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Variations of the ovation model

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VARIATIONS OF THE OVATION MODEL

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ABSTRACT

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In nature, many dynamical systems exhibit the behavior of self-organization. Self-organization can manifest itself in many ways, one of them being the synchroneity of the individual components of the system. The "Ovation model" is a mathematical model of systems that exhibit synchroneity. The model consists of individual components which are influenced by neighboring components as the system evolves over time. The size and topology of the neighborhoods affect the system's evolution.

This thesis will examine variations of the Ovation model by altering the concept of neighborhood. Analytical results and computer experiments will be presented.

Also provided is computer-generated experimental evidence for a standing conjecture, the Road-Coloring Conjecture (RCC). Additional evidence is provided for two, more general, conjectures: Probabilistic RCC and Strong RCC.

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TABLE OF CONTENTS

Chapter

CHAPTER 1

INTRODUCTION

This thesis addresses variations of a mathematical model known as the *Ovation model.* The Ovation model was introduced in *Consensus in Small and Large Audiences* (Kanevsky, Garcia, and Naroditsky 1992) in an attempt to understand the dynamics of self-organizing systems. The systems modelled consist of individual entities which are influenced by neighboring entities. The Ovation model's application domain includes both biological and physical systems.

A distant relative of the Ovation model turns out to be an outstanding problem in graph theory, conjectured in 1977. This conjecture is known as the *road-coloring conjecture* (Adler, Goodwyn, and Weiss 1977), and it will be described in chapter 4. Compelling experimental evidence was presented in *Experimental Investigation of the Road-Coloring Conjecture* (Gocka, Kirchherr, and Schmeichel 1994) that implies that not only is the road-coloring conjecture true but also a stronger result is the case.

The unifying theme throughout the thesis is the investigation of how alterations of the Ovation model's concept of neighborhood affect the probability of reaching unison or the expected time to unison.

Chapter Two reviews the Ovation model and some of the results achieved in

Consensus in Small and Large Audiences. An extension to the theoretical results for the discrete case is presented. An experimental investigation, using computer simulation, of the Ovation model with an audience whose members reside on fractional dimensional lattice is also provided.

Chapter Three analyzes a variant of Ovation model presented in *Consensus with Probability One on a Digraph* (Kirchherr, Naroditsky, and Schmeichel 1992). The results of this paper are presented using the tools of graph theory. An alternate proof of one of the paper's results is given using Perron-Frobenius theorem from matrix theory.

Chapter Four reviews the results from *Experimental Investigation of the Road-Coloring Conjecture.* The rationale for the strengthening of the conjecture is discussed.

The appendix establishes the Perron-Frobenius theorem. Results from this appendix are used in chapters 2 and 3.

New results in this thesis are the following:

1. Evidence of two new conjectures related to the road-coloring conjecture. These results were first presented in *Experimental Investigation of the Road-Coloring Conjecture.*

2. Discrete solution of the Ovation model in one dimension when the neighborhood size is N-1 (chapter 2, section 3) .

3. Experimental results on expected time to absorption on a neighborhood of fractional dimension using percolation clusters.

4. Matrix representation of the road-coloring conjecture (chapter 4, section 5).

5. Description of the "worst-case" digraph observed with respect to the strong road-coloring conjecture (chapter 4, section 3)

1.1 Definitions and Notation

The definitions in this thesis are derived from (Aho, Hopcroft, and Ullman 1974) and (Horn and Johnson 1985).

1.1.1 Graph Theory

A graph Γ is a finite, nonempty, collection of "vertices" (or nodes), denoted by $V(\Gamma)$, and a collection of pairs of vertices called "edges," which will be denoted by $E(\Gamma)$. The elements of $V(\Gamma)$ will be denoted by v_i, where i is an integer from 1 to $|V(\Gamma)|$. If the collection of edges consists of ordered pairs, the graph is a directed graph, or a "digraph" for short. This thesis is only concerned with digraphs.

A vertex v_j is "adjacent" to v_i, if (v_i, v_j) is an element of $E(\Gamma)$. The number of adjacent vertices of v_i is the "outdegree" of v_i.

An "adjacency" matrix, A, for a digraph Γ is defined as $a_{ij} = 1$, if v_i is adjacent to v_j. Otherwise, $a_{ij} = 0$.

A *path* P from v_{i_1} to v_{i_n}. in a digraph Γ is a sequence of edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{n-1}}, v_{i_n})$. The length of a path is the number of edges in the sequence. This definition of path implies that all paths are of finite length. A *closed path*, is a path that starts and ends on the same vertex. A *cycle* is a closed path in which no vertex appears more than once in the sequence, with exception of the vertex that starts and ends the sequence. The term cycle will be used interchangeably with *simple directed cycle.* A cycle of length 1 is called a *loop.*

A digraph Γ is said to be *strongly connected*, if there exists a path from v_i to v_j for every pair of integers i, j with $1 \leq i, j \leq |V(\Gamma)|$. The *period* of a digraph is the greatest common divisor, or gcd, of the set of lengths of all of the cycles in the graph. A graph of period 1 is said to be *aperiodic.*

The *outdegree* of a vertex v, denoted by $d^+(v)$, is the number of edges in $E(\Gamma)$ in which v is the first vertex in the ordered pair representing the edge. That is, the number of vertices that have an edge directed from v.

The following is a standard definition in Number Theory (Gilbert and Gilbert 1988):

For integers d and a, d *divides* a, denoted by $d|a$, if there exists an integer q such that $d \cdot q = a$. Integer d is the *greatest common divisor*, or gcd, of integers a and b, if

1. d is positive, and

2. $d|a$ and $d|b$, and

3. $c|a$ and $c|b \Rightarrow c|d$, where c is an integer.

Denote J as the set of all integers.

1.1.2 Matrix Theory

Denote by $M_{n,r}$ the set of n-by-r matrices over the complex field. If $n = r$, abbreviate this to M_n. Let $\mathcal{C}^n$ be the set of n-dimensional vectors over the field of complex numbers, and $\mathcal{R}^n$ be the set of n-dimensional vectors over the field of real numbers. Let $A, B \in M_{n,r}$. The symbol 0 can mean the scalar, the zero vector, or the zero matrix. The meaning will be evident from the context.

A function $\|\cdot\| : M_n \longrightarrow \mathcal{R}$ is a *matrix norm* if for all $A, B \in M_n$ it satisfies the following axioms:

1. $\|A\| \geq 0$

2. $\|A\| = 0$ if and only if $A = 0$

3. $\|cA\| = |c|\|A\|$ for all complex scalars c

4. $\|A + B\| \leq \|A\| + \|B\|$

5. $\|AB\| \leq \|A\|\|B\|$ submultiplicative property.

A function $\|\cdot\| : \mathcal{C}_n \longrightarrow \mathcal{R}$ is a *vector norm* if it satisfies the axioms (1) through (4), where A and B are considered vectors.

The *maximum column sum matrix norm* $\|\cdot\|_1$ is defined on M_n by

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

The *Euclidean norm* $\|\cdot\|_2$ is defined on M_n by

$$\|A\|_2 \equiv \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

The *maximum row sum matrix norm* $\|\cdot\|_\infty$ is defined on M_n by

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

The adjacency matrix A of a graph Γ has *property SC* , if and only if Γ is strongly connected.

A matrix A is said to be *reducible*, if $n = 1$ and $A = 0$, or if $n \ge 2$ and there is a permutation matrix $P \in M_n$ for which there is some integer r with $1 \le r \le n - 1$ such that

$$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where $B \in M_r$, $D \in M_{n-r}$, $C \in M_{r,n-r}$, and $0 \in M_{n-r,r}$. A matrix $A \in M_n$ is *irreducible* if it is not reducible.

Definition (Horn and Johnson 1985) 1.1 *Let $A, B \in M_{n,r}$. Then*

1. *$A \ge 0$ if all $a_{ij} \ge 0$*

2. *$A > 0$ if all $a_{ij} > 0$*

3. *$A \ge B$ if $A - B \ge 0$*

4. $A > B$ *if* $A - B > 0$

The relations $\leq$ *and* $<$ *are defined in a similar fashion. If* $A \geq 0$*, we call* A *a nonnegative matrix, and if* $A > 0$*, we call* A *a positive matrix. Define* $|A| \equiv [|a_{ij}|]$*. Note the* $|\cdot|$ *also refers to the cardinality of a set, if the symbol for a set is between the vertical bars. This will always be clear by the context.*

The *spectral radius* $\rho(A)$ of a matrix $A \in M_n$ is

$$\rho(A) \equiv \max\{|\lambda| : \lambda \text{ is an eigenvalue of A}\}$$

.

CHAPTER 2

THE OVATION MODEL

The process of self-organization is observable in many dynamical systems in the natural and social sciences. One model of self-organization can be described of as follows: A system is considered as a collection of individual entities, each entity having the control over the value of some attribute. The value of an individual's attribute can evolve over time. The system self-organizes, if the individuals, over time, all select a single attribute value. Of scientific interest is how and at what rate the choice of the common attribute value is communicated among the individuals.

The Ovation model (Kanevsky, Garcia, and Naroditsky 1992) is a mathematical idealization of the above model of self-organization. The model was inspired when V. Kanevsky observed the rate at which audience members of a communist party meeting synchronized their clapping. Hence the model was called the *Ovation model.*

The Ovation model consists of an audience of individuals. An attribute with a finite number of values is associated with each individual. For our purposes, the attribute will be *color*. Thus, each individual is associated with a particular color (e.g. red, blue, etc.), selected from a finite set of colors.

Initially, each individual will be assigned a color at random. This is the *initial distribution* of colors. At each time step, the individual is allowed to choose a color. This choice is a function of the color values of a set of individuals at the prior time step. This set of individuals is called the *neighborhood* of the individual. An individual will choose a color with probability equal to the proportion of its neighbors choosing this color at the prior time step.

The members of a neighborhood must be connected in some topological sense. Also, each neighborhood must intersect another neighborhood. The number of members of a neighborhood, and the neighborhood's topology can vary.

We are interested in the evolution of the audience to a monochromatic state of *unison*. Other terms used for unison are *consensus* or *absorption*. From a mathematical point of view, we will look at the probablity of reaching unison and the expected time to unison.

2.1 Mathematical Description

This section is a review of some of the results from Kanevsky, Garcia, and Naroditsky (1992). We will start off by providing a mathematical formalism for the Ovation model.

Let $A = \{a_1, \ldots, a_N\}$ be an arbitrary set, with $|A| = N$. Each element of this set is considered an individual member of audience A. A neighborhood $U_a \subseteq A$ of each $a \in A$ is defined such that $|U_a| = n$ for all $a \in A$. The definition of the actual members of U_a depends on which variant of the model that we are using. Each a_i

takes one of the m colors from the set $\Lambda = \{\lambda_0, \ldots, \lambda_{m-1}\}$.

The dynamics of the system is stochastic, and is Markovian in nature. Each a_i assumes the color λ_j at time step $t + \Delta t$ with a probability proportional to the number of neighbors with this color at time step t. Let the random variable $\lambda(a_i, t)$ be the color of a_i at time t. The dynamics of $\lambda(a_i, t)$ is described by the conditional probability

$$\Pr(\lambda(a, t + \Delta t) = \lambda_j | \lambda(a_i, t), i = 1, \ldots, N) = \frac{\alpha(j, a, t)}{|U_a|} = \frac{\alpha(j, a, t)}{n}$$

where $\alpha(j, a, t)$ is the number of individuals in U_a selecting color λ_j at time step t. If the audience reaches a state where it is monochromatic, that state is absorbing. There are m absorbing states.

In all of the variations of the model explored in Kanevsky, Garcia, and Naroditsky (1992), $a \in U_a$. Therefore, a state of absorption is reached with probability 1. In the language of chapter 3, each a can be thought of as a vertex of a strongly connected graph of period 1. Note that the period is 1, since every a has itself as a neighbor, implying every vertex has a loop.

2.2 Neighborhood size $= N$

Let the neighborhood be the entire audience, and set the number of colors to two. In Kanevsky, Garcia, and Naroditsky (1992), this case was handled as follows:

From the set of colors $\Lambda = \{\lambda_0, \lambda_1\}$, assign each color a number as follows:

$\lambda_i = i$. Let

$$x(t) = \sum_{k=1}^{N} \lambda(a_k)$$

where $t = 0, 1, 2, \ldots,$ represents the discrete time steps during the evolution of the process. Note that the function $x(t)$ is simply a count of the members of the audience set to color λ_1 at time step t. Also note that the range of $x(t)$ is $\{0, 1, 2, \ldots, N\}$ and that $x(t) = 0$ and $x(t) = N$ are the two absorbing states.

Since the neighborhood is the entire audience, the probability that a given individual a_i will transition to λ_1 at time step t is $\frac{x(t)}{N}$. The probability transistion rule for x follows the binomial distribution (DeGroot 1975):

$$\Pr(x(t+1)) = j | x(t) = i) = \binom{N}{j} (i/N)^j (1 - i/N)^{N-j}$$

Thus, the process is described by a Markov Chain, with states $x(t) = \{0, 1, 2, \ldots, N\}$. The matrix P representing the chain, has entries

$$p_{ij} = \Pr(x(t+1)) = j | x(t) = i)$$

and is of the form:

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ q_1 & & Q & & q_2 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

where $P \in M_{N+1}$, q_1 and q_2 are $N-1$ dimensional column vectors, and $Q \in M_{N-1}$.

From the theory of Markov Chains (Bharucha-Reid 1960), P^k is the matrix of transition probabilities for time step k. The form of the matrix P^k is the same as the form for P. That is, there is a matrix $Q^{(k)}$ located in the same position as Q. Because of the zeros located in rows 1 and $N+1$ of P, $Q^k = Q^{(k)}$.

When the process reaches a state of absorption, say, at time step τ, then Q^k will equal the zero matrix.

Now we want to compute the expected time to absorption. Let p_k denote the probability of absorption at time step k starting from initial state $x(0) = i$. Note that the probability ξ_k of *not* being in an absorbing state at time k, starting from initial state $x(0) = i$, is

$$\xi_k = 1 - p_k = \sum_{j=1}^{N-1} q_{ij}^{(k)}$$

where $q_{ij}^{(k)}$ is the element from row i and column j of Q^k. In other words, ξ_k is the sum of row i of Q^k.

The expected time to absorption is:

$$E(t) = \sum_{k=0}^{\infty} k \cdot p_k$$

However, note that an argument by induction shows that $E(t)$ can also be expressed as

$$E(t) = \sum_{k=0}^{\infty} \Pr(t \geq k)$$

This is easily seen by observing:

$$\begin{array}{rcl} \Pr(t \geq 1) & = & p_1 + \; p_2 + \cdots + \; p_k + \cdots \\ \Pr(t \geq 2) & = & \qquad p_2 + \cdots + \; p_k + \cdots \\ \vdots & & \\ \Pr(t \geq k) & = & \qquad\qquad\qquad p_k + \cdots \\ \vdots & & \end{array}$$

and then summing up the equations.

Notice that the negation of $\Pr(t < k)$ is ξ_k. Therefore,

$$\Pr(t \geq k) \quad = \quad 1 - \Pr(t < k)$$

$$= 1 - (1 - \xi_k)$$

$$= \xi_k$$

So,

$$E(t) = \sum_{k=0}^{\infty} \xi_k$$

Let

$$V = I + Q + Q^2 + Q^3 + \cdots$$

and note that starting from initial state $x(0) = i$,

$$E(t) = \sum_{j=1}^{N-1} [V]_{ij}$$

To show that the series representing V is well defined, note that $\|Q\|_\infty < 1$ because each row in the original transition matrix P has a nonzero entry in columns 1 and $N + 1$, and that the sum of the rows of P is 1.

Now we digress to obtain a solution for the series. First we proof a theorem (Horn and Johnson 1985).

Theorem 2.1 *Let $A \in M_n$ be such that $\|I - A\|_\infty < 1$. Then $A^{-1} = \sum_{k=0}^{\infty}(I - A)^k$.*

Proof First note that if $\|I - A\|_\infty < 1$, then $\sum_{k=0}^{\infty} \|I - A\|_\infty^k$ is a convergent geometric series. By definition of $\|\cdot\|_\infty$ and the submultiplicative property of matrix norms which implies $\|A^k\| \leq \|A\|^k$, we get the fact that each element of $(I - A)^k$ is less than or equal to

$$\|(I - A)^k\|_\infty \leq \|I - A\|_\infty^k$$

This implies that

$$\sum_{k=0}^{\infty}(I - A)^k$$

converges. Now from the following identities:

$$A\sum_{k=0}^{N}(I-A)^k = [I-(I-A)]\sum_{k=0}^{N}(I-A)^k = I-(I-A)^{N+1}$$

and the fact that $\rho(I-A) \leq \|I-A\| < 1$ by theorem A.5, we get

$$I-(I-A)^{N+1} \to I$$

as $N \to \infty$ by theorem A.7. □

By substituting Q for $I - A$ in that above theorem, we get:

$$(I-Q)^{-1} = \sum_{k=0}^{\infty} Q^k$$

Thus, the expected time to unison is

$$E(t) = \sum_{j=1}^{N-1}[(I-Q)^{-1}]_{ij}$$

when $x(0) = i$.

2.3 Neighborhood size $= N - 1$

We now provide a generalization of the result in section 2.2.

We will make $|U_a| = N - 1$, for all $a \in A$, where N is an even integer. Now whenever $|U_a| < N$, we must explicitly define a neighborhood. Let the audience of individuals reside on a 1-dimensional torus (circle). For all $a \in A$, each neighborhood U_a consists of a and $a's$ $N/2$ closest neighbors on the left and $a's$ $N/2$ closest neighbors on the right. Thus, the only $b \in A$ such that $b \notin U_a$ is the individual on the opposite side of the torus, so to speak.

Define $\Lambda = \{0, 1\}$, $\lambda(a, t)$, and $x(t)$ as in section 2.1.

Theorem 2.2 *Let* $p = x(t)$, $t = 0, 1, 2, \ldots$. *There are*

$$\begin{array}{lll} N - p & \textit{neighborhoods } \tilde{U} \textit{ with} & \sum_{a_i \in \tilde{U}} \lambda(a_i, t) = p \\ & \textit{and} & \\ p & \textit{neighborhoods } \hat{U} \textit{ with} & \sum_{a_i \in \hat{U}} \lambda(a_i, t) = p - 1 \end{array}$$

Proof First note that there is a one-to-one correspondence between each neighborhood U_a and the single individual in the complement of U_a. Denote that individual by $C(U_a)$.

If $\lambda(C(U_a)) = 0$, then U_a must possess all of the $a_i \in A$ such that $\lambda(a_i, t) = 1$. There are p such a_i. Since there are $N - p$ individuals b such that $\lambda(b) = 0$, there are $N - p$ neighborhoods, each with the color "1" occurring p times.

If $\lambda(C(U_a)) = 1$, then color "1" occurs $p - 1$ times in U_a. Since there are p individuals b such that $\lambda(b, t) = 1$, there are p neighborhoods, each with the color "1" occurring $p - 1$ times. □

Now we want to obtain the relation

$$p_{ij} = \Pr(x(t + 1) = j | x(t) = i)$$

So, assume that $x(t) = i$. Recall that $x(t) = \sum_{a \in A} \lambda(a, t)$ and note that for a given time $t = T$ there are exactly N $\lambda(a, T)$ random variables—one for each $a \in A$. By theorem 2.2, for exactly i of these random variables, at time $t + 1$, we have the following conditional probability:

$$\Pr_{x(t)=i,1}(\lambda(a,t+1)=1|\sum_{a\in A}\lambda(a,t)=x(t)=i)=\frac{i-1}{N-1}$$

The probability distribution for $\Pr_{x(t)=i,1}$ is $\{1-\frac{i-1}{N-1},\frac{i-1}{N-1}\}$. Also by theorem 2.2, the remaining $N-i$ random variables, at time $t+1$, have the following conditional probability:

$$\Pr_{x(t)=i,2}(\lambda(a,t+1)=1|\sum_{a\in A}\lambda(a,t)=x(t)=i)=\frac{i}{N-1}$$

The probability distribution for $\Pr_{x(t)=i,2}$ is $\{1-\frac{i}{N-1},\frac{i}{N-1}\}$. Note that both distributions $\Pr_{x(t)=i,1}$ and $\Pr_{x(t)=i,2}$ are independent of time t.

Finally note that $x(t)$ is the sum of N independent random variables $\lambda(a,t)$. From probability theory (Bharucha-Reid 1960) the conditional distribution of such a sum is expressed in the form of a *convolution*:

$$p_{ij}=\Pr(x(t+1))=j|x(t)=i)=\left([\Pr_{x(t)=i,1}]^{i^*}*[\Pr_{x(t)=i,2}]^{(N-i)^*}\right)(j)$$

where $[p]^{i^*}$ is the i-fold convolution of the probability p, and $*$ is the convolution operator between two distributions. Note that this form is legitimate, because the convolution operation is associative (Feller 1966). The results from the prior section can now be employed to obtain the expectation.

2.4 Computer Investigation

In Kanevsky, Garcia, and Naroditsky (1992), computer simulations of the Ovation model were run on one- and two-dimensional audiences. In this section we present results of a computer simulation run on what is called a *percolation cluster*, although no attempt is made to interpret the results. A percolation cluster has a fractional dimension of 1.896 (Gould and Tobochnik 1988).

To understand what a percolation cluster is, imagine a two-dimensional lattice. The size of the lattice is L, which is the number of sites on the side of the lattice. For example, a chess board can be considered a lattice of size 8. A square on the chess board corresponds to a site of the lattice. A random number between 0 and 1 is generated for each site on the lattice. A site is considered *occupied* if the site's random number is less than a threshold value p. Of course, if $p = 1$, every site will be occupied, and if $p = 0$, none of the sites will be occupied.

Once each site's occupancy has been determined, imagine someone trying to walk across the lattice, from left to right, or bottom to top. This person must only step on occupied sites, and can move left, right, up or down one square. Diagonal moves are illegal. If the person can move all the way across the board from left to right, or bottom to top, a *spanning cluster* is said to exist. Again, it is obvious that such a spanning cluster exists for $p = 1$, and that one does not exist for $p = 0$.

It is known that as $L \to \infty$, there is a critical value $p = p_c$ such that if $p \geq p_c$, a spanning cluster exists, and if $p < p_c$ a spanning cluster does not exist (Gould and Tobochnik 1988). Call this spanning cluster a percolation cluster. The percolation cluster has a fractional dimension of 1.896. The value of $p_c = 0.5927$.

The Ovation model on a percolation cluster is defined as follows: Each individual resides on a occupied site. The neighborhood for the individual consists of the individual's site and the nearest site to the left, right, above and below the individual's site. If these nearest site's are not occupied, they are not included in the neighborhood. Therefore, an individual can have from 1 to 5 neighbors. Also note that the lattice will be considered a two-dimensional torus where sites on one edge of the lattice are adjacent to edges on the opposite side.

To simulate the Ovation model on a percolation cluster, with the two colors red and blue, the following algorithm was used. A trial consists of the following steps:

1. Assign each site a random number. The site is occupied if its associated random number is less than or equal to p_c.

2. Determine the size of the neighborhood for each site and store it.

3. Initialize each occupied site with a color.

4. At time step $t + 1$, each site changes to color red with probability q equal to the number of the site's neighbors with color red, at time step t, divided by the size of the site's neighborhood. The site changes to color blue with probability $1 - q$.

5. Repeat step 3 until stability is reached. Stability is defined as having all of the sites remain the same color for 10 iterations.

This simulation does not determine if a spanning cluster is actually produced during step 1. However, given the value of p_c, the hope is that on average a spanning cluster will be produced.

Each iteration of the algorithm is considered a trial. The algorithm was run for different audience sizes with 100 trials each. The results of this simulation are in table 2.1.

Table 2.1: Mean Time to Absorption

Audience Size (N)	Average Time to Unison
25	27
100	179
225	776
400	1,502
625	2,881
900	5,610
1,225	7,010
1,600	10,826

The average time to absorption is roughly $N^{3/2}$, where N is the audience size. The fit is pretty good for $\frac{1}{5}N^{3/2}$ for neighborhood sizes up to 900. This data matches up with expectations based on the 1-dimensional results (average time to absorption goes as N^2) and the 2-dimensional results (average time to absorption goes as N).

CHAPTER 3

CONSENSUS ON A DIGRAPH

The paper *Consensus with Probability One on a Digraph* (Kirchherr, Naroditsky, and Schmeichel 1992), provides a connection between the Ovation problem and the road-coloring problem. The paper solves the probabilistic convergence issue for a certain type of Ovation problem. This type of Ovation problem is characterized by placing the audience on a digraph. That is, each individual resides on a vertex of a directed graph with the properties that the graph is strongly connected and aperiodic. The neighborhood of an individual consists of all individuals that possess an edge directed toward the individual in question. The problems in this paper are an interesting extension of the Ovation problem's assumptions regarding neighborhood geometry.

Neighborhoods, as defined above, do not seem to have a definable dimension. Any vertex can be a neighbor of any other vertex, as long as the assumptions of strong connectivity and aperiodicity are maintained. Thus, neighbors do not have to be "close," in the sense of any metric. If individual A has individual B as a neighbor, individual B may not have individual A as a neighbor. Also, the cardinality of neighborhoods is not necessarily uniform.

The problems in Kirchherr, Naroditsky, and Schmeichel (1992) are related to the road-coloring problem in the sense that the graphs discussed must fit the criteria of being strongly connected and aperiodic.

The next section reviews the results from Kirchherr, Naroditsky, and Schmeichel (1992). The following section provides an alternate proof to one of the results in Kirchherr, Naroditsky, and Schmeichel (1992).

3.1 Review of Paper

In this section we review the main results of the paper *Consensus with Probability One on a Digraph* (Kirchherr, Naroditsky, and Schmeichel 1992). Note that all of the results in this section stem from this paper.

The paper treats a modification of the Ovation problem in which the neighborhood topology is specified by a digraph. Each individual in this modified problem corresponds to a vertex. The neighborhood of a vertex v_i consists of all vertices with an edge directed towards v_i. The number of colors is two. The main result of the paper is to classify which strongly connected digraphs and which initial distributions lead to a monochromatic state of unison.

Labelling the two colors 0 and 1, let $f : V(\Gamma) \to \{0, 1\}$ specify the initial distribution of the colors. Let $\Pr(\Gamma, f, t)$ denote the probability of reaching the monochromatic state of unison at time step t.

The first theorem provides a sufficient condition that a strongly connected digraph will reach unison with probability one, for all f. This condition is that the digraph be aperiodic. The proof will require the next three lemmata.

Let $l_1, l_2, \ldots, l_k$ be the lengths of the simple directed cycles $C_1, C_2, \ldots, C_k$ in Γ. Let v_i denote a vertex on C_i, $1 \leq i \leq k$. It does not matter which vertex on C_i is picked. Also note that the vertices $v_1, v_2, \ldots, v_k$ need not be distinct. Since Γ is aperiodic, we have $\gcd(l_1, l_2, \ldots, l_k) = 1$. The first lemma is:

Lemma 3.1 *There exists integers $\beta_1, \beta_2, \ldots, \beta_k$ such that*

$$1 = \gcd(l_1, l_2, \ldots, l_k) = \sum_{i=1}^{k} \beta_i l_i$$

Proof This is a standard result in Number Theory (Gilbert and Gilbert 1988). For the case $k = 2$, assume one of l_1, l_2 is nonzero. If $l_2 = 0$, $l_1 \neq 0$ and $\gcd(l_1, l_2) = |l_1|$. If both l_1, l_2 are nonzero, define the set $S = \beta_1 l_1 + \beta_2 l_2 : \beta_1, \beta_2 \in J$ and the set $S^+ = s \in S : s > 0$. S^+ is nonempty because $l_2 = l_1 \cdot 0 + l_2 \cdot 1 \in S$ and $-l_2 = l_1 \cdot 0 + l_2 \cdot -1 \in S$. By the Well-Ordering theorem, S^+ contains a least element $d = \beta_1 l_1 + \beta_2 l_2$. By the Division Algorithm, there exists integers q,r such that $l_1 = dq + r$ with $0 \leq r < d$ so we have:

$$\begin{aligned} r &= l_1 - dq \\ & l_1 - (\beta_1 l_1 + \beta_2 l_2) \cdot q \\ & l_1(1 - \beta_1 q) + l_2(-\beta_2 q) \end{aligned}$$

Therefore, $r \in S$ and $0 \leq r < d$. Since d is the least element of S^+, we have $r = 0$. Thus, $d | l_1$. Similarly, one can argue that $d | l_2$. Finally, if there exists an integer c

such that $c|l_1$ and $c|l_2$, there exists integers h,k such that $l_1 = ch$, $l_2 = ck$. So,

$$\begin{aligned} d &= \beta_1 l_1 + \beta_2 l_2 \\ & \beta_1 ch + \beta_2 ck \\ & c \cdot (\beta_1 h + \beta_2 k) \end{aligned}$$

which implies $c|d$. The rest follows easily by induction. □

Lemma 3.2 *There exists an integer M_0 such that for any integer $m \geq M_0$ there exists non-negative integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that the following representation can be made:*

$$m = \sum_{i=1}^{k} \alpha_i l_i$$

Proof Let $l_0 = \min\{l_1, l_2, \ldots, l_k\} \geq 1$. Let $B = \max\{|\beta_i| : \beta_i \leq 0\}$, using β_i from the prior lemma. Set

$$M_0 = \sum_{i=1}^{k} B \cdot (l_0 - 1) \cdot l_i \geq 0$$

Each coefficient in this summation is a constant equal to $B \cdot (l_0 - 1)$ and is a suitable α_i. Therefore, for all i, $1 \leq i \leq k$, β_i can be added to α_i, $(l_0 - 1)$ times, with each sum being non-negative. Therefore, iteratively adding

$$1 = \sum_{i=1}^{k} \beta_i l_i$$

$(l_0 - 1)$ times to the equation for M_0 produces the desired representation for the integers $M_0, M_0{+}1, \ldots, M_0{+}l_0{-}1$. For these $(l_0{-}1)$ representations, adding 1 to one of the coefficients with minimum cycle length (l_0) obtains the desired representation for the integers $M_0 + l_0, M_0 + l_0 + 1, \ldots, M_0 + 2l_0 - 1$. Further multiples of this coefficient produce the representation for all integers $m \geq M_0$. $\square$

Lemma 3.3 *Let $v_0 \in V(\Gamma)$. There exists an integer T such that for any $w \in V(\Gamma)$ there is a path of* exactly *length T from v_0 to w.*

Proof Since Γ is strongly connected, there exists a path P_w from v_0 to w that passes through $v_1, v_2, \ldots, v_k$. Let $L = \max_w |P_w|$. Choose the integer T such that $T \geq L + M_0$ where M_0 is defined in the prior lemma. For any $w \in V(\Gamma)$,

$$\begin{aligned} T - |P_w| &\geq T - L \\ &\geq M_0 \end{aligned}$$

Therefore, $T - |P_w|$ can be represented as

$$T - |P_w| = \sum_{i=1}^{k} \alpha_i l_i$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative integers.

Now examine a path from v_0 to w following P_w, with the exception that when v_i is reached, the path "detours" around C_i, α_i times. This path has length

$$|P_w| + \sum_{i=1}^{k} \alpha_i l_i = T$$

$\square$

The next theorem establishes aperiodicity as a sufficient condition for strongly connected digraphs to reach unison.

Theorem 3.1 *Let Γ be a strongly connected, aperiodic digraph. For all f,*

$$\lim_{t \to \infty} \Pr(\Gamma, f, t) = 1$$

Proof Lemma 3.3 implies that after T time steps, every vertex will "hear" vertex v_0. So, there exists a nonzero probability ε that all $w \in V(\Gamma)$ will possess the same color as the *current* color of v_0. Thus, the probability of not reaching consensus at time T is $\leq 1 - \varepsilon$. The relation is an inequality because there are other ways to reach consensus. For each time kT, where k is a positive integer, the probability of not reaching consensus is $\leq (1 - \varepsilon)^k$. Therefore,

$$\Pr(\Gamma, f, kT) \geq 1 - (1 - \varepsilon)^k \to 1 \text{ as } k \to \infty.$$

So, we have $lim_{t \to \infty} \Pr(\Gamma, f, t) = 1$. □

The next theorem establishes aperiodicity as a necessary condition for the prior theorem, unless, of course, the initial distribution starts in unison. That is, if a strongly connected digraph is periodic with period $d > 1$, we cannot be certain that unison is reached, if f is nonconstant. Before we state the theorem, we will start off with a few more lemmata.

Lemma 3.4 *A closed path on a digraph can be expressed as a union of directed simple cycles.*

Proof Assume the closed path starts with vertex v_0. Follow the closed path, recording vertices along the way in an ordered list. Label this list 1. If a vertex, say v_i, is repeated, remove those vertices in the list that appear after the first occurrence of v_i. Create a new list starting with vertex v_i followed by the removed vertices in their original order, labelling the new list with the next consecutive number. Eventually, the closed path will terminate at v_0. List 1 will form a directed simple cycle because only vertex v_0 has been repeated. This argument can be applied to all of the other lists because each list forms a closed path. This process must terminate, because a closed path has finite length. □

Lemma 3.5 *The length any closed path on* Γ *is divisible by* d.

Proof Since a closed path is an edge-disjoint union of directed simple cycles, all of which are divisible by d, then the length of the closed path must be divisible by d.

□

Lemma 3.6 *For any* $v, w \in V(\Gamma)$, *and any two paths* P_1, P_2, *from* v *to* w, *we have* $|P_1| \equiv |P_2| \pmod{d}$.

Proof Let $v \xrightarrow{P} w$. Then a path that follows P_1 then P is a closed path with length $|P_1| + |P|$. Likewise, a path that follows P_2 then P is a closed path with length $|P_2| + |P|$. By lemma 1, both of these closed paths are divisible by d. So, $|P_1| + |P| \equiv |P_2| + |P| \equiv 0 \pmod{d}$. Therefore, $|P_1| \equiv |P_2| \pmod{d}$.

The remaining lemmas and theorem establish that if Γ is a strongly digraph with period $d > 1$, $V(\Gamma)$ can be expressed as the union of d mutually disjoint vertex sets $V_0(\Gamma), V_1(\Gamma), \ldots, V_{d-1}(\Gamma)$, with the following property: if $v \in V_i(\Gamma)$, and $(v, w) \in E(\Gamma)$, then $w \in V_{i+1}(\Gamma)$. To prove this, the concept of "collision" is required.

Define the relation $n : V(\Gamma) \rightarrow \{0, \ldots d\}$ as follows: Start with any $v_0 \in V(\Gamma)$ and traverse the graph in a breadth first manner, making the following assignments for n:

$$n(w) = \begin{cases} 0 & \text{for } v = v_0 \\ i+1 & \text{for } (v, w) \in E(\Gamma) \text{ and } n(v) = i \end{cases} \tag{3.1}$$

A "collision" is said to occur if w has already been assigned j, where $i + 1 \not\equiv j \pmod{d}$.

Lemma 3.7 *There are no collisions. In other words, the relation n, as defined above, is a function.*

Proof Assume collisions occur, and that the first collision occurs at vertex w. This implies that two distinct paths $v_0 \xrightarrow{P_1} w$ and $v_0 \xrightarrow{P_2} w$ exist, such that $|P_1| = i$ and $|P_1| = j$ with $i \not\equiv j \pmod{d}$. This contradicts lemma 3.6. □

This proves that $V(\Gamma)$ can be partitioned as the union of d mutually disjoint vertex sets $V_0(\Gamma), V_1(\Gamma), \ldots, V_{d-1}(\Gamma)$, with the property that if $v \in V_i(\Gamma)$, and $(v, w) \in E(\Gamma)$, then $w \in V_{i+1}(\Gamma)$. Vertex $v \in V_i(\Gamma)$ if $n(v) = i$.

Lemma 3.8 *If* $\gcd(l_1, l_2, \ldots, l_k) = d > 1$, $\gcd(\frac{l_1}{d}, \frac{l_2}{d}, \ldots, \frac{l_k}{d}) = 1$.

Proof By the definition of gcd, there exist integers $m_1, m_2, \ldots, m_k$ such that $d \cdot m_i = l_i, 1 \leq i \leq k$. Let $\gcd(\frac{l_1}{d}, \frac{l_2}{d}, \ldots, \frac{l_k}{d}) = p$. There exist integers $s_1, s_2, \ldots, s_k$ such that

$p \cdot s_i = \frac{l_i}{d}, 1 \leq i \leq k$. Therefore, $p \cdot d \cdot s_i = l_i, 1 \leq i \leq k$, which implies $p \cdot d | l_i, 1 \leq i \leq k$. Since $d | p \cdot d$, d being the gcd of $l_1, l_2, \ldots, l_k$ is contradicted, unless p = 1. □

Lemma 3.9 *Let $v_0 \in V(\Gamma)$. There exists an integer T such that for all $w \in V(\Gamma)$ with $n(w) = n(v_0)$, there exists a path P such that $v \xrightarrow{P} w$ and $|P| = d \cdot T$.*

Proof Again, let $l_1, l_2, \ldots, l_k$ be the lengths of the simple directed cycles $C_1, C_2, \ldots, C_k$ in Γ. Let $v_1, v_2, \ldots, v_k$ denote vertices on those respective cycles, where $n(v_i) = n(v_0)$. Such vertices exist because as you progress through any of the cycles, at least d distinct values of the function n will be seen when n is applied to the vertices of the cycle.

Let $w \in V(\Gamma)$ such that $n(w) = n(v_0)$. Define path P_w such that $v \xrightarrow{P} w$ and P_w passes $v_1, v_2, \ldots, v_k$ in order of $v_i's$ subscripts. Since $n(v_0) = n(v_1) = \cdots = n(v_k) = n(w)$, and d divides the portion of the path between each vertex, there exists an integer s, such that $d \cdot s = |P_w|$. Let $L = \frac{1}{d} \max_w |P_w|$. By lemma 3.2, there exists an integer M_0 such that for any integer $m \geq M_0$ there exists non-negative integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that the following representation can be made:

$$m = \sum_{i=1}^{k} \alpha_i \left(\frac{l_i}{d} \right)$$

Pick integer T such that $T \geq M_0 + L$. The fact that $T - s \geq T - L \geq M_0$ allows us to use the above representation for $T - s$. Since $T - |P_w| = d(T - s)$, we have

$$d \cdot T \;\; = \;\; d \cdot s + d \cdot T - d \cdot s$$

$$= d \cdot s + d(T - s)$$
$$= d \cdot s + d\sum_{i=1}^{k} \alpha_i \left(\frac{l_i}{d}\right)$$
$$= |P_w| + \sum_{i=1}^{k} \alpha_i l_i$$

Just as the argument went in theorem 3.1, there is a path P, $v \xrightarrow{P} w$ following P_w and the cycles C_i, α_i times, and P has length exactly equal to $d \cdot T$. □

Theorem 3.2 *Let* Γ *be a strongly connected digraph with period* $d > 1$. *If* f *is nonconstant,*

$$\lim_{t \to \infty} \Pr(\Gamma, f, t) < 1$$

Proof Let $f : V(\Gamma) \to \{0, 1\}$ be a any nonconstant initial distribution. Choose vertices $v_1, v_2, \ldots, v_{d-1} \in V(\Gamma)$ such that $v_j \in V_j(\Gamma), 0 \leq j \leq d - 1$, and $f(v_i) \neq f(v_j)$ for at least one pair (i,j). By lemma 3.9, there exists integers $T_1, T_2, \ldots, T_{d-1}$ such that there is a path of length $d \cdot T_j$ from v_j to any vertex $w \in V_j(\Gamma)$. Let $T = \max\{T_1, T_2, \ldots, T_{d-1}\}$. After $d \cdot T$ time steps, the probability is greater than 0 that all vertices $w \in V_j(\Gamma)$ will have the same color $f(v_j)$. If this event is realized, the color $f(v_j)$ will cycle through the digraph between the vertex sets $V_0(\Gamma), V_1(\Gamma), \ldots, V_{d-1}(\Gamma)$. Since $f(v_i) \neq f(v_j)$, there is positive probability that all vertices $w \in V_i(\Gamma)$ will have the same color $f(v_i)$, and this different color will cycle through the vertex sets. Thus, there exists a positive probability that unison will not be reached. □

3.2 Alternate Proof of Lemma 3.3

In this section we provide an alternate proof of lemma 3.3 using the results of matrix theory (Horn and Johnson 1985). Lemma 3.3 is the key to proving theorem 3.1. We need the next three theorems, but first a definition:

Definition 3.1 Horn and Johnson *A nonnegative matrix $A \in M_n$ is said to be primitive if it is irreducible and has only one eigenvalue of maximum modulus.*

Theorem 3.3 Horn and Johnson *If $A \in M_n$ is nonnegative and primitive, then*

$$\lim_{m\to\infty}[\rho(A)^{-1}A]^m = L > 0$$

where $L \equiv xy^T$, $Ax = \rho(A)x$, $A^Ty = \rho(A)y$, $x > 0$, $y > 0$, $x^Ty = 1$.

Proof The matrix meets the conditions stipulated in lemma A.6. Therefore, the proof is identical to theorem A.14. □

Theorem 3.4 Horn and Johnson *If $A \in M_n$ is nonnegative, then A is primitive if and only if $A^m > 0$ for some $m > 1$.*

Proof If the positive entries of A are set to 1, the resulting matrix can be thought of as an adjacency matrix for some digraph. Call Γ the name of this graph associated with A. If $A \geq 0$ and $A^m > 0$, then from every vertex v_i of the digraph Γ to every other vertex v_j there must be a path of exact length m, by Corollary A.1. This implies that A is strongly connected which is equivalent to A being irreducible.

Now if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A then $\lambda_1^k, \dots, \lambda_n^k$ are the eigenvalues of A^m. We know that $\rho(A)$ is an eigenvalue of A by Theorem A.16, so if $\rho(A)$ were a multiple eigenvalue of A, then $\rho(A)^m = \rho(A^m)$ would be a multiple eigenvalue of A^k. This contradicts the fact that $\rho(A^m)$ is a simple eigenvalue of A^m by Theorem A.15. Thus, $\rho(A)$ is a simple eigenvalue of A. Now assume there exists $\lambda \neq \rho(A)$ such that $|\lambda| = \rho(A)$. By the above argument, we have $\lambda^k \neq \rho(A^k)$ such that $|\lambda^k| = \rho(A)^k$ which contradicts theorem A.12. Thus, A is primitive.

Conversely, if A is primitive, then

$$\lim_{m\to\infty} [\rho(A)^{-1}A]^m = L > 0$$

by theorem 3.3. Let α equal the minimum value of the vector L. Take $\epsilon = \alpha/2$. There exists an integer n such that $|[\rho(A)^{-1}A]^n - L| < \epsilon$ which implies $[\rho(A)^{-1}A]^n > 0$. □

Theorem 3.5 Horn and Johnson *If $A \in M_n$ is nonnegative and irreducible, and let Γ be the associated digraph of A as discussed in theorem 3.3. Denote by $\{l_1, l_2, \dots, l_n\}$ the set of lengths of all simple directed cycles for Γ. Then A is primitive if and only if Γ is aperiodic.*

Proof Assume A is primitive. Since this implies that A is irreducible, and thus strongly connected, each $v_i \in V(\Gamma)$ has a cycle. Also, $A^m > 0$ implies that $A^k > 0$ for $k \geq m$. This implies that there are closed paths for all $v_i \in V(\Gamma)$ of length $m, m+1, m+2, \dots$. Since $\gcd(\{l_1, l_2, \dots, l_n\})$ divides the length of every closed path, by lemma 3.5, then $1 = \gcd(\{l_1, l_2, \dots, l_n\})$.

Now suppose that A is not primitive. If A has exactly $k > 1$ eigenvalues of maximum modulus, then by A.1 we know that $a_{ii}^m \equiv 0$ for all $i = 1, \ldots, n$ and for all such m such that m is not an integral multiple of k. Therefore, all closed paths, including simple directed cycles, have lengths that are a multiple of k. Thus, $1 < k \leq \gcd(\{l_1, l_2, \ldots, l_n\})$. $\Box$

Now note that one direction of theorem 3.5, assumes that A is irreducible, hence Γ is strongly connected, and that Γ is aperiodic. This implies that A is primitive. Thus, by theorem 3.4, $A^m > 0$ for some $m > 1$. This fact is identical to the result from lemma 3.3, which now can be used to prove theorem 3.1. It is readily apparent that clearer results are derived from the cycle structure of the graph (section 3.1) than from deriving the same results from the adjacency matrix (cross reference the appendix where the Perron-Frobenius theorem, required for the above three theorems, is derived).

CHAPTER 4

ROAD-COLORING CONJECTURE

In this chapter we review the main results of the paper *Experimental Investigations of the Road-Coloring Conjecture* (Gocka, Kirchherr, and Schmeichel 1994), and elaborate on some of the results. For the sake of proper citation, it must be noted that most of the results in the paper, and those presented here, are due to W. Kirchherr and E. Schmeichel.

The road-coloring conjecture (Adler, Goodwyn, and Wiess 1977) originated in the field of ergodic theory, but the conjecture is expressible completely in graph-theoretic terms. Before stating the conjecture, we will start out with a few definitions.

Let Γ be a strongly connected digraph with $d^+(v) = 2$ for all $v \in V(\Gamma)$. A vertex v has a loop if $(v, v) \in E(\Gamma)$. A pair of vertices v and w has multiple edges if there is more than one occurrence of (v, w) in the set $E(\Gamma)$. In this chapter, digraphs with loops and multiple edges will *not* be considered.

Let $\chi : E(\Gamma) \rightarrow \{R, B\}$ be an edge coloring of Γ such that for each $v \in V(\Gamma)$, v has exactly one red edge, labelled "R," and exactly one blue edge, labelled "B," with both edges directed out of v. The function χ is called a *road-coloring* of Γ. A string $I \in \{R, B\}^*$ will be called a *set of instructions* or a *map*. Given $v \in V(\Gamma)$

and χ, let I(v) designate that $w \in V(\Gamma)$ which is arrived at if one begins at v and follows the path labelled by I.

Definition 4.1 *Let $v \in V(\Gamma)$, and let χ be a road-coloring of Γ. We call χ a resolving road-coloring for v if and only if there exists an $I \in \{R, B\}^*$ such that for all $w \in V(\Gamma), I(w) = v$. Call I a universal map to v.*

In Adler, Goodwyn, and Wiess (1977), the above situation is described in layman's terms with respect to cities and roads: Imagine a map of cities, with each city having two one-way roads coming out. For each city, we can paint one road coming out red and the other road blue. Now assume that we want to supply generic directions (a universal map) to a destination city. That is, we will supply a traveller with a sequence of colors (e.g. red, blue, blue, red, red, red, ...) such that if the traveller follows the path indicated by this sequence, then the traveller will reach the destination city regardless of the source city. Now the question is: Can we paint the roads in such a way that a sequence of generic instructions exists?

Note that the universal map may be inefficient in terms of representing the shortest path between cities, and the map may force the traveller to enter a city multiple times (Adler, Goodwyn, and Wiess 1977).

Adler, Goodwyn, and Weiss (1977) proved the next theorem.

Theorem 4.1 *Let Γ be a strongly connected digraph such that $d^+(v) = \delta$ for all*

$v \in V(\Gamma)$. *Then*

$$\Gamma \text{ has a resolving road coloring} \Longrightarrow \Gamma \text{ is aperiodic}$$

Proof Assume vertex v has a resolving road coloring with universal map I. Let n be the length of string I. Starting at v, follow the path indicated by string I. Call this path P_1. Since $I(w) = v$ for all $w \in V(\Gamma)$, $I(v) = v$. Therefore, P_1 is a closed path with $|P_1| = n$. Now start from v again and move to any one of the two adjacent vertices. Call the vertex p. Since $I(p) = v$, we now have path P_2 which is closed and $|P_2| = n + 1$. Since two closed paths originating from the same vertex are divisible by the period of the digraph, and the only number that divides n and n+1 is 1, Γ is aperiodic by lemma 3.5. □

The converse of the above theorem has been outstanding for 18 years and is known as the road-coloring conjecture. We will refer to it as RCC.

Conjecture 4.1 *Let* Γ *be a strongly connected digraph such that* $d^+(v) = \delta$ *for all* $v \in V(\Gamma)$. *Then*

$$\Gamma \text{ is aperiodic} \Longrightarrow \Gamma \text{ has a resolving road coloring.}$$

The following two related theorems have been proven by O'Brien (O'Brien 1981):

Theorem 4.2 *RCC is true for* $\delta = 2 \Longrightarrow$ *RCC is true for all* $\delta \geq 2$.

Theorem 4.3 *Let Γ be a strongly connected digraph such that $d^+(v) = \delta \geq 2$ for all $v \in V(\Gamma)$ and such that Γ contains a prime-length directed cycle. Then*

$$\Gamma \text{ is aperiodic} \implies \Gamma \text{ has a resolving road coloring.}$$

We point out that there exist strongly connected, aperiodic digraphs such that some road colorings admit a universal map while other colorings do not. The simplest example, given by O'Brien (1981), is three vertices where each vertex has an edge directed toward the other two vertices, and each vertex has an edge directed inward from the other two vertices. First off, this digraph is obviously strongly connected with $d^+(v) = 2$ for all $v \in V(\Gamma)$. To see that the digraph is aperiodic, note that there are cycles of length 2 and 3, and that $\gcd(2,3) = 1$.

Now color edges (1,2),(2,3), and (3,1) red. The other edges must be colored blue. This coloring does not have a resolving set of instructions. To see this, imagine a coin placed on each vertex. Also imagine that each coin moves from vertex to vertex as specified by a universal map. Note that the coins must simply cycle around the graph, regardless of the map. This type of coloring will be referred to as a *permutation coloring*.

Now color edges (1,2),(3,2), and (2,3) red, and the other edges blue. The string $\{RB\}$ is a set of instructions that brings the coins together.

Thus note that for a coloring to have a universal map, the map must make the coins come together. This process must continue until, finally, all of the coins are stacked on one vertex. The process of two or more coins coming together at vertex

v will be referred to as *collapsing* at vertex v.

Given the difficulty of finding a proof of RCC, the hope was that computer-based experiments might lend insight into the problem. To carry out such experimentation, two types of algorithms were required. One algorithm was needed to generate a strongly connected, aperiodic digraph with $d^+(v) = 2$ for all $v \in V(\Gamma)$. If possible, this algorithm should randomly select digraphs from the set of all such digraphs. The second algorithm must determine if the digraph possesses a resolving color, or better still, what percent of the road colorings are resolving. In the next section we present two algorithms. The second algorithm is proved to work using Formal Language theory. The first algorithm is more heuristic. We will discuss its merits.

4.1 The RCC in Formal Language Theory

From here on, we will use $\mathcal{M}$ instead of Γ to indicate a strongly connected digraph such that $d^+(v) = \delta \geq 2$ for all $v \in V(\mathcal{M})$. Consider $\mathcal{M}$ as a finite state automata with n vertices labelled $\{1, 2, \ldots, n\}$

Lemma 4.1 *$\mathcal{M}$ has a resolving road-coloring for a particular vertex i if and only if $\mathcal{M}$ has a resolving road-coloring for every vertex $j \in V(\mathcal{M})$.*

Proof Assume $\mathcal{M}$ has a resolving road-coloring for every vertex $j \in V(\mathcal{M})$. Then obviously $\mathcal{M}$ has a resolving road-coloring for vertex i.

Now assume that $\mathcal{M}$ has a resolving road-coloring for a particular vertex i. Since $\mathcal{M}$ is strongly connected, for any vertex $j \in V(\mathcal{M})$ there is a sequence of colors that lead from i to j. Therefore, for any vertex k $k \in V(\mathcal{M})$ direct k to i, and then to j. □

Therefore, without loss of generality, we can assume that any vertex is the target vertex. This allows us to refer to a resolving road-coloring as an attribute of $\mathcal{M}$, itself, and not just one of $\mathcal{M}'\int$, vertices.

Let $\mathcal{S} = \{i_1, i_2, \ldots, i_k\}$ be a subset of $V(\mathcal{M})$. Call a set of instructions I such that $I(i_1) = I(i_2) = \cdots = I(i_k) = n$ a resolving road-coloring for $\mathcal{S}$. We will say that such a set of instructions *directs* vertices $\{i_1, i_2, \ldots, i_k\}$ to n.

Lemma 4.2 *Let*

$$\mathcal{S} = \{\mathcal{S} \subseteq V(\mathcal{M}) : |\mathcal{S}| = k\}$$

If, for all $i \in V(\mathcal{M})$, there exists a corresponding set of instructions, I_i, such that

$$I_i(i) = I_i(n) = n$$

then $\mathcal{M}$, has a resolving road-coloring for $\mathcal{S}$.

Proof Prove by induction. For k = 1, such an I exists for every i. Assume the hypothesis is true for k-1. Pick any $i \in \mathcal{S}$. Start with $I_i(i) = I_i(n) = n$. This leaves at most k-1 vertices to be directed to n. By induction, there exist a set on instructions, I, that directs these k-1 vertices to n. Just concatenate the string for I_i with the string for I to get the necessary set of instructions. □

Lemma 4.3 *The road-coloring* χ *is a resolving road-coloring for* $\mathcal{M}$ *if and only if for all* $i \in V(\mathcal{M})$ *there exists a corresponding set of instructions, I, such that*

$$I(i) = I(n) = n$$

Proof If χ is a resolving road-coloring for $\mathcal{M}$, the implication comes directly from the definition. The converse holds if we take $\mathcal{S} = \mathcal{M}$ in lemma 4.2 □

To consider $\mathcal{M}$ a finite automata we need to name an initial state and a set of final states with χ describing the transition function. Let $\mathcal{M}_i$ be an automaton with initial state i and final state n. Let L_i denote the language accepted by $\mathcal{M}_i$.

Theorem 4.4 *The road-coloring* χ *is a resolving road-coloring for* $\mathcal{M}$ *if and only if for all* $i \in \{1, 2, \ldots, n-1\}$, $L_i \cap L_n \neq \emptyset$.

Proof Assume the road-coloring χ is a resolving road-coloring. By lemma 4.3, for all $i \in V(\mathcal{M})$, there is a set of instructions $I = I(i) = I(n)$. Let $L_i = L_n = I$. Note that L_i is accepted by $\mathcal{M}_i$ and L_n is accepted by $\mathcal{M}_n$.

Now assume for all $i \in \{1, 2, \ldots, n-1\}$, $L_i \cap L_n \neq \emptyset$. Let $I = L_i = L_n$. By definition, $I(i) = I(n) = n$ which implies that χ is resolving. □

Construct $\mathcal{M}_\chi$, a finite automaton that accepts language $L_i \cap L_n$ for all $i \in \{1, 2, \ldots, n-1\}$, as follows (Gocka, Kirchherr, and Schmeichel 1994): The states of $\mathcal{M}_\chi$ consist of the cross-product $V(\mathcal{M}) \times (\mathcal{M})$. There is an edge labelled "R" ("B") from (j, k) to (l, m) if and only if in $\mathcal{M}$ there is an edge labelled "R" ("B") from j to l and k to m. The initial state of $\mathcal{M}_\chi$ is (i, n) and the final state is (n, n).

A language is accepted by $\mathcal{M}_\chi$ if and only if there is a directed path from (i, n) to (n, n). That $L_i \cap L_n$ is accepted by $\mathcal{M}_\chi$ is easily seen by observing the following: simultaneously start "following" the transitions made by $\mathcal{M}_i$ in response to L_i, $\mathcal{M}_n$ in response to L_n, and $\mathcal{M}_\chi$ in response to $L_i \cap L_n$. Then $\mathcal{M}_\chi$ is at state (p, q) if and only if $\mathcal{M}_i$ is at state p and $\mathcal{M}_n$ is at state q. Of course, the transitions made by $\mathcal{M}_i$ in response to L_i and $\mathcal{M}_n$ in response to L_n will terminate at state n.

Let a *root-directed arborescence* be a directed tree in which all paths are directed to the root.

Theorem 4.5 *The road-coloring χ is a resolving road-coloring for $\mathcal{M}$ if and only if $\mathcal{M}_\chi^2$ contains a root-directed arborescence rooted at (n,n) which includes nodes (i,n) for all $1 \leq i \leq n-1$.*

Proof Proof follows directly from the above comments. □

Now we define an algorithm that determines if a road-coloring χ is resolving for a digraph $\mathcal{M}$.

Algorithm 4.1 *Compute the function*

$$f(\mathcal{M}, \chi) = \begin{cases} 1, & \text{if } \chi \text{ is a resolving road-coloring for } \mathcal{M} \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

where $\mathcal{M}$ is a digraph such that, for all $i \in V(\mathcal{M})$, $d^+(i) = 2$ and χ is a road-coloring of $\mathcal{M}$.

1. *Create $\mathcal{M}_\chi^2$ from $\mathcal{M}$ and χ.*

2. *Perform a depth-first search from vertex (n,n) in $\mathcal{M}_\chi^2$ against the direction of the edges and return "1" if every vertex of the form (i,n) for $1 \leq i \leq n-1$ is encountered. Otherwise, return 0.*

□

Based on the running time of depth-first search and the fact that $E(\mathcal{M}_\chi^2) = 2n^2$, the above algorithm has running time $\mathcal{O}(n^2)$.

We must produce a randomly generated strongly connected digraph as input to Algorithm 4.1. This is the job of the next algorithm. This algorithm uses the concept of a Prüfer code. Prüfer showed that there is a one-to-one correspondence between sequences of length $n-2$, taking values on $\{1, 2, \ldots, n\}$ and labelled trees on n vertices (Moon 1967). Thus one may generate a random tree with n vertices by generating a length $n-2$ random sequence.

In order to make the generated tree a root-directed arborescence, direct all paths to a chosen root. This root can be the initial vertex in the Prüfer code.

The root-directed arborescence, $\mathcal{T}$, may be converted to a digraph Γ such that every vertex has outdegree 2 by doing the following for every vertex $v \in V(\Gamma)$ except the root: Suppose (i,j) is the edge in $\mathcal{T}$ originating at i; let k be a randomly selected element of $\{1, 2, \ldots, n\} - \{i, j\}$, and add edge (i,j) to $\mathcal{T}$. For the two edges originating at the root, r, one need only to randomly select two elements from $\{1, 2, \ldots, n\} - \{r\}$.

The resulting digraph must be checked for strong connectivity and aperiodicity; if these properties do not hold, throw the graph out. Strong connectivity can be checked by doing a depth-first search on each of the vertices, making sure on each scan that all of the vertices are reached.

To check for aperiodicity, construct the adjacency matrix A from Γ and raise the matrix to consecutive powers of $i = 1, 2, \ldots, n$. From Kirchherr, Naroditsky, and Schmeichel (1992) we get the following result:

For each A_m where $m = 1, 2, \ldots, n$, check if $[a_{ii}^{(m)}] \neq 0$ for any $i = 1, 2, \ldots, n$. If so, record the power m in a set CP. The power $j \in CP$ if and only if Γ contains a closed path of length j.

Let $l_1, l_2, \ldots, l_n$ be the lengths of the simple directed cycles in Γ. By lemma 3.5, we have

$$\gcd(l_1, l_2, \ldots, l_n) = \gcd(\{j | j \in CP\})$$

Thus, if the numbers in CP have a common divisor > 1, then Γ is not aperiodic. To test for a common divisor, note that $m > 1$ can be expressed by a unique product of prime integers (except for the order of factors), by the virtue of the Fundamental Theorem of Arithmetic (Gilbert and Gilbert 1988). Therefore, for each prime number p from 2 to $n/2$, divide each element of CP by p. If any p evenly divides all of the elements of CP, then Γ is not aperiodic.

Algorithm 4.2 *Algorithm 2 generates a random strongly connected, aperiodic digraph with n vertices, where each vertex has outdegree 2.*

1. *Generate a random tree $\mathcal{T}$ with n vertices.*

2. *Orient the edges of $\mathcal{T}$ to get a root-directed arborescence.*

3. *Generate digraph Γ from $\mathcal{T}$ so that each vertex of Γ has outdegree 2.*

4. *Test Γ to see if it is strongly connected and aperiodic. If so, return Γ. Otherwise, start from step (1) again.*

□

4.2 Experimental Results

Algorithms 4.1 and 4.2 were implemented for the purpose of trying to generate a counter-example to RCC. No such counter-example was found. In fact, the evidence led us to make two stronger conjectures.

First algorithm 4.2 was used to generate a strongly connected, aperiodic digraph with each vertex having outdegree 2. The digraph was supplied with a road-coloring and then algorithm 4.1 was employed to test if the road-coloring was resolving.

Two types of experiments were performed. The first experiment took each graph and exhaustively tested each road-coloring to see if it was resolving. Note that although there are 2^n possible road-colorings, only 2^{n-1} are unique. To see this, assume that a particular coloring is resolving, with universal map I_1. Change every edge colored blue to red, and every edge colored red to blue. This coloring is also resolving with a universal map I_2 where each instruction is the opposite of I_1. That is, if $I_1 = RBBBRRBR$, then $I_2 = BRRRBBRB$.

The results from the first experiment are summarized in table 4.1.

Table 4.1: All Possible Road-Colorings

No. Vertices	No. Graphs	No. Colorings	Average No. Resolving	Percentage Resolving
10	10,000	512	511.25	99.8535
15	500	16,384	16,383.37	99.9962
20	100	524,288	524,287.50	99.9999

These results simply state that more than 99 percent of a typical digraph's road-colorings are resolving. Needless to say, these results are striking when one considers that RCC only requires 1 coloring to be resolving. Even more intriguing is the spread of the distribution. For example, each of the 20 vertex digraphs had at least 99.9992 percent of their colorings as resolving. Also, the number of graphs that had all of their colorings resolving is impressive. These results are in table 4.2.

Table 4.2: All Road-Colorings Resolving

No. Vertices	Percentage of Graphs in Table 4.1 with all Colorings Resolving
10	48.91
15	65.00
20	74.00

Since exhaustive checking of all possible colorings of a digraph with a large vertex set is intractable, the second experiment went as follows: Each digraph was colored a certain number of times with randomly selected colorings. Each coloring was then tested to determine if it was a resolving road coloring by algorithm 4.1. The results here were equally impressive. Table 4.3 simply lists the number of digraphs and colors tested. The number of resolving colorings is not listed because *every* random coloring in this experiment was resolving.

Table 4.3: Random Road-Colorings

No. Vertices	No. Graphs	No. Random Colorings
50	1,000	10,000
80	10,000	1

Although we did not collect any data, one of earlier versions of the implementation of the experiment suggest that, for many of the colorings, the length of the universal map need not be very long. In this version, algorithm 4.1 was not used. Instead, all possible universal maps, up to length n were applied to the n-vertex digraph. If one of the universal maps brought *all* of the vertices together, a count of colorings that fit this criteria was incremented, and we moved on to the next coloring. Note, of course, this criteria is a sufficient, but not necessary, condition for a resolving color. Thus, the number of colorings that fit this criteria was less than the number of resolving colorings. This is, of course, because algorithm 4.1 can detect if a color is resolving, regardless of the length of the universal map required to bring the vertices together. But, even with this stronger criteria, the percentage of colorings fitting this criteria was close, in average, to the number of resolving colorings.

The fact that a counter-example to RCC was not found certainly lends more evidence for the conjecture. Our data suggests that a proof of RCC would have a great deal of freedom in choosing the coloring proposed to be resolving. Hopefully this information can help toward the goal of obtaining a proof.

4.3 New Conjectures

The evidence from the computer experiments led to two new conjectures:

Let Ω_n be the set of all strongly connected, aperiodic digraphs without loops or multiple edges, on n vertices in which each vertex has outdegree 2. Let χ be a road-coloring for $\Gamma \in \Omega_n$. The pair $(\Gamma, \chi) \in \Omega_n \times \{R, B\}^n$ is a resolved pair if χ is a resolving road-coloring for Γ.

Conjecture (Probabilistic RCC) 4.1

$$\Pr((\Gamma, \chi) \textit{ is a resolved pair}) \to 1, \quad \textit{as } n \to \infty$$

This just states that the probability that a randomly selected digraph with a randomly selected road-coloring is a resolved pair that tends to one as the number of vertices goes to infinity.

Now for $\Gamma \in \Omega_n$, define

$$f(\Gamma) = \frac{1}{2^{n-1}} \sum_{\text{Road Colorings } \chi \text{ of } \Gamma} f(\Gamma, \chi)$$

where $f(\Gamma, \chi)$ is defined as in algorithm 4.1. $f(\Gamma)$ is the fraction of the road-colorings of Γ which are resolving. Define

$$f_n = \min_{\Gamma \in \Omega_n} f(\Gamma)$$

Conjecture (Strong RCC) 4.1 *1. For any $n \geq 3$, we have $f_n > 0$*

2. $\lim_{n\to\infty} f_n = 1$

Statement (1) of Strong RCC is equivalent to RCC. Statement (2) implies Probabilistic RCC.

The evidence for Strong RCC is the observation that the distribution of the fraction of resolving colorings over all colorings is tightly clustered around 1. However, it should be pointed out that one graph was observed that makes us less sure about Strong RCC than we are about RCC and Probabilistic RCC. The graph has six vertices, labelled 1 through 6. The graph can be drawn in planar form by stacking the vertices as such:

1	4
2	5
3	6

and connecting the following edge list. {(1,4), (4,1), (2,5), (5,2), (3,6), (6,3), (1,2), (2,3), (3,1), (4,6), (6,5), (5,4)}. Only 15 of the possible 32 colorings of this graph are resolving. Note that for each vertex, indegree = outdegree = 2. This means that some of the nonresolving colorings will be permutation coloring.

4.4 Comments on Sampling Method

One criticism of algorithm 4.2 is that we do not know if all $\Gamma \in \Omega_n$ are equally likely to be picked by algorithm 4.2. For example, algorithm 4.2 may be sampling from certain clusters of graphs in Ω_n with a higher probability than other clusters. It would especially skew the results if algorithm 4.2 sampled *only* a subset of Ω_n.

The fact is, we do not know if algorithm 4.2 uniformly samples Ω_n. On the other hand, by Prüfer's code, we know that we are uniformly sampling the set of all labelled trees, which is a superset of Ω_n. We see no evidence that the construction of Γ from $\mathcal{T}$ restricts the selection of Γ to a subset of Ω_n.

Finally, even if algorithm 4.2 sampled only a subset of Ω_n, we know from observing the output of algorithm 4.2 that this subset appears, on the surface, diverse. We also know that the data obtained from this subset of digraphs implies that Strong RCC holds. Therefore, it would be of great interest to know what property these digraphs possess such that almost of their road-colorings are resolving.

4.5 Alternate Description of RCC

Let $\Gamma \in \Omega_n$ and $A = [a_{ij}]$ be the adjacency matrix for Γ. Let χ be a road-coloring of Γ. Note that each row of A has exactly two "1's," with the rest of the entries in the row being zero. *Color* the matrix by associating exactly one of the "1's" in each row with red, based on the road-coloring χ. For each row, the "1" not associated with red is associated with blue. Consider the matrices $\tilde{R} = [r_{ij}]$ where

$$r_{ij} = \begin{cases} 1 & \text{if } \chi[(v_i, v_j)] = R \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

and $\tilde{B} = [b_{ij}]$ where

$$b_{ij} = \begin{cases} 1 & \text{if } \chi[(v_i, v_j)] = B \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

Again, imagine a single coin placed on each vertex of Γ. If the character R shows up in a set of instructions, a coin on vertex v_i will transition to vertex v_j if and only if $r_{ij} = 1$. Thus the matrix $\tilde{R}$ represents the transitions made by the coins when the current instruction is R. Likewise, for the character B and the matrix $\tilde{B}$.

Therefore, to determine where the coins end up after a set of instructions are applied to Γ, simply multiply the matrices $\tilde{R}$ and $\tilde{B}$ in the exact order of the instruction set. For example, if the instruction set is $\{RBBR\}$, the matrices are multiplied in order $\tilde{R}\tilde{B}\tilde{B}\tilde{R}$.

Now note that if a coloring is resolving, a set of instructions, I, exists such that all of the coins will eventually be stacked on top of one another. This corresponds to the product of the $\tilde{R}$ and $\tilde{B}$ matrices to be a matrix with all ones in one column and zero everywhere else.

Thus, a coloring is resolving if the adjacency matrix A can be partitioned as the sum of two matricies, as described above, such that there exists some multiplicative combination of these matrices such that the product is a rank 1 matrix with all the ones in one column and zero everywhere else. Note that if such a product has rank 1, the product matrix must have all 1's in one column because each row of the product has exactly one 1.

An interesting feature of this representation is that

$$A^N = (\tilde{R} + \tilde{B})^N$$

is a multinomial where the coefficients form all possible combinations of the products of length N of $\tilde{R}$ and $\tilde{B}$. This is because matrix multiplication is, in general, not commutative. Thus, all universal maps are imbedded in this multinomial form.

Somewhat intriguing is that by theorem A.14, we know that

$$lim_{N\to\infty}[\rho(A)]^{-1}A^N \to L$$

where L is of rank 1. Of course, this does not imply that any of the coefficients of the multinomial form have rank 1.

4.6 RCC's relationship to the Ovation Model

There is not a true relationship between RCC and the Ovation Model, but the following discussion relates RCC to the Ovation Model variant discussed in chapter 3.

Let $\Gamma \in \Omega_n$. Let χ be a road-coloring of Γ. Now change direction of all of the edges of Γ. Now each vertex has an indegree of 2, with each edge colored red or blue. It is easy to show that this graph is strongly connected and aperiodic.

Again, imagine a set of n coins, but this time place all of the coins on top of one vertex, any vertex will do. If an instruction indicates to follow a blue edge, for example, note that for a given vertex there may be x blue out-edges, where $x = 1, \dots, n$. When such a vertex and instruction occurs, break out x coins along the x edges. Where the balance of the coins go is mentioned subsequently.

Now it is also easy to see that χ has a resolving coloring if an instruction set causes the entire stack can be spread out to all n vertices. Now we will assume that just the right number of coins are x-furicated at each point.

This is similar to the crucial result in Kirchherr, Naroditsky, and Schmeichel (1992) that determines if a bit value can be propagated to all of the vertices at the same time point T. Here the neighborhood of a vertex (individual) consists of exactly the two vertices with edges directed in toward the vertex. The choice to which bit value from which neighbor a vertex will transition is not a stochastic one but is determined by an instruction set that is *globally* applied to all vertices. This is fundamentally different from the Ovation model, where the transition function for an individual is solely dependent on the individual's neighborhood. Thus, this characterization of a relationship between the problems puts them a fair distance apart.

CHAPTER 5

CONCLUSION

One strong impression obtained from the topics reviewed in this thesis is that processes modelled with relatively simple rules of transition appear to be very difficult to handle analytically. Even the startling results leading to Strong RCC do not provide an immediate path around the difficulties of proving RCC. This might lead one to two conclusions:

1. More mathematical machinery needs to be developed to handle processes beyond the most trivial. For example, stationary Markov Processes are relatively well understood, but little literature exists for non-stationary Markov Processes.

2. Computer investigations are extremely valuable in the understanding of such processes. As such, more formal rules need to be developed about how computer experiments should be carried out. These rules would be analogous to the experimental protocols of the natural sciences.

Future work in this area would include the computational experiments mentioned in *Consensus in Small and Large Audiences* (Kanevsky, Garcia, and Naroditsky 1992), as well as work trying to analytically solve the Ovation problem for arbitrary neighborhood sizes. For RCC, one still hopes that the results leading to Strong RCC lead to a proof of RCC. However, even a proof of RCC, without further understanding of Strong RCC, would still make us feel that we do not understand the taproot of the problem.

APPENDIX

PERRON-FROBENIUS THEOREM

In this appendix we establish the Perron-Frobenius theorem. This theorem serves as the basis of an alternate proof of a result in chapter 3. Assumed is a basic knowledge of matrices from linear algebra.

All of the results in this appendix were extracted from *Matrix Analysis* (Horn and Johnson 1985), with some of the details of the proofs fleshed out.

Assume the next three theorems:

Theorem (Horn and Johnson 1985) A.1 *For a finite dimensional space, if a vector sequence converges to a vector x with respect to one vector norm, it will converge to vector x with respect to any vector norm. Thus vector norms are equivalent on a finite-dimensional space.*

Theorem (Horn and Johnson 1985) A.2 *Suppose $A \in M_n$ is irreducible and nonnegative. Denote $A^m \equiv [a_{ij}^{(m)}]$ for $m = 1, 2, 3, \ldots$. If there is exactly $k > 1$ eigenvalues of A of maximum modulus, then $a_{ii}^{(m)} \equiv 0$ for all $i = 1, 2, 3, \ldots$ whenever m is not an integral multiple of k.*

This next theorem is known as Schur's theorem.

Theorem (Horn and Johnson 1985) A.3 *Suppose* $A \in M_n$ *with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *in any prescribed order, there is a unitary matrix* $U \in M_n$ *such that* $U^*AU = T = [t_{ij}]$ *is upper triangular with diagonal entries* $t_{ii} = \lambda_i$, $i = 1, 2, 3, \ldots, n$.

A.1 Adjacency Matrices

Theorem (Horn and Johnson 1985) A.4 *Let* v_i, v_j *be vertices of digraph* Γ *and* $A \in M_n$ *the adjacency matrix of* Γ. *There exists a directed path,* $v_i \xrightarrow{P} v_j$ *with* $|P| = m$ *if and only if* $[A^m]_{ij} \neq 0$.

Proof By induction: The case $m = 1$ is obvious. For $m = 2$,

$$[A^2]_{ij} = \sum_{k=1}^{n} [A]_{ik}[A]_{kj} = \sum_{k=1}^{n} a_{ik}a_{kj}$$

so that $[A^2]_{ij} \neq 0$, if and only if for at least one value of k, a_{ik} and a_{kj} are both nonzero. This is the case if and only if there exists a path of length two from v_i to v_j. Now assume the assertion is true for $m = q$. Then

$$[A^{q+1}]_{ij} = \sum_{k=1}^{n} [A^q]_{ik}[A]_{kj} = \sum_{k=1}^{n} [A^q]_{ik}a_{kj} \neq 0$$

if and only if for at least one value of k, $[A^q]_{ik}$, and a_{kj} are both nonzero. This is equivalent to having a path from v_i to v_k of length q, and one from v_k to v_j of length 1. This is the case if and only if there is a path from v_i to v_j of length q+1. □

Corollary (Horn and Johnson 1985) A.1 *Let* v_i, v_j *be vertices of digraph* Γ *and* $A \in M_n$ *the adjacency matrix of* Γ. *Then* $A^m > 0$ *if and only if from each vertex* v_i *to each vertex* v_j *there is a directed path in* Γ *of exactly length* m.

Corollary (Horn and Johnson 1985) A.2 *Let v_i, v_j be vertices of digraph Γ and $A \in M_n$ the adjacency matrix of Γ. Then A is strongly connected if and only if $(I + A)^{n-1} > 0$.*

Proof

$$(I + A)^{n-1} = I + (n-1)A + \binom{n-1}{2} A^2 + \cdots + \binom{n-1}{n-2} A^{n-1} > 0$$

if and only if for each pair v_i, v_j such that $i \neq j$ at least one of the terms $A, A^2, \ldots,$ A^{n-1} has a positive (i, j) entry. But Theorem A.3 says this happens if and only if there is some directed path in Γ from v_i, v_j. This is equivalent to Γ being strongly connected, which is equivalent to A having property SC. □

Theorem (Horn and Johnson 1985) A.5 *An adjacency matrix $A \in M_n$, of Γ, is irreducible if and only if*

$$(I + A)^{n-1} > 0$$

Proof Assume A is reducible and that for some permutation matrix P we have

$$A = P \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} P^T = P\tilde{A}P^T$$

where $1 \leq r \leq n-1$, $B \in M_r$, $D \in M_{n-r}$, $C \in M_{r,n-r}$, and $0 \in M_{n-r,r}$ is a zero matrix. Notice that $\tilde{A}^2, \tilde{A}^3, \ldots, \tilde{A}^{n-1}$ all have the same (n-r) by r block of 0's in the lower left corner as $\tilde{A}$. Thus,

$$(I + A)^{n-1} \quad = \quad (I + P\tilde{A}P^T)^{n-1}$$

$$
\begin{aligned}
&= (P[I+\tilde{A}]P^T)^{n-1} \\
&= (P[I+\tilde{A}]^{n-1}P^T) \\
&= P\left[I+(n-1)\tilde{A}+\binom{n-1}{2}\tilde{A}^2+\cdots+\binom{n}{n-1}\tilde{A}^{n-1}\right]P^T
\end{aligned}
$$

and all of the terms in the square brackets have an (n-r) by r block of 0's in the lower left corner. Thus, $(I+A)^{n-1}$ is reducible and hence it cannot have all nonzero entries.

Conversely, suppose that for some $p \neq q$ that the (p,q) entry of $(I+A)^{n-1}$ is zero. Then we know that there is no directed path in Γ from v_p to v_q. Define the set of vertices

$$V_1 = \{v_i : v_i = v_q \text{ or there is a path from } v_i \text{ to } v_q\}$$

Define $V_2 = \{v_i : v_i \notin V_1\}$. We have $v_q \in V_1 \neq \emptyset$ and no path from a vertex in V_2 leads to a vertex in V_1. Relabel the vertices $V_1 = \{\tilde{v}_1, \ldots, \tilde{v}_r\}$ and $V_2 = \{\tilde{v}_{r+1}, \ldots, \tilde{v}_n\}$ and notice that

$$\tilde{A} = P^TAP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where $1 \leq r \leq n-1$, $B \in M_r$, $D \in M_{n-r}$, $C \in M_{r,n-r}$, and $0 \in M_{n-r,r}$ is a zero matrix. Therefore, A is reducible. □

A.2 Matrix Norms

Theorem (Horn and Johnson 1985) A.6 *If $\|\cdot\|$ is any matrix norm and if $A \in M_n$, then $\rho(A) \leq \|A\|$.*

Proof If $Ax = \lambda x, x \neq 0$, and $|\lambda| = \rho(A)$, consider the matrix $X \in M_n$, all columns of which are equal to the eigenvector x, and observe that $AX = \lambda X$. If $\|\cdot\|$ is any matrix norm,

$$|\lambda|\|X\| = \|\lambda X\| = \|AX\| \leq \|A\|\|X\|$$

and therefore $|\lambda| = \rho(A) \leq \|A\|$. □

Theorem (Horn and Johnson 1985) A.7 *If $\|\cdot\|$ is any matrix norm on M_n and if $S \in M_n$ is nonsingular, then $\|A\|_S \equiv \|S^{-1}AS\|$ for all $A \in M_n$ is a matrix norm.*

Proof For axiom 1, $\|A\|_S \equiv \|S^{-1}AS\| \geq 0$.

For axiom 2, if A = 0, $\|0\|_S \equiv \|0\| = 0$. If $\|A\|_S = 0$, then $\|S^{-1}AS\| = 0$ which implies that A = 0 because S is nonsingular.

Axiom 3: $\|cA\|_S \equiv \|S^{-1}cAS\| = c\|S^{-1}AS\| \equiv c\|A\|_S$.

Axiom 4: $\|A+B\|_S \equiv \|S^{-1}(A+B)S\| = \|S^{-1}AS+S^{-1}BS\| \leq \|S^{-1}(A)S\| + \|S^{-1}(B)S\| \equiv \|A\|_S + \|B\|_S$.

Finally, axiom 5, follows from

$$\begin{aligned}\|AB\|_S &= \|S^{-1}ABS\| \\ &= \|(S^{-1}AS)(S^{-1}BS)\| \\ &\leq \|(S^{-1}AS\|\|S^{-1}BS)\| \\ &= \|A\|_S\|B\|_S.\end{aligned}$$

Lemma (Horn and Johnson 1985) A.1 *Let $A \in M_n$ and $\epsilon > 0$ be given. There is matrix norm $\|\cdot\|$ such that $\rho(A) \leq \|A\| \leq \rho(A) + \epsilon$.*

Proof By the Schur triangularization theorem, there is a unitary matrix U and an upper triangular matrix Δ such that $A = U\Delta U^*$. Set $D_t \equiv diag(t, t^2, t^3, \ldots, t^n$ and compute

$$D_t \Delta D_t^{-1} = \begin{bmatrix} \lambda_1 & t^{-1}d_{12} & t^{-2}d_{13} & \cdots & t^{-n+1}d_{1n} \\ 0 & \lambda_2 & t^{-1}d_{23} & \cdots & t^{-n+2}d_{2n} \\ 0 & 0 & \lambda_3 & \cdots & t^{-n+3}d_{3n} \\ 0 & 0 & 0 & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & t^{-1}d_{n-1,n} \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Thus, for $t > 0$ large enough, we can be certain that the sum of all the absolute values of the off-diagonal entries of $D_t \Delta D_t^{-1}$ is less than ϵ. In particular, we can be sure that $\|D_t \Delta D_t^{-1}\|_1 \leq \rho(A) + \epsilon$ for large enough t. Thus, for a given $\epsilon > 0$, if we define the matrix norm $\|\cdot\|$ by

$$\|B\| \equiv \|\|D_t U^* B U D_t^{-1}\|_1 = \|(UD_t^{-1})^{-1} B (UD_t^{-1})\|_1$$

for any $B \in M_n$, and if we choose t large enough, then we will have constructed a matrix norm such that $\|A\| \leq \rho(A) + \epsilon$. Note that this is a norm because UD_t^{-1} is nonsingular and theorem A.6. For the other inequality, $\|A\| \geq \rho(A)$ for any matrix norm, by theorem A.5. □

Lemma (Horn and Johnson 1985) A.2 *Let $A \in M_n$ be a given matrix. If there is a matrix norm $\|\cdot\|$ such that $\|A\| < 1$, then $\lim_{k\to\infty} A^k = 0$.*

Proof If $\|A\| < 1$, then $\|A^k\| \leq \|A\|^k \to 0$ as $k \to \infty$. Since all vector norms on M_n are equivalent, then $A^k \to 0$ with respect to vector norm $\|\cdot\|_\infty$. □

Theorem (Horn and Johnson 1985) A.8 *Let $A \in M_n$. Then $\lim_{k\to\infty} A^k = 0$ if and only if $\rho(A) < 1$*

Proof If $A^k \to 0$ and if $x \neq 0$ is a vector such that $Ax = \lambda x$, then $A^k x = \lambda^k x \to 0$, only if $|\lambda| < 1$. Since this inequality must hold for every eigenvalue of A, $\rho(A) < 1$. Conversely, if $\rho(A) < 1$, then by lemma A.1 there is some matrix norm $\|\cdot\|$ such that $\|A\| < 1$. Thus, $A^k \to 0$ as $k \to \infty$ by lemma A.2. □

Corollary (Horn and Johnson 1985) A.3 *Let $A \in M_n$ and $\epsilon > 0$ be given. There is constant $C = C(A, \epsilon)$ such that*

$$|(A^k)_{ij}| \leq C(\rho(A) + \epsilon)^k$$

for all $k = 1, 2, 3, \ldots$ and all $i, j = 1, 2, 3, \ldots, n$.

Proof Note that if $Ax = \lambda x, x \neq 0$, then if $\tilde{A} = cA$, $\tilde{A}x = cAx = c\lambda x, x \neq 0$ implies that $c\lambda$ is an eigenvalue of $\tilde{A}$. Therefore, $\tilde{A} \equiv [\rho(a) + \epsilon]^{-1}A$ has spectral radius strictly less than 1 and is convergent by theorem A.7. Hence, $A^k \to 0$ as $k \to \infty$. This implies that $\{\tilde{A}^k\}$ is bounded, and thus, $|(\tilde{A}^k)_{ij}| \leq C$, $k = 1, 2, 3, \ldots$ and all $i, j = 1, 2, 3, \ldots, n$. □

Corollary (Horn and Johnson 1985) A.4 *Let $\|\cdot\|$ be a matrix norm on M_n. Then*

$$\rho(A) = \lim_{k \to \infty} \|A^k\|^{1/k}$$

for all $A \in M_n$

Proof First note that if $Ax = \lambda x, x \neq 0$, then

$$A^2 x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$$

when $x \neq 0$. Now assume that

$$A^k x = \lambda^k x$$

for any integer k. For $k+1$ we have

$$A^{k+1}x = A(A^k x) = A(\lambda^k x) = \lambda^k Ax = \lambda^{k+1}x$$

This, and theorem A.5 imply that $\rho(A)^k = \rho(A^k) \leq \|A^k\|$. Hence, $\rho(A) \leq \|A^k\|^{1/k}$ for all $k = 1, 2, \ldots$.

Given $\epsilon > 0$, the matrix $\tilde{A} \equiv [\rho(a) + \epsilon]^{-1}A$ has spectral radius strictly less than 1 and is convergent by theorem A.7. Thus, $\|\tilde{A}^k\| \to 0$ as $k \to \infty$ and there is some $N = N(\epsilon, A)$ such that $\|\tilde{A}^k\| < 1$ for all $k \geq N$. This implies $\|A^k\| < [\rho(A)+\epsilon]^k$ for all $k \geq N$, or that $\|A^k\|^{1/k} < \rho(A) + \epsilon$ for all $k \geq N$. Since,

$$\rho(A) \leq \|A^k\|^{1/k} < \rho(a) + \epsilon$$

for all k and arbitrary $\epsilon > 0$, the limit exists and is $\rho(A)$. □

A.3 Positive and Nonnegative Matrices

Claim A.1 $|A^m| \leq |A|^m$ *for all* $m = 1, 2, \ldots$.

Proof It is obvious for $m = 1$. Assume true for some m. We have

$$|A^{m+1}| = \left|\sum_{j=1}^{n} a_{ij}^{(m)} a_{ji}\right| \leq \sum |a_ij^{(m)}||a_{ji}| = |A^m||A| = |A|^m|A| = |A|^{m+1}$$

for $i = 1, \ldots, n$. The inequality comes from the triangle inequality for complex numbers. □

Claim A.2 *If* $0 \leq A \leq B$, *then* $0 \leq A^m \leq B^m$ *for all* $m = 1, 2, \ldots$.

Proof Show $0 \leq A$ implies $0 \leq A^m$ for all $m = 1, 2, \ldots$. The case $m = 1$ is obvious. Assume true for some m. $A^{m+1} = \sum_{j=1}^{n} a_{ij}^{(m)} a_{ji}$ for $i = 1, \ldots, n$ Now note that each $a_{ij}^{(m)} \geq 0$ and $a_{ji} \geq 0$ $i, j = 1, \ldots, n$ which implies that the sum is greater than or equal to zero.

Now show $0 \leq A \leq B$ implies $A^m \leq B^m$ We have $m = 1$ by hypothesis. Assume true for some m

$$B^{m+1} - A^{m+1} = \sum_{j=1}^{n} b_{ij}^{m} b_{ji} - \sum_{j=1}^{n} a_{ij}^{m} a_{ji}$$

$i = 1, \ldots, n$. Now just note that each difference is ≥ 0. □

Claim A.3 *If* $A > 0$, $x \geq 0$, *and* $x \neq 0$, *then* $Ax > 0$.

Proof

$$Ax = \sum_{j=1}^{n} a_{ij} xj$$

$i = 1, \ldots, n$. Since each $a_{ij} > 0$, at least one $x_j > 0$ then the sum is greater than zero. □

Claim A.4 *If* $|A| \leq |B|$, *then* $\|A\|_2 \leq \|B\|_2$.

Proof $|A| \leq |B|$ implies that

$$|a_{ij}| \leq |b_{ij}| \Rightarrow |a_{ij}|^2 \leq |b_{ij}|^2 \Rightarrow \|A\|_2 \leq \|B\|_2$$

□

Claim A.5 $\|A\|_2 = \||A|\|_2$.

Proof This is immediate from the definition of $\|\cdot\|_2$ and the fact that taking the modulus of the modulus of a number is the same as the modulus of the number. □

Theorem (Horn and Johnson 1985) A.9 *Let* $A, B \in M_n$. *If* $|A| \leq B$, *then* $\rho(A) \leq \rho(|A|) \leq \rho(B)$.

Proof For all $m = 1, 2, \ldots$, we have $|A^m| \leq |A|^m \leq B^m$ by claims A.1 and A.2. Thus, by claims A.4 and A.5 we have

$$\|A^m\|_2 \leq \||A|^m\|_2 \leq \|B^m\|_2$$

and

$$\|A^m\|_2^{1/m} \leq \||A|^m\|_2^{1/m} \leq \|B^m\|_2^{1/m}$$

for all $m = 1, 2, \ldots$. If we now let $m \to \infty$ and apply theorem A.5, we have $\rho(A) \leq \rho(|A|) \leq \rho(B)$. □

Corollary (Horn and Johnson 1985) A.5 *Let* $A, B \in M_n$. *If* $0 \leq A \leq B$, *then* $\rho(A) \leq \rho(B)$.

Proof $0 \leq A$ implies $A \equiv |A|$. By the above theorem, $\rho(A) \leq \rho(b)$. □

Lemma (Horn and Johnson 1985) A.3 *Let* $A \in M_n$ *and suppose* $0 \leq A$. *If the row sums of* A *are constant, then* $\rho(A) = \|A\|_\infty$. *If the column sums of* A *are constant, then* $\rho(A) = \|A\|_1$.

Proof By theorem A.5, $\rho(A) \leq \|A\|$ for any matrix norm $\|\cdot\|$, but if the row sums are constant, $x = [1, \ldots, 1]^T$ is an eigenvector with eigenvalue $\|A\|_\infty$ so $\rho(A) = \|A\|_\infty$. To get $\rho(A) = \|A\|_1$, apply the same argument to A^T. □

Theorem (Horn and Johnson 1985) A.10 *Let $A \in M_n$ and suppose $0 \leq A$. Then*

$$\min_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij} \leq \rho(A) \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij}$$

and

$$\min_{1\leq j\leq n} \sum_{i=1}^{n} a_{ij} \leq \rho(A) \leq \max_{1\leq j\leq n} \sum_{i=1}^{n} a_{ij}$$

Proof Let $\alpha = \min_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij}$ and construct a new matrix B with $0 \leq B \leq A$ and $\sum_{j=1}^{n} b_{ij} \equiv \alpha$ for all $i = 1, 2, \ldots, n$. For example, $\alpha = 0$, set $B = 0$. If $\alpha > 0$, set $b_{ij} = \alpha a_{ij} (\sum_{j=1}^{n} a_{ij})^{-1}$.

By lemma A.3, $\rho(B) = \alpha$. Also, $\rho(B) \leq \rho(A)$ by Corollary A.5. The upper bound case can be handled in a similar fashion. To handle the column sum bounds, use A^T. □

Corollary (Horn and Johnson 1985) A.6 *Let $A \in M_n$. If $0 \leq A$ and $\sum_{j=1}^{n} a_{ij} > 0$, $i = 1, 2, \ldots, n$, then $\rho(A) > 0$. Note that this implies $\rho(A) > 0$ if A is irreducible and nonnegative.*

Proof Immediately follows from the prior theorem. □

Theorem (Horn and Johnson 1985) A.11 *Let $A \in M_n$ and suppose $0 \leq A$. Then for any positive vector $x \in \mathcal{C}^n$.*

$$\min_{1\leq i\leq n} \frac{1}{x_i}\sum_{j=1}^{n} a_{ij}x_j \leq \rho(A) \leq \max_{1\leq i\leq n} \frac{1}{x_i}\sum_{j=1}^{n} a_{ij}x_j$$

and

$$\min_{1\leq j\leq n} x_j\sum_{i=1}^{n} \frac{a_{ij}}{x_i} \leq \rho(A) \leq \max_{1\leq j\leq n} x_j\sum_{i=1}^{n} \frac{a_{ij}}{x_i}$$

Proof Let $S = diag(x_1, \ldots, x_n)$ and if all $x_i > 0$, then $S^{-1}AS \geq 0$ if $A \geq 0$. Thus, the matrix $S^{-1}AS$ satisfies the conditions of theorem A.9. Apply this theorem, noting that $\rho(A) = \rho(S^{-1}AS)$, gives us the desired result by theorem A.6 and the fact that S is nonsingular. □

Corollary (Horn and Johnson 1985) A.7 *Let $A \in M_n$, $x \in \mathcal{R}^n$. Suppose $A \geq 0$ and $x > 0$. If $\alpha, \beta \geq 0$ are such that $\alpha x \leq Ax \leq \beta x$, then $\alpha \leq \rho(A) \leq \beta$. If $\alpha x < Ax$, then $\alpha < \rho(A)$. If $Ax < \beta x$, then $\rho(A) < \beta$.*

Proof If $\alpha x \leq Ax$, then $\alpha x \leq \min_{1\leq i\leq n} x_i^{-1}\sum_{j=1}^{n} a_{ij}x_j$. Then $\alpha \leq \rho(A)$ by the above theorem. If $\alpha x < Ax$, then there is some $\eta > \alpha$ such that $\eta x < Ax$. Again, by the theorem, $\eta \leq \rho(A)$, so α is strictly less than $\rho(A)$. The proof the upper bounds is similar. □

Corollary (Horn and Johnson 1985) A.8 *Let $A \in M_n$ and suppose that A is nonnegative. If A has a positive eigenvector, then the corresponding eigenvalue is $\rho(A)$.*

Proof If $x > 0$ and $Ax = \lambda x$, then $\lambda \geq 0$ and $\lambda x \leq Ax \leq \lambda x$. Then $\lambda \leq \rho(A) \leq \lambda$, by the above Corollary. □

Lemma (Horn and Johnson 1985) A.4 *Let $A \in M_n$ and suppose $0 < A$, $Ax = \lambda x$, $x \neq 0$, and $|\lambda| = \rho(A)$. Then $A|x| = \rho(A)|x|$ and $|x| > 0$.*

Proof

$$\begin{aligned} \rho(A)|x| &= |\lambda||x| \\ &= |\lambda x| \\ &= |Ax| \\ &\leq |A||x| \\ &= A|x| \end{aligned}$$

Set $y \equiv A|x| - \rho(A)|x| \geq 0$. Since $|x| \geq 0$ and $|x| \neq 0$, claim A.3 implies $A|x| > 0$. Corollary A.6 guarantees that $\rho(A) > 0$.

Now if $y = 0$, we have $A|x| = \rho(A)|x|$ and $|x| = \rho(A)^{-1}A|x| > 0$.

If $y \neq 0$, set $z \equiv A|x| > 0$ and apply claim A.3 again. Then $0 < Ay = Az - \rho(A)z$ which implies $Az > \rho(A)z$. However, by Corollary A.7 we have the contradiction $\rho(A) > \rho(A)$. Therefore, $y = 0$. □

Theorem (Horn and Johnson 1985) A.12 *Let $A \in M_n$ and suppose $0 < A$. Then $\rho(A) > 0$, $\rho(A)$ is an eigenvalue of A, and there is a positive vector x such that $Ax = \rho(A)x$.*

Proof By definition, there is an eigenvalue λ with $|\lambda| = \rho(A) > 0$ and an associated eigenvector $x \neq 0$. By Lemma A.4, the required vector is $|x|$. □

Lemma (Horn and Johnson 1985) A.5 *Let $A \in M_n$ and suppose $0 < A$, $Ax = \lambda x$, $x \neq 0$, and $|\lambda| = \rho(A)$. Then for some $\theta \in \mathcal{R}$, $e^{-i\theta}x = |x| > 0$.*

Proof We have $|Ax| = |\lambda x| = \rho(A)|x|$ and from lemma A.4 we know that $A|x| = \rho(A)|x|$ and $|x| > 0$. These identities and the triangle inequality give us

$$\begin{aligned} \rho(A)|x_k| &= |\lambda||x_k| = |\lambda x_k| = \left|\sum_{p=1}^{n} a_{kp}x_p\right| \\ &\leq \sum_{p=1}^{n} |a_{kp}||x_p| = \sum_{p=1}^{n} a_{kp}|x_p| \\ &= \rho(A)|x_k| \end{aligned}$$

for $k = 1, \ldots, n$.

Thus, the above inequality is an equality, so the numbers $a_{kp}x_p$ for $p = 1, \ldots, n$ have the same arg, say θ. Then $e^{-i\theta}a_{kp}x_p > 0$ for $p = 1, \ldots, n$. Since $a_{kp} > 0$ we have $e^{-i\theta}x > 0$. □

Theorem (Horn and Johnson 1985) A.13 *Let $A \in M_n$ and suppose $0 < A$. Then $|\lambda| < \rho(A)$ for every eigenvalue $|\lambda| \neq \rho(A)$.*

Proof By definition, $|\lambda| \leq \rho(A)$ for all eigenvalues $|\lambda|$ of A. Suppose $|\lambda| = \rho(A)$ and $Ax = \lambda x$, $x \neq 0$. By lemma A.5, there exists $w \equiv e^{-i\theta}x > 0$ for some $\theta \in \mathcal{R}$, so w is an eigenvector of A. Hence, $\lambda = \rho(A)$ by Corollary A.8. □

Theorem (Horn and Johnson 1985) A.14 *Let $A \in M_n$ and suppose $0 < A$ and that w and z are nonzero vectors such that $Aw = \rho(A)w$ and $Az = \rho(A)z$. Then there exists some $\alpha \in C$ such that $w = \alpha z$.*

Proof By lemma A.5 there exists real numbers θ_1 and θ_2 such that $p \equiv e^{-i\theta_1}z > 0$ and $q \equiv e^{-i\theta_2}z > 0$. Set $\beta = \min_{1 \leq i \leq n} q_i p_i^{-1}$ and define $r \equiv q - \beta p$. Notice that $r \geq 0$ and at least one coordinate of r is 0, so r is not a positive vector. But $Ar = Aq - \beta Ap = \rho(A)q - \beta\rho(A)p = \rho(A)r$, so if $r \neq 0$, we know by claim A.3 that $r = \rho(A)^{-1}Ar > 0$. Since this is not true, $r = 0$ and hence $q = \beta p$ and $w = \beta e^{i(\theta_2 - \theta_1)}z$. □

Corollary (Horn and Johnson 1985) A.9 *Let $A \in M_n$ and suppose $0 < A$. There exists a unique vector x such that $Ax = \rho(A)x$, $x > 0$, and $\sum_{i=1}^{n} x_i = 1$. The unique normalized eigenvector is called the* Perron *vector.*

Proof A vector x, such that $Ax = \rho(A)x$, $x > 0$, exists by Theorem A.11. By Theorem A.13, any other such vector must be a multiple of x. Thus, if the vector x is normalized, it must be unique. □

Lemma (Horn and Johnson 1985) A.6 *Let $A \in M_n$ and $\lambda \in C$ be given. Suppose x and y are vectors, with $L \equiv xy^T$. We will make the following assumptions:*

1. $Ax = \lambda x$

2. $A^T y = \lambda y$

3. $x^T y = 1$

4. $\lambda \neq 0$

5. λ *is an eigenvalue of* A *with geometric multiplicity 1*

6. $|\lambda| = \rho(A) > 0$

7. λ *is the only eigenvalue of* A *with modulus* $\rho(A)$. *Also, the eigenvalues of* A *are ordered as such:*

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_{n-1}| \leq |\lambda_n| = |\lambda| = \rho(A).$$

The ten parts of lemma A.6 make use of some or all of these assumptions:

(A) A.1 *Assume statements (1),(2),(3).* $Lx = x$ *and* $y^T L = y^T$.

Proof *Note that taking the transpose of (3) implies* $yx^T = 1$ *Thus,* $Lx = (xy^T x = x$ *and* $y^T L = y^T x y^T = y^T$. □

(B) A.1 *Assume statements (1),(2),(3).* $L^m = L$ *for all* $m = 1, 2, \ldots$.

Proof *Note that* $L^2 = x(y^T x)y^T = xy^T = L$. *The induction step trivially follows.* □

(C) A.1 *Assume statements (1),(2),(3)* $A^m L = LA^m = \lambda^m$ *for all* $m = 1, 2, \ldots$.

Proof *For the first part,* $A^m L = A^m xy^T = \lambda^m xy^T = \lambda^m L$ *For the second part, note that (2) implies* $y^T A = \lambda y^T$. *We then have* $LA^m = xy^T A^m = xy^T \lambda^m = L\lambda^m$. □

(D) A.1 *Assume statements (1),(2),(3).* $L(A - \lambda L) = 0$.

Proof *By (d) and (c), we get*

$$\begin{aligned} L(A - \lambda L) &= LA - \lambda L \\ &= (A - \lambda)L \\ &= (A - \lambda)xy^T \\ &= (Ax - \lambda x)y^T \\ &= 0 \end{aligned}$$

□

(E) A.1 *Assume statements (1),(2),(3).* $(A - \lambda L)^m = A^m - \lambda^m L$ *for all* $m = 1, 2, \ldots$.

Proof *By (d) and (c), we get for* $m = 2$

$$\begin{aligned} (A - \lambda L)^2 &= A^2 - \lambda LA - \lambda AL + \lambda^2 L^2 \\ &= A^2 - 2\lambda AL + \lambda^2 L^2 \\ &= A^2 - 2\lambda^2 L + \lambda^2 L \\ &= A^2 - \lambda^2 L \end{aligned}$$

The induction step trivially follows. □

(F) A.1 *Assume statements (1),(2),(3). Every nonzero eigenvalue of* $A - \lambda L$ *is also an eigenvalue of* A.

Proof *If* $\mu \neq 0$ *is an eigenvalue of* $A - \lambda L$ *and if* $(A - \lambda L)w = \mu w$ *for some* $w \neq 0$*, then by (d),* $L(A - \lambda L)w = 0 \cdot w = 0 = \mu L w$*. Hence,* $Lw = 0$*. Thus,* $(A - \lambda L)w = Aw = \mu w$*, so* μ *is also an eigenvalue of* A*.* □

(G) A.1 *Assume statements (1),(2),(3),(4),(5).* λ *is not an eigenvalue of* $A - \lambda L$*.*

Proof *Let* $\mu = \lambda$ *If* w *were a* λ *eigenvector of* $A - \lambda L$*, then it would also be a* λ *eigenvector of* A*. But (5) implies that* $w = \alpha x$ *for some* $\alpha \neq 0$*. But then*

$$\begin{aligned} \mu w &= \lambda w \\ &= (A - \lambda L)w \\ &= (A - \lambda L)\alpha x \\ &= \alpha\lambda x - \lambda\alpha x \\ &= 0 \end{aligned}$$

which is impossible since $\lambda \neq 0$ *and* $w \neq 0$*. Therefore we have a contradiction.* □

(H) A.1 *Assume statements (1),(2),(3),(4),(5),(6),(7).* $\rho(A - \lambda L) \leq |\lambda_{n-1}| < \rho(A)$*.*

Proof *Because of (f), we know either that* $\rho(A - \lambda L) = |\lambda_k|$ *for some eigenvalue* λ_k *of* A *or that* $\rho(A - \lambda L) = 0$*. Since we have ordered the eigenvalues of* A *by increasing modulus and* $|\lambda_n| = |\lambda| = \rho(A)$*, we know that in either event from (g) that* $\rho(A - \lambda L) \leq |\lambda_{n-1}|$*. The inequality now follows directly from (7).* □

(I) A.1 *Assume statements (1),(2),(3),(4),(5),(6),(7).* $(\lambda^{-1}A)^m = L + (\lambda^{-1}A - L)^m \to L$ *as* $m \to \infty$*.*

Proof *Combine (H) and (E) to get* $(\lambda^{-1}A - L)^m = (\lambda^{-1}A)^m - L \to 0$ *as* $m \to \infty$. *This is because*

$$\begin{aligned}\rho(\lambda^{-1}A - L) &= \rho(A - \lambda L)/\rho(A) \\ &\leq |\lambda_{n-1}|/\rho(A) \\ &< 1\end{aligned}$$

□

(J) A.1 *Assume statements (1), (2), (3), (4), (5), (6), (7). For every* r *such that* $[|\lambda_{n-1}|/\rho(A)] < r < 1$ *there exists some* $C = C(r, a)$ *such that* $\|(\lambda^{-1}A)^m - L\|_\infty < Cr^m$ *for all* $m = 1, 2, \ldots$.

Proof *This is a direct consequence of Corollary A.4 applied to the matrix* $\lambda^{-1}A - L$ *with* ϵ *chosen so that*

$$\begin{aligned}\rho(\lambda^{-1}A - L) + \epsilon &\leq [|\lambda_{n-1}|/\rho(A)] + \epsilon \\ &< r \\ &< 1\end{aligned}$$

□

□ (End of lemma A.6)

Theorem (Horn and Johnson 1985) A.15 *Let* $A \in M_n$ *and suppose* $0 < A$. *Then*

$$\lim_{m\to\infty} [\rho(A)^{-1}A]^m = L$$

where $L \equiv xy^T$, $Ax = \rho(A)x$, $x > 0$, $y > 0$, $x^Ty = 1$.

Proof The assumptions (1) through (7) of the Lemma A.6 are met with $\lambda = \rho(A)$, x the Perron vector of A, and $y = (x^Tz)^{-1}z$ where z is the Perron vector of A^T. Thus, (I) in Lemma A.6 implies the conclusion. □

Theorem (Horn and Johnson 1985) A.16 *Let* $A \in M_n$ *and suppose* $0 < A$. *Then* $\rho(A)$ *is an eigenvalue of algebraic multiplicity 1.*

Proof By the Shur triangularization theorem, $A = U\delta U^*$, where U is unitary, δ is an upper triangular matrix with main diagonal entries $\rho, \ldots, \rho, \lambda_{k+1}, \ldots, \lambda_n$ and $\rho = \rho(A)$ is an eigenvalue of algebraic multiplicity $k \geq 1$. The eigenvalues λ_i all have modulus strictly less than $\rho(A)$ for all $i = k+1, \ldots, n$. Thus, $L = \lim_{m\to\infty}[\rho(A)^{-1}A]^m$ which equals

$$U \lim_{m\to\infty} \begin{bmatrix} 1 & & & & & \\ & \ddots & & * & & \\ & & 1 & & & \\ & & & \frac{\lambda_{k+1}}{\rho} & & \\ & 0 & & & \ddots & \\ & & & & & \frac{\lambda_n}{\rho} \end{bmatrix}^m U^*$$

which equals

$$U \begin{bmatrix} 1 & & & & & \\ & \ddots & & * & & \\ & & 1 & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{bmatrix}^m U^*$$

Since this last matrix has rank k, and the matrix L has rank 1, $k > 1$ results in a contradiction. □

Theorem (Horn and Johnson 1985) A.17 *Let $A \in M_n$ and suppose $0 \leq A$. Then $\rho(A)$ is an eigenvalue A and there is a nonnegative vector $x \geq 0$, $x \neq 0$, such that $Ax = \rho(A)x$.*

Proof For any $\epsilon > 0$, define $A(\epsilon) \equiv [a_{ij} + \epsilon] > 0$. Denote by $x(\epsilon)$ the Perron vector of $A(\epsilon)$, so $x(\epsilon) > 0$ and $\sum_{i=1}^{n} x(\epsilon)_i = 1$. Since the set of vectors $\{x(\epsilon) : \epsilon > 0\}$ is contained in the compact set $\{x : x \in C^n, \|x\|_1 \leq 1\}$, there is a monotone decreasing sequence $\epsilon_1, \epsilon_2, \ldots$ with $\lim_{k\to\infty} \epsilon_k = 0$ such that $\lim_{k\to\infty} x(\epsilon_k) \equiv x$ exists. Since $x(\epsilon_k) > 0$ for all $k = 1, 2, \ldots$, it must be that $x = \lim_{k\to\infty} x(\epsilon_k) \geq 0$. Now $x = 0$ is impossible because

$$\sum_{i=1}^{n} x_i = \lim_{k\to\infty} \sum_{i=1}^{n} x(\epsilon_k)_i \equiv 1$$

By Theorem A.8, $\rho(A(\epsilon_k)) \geq \rho(A(\epsilon_{k+1})) \geq \cdots \geq \rho(A)$ for all $k = 1, 2, \ldots$, so this sequence is a monotone decreasing sequence. Thus, $\rho \equiv \lim_{k\to\infty} \rho(A(\epsilon_k))$ exists and $\rho \geq \rho(A)$. But from the fact that

$$\begin{aligned} Ax &= \lim_{k\to\infty} A(\epsilon_k)x(\epsilon_k) = \lim_{k\to\infty} \rho(A(\epsilon_k))x(\epsilon_k) \\ &= \lim_{k\to\infty} \rho(A(\epsilon_k)) \lim_{k\to\infty} x(\epsilon_k) = \rho x \end{aligned}$$

and the fact that $x \neq 0$, we deduce that ρ is an eigenvalue of A. But then $\rho \leq \rho(A)$, so it must be that $\rho = \rho(A)$. □

Lemma (Horn and Johnson 1985) A.7 *Let $A \in M_n$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A (including multiplicities). Then $\lambda_1+1, \ldots, \lambda_n+1$ are the eigenvalues of $I + A$ and $\rho(I + A) \leq 1 + \rho(A)$. If $A \geq 0$, then $\rho(I + A) = 1 + \rho(A)$.*

Proof If $\lambda \in \sigma(A)$ has multiplicity k, then λ is a root of the characteristic equation $p_A(t) = det[tI - A] = 0$ of multiplicity k. But then $\lambda + 1$ is a root of $p_{A+I}(s) = det[sI - (A+I)] = 0$ of multiplicity k because $det[tI - A] = det[(t+1)I - (A+I)]$. Thus, $\lambda_1 + 1, \ldots, \lambda_n + 1$ are the eigenvalues of $I + A$. Therefore,

$$\rho(I + A) = \max_{1 \leq i \leq n} |\lambda_i + 1| \leq \max_{1 \leq i \leq n} |\lambda_i| + 1 = 1 + \rho(A).$$

However, by theorem A.16, $1 + \rho(A)$ is an eigenvalue of $I + A$ when $A \geq 0$, so that $\rho(I + A) = 1 + \rho(A)$. □

Lemma (Horn and Johnson 1985) A.8 *Let $A \in M_n$, $A \geq 0$, and $A^k > 0$ for some $k > 1$, then $\rho(A)$ is an algebraically simple eigenvalue of A.*

Proof If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k. We know that $\rho(A)$ is an eigenvalue of A by Theorem A.16, so if $\rho(A)$ were a multiple eigenvalue of A, then $\rho(A)^k = \rho(A^k)$ would be a multiple eigenvalue of A^k. This contradicts the fact that $\rho(A^k)$ is a simple eigenvalue of A^k by Theorem A.15.

□

The next theorem is the Perron-Frobenius theorem.

Theorem (Horn and Johnson 1985) A.18 *Let $A \in M_n$, irreducible and non-negative. Then*

1. $\rho(A) > 0$.

2. *$\rho(A)$ is an eigenvalue of A.*

3. There is a positive vector x such that $Ax = \rho(A)x$.

4. $\rho(A)$ is an algebraically simple eigenvalue of A.

Proof Corollary A.6 shows that (1) is true. Theorem A.16 implies (2). Theorem A.16 also implies that there exists a nonnegative vector $x \neq 0$ such that $Ax = \rho(A)x$. By lemma A.7, $(I + A)x = [1 + \rho(A)]x$ so, $(I + A)^{n-1}x = [1 + \rho(A)]^{n-1}x$. By lemma A.4, we have $(I + A)^{n-1} > 0$, and therefore, $(I + A)^{n-1}x > 0$ is positive by lemma A.3. Thus,

$$x = [1 + \rho(A)]^{1-n}(I + A)^{n-1}x > 0$$

which proves (3). If $\rho(A)$ is a multiple eigenvalue of A, then $1 + \rho(A) = \rho(I + A)$ is a multiple eigenvalue of $(I+A)$ by lemma A.7. However, $(I+A) \geq 0$ and $(I+A)^{n-1} > 0$ by lemma 8.4.1, so $1 + \rho(A)$ is a simple eigenvalue of $(I + A)$. This contradiction establishes (4). □

Note that the unique normalized and positive eigenvector of an irreducible and nonnegative matrix is called the *Perron vector.*

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