corresponding groups of permutations which, respectively, leave the rows and columns of T invariant. If two numbers which occur in the same row in the table T never occur in the same column in the table $T' = rT$, then $r = pq$ for some $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$.

Proof: The numbers which occupy column one of T' are found in different rows of T. Thus these numbers can be brought into the first column by a permutation p_1 which leaves the rows of T invariant. Next, the numbers which occupy column two of T' are found in different rows of T and thus p_1T. These numbers can be brought into the second column by a permutation p_2 which leaves the first number in each row of p_1T fixed and leaves each row of T invariant. We continue in this manner until all of the numbers are placed in the correct columns. Now a permutation q' may be applied which brings each of these numbers into the correct rows of T'. The permutation q' leaves each column of pT invariant, where p is the composition of the p_i's. Since each $p_i \in \mathfrak{P}$ we have $p \in \mathfrak{P}$. Let permutation $q \in \mathfrak{Q}$ be such that q' corresponds to q for pT. Then the table $T' = pqT$ with $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$.∎

We see, from the previous two theorems, that any permutation *which can be written in the form pq is* characterized by the following property: The permutation transforms the table T into a table T' in such a way that any two elements which occupy the same row of T can never occupy the same column of T'.

While the previous theorems and subsequent remark deal with the transformation of different Young tables, only a single diagram has been involved. We will now broaden the discussion and consider different diagrams.

A Young diagram for the group S_n is characterized by a set of integers $m = \{m_1,\ldots,m_r\}$ which satisfy the following conditions $\sum_{i=1}^{r} m_i = n$ and $m_i \geq m_{i+1} > 0$ for all $i < r$. We order these sets, and so order the Young diagrams, by writing $m > m'$ if the first non-zero difference $m_i - m'_i$ is positive.

Theorem 2.19. Let T and T' be Young tables which belong to the diagrams determined by the sets m and m', respectively. If $m > m'$, then there exists two numbers which occupy one row in T and one column in T'.

Proof: Assume that there do not exist two numbers which occupy one row in T and one column of T'. Then the m_1

numbers in the first row of T are found in different columns in T'. The table T', then, must possess at least m_1 columns and so $m_1' \geq m_1$. But by assumption $m_1 \geq m_1'$, therefore we have $m_1 = m_1'$. Now, we may apply a permutation q' to the table T' which brings each of these m_1 numbers into the first row of T' while leaving each column of T' invariant. Next, we consider the tables T and $q'T'$ minus their first rows. The m_2 numbers in row two of T are found in different columns of $q'T'$ and again, as above, we conclude that $m_2 = m_2'$. We continue in this manner until we have $m = m'$, in contradiction with the assumption.∎

Theorem 2.20. Let T be a Young table and let $\mathfrak{P}$ and $\mathfrak{Q}$ be the corresponding groups of permutations which, respectively, leave the rows and columns of T invariant. let $P = \sum_{p\in\mathfrak{P}} p$ and $Q = \sum_{q\in\mathfrak{Q}} \varepsilon_q q$ where $\varepsilon_q = \pm 1$ according to whether q is even or odd, and let $e = PQ$, then for $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$

$$pP = Pp = P \;,\; qQ = Qq = \varepsilon_q Q \text{ and } peq = \varepsilon_q e.$$

Proof: Since $\mathfrak{P}$ and $\mathfrak{Q}$ are groups we have, for $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$, that $p\mathfrak{P} = \mathfrak{P}p = \mathfrak{P}$ and $q\mathfrak{Q} = \mathfrak{Q}q = \mathfrak{Q}$. In particular, for p a fixed group element we have as r varies through the whole group then so does pr and rp. A similar comment may be made regarding the fixed element q in $\mathfrak{Q}$. Note that when permutation q is odd the product qr is even (odd) when r is odd (even). We therefore have

$$_pP = p\sum_{r\in\mathfrak{P}} r = \sum_{r\in\mathfrak{P}} pr = \sum_{pr\in\mathfrak{P}} pr = P = \sum_{rp\in\mathfrak{P}} rp = \sum_{r\in\mathfrak{P}} rp = P_p \text{ and}$$

$$_qQ = q\sum_{r\in\mathfrak{Q}} \varepsilon_r r = \sum_{r\in\mathfrak{Q}} \varepsilon_r qr = \sum_{qr\in\mathfrak{Q}} \varepsilon_q \varepsilon_{qr} qr = \varepsilon_q Q = \sum_{rq\in\mathfrak{Q}} \varepsilon_q \varepsilon_{rq} rq$$

$$= \sum_{r\in\mathfrak{Q}} \varepsilon_r rq = Q_q.$$

Finally, $peq = pPQq = ({_pP})\cdot(Q_q) = (P)(\varepsilon_q Q) = \varepsilon_q PQ = \varepsilon_q e$, as was to be shown.∎

Theorem 2.21. Let T and T' be Young tables. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be the corresponding groups of permutations which, respectively, leave the rows and columns of T invariant. Similarly let $\mathfrak{P}'$ and $\mathfrak{Q}'$ be the corresponding groups for T'. Let the quantities P, Q, P' and Q' be defined in the usual way and let $e = PQ$ and $e' = P'Q'$. If there exists two numbers which occur in one row in T, and in one column in T', then $Q'P = 0$ and consequently $e'e = 0$.

Proof: Let t be the transposition of the two numbers which occupy one row of T and one column of T', then clearly $t \in \mathfrak{P}$ and $t \in \mathfrak{Q}'$. Also, it is clear that $\varepsilon_t = -1$. Now, due to theorem 2.20, we have $tP = P$ and $Q't = \varepsilon_t Q' = -Q'$, thus $Q'P = Q'(tP) = (Q't)P = -Q'P$. This implies that $Q'P = 0$ and consequently $e'e = (P'Q')(PQ) = P(Q'P)Q = 0$.∎

In theorem 2.21 we made no assumption regarding the diagrams of the tables T and T'. Let m and m' be the set of integers which characterize the diagrams for T and T'. Then according to theorem 2.19, if $m > m'$ there exist two numbers which occupy one row of T and one column of T'. Therefore, we have $e'e = 0$ for all tables T and T' whenever their diagrams are different.

Theorem 2.22. Let T be a Young table and let $\mathfrak{P}$ and $\mathfrak{Q}$ be the corresponding groups of permutations which, respectively, leave the rows and columns of T invariant. Let $T' = pqT$ with $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$. Then $Q'pqP = \varepsilon_q Q'P$.

Proof: Let $q' = pqp^{-1}$ then $pq = (pqp^{-1})p = q'p$. By the remark preceding theorem 2.17, the permutation q' belongs to the group $\mathfrak{Q}'$ of permutations which leaves the columns of $T' = pqT$ invariant. Thus, according to theorem 2.20, we have $Q'q' = \varepsilon_{q'}Q'$ and $pP = P$. Finally, since q and q' are conjugate we have $\varepsilon_{q'} = \varepsilon_q$, and so

$$Q'pqP = Q'q'pP = \varepsilon_{q'}Q'P = \varepsilon_q Q'P. \blacksquare$$

Theorem 2.23. Let T be a Young table and let $\mathfrak{P}$ and $\mathfrak{Q}$ be the corresponding groups of permutations which, respectively, leave the rows and columns of T invariant. If a permutation σ cannot be written in the form pq with $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$, then there exist a transposition $p \in \mathfrak{P}$ and a transposition $q \in \mathfrak{Q}$ such that $p\sigma q = \sigma$.

Proof: According to theorem 2.18 there exist two numbers which occur in the same row in T and the same column in $T' = \sigma T$. Let τ be the transposition of these two numbers. Clearly τ lies in $\mathfrak{P}$ and $\mathfrak{Q}'$. Let $p = \tau$ and let $q \in \mathfrak{Q}$ be such that τ is the permutation corresponding to q for $T' = \sigma T$. We then have $\tau = \sigma q \sigma^{-1}$, thus $q = \sigma^{-1}\tau\sigma$ is also a transposition. Therefore we have $p\sigma q = (\tau)\sigma(\sigma^{-1}\tau\sigma) = \tau\tau\sigma = 1_{S_n}\cdot\sigma = \sigma$.■

According to theorem 2.20 the element e belonging to a given table T has the property that $peq = \varepsilon_q e$ for every $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$. It is clear that this property holds also for numerical multiples λe of the element e. In the following theorem we see that the converse is also true.

Theorem 2.24. Let T be a Young table and let $\mathfrak{P}$ and $\mathfrak{Q}$ be the corresponding groups of permutations which, respectively, leave the rows and columns of T invariant. Let an element $a = \sum_{\sigma\in S_n} \alpha_\sigma\cdot\sigma$ of the group ring $\mathbb{C}_{S_n}$ have the property that $paq = \varepsilon_q a$ for all $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$. Then there exists a number λ such that $a = \lambda e$.

Proof: It suffices to show the following: Whenever the permutation σ can be expressed in the form pq with $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$ then $\alpha_\sigma = \alpha_{pq} = \lambda\varepsilon_q$, otherwise $\alpha_\sigma = 0$. For the fixed elements p and q of S_n we have that $p\sigma q$ runs through all of S_n as σ does. Therefore, in the following equation

$$\sum_{\sigma\in S_n} \alpha_\sigma\cdot p\sigma q = paq = \varepsilon_q a = \varepsilon_q \sum_{\sigma\in S_n} \alpha_\sigma\cdot\sigma$$

every element of S_n occurs exactly one time in each sum. The element pq occurs on the left when $\sigma = 1_{S_n}$ and on the right when $\sigma = pq$. Therefore $\alpha_{1_{S_n}} = \varepsilon_q\alpha_{pq}$ for every element of the form pq. We then let $\lambda = \alpha_{1_{S_n}}$. If σ cannot be expressed in the form pq with $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$, then by theorem 2.23 there exist a transposition p and a transposition q such that $p\sigma q = \sigma$. Note that since q is a transposition we have $\varepsilon_q = -1$. Then from $paq = \varepsilon_q a$ follows $\alpha_\sigma = \varepsilon_q \alpha_\sigma = - \alpha_\sigma$ and so $\alpha_\sigma = 0$ whenever σ cannot be expressed in the form pq.∎

Theorem 2.25. Let T be a Young table and let $\mathfrak{P}$ and $\mathfrak{Q}$ be the corresponding groups of permutations which, respectively, leave the rows and columns of T invariant. Then the element $e = \sum_{pq \in S_n} \varepsilon_q pq$ is semiidempotent and the left ideal $\mathbb{C}_{S_n} \cdot e$ generated by e is minimal. Therefore $\mathbb{C}_{S_n} \cdot e$ yields an irreducible representation of $\mathbb{C}_{S_n}$. The dimension of $\mathbb{C}_{S_n} \cdot e$ and thus the degree of the representation is a factor of n!.

Proof: Let $p \in \mathfrak{P}$ and $q \in \mathfrak{Q}$ then $pe^2q = \varepsilon_q e^2$ since $pe^2q = p(PQ)^2q = PQP(\varepsilon_q Q) = \varepsilon_q (PQ)^2 = \varepsilon_q e^2$. Thus e^2 has the property of theorem 2.24 and we have $e^2 = \kappa e$. The element e^2 is semiidempotent if and only if $\kappa \neq 0$, we must therefore show this to be the case.

Regardless of whether $\kappa = 0$ or not, the left ideal $\mathbb{C}_{S_n} \cdot e$ is defined and its dimension d_e is at least one since $e \neq 0$. Right multiplication with e, denoted r_e, is a linear transformation in $\mathbb{C}_{S_n}$ which maps every element a into an element ae of $\mathbb{C}_{S_n}\cdot e$ and multiplies every element ae of $\mathbb{C}_{S_n}\cdot e$ by the number κ. Now, the trace of the transformation r_e is independent of the coordinate system, thus we may choose the natural coordinate system for $\mathbb{C}_{S_n}$. Denote by ε_τ the components of e, then $r_e(a) = ae = \sum_{\sigma\in S_n}\sum_{\tau\in S_n}\alpha_{\sigma\tau^{-1}}\,\varepsilon_\tau\cdot\sigma$ with $a = \sum_{\rho\in S_n}\alpha_\rho\cdot\rho$ and $\sigma = \rho\tau$. In particular the component found in the σ-th position of $r_e(a)$ is given by $\sum_{\tau\in S_n}\alpha_{\sigma\tau^{-1}}\,\varepsilon_\tau = \sum_{\rho\in S_n}\alpha_\rho\varepsilon_{\rho^{-1}\sigma}$. Thus, if E is the matrix representing r_e with respect to the natural basis, the element in the ρ-th row and σ-th column of E is $\varepsilon_{\rho^{-1}\sigma}$. The trace is therefore given by $\sum_{\sigma\in S_n}\varepsilon_{\sigma^{-1}\sigma} = \sum_{\sigma\in S_n}\varepsilon_{1_{S_n}} = n!\varepsilon_{1_{S_n}} = n!$. Let us now compute the trace in a coordinate system chosen so that the first d_e basis elements span the linear subspace $\mathbb{C}_{S_n}\cdot e$. Let E' be the matrix representing r_e with respect to this basis. In this system, the linear transformation r_e maps an arbitrary element a with components $(\alpha_1,\ldots,\alpha_{n!})$ to the element ae in $\mathbb{C}_{S_n}\cdot e$ with components $(\alpha'_1,\ldots,\alpha'_{d_e},0,\ldots,0)$. Thus the last $(n! - d_e)$ rows of the transformation matrix E' consist of zeros. For any element ae in $\mathbb{C}_{S_n}\cdot e$ we have $r_e(ae) = ae^2 = \kappa(ae)$, thus the upper left block of the matrix E' consists of κ times the identity matrix of dimension d_e. The trace of the matrix E' is therefore $\kappa\cdot d_e$. Since $\mathrm{Tr}(E) = \mathrm{Tr}(E')$ we have $\kappa \cdot d_e = n!$, and so $\kappa = n!/d_e \neq 0$. We may then conclude that $e = \sum_{pq \in S_n} \varepsilon_q pq$ is semiidempotent.

Note that the element e has components $\varepsilon_q = \pm 1$, thus the product e^2 must have integral components. Now since $e^2 = \kappa e$, we have that the number κ is an integer. Therefore the dimension d_e of the linear subspace $\mathbb{C}_{S_n} \cdot e$ must divide $n!$, since $\kappa = n!/d_e$.

We will now prove that the linear subspace $\mathbb{C}_{S_n} \cdot e$ generated by the semiidempotent e is minimal. It is sufficient, according to theorem 2.7, to show that e/κ is primitive. Let a be an arbitrary element of the group ring $\mathbb{C}_{S_n}$, the product eae has the property that $p(eae)q = \varepsilon_q eae$ for all $p \in \mathfrak{P}$ and $q \in \Omega$. This is clear since $p(eae)q = p(PQaPQ)q = (pP)QaP(Qq) = (P)QaP(\varepsilon_q Q) = \varepsilon_q PQaPQ = \varepsilon_q eae$. Then, according to theorem 2.24, there exists a number λ such that the element $eae = \lambda e$. Finally, as a result of theorem 2.11, we are able to conclude that e is primitive and so also e/κ.

Thus every Young table determines a primitive semiidempotent which in turn determines a minimal left ideal. Each minimal left ideal is an invariant subspace under the action of the group S_n and therefore yields an irreducible representation as was to be shown.∎

Theorem 2.26. Young tables which belong to the same Young diagram yield equivalent representations. Young tables with different diagrams yield nonequivalent representations.

Proof: Let T and T' be Young tables which belong to the same Young diagram. Then table T is transformed into table T' by a permutation σ, that is $T' = \sigma T$. Let e be the semiidempotent belonging to the table T, then $e' = \sigma e \sigma^{-1}$ is the corresponding semiidempotent for T'. According to theorem 2.10, two minimal left ideals are equivalent if and only if there exist non-zero elements of the form exe'. It therefore suffices to show that exe' is not zero for all x. Choose $x = \sigma^{-1}$ then $e\sigma^{-1}e' = e\sigma^{-1}\sigma e\sigma^{-1} = e^2\sigma^{-1} = \kappa e\sigma^{-1} \neq 0$. The equivalence mapping in this case is given by right multiplication with the element $\kappa e\sigma^{-1}$.

Let us now assume that the Young tables T and T' belong to different Young diagrams. Let m and m′ be the set of integers which characterize the diagrams for T and T', respectively. Let e and e' be the semiidempotents belonging to the tables T and T'. We may assume, without loss of generality, that $m < m'$, then according to the remark following theorem 2.21 we have $e'e = 0$.

If we now consider the table σT with its semiidempotent $\sigma e\sigma^{-1}$, we also have by the previous remark that $e'\sigma e\sigma^{-1} = 0$. Now multiplying, each side, on the right by σ leads to the following property: $e'\sigma e = 0$ for all $\sigma \in S_n$. Let $a \in \mathbb{C}_{S_n}$ be arbitrary then $e'ae = e'(\sum_{\sigma\in S_n}\alpha_\sigma\cdot\sigma)e = \sum_{\sigma\in S_n}\alpha_\sigma e'\sigma e = 0$, thus there do not exist non-zero elements of the form $e'ae$. Therefore according to theorem 2.10 the tables T and T' are nonequivalent.∎

Recall that the group ring $\mathbb{C}_{S_n}$ is a direct sum of simple two-sided ideals A^i which annihilate each other on the left and right. Also each A^i is spanned by a collection of equivalent minimal left ideals. The minimal left ideals N and N' occur in different simple two-sided ideals if and only if they are nonequivalent. In the terminology of Young diagrams and Young tables the above comments translate as follows. Each simple two-sided ideal A^i belongs to one diagram. Each A^i contains the minimal left ideals $\mathbb{C}_{S_n}\cdot e$ defined by the n! tables which belong to the diagram for A^i. We will show in fact that each A^i is spanned by the n! minimal left ideals $\mathbb{C}_{S_n}\cdot e$.

Theorem 2.27. Let D be a Young diagram and let A be the corresponding simple two-sided ideal. Denote by $T_1,\ldots,T_{n!}$ the n! tables belonging to D. Let e_i be the semiidempotent defined by the table T_i. Then the n! minimal left ideals $\mathbb{C}_{S_n}\cdot e_i$ span the entire corresponding two-sided ideal A.

Proof: Let A^0 be the linear subspace spanned by the n!

left ideals $\mathbb{C}_{S_n}\cdot e_i$. Clearly $A^0 \leq A$ is a left ideal. If we can show that A^0 is also a right ideal, and thus two-sided, we are done since A is simple implies that the only non-zero two-sided ideal contained in A is A itself. Let $x \in A^0$ then x has the form $x = x_1 e_1 + \cdots + x_{n!} e_{n!}$. Let $a = \sum_{\sigma \in S_n} \alpha_\sigma \cdot \sigma$ be an arbitrary element of the group ring $\mathbb{C}_{S_n}$, we must show that xa is again in A^0. It is actually enough to show that $x\sigma$ belongs to A^0 for all $\sigma \in S_n$.

When the permutation σ is applied to a table T_i it simply permutes the numbers in T_i to form another table, say T_j. If σ is applied to the entire set of tables T_i it simply permutes the set. Thus we have for each i there exists a j such that $T_j = \sigma T_i$ and $e_j = \sigma e_i \sigma^{-1}$. Then for each term in $x\sigma$ we have $x_j e_j \sigma = x_j(\sigma e_i \sigma^{-1})\sigma = x_j \sigma e_i$. Thus the product $x\sigma$ can be expressed as a linear combination of the e_i's for all $\sigma \in S_n$. Therefore xa belongs to A^0 and A^0 equals the simple two-sided ideal A.∎

While the previous set of semiidempotents e_i generate the minimal left ideals $\mathbb{C}_{S_n}\cdot e_i$ which span A, the set is not linearly independent. Any proper two-sided ideal A must be a proper subspace of $\mathbb{C}_{S_n}$; therefore $\dim(A) < \dim \mathbb{C}_{S_n} = n!$.

We will now consider certain special tables which Young referred to as "standard tableaux." We shall see that a set of standard tables correspond to a linearly independent set of semiidempotents which generate a spanning set of minimal left ideals for the two-sided ideal A.

A table T, for a given young diagram, is called a <u>standard table</u> if the numbers increase in every row of T from left to right and in every column of T downwards.

Note that the number of standard tables for a diagram consisting of a single row or a single column is one. This is in agreement with the degree of the corresponding irreducible unit and alternating representations. This simple observation generalizes as we shall see.

The final two theorems will be stated without proof. The first theorem is not actually necessary for the application of representation theory of the symmetric group to the problem of finding explicit identities for a P.I. algebra. It is offered here for completeness only. The second theorem is necessary for what is to come in part three. However, the proof is tedious and involves several technical lemmas. The interested reader may refer to Boerner (page 138).

Theorem 2.28. The left ideals which arise from the standard tables belonging to one diagram D_i, with characteristic set of integers m_i, are linearly independent and they span the corresponding ideal A^i. Their number is thus equal to the degree d_{m_i} of the corresponding irreducible representation.

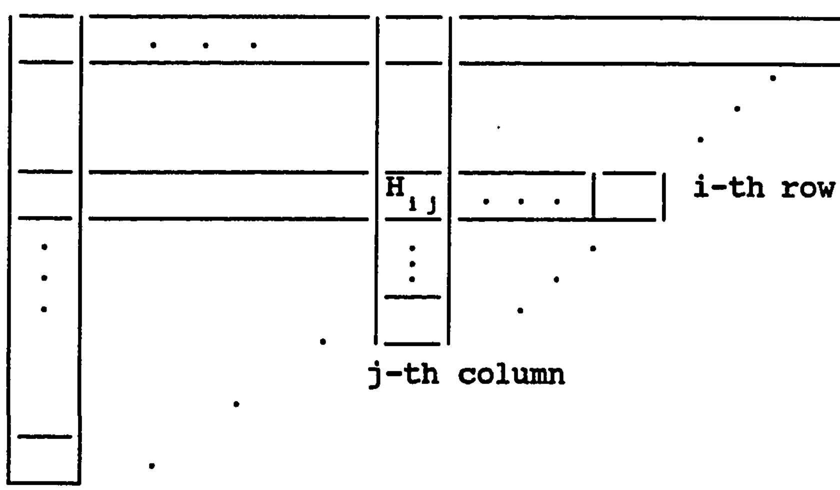
Theorem 2.29. The number of standard tables which belong to the diagram with row lengths $m_1,\ldots,m_r$ is

$$d_m = n!\frac{\prod_{i<k}(\ell_i-\ell_k)}{\ell_1!\cdots\ell_r!}$$

where $\ell_1 = m_1 + r - 1$, $\ell_2 = m_2 + r - 2,\ldots,\ell_r = m_r$.

The number of standard tables and thus the degree of the corresponding irreducible representation may also be computed using "hook numbers."

Let D be a Young diagram characterized by the set of integers $m = m_1,\ldots,m_r$. Let H_{ij} be the hook which is formed by the square in the i-th row and j-th column of D along with all squares to its right and below. Let h_{ij} be the number of squares in the hook H_{ij}.

D =


The number d_m of standard tables which belong to the Young diagram D is then

$$d_m = \frac{n!}{\prod_{i,j} h_{ij}} \quad \text{where } 1 \le i \le r \text{ and } 1 \le j \le m_1.$$

Chapter 3 An Application of the Theory of Representations of the Symmetric Group

It is now time to apply the theory of representations of the symmetric group to the problem of finding explicit identities for a polynomial identity algebra. In section one we introduce the basic definitions of polynomial identity algebras and give some examples. We define such terms a T-ideals and co-dimensions of order n for a given T-ideal.

Section two is entirely devoted to determining explicit identities for a P.I. algebra and the tensor product of two P.I. algebras. The work is based on several recent articles by Amitai Regev.

Section 3.1 Polynomial Identity Algebras

Definition. An algebra A over a field F is said to satisfy a <u>polynomial identity</u> if there exists an $f \neq 0$ in $F[x_1,\ldots,x_n]$, the free algebra over F in the noncommuting indeterminants $x_1,\ldots,x_n$ for some n, such that $f(a_1,\ldots,a_n) = 0$ for all $a_1,\ldots,a_n$ in A. Such an algebra A is called a <u>P.I. algebra</u>. Some examples of P.I. algebras are given below:

1. Any commutative algebra A over a field F is a P.I. algebra since each pair of elements $a_1,a_2 \in A$ satisfies $f(x_1,x_2) = x_1x_2-x_2x_1 = 0$.
2. Let $A = M_2(F)$ be the set of 2×2 matrices over a field F. It is simple to verify that A satisfies the polynomial identity $f(x_1,x_2,x_3) = (x_1x_2-x_2x_1)^2x_3 - x_3(x_1x_2-x_2x_1)^2 = 0$.
3. Let A be a nil algebra of bounded index of nilpotentcy that is $a^k = 0$ holds for every a in A and some integer k where $\{k\}$ is bounded. Then every element of A satisfies $f(x_1) = x_1^m = 0$, for a suitable fixed integer m.
4. More generally let A be an algebra containing an ideal N such that (i) N is a nil algebra of bounded index, (ii) A/N is commutative. Then there exists an integer m such that $f(x_1,x_2) = (x_1x_2-x_2x_1)^m$ vanishes on A.

Definition. In $F[x_1,\ldots,x_n]$ the <u>standard identity</u> of degree n is $[x_1,\ldots,x_n] = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}$ where σ runs over S_n, the symmetric group on n elements, and where $(-1)^\sigma$ is 1 or -1 according as σ is an even or odd permutation.

An algebra A is said to satisfy a standard identity if $[x_1,\dots,x_n]$ vanishes on A for some n. We have already noted that any commutative algebra satisfies the standard identity of degree two $[x_1,x_2] = x_1x_2-x_2x_1$. In this sense we may consider algebras which satisfy a standard identity of degree n to generalize the class of commutative algebras.

We see from the definition of the standard identity that it is multilinear and homogeneous in its variables, and that the identity vanishes if two of its arguments are equal. These properties lead to many general examples of P.I. algebras for if A is any n-dimensional algebra over a field F, it can be shown that the algebra A satisfies a standard identity of degree n+1. To see this, note that an algebra A of dimension n has a basis $\{\beta_1,\dots,\beta_n\}$ over F and each element $a_1,\dots,a_{n+1}$ in A can be expressed as a linear combination of the β_i's. By the multilinearity of $[x_1,\dots,x_{n+1}]$ we have that $[a_1,\dots,a_{n+1}]$ is a linear combination of terms of the form $[\beta_{i_1},\dots,\beta_{i_{n+1}}]$. Now each of these terms vanish in A since each β_{i_j} is one of n possible basis elements and thus two arguments are equal. Since each term vanishes we have that $[a_1,\dots,a_{n+1}]$ vanishes on the algebra A. Therefore any finite dimensional algebra A over a field F is a P.I. algebra. In particular $M_n(F)$ the algebra of n×n matrices over a field F satisfies the standard identity $[x_1,\ldots,x_{n^2+1}]$ of degree n^2+1.

Amitsur has shown that any P.I. algebra A that satisfies a polynomial of degree d, satisfies some identity of the form $s_\ell^k[x]$ where $s_\ell[x] = [x_1,\ldots,x_\ell]$ is the standard polynomial of degree ℓ. Also the exponent k was shown by Amitsur and Robinson to be dependent on d. But they were unable to find an effective way to compute k explicitly and were thus unable to write an explicit identity for A. With the application of representation theory of S_n, Regev has been able to prove Amitsur's $s_\ell^k[x]$ theorem by determining explicit values of ℓ and k. A necessary step in reproducing Regev's arguments is the following theorem about P.I. algebras.

Theorem 3.1. If an algebra A over a field F satisfies a polynomial identity of degree d, then it satisfies a multilinear homogeneous identity of degree less than or equal to d.

Proof. Let the algebra A satisfy the identity $f(x_1,\ldots,x_n)$ of degree d and suppose x_1 has degree $d_1>1$ in the polynomial f. Then A also satisfies $g(x_1,\ldots x_n,x_{n+1}) = f(x_1+x_{n+1},x_2,\ldots,x_n) - f(x_1,x_2,\ldots,x_n) - f(x_{n+1},x_2,\ldots,x_n)$ which is of lower degree in x_1 than is f. If we continue in this manner the result will be an identity which is linear in x_1, so that x_1 will occur in each monomial at most once.

Each time we reduce the degree of x_1 we introduce one new variable. Thus the degree of the new polynomial is at most d. Now go on to x_2. If x_2 has degree $d_2 > 1$ then repeat the procedure, otherwise go to x_3. If we run through all of the variables in this way, we end up with a polynomial g of degree $\leq$ d which is of degree $\leq$ 1 in each of the variables.

Now we may assume that the degree of each variable is exactly one in each term of this polynomial. Otherwise, suppose x_1 does not occur in each term; we then collect the terms of g not involving x_1. If we specialize $x_1 = 0$ in A we see that the remaining terms give a nontrivial identity for the algebra A. After a finite number of these steps we have finally a multilinear homogeneous identity of degree $\leq$ d which vanishes on the algebra A.∎

We need a few more concepts from P.I. algebras before we can begin constructing actual identities in an arbitrary P.I. algebra.

Let A be a P.I. algebra over a field F. Let F[X] be the free ring in noncommutative indeterminants $x_\alpha \in X$. Let Q be the subset of F[X] whose elements are mapped into 0 by every homomorphism of F[X] into A. Then it is clear that Q is a nonzero ideal in F[X] and that Q is invariant under every homomorphism of F[X] into itself. This last point is clear, for if we let $h:F[X] \to F[X]$ and $g:F[X] \to A$ be homomorphisms, then $g \circ h$ is a homomorphism from F[X] into A and so for each $q \in Q$, $g \circ h(q) = g(h(q)) = 0$. Since this must hold for all homomorphisms $g:F[X] \to A$, we must have $h(q) \in Q$.

Definition. Let A be a P.I. algebra over a field F and let Q be a nonzero ideal in $F[X]$ whose elements are mapped into 0 by every homomorphism of $F[X]$ into A. Then the ideal Q is called a T-ideal (in $F[X]$) of polynomial identities of algebra A.

In general an ideal $H \subseteq F[X]$ is a T-ideal if for each $g(x_1,\ldots,x_n) \in H$ and each $h_1,\ldots,h_n \in F[X]$ we have that $g(h_1,\ldots,h_n) \in H$. Thus if H is the T-ideal of identies of A then each $g(x_1,\ldots,x_n) \in H$ must also vanish on A. The relationship between P.I. algebras over a field F and T-ideals in the ring $F[X]$ is made explicit in the following proposition which will be stated without proof.

Proposition 3.2. (1) Let H be any non-zero T-ideal in $F[X]$. Then $A = F[X]/H$ is a P.I. algebra whose ideal of polynomial identies is H.
(2) Every P.I. algebra is a homomorphic image of an algebra of the form $F[X]/H$ where H is a T-ideal $\neq \{0\}$.

Let V_n be the vector space over a field F spanned by the n! monomials $x_{\sigma_1}\cdots x_{\sigma_n}$ where $\sigma_i=\sigma(i)$ and $\sigma \in S_n$ the group of permutations of $\{1,\ldots,n\}$. It is clear that V_n is the subspace of $F[x_1,\ldots,x_n]$ consisting of all multilinear homogeneous polynomials of degree n in $x_1,\ldots,x_n$. If we let F be the field $\mathbb{C}$ of complex numbers and identify each monomial $x_{\sigma_1}\cdots x_{\sigma_n}$ with the permutation $\sigma \in S_n$, then we have in fact identified the vector space V_n with the group algebra $\mathbb{C}_{S_n}$ since $V_n = sp\{x_{\sigma_n}\cdots x_{\sigma_n} | \sigma \in S_n\}$ and $\mathbb{C}_{S_n} = sp\{\sigma | \sigma \in S_n\}$. Also by theorem 3.1 we have that if a P.I. algebra A satisfies a polynomial $f \neq 0$ in $F[X]$ of degree n then A satisfies a multilinear homogeneous polynomial g in vector space V_n. These last two observations allow us to bring to bear the full power of the theory of representations of group S_n into the problem of finding explicit identities for an abitrary P.I. algebra.

Definition. Let A be an algebra over a field F satisfying a polynomial identity of degree d. Let $Q \subset F[X]$ be the T-ideal of polynomial identities of A and let $0 < n$ be an arbitrary integer. The integer

$$c_n = \dim_F \frac{V_n}{Q \cap V_n} = \dim V_n - \dim(Q \cap V_n)$$

is called the <u>co-dimension</u> of order n of Q. The sequence $\{c_n\}$ is called the sequence of co-dimensions of Q. Since we have from the above proposition that the T-ideal Q is so closely related to the algebra A we may refer to $\{c_n\}$ as the co-dimensions of the algebra A. If we let $Q_n = Q \cap V_n$, then $c_n = \dim V_n/Q_n$ for each integer $n > 0$.

In order to construct explicit polynomial identities for arbitrary P.I. algebras over a field F we first identify elements of the group algebra F_{S_n} with polynomials in the vector space V_n over F, as above. We next look at a way of relating polynomials to Young diagrams and show that they are *almost* products of standard polynomials. Recall that, to each Young diagram corresponds an irreducible representation, and therefore, an irreducible character χ_λ where λ is a partition of n. We will show that the sequence $\{d_\lambda\}$ of the degrees of irreducible characters χ_λ, for some sequences of Young diagrams, grows at a faster rate, as $n \to \infty$, than the sequence $\{c_n\}$ of codimensions of algebra A. We will then show that for any P.I. algebra A, there exists a certain two-sided ideal I_λ of identities in the group algebra F_{S_n} for n big enough. And finally we prove Amitsur's $s_\ell^k[x]$ theorem, in character zero, by giving explicit ℓ and k.

Section 3.2 Explicit Identities for a P.I. Algebra

Let V_n be the vector space over a field F of multilinear and homogeneous polynomials in $x_1,\ldots,x_n$ of degree n. Let the monomial $M_\sigma(x) = x_{\sigma_1}\cdots x_{\sigma_n}$ in V_n be identified with the permutation σ in the group algebra F_{S_n}. Let us denote the identification by $M_\sigma(x) \equiv \sigma$. This identification then induces a multiplication on V_n with the following properties:

(1) For $\sigma,\tau \in S_n$,

$$\sigma\tau \equiv \sigma M_\tau(x_1,\ldots,x_n) = M_\tau(x_{\sigma_1},\ldots,x_{\sigma_n})$$

and therefore, if $f(x_1,\ldots,x_n) \in V_n$, then

$$\sigma f(x_1,\ldots,x_n) = f(x_{\sigma_1},\ldots,x_{\sigma_n}).$$

(2) $\sigma\tau \equiv M_\sigma(x)\tau = (x_{\sigma_1}\cdots x_{\sigma_n})\tau = x_{\tau(\sigma_1)}\cdots x_{\tau(\sigma_n)}$.

We see from property (1) that multiplication by a permutation σ on the left is equivalent to a substitution that corresponds to that permutation. That is, for each variable x_i in the polynomial $f(x_1,\ldots,x_n) \in V_n$, permutation σ substitutes the variable $x_{\sigma(i)}$. Property (2), on the other hand, indicates that multiplication by a permutation τ on the right changes the order in each monomial, of polynomial $f \in V_n$, by that permutation. For example if the monomial $x_{i_1}\cdots x_{i_n}$ is a term in $f(x_1,\ldots,x_n) \in V_n$, then τ reorders the factors by replacing each x_{i_j} by $x_{\tau(i_j)}$.

Note that the definition of left multiplication implies that $Q_n = Q \cap V_n$ is a left ideal in vector space V_n, where $Q \leq F[X]$ is a T-ideal. In particular if we let Q be the T-ideal of polynomial identities of a P.I. algebra A, then for each n, Q_n is a left ideal in V_n. Since V_n may be identified with the group algebra F_{S_n}, we have that each Q_n is identified with a left ideal in F_{S_n}. Thus each Q_n corresponds to an invariant subspace in the regular representation space of the group S_n. We shall return to this point later, but for now we turn our attention to the polynomials that correspond to a Young diagram.

Let λ be a partition of n, $D(\lambda)$ the corresponding Young diagram and $T(\lambda)$ a Young table based on $D(\lambda)$. Let $e_\lambda = e_{T(\lambda)}$ be the semiidempotent that corresponds to $T(\lambda)$. If $C \leq S_n$ is the group of column preserving permutations of $T(\lambda)$ and $R \leq S_n$ the row preserving permutations, then $e_\lambda = \sum_{\rho \in R} \rho(\sum_{\sigma \in C} \operatorname{sgn}(\sigma) \cdot \sigma)$. We may write

$$\sum_{\sigma \in C} \operatorname{sgn}(\sigma) \cdot \sigma = \sum_{\sigma \in C} (-1)^\sigma \cdot \sigma \equiv \sum_{\sigma \in C} (-1)^\sigma M_\sigma(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$$

thus $e_\lambda(x_1, \ldots, x_n) = \sum_{\rho \in R} f(x_{\rho_1}, \ldots, x_{\rho_n})$. We shall see that in general $f(x_1, \ldots, x_n)$ is *almost* a product of standard polynomials.

Among the n! tables based on $D(\lambda)$, choose the following standard tableau

$$T_o(\lambda) = \begin{array}{|c|c|ccc|c|} \hline 1 & h_1+1 & \cdot & \cdot & \cdot & \\ \cline{1-2}\cline{6-6} 2 & h_1+2 & & & & \vdots \\ \cline{1-2}\cline{6-6} \vdots & \vdots & & & \cdot & n \\ \cline{2-2}\cline{6-6} & h_1+h_2 & & \cdot & & \\ \cline{1-2} h_1 & & & & & \\ \cline{1-1} \end{array}$$

where the numbers are in increasing order in the columns and the columns are ordered from left to right. This table decomposes the set $P_n = \{1,\ldots,n\}$ into a union of subsets H_i. That is $P_n = H_1 \cup \cdots \cup H_\ell$ where H_i = entries of *ith* column $= \{h_1 + \cdots + h_{i-1} + 1, \ldots, h_1 + \cdots + h_i\}$, $H_0 = 0$, and $\# H_i = h_i$. The group of column preserving permutations C is then given by $S_{h_1}(H_1) \times \cdots \times S_{h_\ell}(H_\ell)$ where

$S_{h_i}(H_i) = \{\sigma \in S_n | \sigma(j) = j \text{ if } j \notin H_i\}$. We then have

$$\sum_{\sigma \in C} (-1)^{\sigma} \cdot \sigma = \Big(\sum_{\sigma^1 \in S_{h_1}(H_1)} (-1)^{\sigma^1} \cdot \sigma^1 \Big) \cdots \Big(\sum_{\sigma^\ell \in S_{h_\ell}(H_\ell)} (-1)^{\sigma^\ell} \cdot \sigma^\ell \Big)$$

$$\equiv s_{h_1}[x_1, \ldots, x_{h_1}] \cdots s_{h_\ell}[x_{h_1 + \cdots + h_{\ell-1} + 1}, \ldots, x_n].$$

If we now let $f(x_1, \ldots, x_n) \equiv \sum_{\sigma \in C} (-1)^{\sigma} \cdot \sigma$, then $f(x)$ in this case is a product of standard polynomials and we have

$$e_{\lambda_0}(x_1, \ldots, x_n) = \sum_{\rho \in R} f(x_{\rho_1}, \ldots x_{\rho_n})$$

where e_{λ_0} is the semiidempotent that corresponds to the Young table $T_0(\lambda)$

Now suppose we rename the variables $x_1, \ldots, x_n$ to reflect the partition λ of n. Since λ determines the columns and rows of the Young table $T_0(\lambda)$, we have

$$(x_1,\ldots,x_n) = (x_{11},\ldots,x_{1h_1},x_{21},\ldots,x_{2h_2},\ldots,x_{\ell 1}\ldots,x_{\ell h_\ell})$$

where x_{ij} corresponds to the position in the *ith* column and *jth* row of $T_0(\lambda)$ and so

$$f(x_1,\ldots,x_n) = f(x_{ij}) = s_{h_1}[x_{11},\ldots,x_{1h_1}]\cdots s_{h_\ell}[x_{\ell 1},\ldots,x_{\ell h_\ell}].$$

As noted above the columns of table $T_0(\lambda)$ are ordered from left to right, that is $h_1 \geq h_2 \geq \cdots \geq h_\ell$, thus we have h_1 rows in $T_0(\lambda)$. To each of these h_1 rows there corresponds a permutation group. The group R of row preserving permutations of $T_0(\lambda)$ is then the direct product of these h_1 permutation groups. That is for $\rho \in R$, we have $\rho = \rho^1\cdots\rho^{h_1}$ where ρ^i is a permutation of the *ith* row of table $T_0(\lambda)$. It is clear that ρ^i permutes only the variables $x_{1i},x_{2i},\ldots$ and so we can effectively compute $\rho\cdot f(x_{ij})$. Since x_{ij} are arbitrary indeterminants we can make the following substitutions

$$x_{11}= x_{21}=\cdots= x_{\ell 1}= y_1,\ x_{12}= x_{22}=\cdots= y_2,\ldots,x_{1h_1}=\cdots= y_{h_1}.$$

We then have for any $\rho \in R$

$$\begin{aligned}\rho\cdot f(x_{ij}) &= \rho\cdot s_{h_1}[x_{11},\ldots,x_{1h_1}]\cdots s_{h_\ell}[x_{\ell 1},\ldots,x_{\ell h_\ell}] \\ &= s_{h_1}[y_1,\ldots,y_{h_1}]\cdots s_{h_\ell}[y_1,\ldots,y_{h_\ell}],\ \text{for } x_{ij} = y_j.\end{aligned}$$

This is clear since we may write $\rho = \rho^1\cdots\rho^{h_1}$, where each ρ^i permutes only the variables $x_{1i},x_{2i},\ldots$ which were all assigned the same value y_i.

Lemma 3.3. Let λ be a partition of n, and let $D(\lambda)$ be the Young diagram determined by λ. Again, as above choose the Young table $T_0(\lambda)$ based on $D(\lambda)$. Rename the variables x_i so that $f(x_1,\ldots,x_n) = f(x_{ij})$, where $f(\vec{x}) \equiv \sum_{\sigma\in C}(-1)^{\sigma}\cdot\sigma$, and so that we have

$$f(x_{ij}) = s_{h_1}[x_{11},\ldots,x_{1h_1}]\cdots s_{h_\ell}[x_{\ell 1},\ldots,x_{\ell h_\ell}].$$

Also let e_{λ_0} be the corresponding semiidempotent of $T_0(\lambda)$ and let m_i denote the length of the *ith* row of $D(\lambda)$. Then $e_{\lambda_0}(x_{ij}) =$

$$m_1!\cdots m_{h_1}!\cdot s_{h_1}[y_1,\ldots y_{h_1}]\cdots s_{h_\ell}[y_1,\ldots,y_{h_\ell}], \text{ for } x_{ij} = y_j.$$

Proof.
$$\begin{aligned} e_{\lambda_0}(x_{ij}) &= \sum_{\rho\in R}\rho(f(x_{ij})) \\ &= \sum_{\rho\in R}\rho\cdot s_{h_1}[x_{11},\ldots,x_{1h_1}]\cdots s_{h_\ell}[x_{\ell 1},\ldots,x_{\ell h_\ell}] \\ &= \sum_{\rho\in R}s_{h_1}[y_1,\ldots,y_{h_1}]\cdots s_{h_\ell}[y_1,\ldots,y_{h_\ell}] \\ &= m_1!\cdots m_{h_1}!\cdot s_{h_1}[y_1,\ldots,y_{h_1}]\cdots s_{h_\ell}[y_1,\ldots,y_{h_\ell}].\blacksquare \end{aligned}$$

If we now let $T(\lambda)$ be an arbitrary table based on $D(\lambda)$ and let e_λ be the corresponding semiidempotent, then there exists $\tau \in S_n$ such that $e_\lambda = \tau e_{\lambda_0}\tau^{-1}$. That is we have

$$e_\lambda \equiv e_\lambda(x_1,\ldots,x_n) = \tau(e_{\lambda_0}(x_1,\ldots,x_n))\tau^{-1}$$

where the right hand side of this equation can be computed from e_{λ_0} by applying the multiplcation properties (1) and (2) to the factors τ and τ^{-1} respectively.

Let F be a field of characteristic zero, and consider λ a partition of n. We know that λ defines, by way of a Young diagram and table, an irreducible representation with degree d_λ. Also to λ corresponds a unique two-sided ideal I_λ in F_{S_n}, and d_λ is the dimension of each minimal one-sided ideal J_λ contained in I_λ. The dimension d_λ can be effectively computed using the "Hook-Fomula" $f^\lambda = n!/\prod_{i,j} h^\lambda_{ij}$ where h^λ_{ij} is the number of nodes in hook H^λ_{ij} of the Young diagram $D(\lambda)$.

We will now study the rate of growth of the degrees of some sequences of Young diagrams. In particular we will compare the rate of growth of d_λ, as $n \to \infty$, for partitions of n corresponding to rectangular diagrams. This rate of growth will then be compared with the rate of growth of co-dimensions of a P.I. algebra A.

Lemma 3.4. Let ℓ, k be integers with $2 \le \ell$, $\ell^2/2 < k$, and $n = \ell \cdot k$. Note that this implies $\ell < k$, and that $n \to \infty$ as $k \to \infty$. Let $\lambda \in \mathrm{Par}(n)$ with $\lambda = (k,\ldots k) = (k^\ell)$ or $\lambda = (\ell,\ldots,\ell) = (\ell^k)$ and let $d_\lambda = d_{\ell k}$ be the degree of χ_λ.

Then $$d_\lambda > A_\ell \cdot \left(\frac{1}{k}\right)^{(\ell^2-1)/2} \cdot \ell^{\ell k} \cdot \left[1 - \frac{(\ell^2-1)(2\ell^2+1)}{12\ell} \cdot \frac{1}{k}\right],$$

where $$A_\ell = (\ell - 1)(\ell - 2)^2 \cdots (2)^{\ell-2} \cdot \left(\frac{1}{(2\pi)^{1/2}}\right)^{\ell-1} \cdot \ell^{1/2}.$$

The proof of lemma 3.4 requires a technical lemma (lemma 3.5) which will be stated below without proof.

Lemma 3.5. Let k,a,b be real numbers such that $k > b \ge a \ge 0$, $a + b - 1 \ge 1$, and $k > (b-a)(a+b-1)/2$. Then

$$\frac{\Gamma(k+a)}{\Gamma(k+b)} \geq \left(\frac{1}{k}\right)^{b-a}\cdot\left[1-\frac{(b-a)(a+b-1)}{2k}\right],$$

where $\Gamma(z)$ is the Gamma function.

Proof of lemma 3.4. Since the partitions $(k)^{\ell}$ and $(\ell)^{k}$ are conjugate, their characters have the same degree. We choose to work with $\lambda = (k)^{\ell}$, whose Young diagram $D(\lambda)$ is an $\ell \times k$ rectangle. If we now fill in this diagram with its "hook numbers" we obtain

$k+\ell-1$		$\ell+1$	ℓ
$\vdots$	$\cdots$	$\vdots$	$\vdots$
$k+1$		3	2
k		2	1

It is clear that the hook numbers are the same on the diagonals of slope + 1 of this diagram and since $\ell < k$ we have

$$d_{\lambda} = d_{\ell k} = f^{\lambda} = \frac{n!}{\prod_{i,j} h^{\lambda}_{ij}}$$

$$= \frac{(k\ell)!}{1^1\cdot 2^2\cdots \ell^{\ell}\cdot(\ell+1)^{\ell}\cdots k^{\ell}\cdot(k+1)^{\ell-1}\cdots(k+\ell-2)^2\cdot(k+\ell-1)}$$

$$= \frac{(k\ell)!\cdot(\ell-1)\cdot(\ell-2)^2\cdots(2)^{\ell-2}\cdot(1)^{\ell-1}}{(k!)^{\ell}\cdot(k+1)^{\ell-1}\cdots(k+\ell-2)^2\cdot(k+\ell-1)}$$

$$= B_{\ell}\cdot\frac{(k\ell)!}{k!\cdot(k+1)!\cdots(k+\ell-1)!}$$

$$= B_{\ell}\cdot\frac{\Gamma(k\ell+1)}{\Gamma(k+1)\cdot\Gamma(k+2)\cdots\Gamma(k+\ell)}$$

where $B_{\ell} = (\ell-1)(\ell-2)^2\cdots(2)^{\ell-2}$.

By Gauss's multiplication formula,

$$\Gamma(mz) = \left(\frac{1}{(2\pi)^{1/2}}\right)^{m-1} \cdot m^{mz-1/2} \cdot \Gamma(z) \cdot (z + 1/m) \cdots \Gamma(z + \frac{m-1}{m}),$$

and the identity $\Gamma(n + 1) = n \cdot \Gamma(n)$ we have d_λ =

$$B_\ell \cdot \ell \cdot k \cdot \left(\frac{1}{(2\pi)^{1/2}}\right)^{\ell-1} \cdot (\ell^{\ell k-1/2}) \cdot \frac{\Gamma(k) \cdot \Gamma(k+\frac{1}{\ell}) \cdots \Gamma(k+\frac{\ell-1}{\ell})}{\Gamma(k+1) \cdot \Gamma(k+2) \cdots \Gamma(k+\ell)}$$

$$= A_\ell \cdot \ell^{\ell k} \cdot \frac{k \cdot \Gamma(k) \cdot \Gamma(k+\frac{1}{\ell}) \cdots \Gamma(k+\frac{\ell-1}{\ell})}{\Gamma(k+1) \cdot \Gamma(k+2) \cdots \Gamma(k+\ell)}$$

$$= A_\ell \cdot \ell^{\ell k} \cdot \prod_{j=1}^{\ell-1} \frac{\Gamma(k + \frac{j}{\ell})}{\Gamma(k+j+1)}, \text{ since } k \cdot \Gamma(k) = \Gamma(k+1).$$

We will now let $a = j/\ell$, and $b = j + 1$. It has already been noted that $k > \ell$ and since $j \in \{1,\ldots,\ell-1\}$ we have $k > \ell \geq j + 1$, thus $k > b > a > 0$. Also we have $a + b - 1 \geq 1$ since $a + b - 1 = j + 1 + (j/\ell) - 1 > j \geq 1$. Finally we have

$$\frac{(b - a)(a + b - 1)}{2} = \frac{j^2 + j + \frac{j}{\ell} - (\frac{j}{\ell})^2}{2}$$

$$< \frac{j^2 + j + 1}{2}, \text{ since } 0 < \frac{j}{\ell} < 1$$

$$\leq \frac{\ell^2 - \ell + 2}{2}, \text{ since } j \leq \ell - 1$$

$$\leq \frac{\ell^2}{2}, \text{ since by assumption } \ell \geq 2$$

$$\leq k, \text{ since by assumption } \frac{\ell^2}{2} \leq k.$$

We have thus met the conditions of lemma 3.5 and may therefore apply this lemma to each factor $\frac{\Gamma(k + \frac{j}{k})}{\Gamma(k+j+1)}$, to get

$$\frac{\Gamma(k + \frac{j}{\ell})}{\Gamma(k+j+1)} \geq \left(\frac{1}{k}\right)^{j+1-(j/\ell)} \cdot \left[1 - \frac{j^2 + j + \frac{j}{\ell} - (\frac{j}{\ell})^2}{2k}\right].$$

It is clear that, $0 < \frac{j^2 + j + \frac{j}{\ell} - (\frac{j}{\ell})^2}{2k} < 1$. Therefore

$$\prod_{j=1}^{\ell-1}\left[1 - \frac{j^2 + j + \frac{j}{\ell} - (\frac{j}{\ell})^2}{2k}\right] \geq 1 - \sum_{j=1}^{\ell-1}\frac{j^2 + j + \frac{j}{\ell} - (\frac{j}{\ell})^2}{2k}$$

$$= 1 - \frac{(\ell^2 - 1)(2\ell^2 + 1)}{12\ell}\cdot\frac{1}{k} .$$

The inequality follows from $\prod_{i=1}^{n}(1 - a_i) \geq 1 - \sum_{i=1}^{n} a_i$ whenever $0 < a_i < 1$, a fact which is easily shown by induction. The equality is due to the finite series formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}.$$

Also $\prod_{j=1}^{\ell-1}(\frac{1}{k})^{j+1-j/\ell} = (\frac{1}{k})^{\Sigma_{j=1}^{\ell-1}(j+1-j/\ell)} = (\frac{1}{k})^{(\ell^2-1)/2}$,

and so we have

$$d_\lambda = d_{\ell k} > A_\ell\cdot\ell^{\ell k}\cdot(\tfrac{1}{k})^{(\ell^2-1)/2}\cdot[1 - \frac{(\ell^2 - 1)(2\ell^2 + 1)}{12\ell}\cdot\frac{1}{k}].\blacksquare$$

Several authors have shown that there is an exponential bound on the co-dimensions c_n of a P.I. algebra A. In fact Regev was able to show that if an algebra A satifies an identity of degree d, then $c_n \leq \alpha^n$ where $\alpha \leq 3\cdot 4^{d-3}$. In a later paper, by Klein and Regev, this estimate was improved to $\alpha < 3\cdot(d^2-7d+16)$. A simpler proof of the exponential bound of the co-dimensions c_n was given by Latyshev, along with the improved estimate of $\alpha \leq (d-1)^2$. We will now state Latyshev's theorem without proof.

Theorem 3.6. Let A be an algebra satisfying a polynomial identity of degree d and let c_n be its codimensions, then

$$c_n \le \frac{1}{(d-1)!}\cdot(d-1)^{2n}.$$

We see in lemma 3.4 that the sequence $\{d_\lambda\}$, of the degree of irreducible representations corresponding to a particular sequence of partitions $\{\lambda\}$, grows faster than a particular exponential sequence. On the otherhand, theorem 3.6 implies that the sequence of co-dimensions $\{c_n\}$ is bounded by an exponential sequence. We will now show that there exists some partition λ of n such that $d_\lambda > c_n$.

Theorem 3.7. Let A be a P.I. algebra satisfying an identity of degree d, and let c_n be its co-dimensions. Let $\ell = (d-1)^2 + 1$, and let $k_0 = k_0(\ell)$ be a number such that for any $k > k_0$, $k/\ln(k) \ge (\ell^2-1)/2$. If we now let $K(d) = \max\{\ell^3/3, k_0\}$, then for any $k > K(d)$ we have $d_\lambda = d_{\ell k} > c_{\ell k}$.

Proof. If $d = 2$ then $c_n \le 1$ and there is nothing to prove. Assume $3 \le d$, so $5 \le \ell$, and let $k \ge K(d)$. Then we have $k \ge \ell^3/3$, so that

$$1 - \frac{(\ell^2-1)(2\ell^2+1)}{12\ell}\cdot\frac{1}{k} \ge \frac{1}{2}$$

and

$$d_{\ell k} \ge \left(\frac{A_\ell}{2}\right)\cdot\left(\frac{1}{k}\right)^{(\ell^2-1)/2}\cdot\ell^{\ell k}.$$

Since $\ell \ge 5$ we have $A_\ell > 2$ and so $d_{\ell k} > (1/k)^{(\ell^2-1)/2}\cdot\ell^{\ell k}$. By theorem 3.6, we have $(d-1)^{2n} > c_n$. Hence it will suffice for us to show, that $\left(\frac{1}{k}\right)^{(\ell^2-1)/2}\cdot(\ell^\ell)^k \ge (d-1)^{2\ell k}$, or equivalently that $\left(\frac{\ell}{(d-1)^2}\right)^{\ell k} \geq (k)^{(\ell^2-1)/2}$. If we now take the natural log on each side of the inequality we have

$$k\cdot\ln\left(\frac{\ell}{(d-1)^2}\right)^{\ell} \geq \frac{\ell^2-1}{2}\cdot\ln(k).$$

Now, since $\ell = (d-1)^2 + 1$ we have

$$(\ell/(d-1)^2)^{\ell} = [1 + (1/(d-1)^2)]^{(d-1)^2+1} > e \text{ , and}$$

$$\ln(\ell/(d-1)^2)^{\ell} > \ln(e) = 1, \text{ thus } k\cdot\ln(\ell/(d-1)^2)^{\ell} > k.$$

If we now recall that k_0 was to be chosen so that $k > k_0$ implied that $k/\ln(k) \geq (\ell^2-1)/2$, we then have

$$k\cdot\ln\left(\frac{\ell}{(d-1)^2}\right)^{\ell} > k \geq \frac{\ell^2-1}{2}\cdot\ln(k).$$

Finally we have

$$\left(\frac{\ell}{(d-1)^2}\right)^{\ell k} \geq (k)^{(\ell^2-1)/2}$$

as was to be shown.∎

We will now apply the previous results to construct actual identities in an arbitrary P.I. algebra A and in particular in $A \otimes_F B$ where A, B are two P.I. algebras. We must first show the existence of a (big) two-sided ideal inside the left ideal $Q_n \subseteq V_n$. Recall that V_n is the vector space of multilinear and homogeneous polynomials in $x_1,\ldots,x_n$ over a field F, and $Q_n = Q \cap V_n$ where Q is a T-ideal in F[X].

Lemma 3.8. Let F be a field and $Q \subseteq F[X]$ be a T-ideal. Let A be an arbitrary P.I. algebra and let the sequence $\{c_n\}$

be its co-dimensions. Assume that $\lambda \in Par(n)$ satisfies $d_\lambda > c_n$ and that the corresponding representation of F_{S_n} is irreducible. Then there exists $I_\lambda \subseteq Q_n$, where I_λ is the two-sided ideal in V_n that corresponds to the partition λ.

Proof. Let $J_\lambda \neq (0)$ be any minimal left ideal contained in I_λ. Since χ_λ is irreducible we have $\dim(J_\lambda) = d_\lambda$. If we assume that J_λ is not contained in Q_n, then $J_\lambda \cap Q_n = (0)$ and we have $d_\lambda = \dim(J_\lambda) \leq \dim(V_n/Q_n) = c_n$. This contradicts our assumption that $d_\lambda > c_n$, therefore we have $J_\lambda \subseteq Q_n$. Now since $J_\lambda \subseteq I_\lambda$ was an arbitrary minimal left ideal in I_λ, and I_λ is the sum of its minimal left ideals we have $I_\lambda \subseteq Q_n$.∎

We are now prepared to obtain our main result.

Theorem 3.9. Let F be a field of characteristic zero and let $Q \subseteq F[X]$ be a nonzero T-ideal. Assume there is a polynomial $f(\vec{x}) \subseteq Q$ of degree d with $f(\vec{x}) \neq 0$. Then there exists $n = n(d)$ and $\lambda \in Par(n)$ such that $Q_n \geq I_\lambda$ where $Q_n = Q \cap V_n$ and I_λ is the two-sided ideal in V_n which corresponds to the partition λ.

Proof. Let the sequence $\{c_n\}$ be the codimensions of Q. By theorem 3.6 we have $c_n \leq [1/(d-1)!](d-1)^{2n}$. Since char(F) = 0, the representation which corresponds to the partition λ is irreducible for every $\lambda \in Par(n)$. Let $\alpha = (d-1)^2$, $\ell = \alpha + 1$, and $k \geq K(\ell,\alpha) = K(d)$ as in theorem 3.7. We then have for $\lambda = (k^\ell)$ or (ℓ^k), that $d_\lambda = d_{\ell k} >$

$\alpha^{\ell k}$. Thus we conclude, by lemma 3.8, that there exists a two-sided ideal $I_\lambda \le Q_n$ for $n = \ell k$.■

A theorem by Amitsur states that if an algebra A satisfies a polynomial identity, then algebra A satisfies $s_\ell^k[\vec{x}] = 0$ for some ℓ and k. The next theorem gives an effective way of computing the values of ℓ and k.

Theorem 3.10. Let F be a field of characteristic zero and let A be an algebra satisfying an identity of degree d. Let $\ell = (d - 1)^2 + 1$ and $K = K(d) = \max\{\ell^3/3, k_0\}$ where k_0 is a number such that for any $k > k_0$ we have $k/\ln(k) \ge (\ell^2-1)/2$. Then for any $k \ge K$, the algebra A satisfies the identities $s_\ell^k[\vec{x}] = 0$ and $s_k^\ell[\vec{x}] = 0$. In particular, we can choose $k = \ell^4/4$.

Proof. Let $\ell = (d-1)^2 + 1$, $k \ge K(d)$, and $\lambda = (k^\ell)$ or $\lambda = (\ell^k)$. We may choose $k = \ell^4/4$, so that $k^{1/2} = \ell^2/2 > (\ell^2-1)/2$. Now since $k/\ln(k) > k^{1/2} > (\ell^2-1)/2$ and $\ell^4/4 > \ell^3/3$ we have $k = \ell^4/4 \ge K(d)$. By theorem 3.9, we have $Q_{\ell k} \ge I_\lambda$. Consider, for example, $\lambda = (k^\ell)$. Then $D(\lambda)$ is an $\ell \times k$ rectangle. Since I_λ is two sided, we can fill in $D(\lambda)$ arbitrarily with the numbers $1,\ldots,\ell k$, and the semiidempotent that corresponds to the resulting Young table will belong to I_λ, and hence to $Q_{\ell k}$. As was done earlier we choose

$$T_0(\lambda) = \begin{array}{|c|c|c|c|} \hline 1 & \ell + 1 & \cdots & \ell(k-1) + 1 \\ \vdots & \vdots & \cdots & \vdots \\ \ell & 2\ell & & \ell k \\ \hline \end{array}$$

and use lemma 3.3 to conclude that

$$(k!)^{\ell} \cdot s_{\ell}^{k}[\vec{x}] \in Q_{\ell k} .$$

Since char(F) = 0, we have

$$s_{\ell}^{k}[\vec{x}] \in Q_{\ell k} . \blacksquare$$

We will next show that the tensor product of two P.I. algebras is again a P.I. algebra. The following proof does not rely on the representation theory of the symmetric group S_n. It is, nevertheless, interesting to see the results of an earlier paper by Regev, in which values of k and ℓ were not explicitly determined. Also in the discussion many of the concepts mentioned above will be given more attention.

let A and B be two P.I. algebras over a field F. The elements $\{a_i \otimes b_i \mid a_i \in A, b_i \in B\}$ are linear generators of $A \otimes_F B$ the tensor product of algebras A and B. If $g(x_1, \ldots, x_n)$ is a multilinear polynomial in $x_1, \ldots, x_n$ with coefficients in F, then $g(\vec{x})$ is an identity for $A \otimes_F B$ if and only if for any sets $\{a_1, \ldots, a_n\} \subseteq A$ and $\{b_1, \ldots, b_n\} \subseteq B$ we have $g(a_1 \otimes b_1, \ldots, a_n \otimes b_n) = 0$.

let $Q \subseteq F[X]$ be the T-ideal of identities of a P.I. algebra A, and let $\{c_n\}$ be its sequence of co-dimensions. If we write $Q_n = Q \cap V_n$ then, by definition we have

$c_n = \dim_F V_n/Q_n$. Now the n! monomials $\{x_{\sigma_1}\cdots x_{\sigma_n} \mid \sigma \in S_n\}$ span V_n, hence they also generate V_n modulo Q_n. Since the dimension of V_n modulo Q_n is c_n there exists c_n monomials $M_1(x_1,\ldots,x_n), \ldots, M_{c_n}(x_1,\ldots,x_n)$ which form a basis for V_n over Q_n. Thus we have for any monomial $x_{\sigma_1}\cdots x_{\sigma_n} \in V_n$ there exists $\phi_i(\sigma) \in F$, $i = 1,\ldots,c_n$ such that

$$x_{\sigma_1}\cdots x_{\sigma_n} \equiv \sum_{i=1}^{c_n} \phi_i(\sigma) M_i(x_1,\ldots,x_n) \quad (\text{modulo } Q_n).$$

Since $Q_n \subseteq Q$ its elements are identities for algebra A. It follows therefore that for any substitution $a_1,\ldots,a_n \in A$ we have the equality

$$a_{\sigma_1}\cdots a_{\sigma_n} = \sum_{i=1}^{c_n} \phi_i(\sigma) M_i(a_1,\ldots,a_n), \quad \sigma \in S_n.$$

With these remarks we are now ready to prove the following theorem.

Theorem 3.11. Let A and B be two P.I. algebras over a field F, then $A\otimes_F B$ is a P.I. algebra.

Proof. Let $\{a_n\}$ be the sequence of co-dimensions of A, and $\{b_n\}$ that of B. By theorem 3.6 there exist real positive numbers α and β such that for all n, $a_n \le \alpha^n$ and $b_n \le \beta^n$. It is well known that there exists n such that $\alpha^n \cdot \beta^n < n!$, hence $a_n \cdot b_n < n!$. We will prove that for this n, $A\otimes_F B$ satisfies a non trivial multilinear homogeneous identity of degree n.

Let us denote the T-ideals for the P.I. algebras A and

B by Q^A and Q^B respectively. And denote the complement of $(V_n \cap Q^A)$ by A_n and the complement of $(V_n \cap Q^B)$ by B_n.

Now let $M_1(x_1,\dots,x_n),\dots,M_{a_n}(x_1,\dots,x_n)$ be a_n monomials in variables $x_1,\dots,x_n$ which generate A_n. Let $\phi_i(\sigma) \in F$ for $1 \le i \le a_n$, and $\sigma \in S_n$, such that for any substitution $a_1,\dots,a_n \in A$ and $\sigma \in S_n$ we have

$$a_{\sigma_1}\cdots a_{\sigma_n} = \sum_{i=1}^{a_n} \phi_i(\sigma) M_i(a_1,\dots,a_n).$$

This is clearly possible by the remark preceeding this theorem.

Similarly, let $N_1(x_1,\dots,x_n),\dots,N_{b_n}(x_1,\dots,x_n)$ be b_n monomials in variables $x_1,\dots,x_n$ which generate B_n. For $1 \le j \le b_n$, let $\psi_j(\sigma) \in F$ with $\sigma \in S_n$, be coefficients in the field F such that for any substitution $b_1,\dots,b_n \in B$ and $\sigma \in S_n$ we have $b_{\sigma_1}\cdots b_{\sigma_n} = \sum_{j=1}^{b_n} \psi_j(\sigma) N_j(b_1,\dots,b_n)$.

Let $g(x_1,\dots,x_n) = \sum_{\sigma\in S_n} \gamma_\sigma \cdot x_{\sigma_1}\cdots x_{\sigma_n}$ be any multilinear polynomial with arbitrary coefficients $\{\gamma_\sigma\} \subseteq F$. Let $\{a_1,\dots,a_n\} \subseteq A$ and $\{b_1,\dots,b_n\} \subseteq B$, we may then write

$$g(a_1\otimes b_1,\dots,a_n\otimes b_n)$$
$$= \sum_{\sigma\in S_n} \gamma_\sigma (a_{\sigma_1}\otimes b_{\sigma_1})\cdots(a_{\sigma_n}\otimes b_{\sigma_n})$$
$$= \sum_{\sigma\in S_n} \gamma_\sigma (a_{\sigma_1}\cdots a_{\sigma_n}) \otimes (b_{\sigma_n}\cdots b_{\sigma_n})$$
$$= \sum_{\sigma\in S_n} \gamma_\sigma \Big(\sum_{i=1}^{a_n} \phi_i(\sigma) M_i(a_1,\dots,a_n)\Big)\Big(\sum_{j=1}^{b_n} \psi_j(\sigma) N_j(b_1,\dots,b_n)\Big)$$

$$= \sum_{i=1}^{a_n} \sum_{j=1}^{b_n} \Big(\sum_{\sigma\in S_n} \phi_i(\sigma)\psi_j(\sigma)\gamma_\sigma \Big)\cdot M_i(a_1,\ldots,a_n)\otimes N_j(b_1,\ldots,b_n).$$

Now consider the system of homogeneous equations $\sum_{\sigma\in S_n} \phi_i(\sigma)\psi_j(\sigma)\gamma_\sigma = 0$ for all i and j. This is a set of $a_n \cdot b_n$ equations with coefficients $\phi_i(\sigma)\psi_j(\sigma)$ and n! unknown indeterminants γ_σ's. Since n was chosen so that $a_n \cdot b_n < n!$, there exists a non trivial solution $\{\gamma_\sigma\}$ for this homogeneous system. If we now chose one such non trivial solution $\{\gamma_\sigma\}$ and define $g(x_1,\ldots,x_n) = \sum_{\sigma\in S_n} \gamma_\sigma \cdot x_{\sigma_1}\cdots x_{\sigma_n}$, then it is clear from above that we have $g(a_1\otimes b_1,\ldots,a_n\otimes b_n) =$

$$\sum_{i=1}^{a_n} \sum_{j=1}^{b_n} \Big(\sum_{\sigma\in S_n} \phi_i(\sigma)\psi_j(\sigma)\gamma_\sigma \Big)\cdot M_i(a_1,\ldots,a_n) \otimes N_j(b_1,\ldots,b_n) = 0.$$

Therefore, by the remark preceding this theorem, we have $g(x_1,\ldots,x_n) = \sum_{\sigma\in S_n} \gamma_\sigma \cdot x_{\sigma_1}\cdots x_{\sigma_n}$ is a non trivial identity for $A\otimes_F B$ and thus $A\otimes_F B$ is a P.I. algebra.∎

While theorem 3.11 states the existence of a polynomial identity for the tensor product of two P.I. algebras, it does not explicitly define such an identity. The following theorem will enable us to construct such an identity.

Theorem 3.12. Let A and B be two P.I. algebras, and let a_n and b_n be their co-dimensions. Let c_n be the co-dimensions of $A\otimes_F B$, then $c_n \le a_n \cdot b_n$.

Proof. Let $C = A\otimes_F B$ and let Q^C denote the T-ideal for the P.I. algebra $A\otimes_F B$. Let C_n denote the complement of $(V_n \cap Q^c)$. Now let $g(x_1,\ldots,x_n) = \sum_{\sigma\in S_n} \gamma_\sigma \cdot x_{\sigma_1}\cdots x_{\sigma_n}$ be any multilinear polynomial in C_n, then as in the above argument, we have for all $\{a_1,\ldots,a_n\} \subseteq A$ and $\{b_1,\ldots,b_n\} \subseteq B$

$$g(a_1\otimes b_1,\ldots,a_n\otimes b_n) = \sum_{i=1}^{a_n} \sum_{j=1}^{b_n} \Gamma_{ij} M_i(a_1,\ldots,a_n) \otimes N_j(b_1,\ldots,b_n)$$

where $\Gamma_{ij} = \sum_{\sigma\in S_n} \phi_i(\sigma)\psi_j(\sigma)\gamma_\sigma$. This implies that

$$g(x_1,\ldots,x_n) = \sum_{i=1}^{a_n} \sum_{j=1}^{b_n} \Gamma_{ij} M_i(x_1,\ldots,x_n) \otimes N_j(x_1,\ldots,x_n) + q$$

where $q \in Q^c$. Thus, $\dim C_n \le \dim A_n \cdot \dim B_n$ or $c_n \le a_n \cdot b_n$ as was to be shown.■

We have finally, that if the degrees of some identities of algebras A and B are given, say degree d_A for the algebra A and degree d_B for the algebra B, then by theorem 3.6 $c_n \le a_n b_n \le [(d_A-1)(d_B-1)]^{2n}$ where c_n is the co-dimensions of $A\otimes_F B$. Theorem 3.10 can now be used to construct an explicit identity for $A\otimes_F B$. If char(F) = 0, we may then choose $\ell = [(d_A-1)(d_B-1)]^2 + 1$ and $k = \ell^4/4$ to conclude as before that $s_\ell^k[\vec{x}] = 0$ is an identity for $A\otimes_F B$.

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