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The irrationality of zeta (2) and zeta (3)

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THE IRRATIONALITY OF $\zeta(2)$ AND $\zeta(3)$

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by

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ABSTRACT

THE IRRATIONALITY OF $\zeta(2)$ AND $\zeta(3)$

by Victor Legge

This thesis gives proofs that each of two "naturally occurring numbers" in mathematics, the sums of the two series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ and $\sum_{k=1}^{\infty}\frac{1}{k^3}$ which are called $\zeta(2)$ and $\zeta(3)$ respectively, are both irrational. Two proofs are given for each of these two numbers. The thesis also looks briefly at the history of the subject of the irrationality of naturally occurring numbers in mathematics. The crux of all the proofs is the same and this is examined in detail.

The first proof that $\zeta(3)$ is irrational is taken from Van der Poorten's informal report on Apéry's proof written in 1978. The second proof that $\zeta(3)$ is irrational comes from Beukers' paper, also written in 1978. The first proof that $\zeta(2)$ is irrational uses the proof that π^2 is irrational. The second proof that $\zeta(2)$ is irrational uses Beukers' paper again.

Dedicated to

my beloved wife, Amba Giri

and

my beloved daughter, Tara Giri

CONTENTS

I. Introduction

1.1. History of the irrationality of $\zeta(n)$ for positive integers *n*.

DEFINITION. The expression $\zeta(n)$ is defined to be the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^n}$, n a positive integer greater than 1.

The history of irrational numbers really begins with the Greeks who proved that $\sqrt{2}$ and other algebraic numbers are irrational. The next breakthrough in irrationality proofs happened in 1737 when Euler proved that e is irrational. This proof opened the door for many other proofs. π was proved to be irrational in 1761 by Johann Heinrich Lambert. The first proof that $\zeta(2)$ is irrational was given in 1794 by Adrien-Marie Legendre. He actually proved that π^2 was irrational. This result, together with Euler's famous theorem that $\zeta(2)$ equaled $\frac{\pi^2}{6}$ (proved in 1732) proves the irrationality of $\zeta(2)$.

It has long been known that for n even $\zeta(n)$ is irrational. When this was first discovered is unclear. It has certainly been known since 1882 when the Hermite-Lindemann theorem proved the transcendence of e^α for α algebraic and thus proved the transcendence of all powers of π. This, together with Euler's formula for $\zeta(2k)$, k a positive integer, that $\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}}{2\cdot(2k)!}B_{2k}$ where B_m are the Bernoulli numbers, which are rational, proved the irrationality of $\zeta(2k)$, k a positive integer. For n odd proving the irrationality of $\zeta(n)$ has been difficult. In 1978 Apéry proved that $\zeta(3)$ is irrational for the first time. Since that time there have been other proofs. Two of these proofs, including Apéry's, will be given in this paper. For other n which are odd the only real inroads into this problem has been the claim within the last two years by Rivoal [unpublished] that he has proved that there are infinitely many integers m such that $\zeta(2m-1)$ is irrational.

In this chapter proofs will be given that $\zeta(2)$ is irrational, using the fact that π^2 is irrational, and also that e and π are irrational. Firstly, the crux of all these proofs is examined.

1.2. The Crux of the Irrationality Proofs for $\zeta(2)$ and $\zeta(3)$.

The crux of the proofs in this thesis is the same. The first kind of proof of this form was written by Euler. The proofs that a number ω is irrational involves coming up with a sequence of integer expressions p_n, q_n and a real expression g_n such that

$$\left|\omega - \frac{p_n}{q_n}\right| < \frac{g_n}{q_n},$$

or, alternatively

$$0 < |\omega q_n - p_n| < g_n$$

and $g_n \to 0$ as $n \to \infty$.

Thus, given $\varepsilon > 0$ an n can always be found such that $g_n < \varepsilon$, and therefore a p_n and a q_n can be found such that

(1.2.1) $$0 < \left|\omega - \frac{p_n}{q_n}\right| < \frac{\varepsilon}{q_n},$$

or

(1.2.2.) $$0 < |\omega q_n - p_n| < \varepsilon$$

This will prove the irrationality of ω because, as will be proved in this section, one of the properties of an irrational is that some multiple of the irrational will get within ε of an integer. This is not the case with a rational as it is not possible to get arbitrarily close to an integer if multiples of a rational are taken – the integer is either hit exactly or else the closest they get to the integer is $\frac{1}{b}$ if the rational can be written in lowest terms as $\frac{a}{b}$.

Looking at equation (1.2.1) carefully it can be seen that it is not sufficient to find any $\frac{p_n}{q_n}$ that converges to ω. The $\frac{p_n}{q_n}$ has to converge fairly rapidly.

Equation (1.2.2) will now be proved beginning with some lemmas.

LEMMA 1. Given q_n, a positive integer, and y, a real number, then the inequality $0 < y - \frac{p_n}{q_n} < \frac{1}{q_n^2}$ can have at most 2 solutions for p_n.

Proof. Suppose there exists p_j and p_k such that

$$0 < \left| y - \frac{p_j}{q_j} \right| < \frac{1}{q_j^2} \text{ and } 0 < \left| y - \frac{p_k}{q_j} \right| < \frac{1}{q_j^2},$$

then it follows that $0 < \left| \frac{p_j}{q_j} - \frac{p_k}{q_j} \right| < \frac{2}{q_j^2}$. Let $\left| \frac{p_j}{q_j} - \frac{p_k}{q_j} \right| = \frac{a}{q_j}$, a a positive integer. The solutions of $\frac{a}{q_j} < \frac{2}{q_j^2}$ are $q_j < \frac{2}{a}$. Solutions only exist for $a = 1$ in which case $q_j = 1$. Thus the result follows. In fact if $q_n > 1$ then there is at most one solution for p_n. □

LEMMA 2. Given a rational $\frac{p}{q}$ in lowest terms, $q \geq 1$, there exists only a finite number of rationals $\frac{p_n}{q_n}$, $q_n \geq 1$ such that

$$0 < \left| \frac{p}{q} - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

Proof. Simplifying the inequality produces

$$0 < \left| \frac{p}{q} - \frac{p_n}{q_n} \right| = \left| \frac{pq_n - p_n q}{qq_n} \right| < \frac{1}{q_n^2}.$$

As p_n, q_n, p, q are all integers and $\left|\frac{p}{q} - \frac{p_n}{q_n}\right| > 0$ then

$$\left|\frac{pq_n - p_n q}{qq_n}\right| \geq \frac{1}{qq_n}.$$

If $q_n > q$ then $\frac{1}{qq_n} > \frac{1}{q_n^2}$. Thus the solutions of the inequality $0 < \left|\frac{p}{q} - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$ have the restriction $q_n \leq q$. Therefore, given q, the possibilities for q_n are finite. From Lemma α there are only finite possibilities for p_n and therefore only finite possibilities for $\frac{p_n}{q_n}$.

LEMMA 3. Given an irrational number x, there exists an infinite number of rationals $\frac{p_n}{q_n}$ such that

$$0 < \left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}.$$

Proof. Let $(jx)_f$ equal the fractional part of $j \cdot x$, j an integer, so $\left(3 \cdot \frac{8}{5}\right)_f = \frac{4}{5}$, for example, using a rational example for x. Also, $(p)_f = 0$ if p is an integer. Given some positive integer Q, let

$$A = \{0, (x)_f, (2x)_f, \ldots, (Qx)_f\}.$$

The elements of A are all distinct for if there exists j and k with $j, k \leq Q$ and $(jx)_f = (kx)_f$ then for some integers α and β,

$$jx - \alpha = kx - \beta$$

and this implies $x = \dfrac{\alpha - \beta}{j - k}$ which is not possible as x is irrational. Thus $|A| = n + 1$.

Consider the set of intervals $\left\{\left[0, \frac{1}{Q}\right), \left[\frac{1}{Q}, \frac{2}{Q}\right), \ldots, \left[\frac{Q-1}{Q}, \frac{Q}{Q}\right)\right\}$. The union of these intervals is $[0,1)$, and thus the $n+1$ elements of A must lie in these intervals. Therefore by the pigeonhole principle there exists integers a, b and t all less than or equal to Q such that $(ax)_f$ and $(bx)_f$ lie inside the interval $\left[\frac{t}{Q}, \frac{t+1}{Q}\right)$. Thus,

$$\left|(ax)_f - (bx)_f\right| < \frac{1}{Q}$$

and therefore for some integers χ and δ

$$|ax - \chi - bx + \delta| < \frac{1}{Q}.$$

Dividing through by $|a - b|$ and rearranging produces

$$\left|x - \frac{\chi - \delta}{a - b}\right| < \frac{1}{Q|a - b|} \tag{1.2.3}$$

Now, $|a - b| < Q$ so

$$\left|x - \frac{\chi - \delta}{a - b}\right| < \frac{1}{|a - b|^2}.$$

Letting $p_n = \chi - \delta$ and $|a-b| = q_n$ then

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}.$$

What is needed is to show that the set of all possible $\frac{p_n}{q_n}$ is infinite. Suppose that it is not infinite and let this set be $D = \left\{\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_s}{q_s}\right\}$ then because the real numbers are well ordered there exists $\frac{p_u}{q_u}$ in this set such that $\left|\frac{p_u}{q_u} - x\right| < \left|\frac{p_i}{q_i} - x\right|$ for all $\frac{p_i}{q_i}$ in the set D.

Because the real numbers are dense there exists a rational $\frac{p}{q}$ such that

$\frac{p_u}{q_u} < \frac{p}{q} < x$ or $x < \frac{p}{q} < \frac{p_u}{q_u}$. Letting $Q = qq_u$ then $\left|\frac{p_i}{q_i} - x\right| > \left|\frac{p_u}{q_u} - x\right| > \frac{1}{Q} \geq \frac{1}{Qq_n}$

However, from (1.2.3) it follows that given Q we can find $\frac{p_n}{q_n}$ such that $\left|x - \frac{p_n}{q_n}\right| < \frac{1}{Qq_n}$.

This $\frac{p_n}{q_n}$ cannot be in D but this is a contradiction as this $\frac{p_n}{q_n}$ satisfies $\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$ and therefore is in D. Thus the set of all possible $\frac{p_n}{q_n}$ is infinite. □

COROLLARY 4. Given an irrational number x, and $\varepsilon > 0$, then two infinite sequences, p_n and q_n can be found with $q_n \to \infty$ such that

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}.$$

Proof. From Lemma 3 the set $\left\{ \frac{p_n}{q_n} \middle| \left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} \right\}$ is infinite and from Lemma 1 for any q_n there is at most two p_n's. Therefore there are an infinite number of q_n's. As q_n is an integer q_n is unbounded above. ∎

THEOREM 5. Given an irrational ω and $\varepsilon > 0$ there exist integers q and p such that

$$0 < |q\omega - p| < \varepsilon .$$

Proof. From the corollary above, a p and q can be found with $q > \frac{1}{\varepsilon}$ such that

$$0 < \left|\omega - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Multiplying by q gives

$$0 < |q\omega - p| < \frac{1}{|q|} < \varepsilon .$$ ∎

Once it is known that it is possible to always find integers p_n and q_n and a real number g_n such that

$$0 < |\omega q_n - p_n| < g_n < \varepsilon ,$$

then the task is to find them!!

THEOREM 6. Given a real number ω, and integers p_n and q_n such that for any $\varepsilon > 0$

$$0 < |\omega q_n - p_n| < \varepsilon$$

then ω is irrational.

Proof 1. Suppose ω is rational then there exists integers a and b such that $\omega = \frac{a}{b}$ and

$$0 < \left|\frac{a}{b} q_n - p_n\right| < \varepsilon .$$

Multiplying through by $|b|$ gives

$$0 < |aq_n - bp_n| < |b| \varepsilon .$$

As ε is arbitrarily small n can be found such that $|b|\varepsilon < 1$. But a, b, p_n and q_n are all integers and as $|aq_n - bp_n| > 0$ then $|aq_n - bp_n| \geq 1$ and this is a contradiction. □

Proof 2. (informal) The number of multiples of a rational number $\frac{a}{b}$ between any two consecutive integers is finite and therefore ω must be irrational. □

1.3. The Irrationality of e.

The irrationality of e is proved in two ways: first in the standard way which implicitly incorporates Theorem 6, and secondly by using Theorem 6.

THEOREM 7. e is irrational.

Proof 1. By the power series for e^x, $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$. Assume e is rational, then $e = \frac{a}{b}$, where a and b are integers. Let

$$\alpha = k!\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} \ldots - \frac{1}{k!}\right)$$

Notice $\alpha - \frac{k!a}{b} = -k!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{k!}\right)$ is an integer. If $k \geq b$, then $\frac{k!a}{b}$ is also an integer, since $b \mid k!$ when $k \geq b$ and therefore α is an integer. Hence

$$0 < \alpha = k! \sum_{n=k+1}^{\infty} \frac{1}{n!} = \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \ldots < \frac{1}{k+1} + \frac{1}{(k+1)^2} + \ldots$$

$$= \sum_{r=1}^{\infty} \frac{1}{(k+1)^r} = \frac{\frac{1}{k+1}}{1 - \frac{1}{k+1}} = \frac{\frac{1}{k+1}}{\frac{k+1-1}{k+1}} = \frac{1}{k},$$

and this contradicts α being an integer.

Proof 2.(Using section 1.2.) By the power series for e^x, $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$. Let $\alpha_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}$, the "tail" of the series. Now, as before,

$$n!\alpha_n = n!\sum_{k=n+1}^{\infty}\frac{1}{k!} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots$$

$$= \sum_{r=1}^{\infty}\frac{1}{(n+1)^r} = \frac{\frac{1}{n+1}}{1-\frac{1}{n+1}} = \frac{1}{n+1-1} = \frac{1}{n}.$$

But α_n can be written as $e - \sum_{k=1}^{n}\frac{1}{k!}$. Therefore given $\varepsilon > 0$ an n can be found such that

$$0 < n!\left(e - \sum_{k=1}^{n}\frac{1}{k!}\right) < \frac{1}{n} < \varepsilon.$$

Let $q_n = n!$ and $p_n = n!\sum_{k=1}^{n}\frac{1}{k!}$ and $g_n = \frac{1}{n}$ then p_n and q_n are integers and

$g_n \to 0$ as $n \to \infty$ and it follows that

$$0 < q_n e - p_n < g_n < \varepsilon$$

which proves that e is irrational by Theorem 6. $\square$

1.4. Proving $\zeta(2)$ is irrational using the fact that π^2 is irrational.

LEMMA 8. Let $g(x) = x^n(1-x)^n$. Then for $k \ge n$ we have $n! \mid g^{(k)}(0)$ and $n! \mid g^{(k)}(1)$.

Proof. By the binomial theorem

$$g(x) = x^n(1-x)^n = \sum_{i=n}^{2n} c_i x^i,$$

where the coefficients c_i are integers. For $k \ge n$, differentiating k times produces

$$g^{(k)}(x) = \sum_{i=n}^{2n} i(i-1)(i-2)\ldots(i-k+1)c_i x^{i-k},$$

$$= \sum_{i=k}^{2n} i(i-1)(i-2)\ldots(i-k+1)c_i x^{i-k},$$

since any term with $i < k$ is zero. Thus

$$g^{(k)}(0) = k(k-1)(k-2)\cdots 1\cdot c_k = k!c_k$$

and since $k \ge n$ it follows that $n! \mid k!$ and hence $n! \mid g^{(k)}(0)$.

Clearly, $g(x) = g(1-x)$. Differentiating k times produces

$$g^{(k)}(x) = (-1)^k g^{(k)}(1-x),$$

which gives $g^{(k)}(1) = (-1)^k g^{(k)}(0)$ and therefore as $n! \mid g^{(k)}(0)$ then $n! \mid g^{(k)}(1)$. $\square$

LEMMA 9. For any real number x, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

Proof. Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and this series converges by the ratio test for all x. Thus the n^{th} term, $\frac{x^n}{n!}$, must go to zero as n goes to infinity. $\square$

THEOREM 10. π^2 is irrational

Proof. Suppose π^2 is rational then there exist positive integers a and b such that

$$\pi^2 = \frac{a}{b}.$$

Let

$$G(x) = b^n\left\{\pi^{2n} f(x) - \pi^{2n-2} f''(x) + \pi^{2n-4} f^{(4)}(x) - \ldots + (-1)^n f^{(2n)}(x)\right\} \tag{1.4.1}$$

where $f(x) = \dfrac{x^n(1-x)^n}{n!}$. Now, $f^{(k)}(x)$ for $k < n$ will contain terms with factors x and $1-x$. So for all $k < n$, $f^{(k)}(0) = 0$ and $f^{(k)}(1) = 0$. As a consequence of Lemma 8 $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers, for $k \geq n$. Therefore $G(0)$ and $G(1)$ are integers.

Differentiating $G'(x)\sin(\pi x) - \pi G(x)\cos(\pi x)$ produces

$$\frac{d}{dx}\left(G'(x)\sin(\pi x) - \pi G(x)\cos(\pi x)\right)$$

$$= G''(x)\sin(\pi x) + \pi G'(x)\cos(\pi x) - \pi G'(x)\cos(\pi x) + \pi^2 G(x)\sin(\pi x),$$

$$= \left\{G''(x) + \pi^2 G(x)\right\}\sin(\pi x). \tag{1.4.2}$$

Differentiating (1.4.1), twice, produces

$$G''(x) = b^n\left\{\pi^{2n} f''(x) - \pi^{2n-2} f^{(4)}(x) + \ldots (-1)^n f^{(2n-2)}(x)\right\}.$$

As $f(x)$ is a polynomial of degree $2n$ the last term disappears and thus

$$G''(x) + \pi^2 G(x) = b^n\left\{\pi^{2n-2} f(x)\right\}.$$

Substituting in (1.4.2) gives

$$b^n \pi^{2n-2} f(x)\sin(\pi x) = b^n\left(\pi^2\right)^n \pi^2 f(x)\sin(\pi x),$$

$$= \pi^2 b^n \frac{a^n}{b^n} f(x)\sin(\pi x),$$

$$= \pi^2 a^n f(x)\sin(\pi x).$$

Normalizing by $\frac{1}{\pi}$ and integrating this expression between 0 and 1 gives

$$\pi\int_0^1 a^n \sin(\pi x) f(x)dx = \frac{1}{\pi}\int_0^1 \frac{d}{dx}\{G'(x)\sin(\pi x) - \pi G(x)\cos(\pi x)\}dx.$$

As all the functions in this integrand are continuous, by the fundamental theorem of calculus the integral becomes

$$= \left[\frac{G'(x)\sin(\pi x)}{\pi} - G(x)\cos(\pi x)\right]_0^1$$

$$= \frac{G'(1)\cdot 0}{\pi} - \frac{G'(0)\cdot 0}{\pi} - G(1)(-1) + G(0)\cdot 1$$

(1.4.3) $$= G(1) + G(0).$$

As shown at the beginning of the proof, this result must be an integer. However $0 < f(x) < \frac{1}{n!}$ for $0 < x < 1$ so therefore

$$0 < \pi\int_0^1 a^n \sin(\pi x) f(x)dx < \frac{\pi a^n}{n!}.$$

From Lemma 9 $\frac{a^n}{n!} \to 0$ as $n \to \infty$ so for all sufficiently large n, $\frac{a^n}{n!} < \frac{1}{\pi}$ which implies $\frac{\pi a^n}{n!} < 1$. Thus it follows that, for some n,

$$0 < \pi \int_0^1 a^n \sin(\pi x) f(x) dx < 1.$$

However, from (1.4.3) it is known that $\pi \int_0^1 a^n \sin(\pi x) f(x) dx$ is an integer which is a contradiction. Therefore the original assumption was incorrect and π^2 is irrational. $\square$

COROLLARY 11. π is irrational.

Proof. Suppose π is rational then $\pi = \frac{a}{b}$, where a and b are integers. This implies that $\pi^2 = \frac{a^2}{b^2}$ which would imply that π^2 is rational which contradicts the theorem above. Thus π is irrational. $\square$

Note that the converse deduction is not possible by this argument.

THEOREM 12. $\zeta(2) = \frac{\pi^2}{6}$

Proof. The Fourier series for $f(x) = x^2$ will be constructed. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right),$$

where a_n and b_n are the Fourier coefficients which are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad n \geq 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad n \geq 1.$$

Evaluating a_n for $f(x) = x^2$ produces, after integrating by parts twice,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 d\left[\frac{1}{n}\sin(nx)\right],$$

$$= \frac{x^2}{\pi}\cdot\frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{n}\sin(nx)\cdot 2x dx.$$

$$= 0 - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{2x}{n} d\left(-\frac{1}{n}\cos(nx)\right).$$

$$= \frac{2x}{\pi n^2}\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} -\frac{1}{n}\cos(nx)\cdot\frac{2}{n}dx.$$

$$= (-1)^n\frac{2}{n^2} + (-1)^n\frac{2}{n^2} - \left[\frac{2}{n^3}\sin(nx)\right]_{-\pi}^{\pi}.$$

$$= \frac{4}{n^2}(-1)^n - 0,$$

$$= \frac{4}{n^2}(-1)^n.$$

Evaluating b_n for $f(x) = x^2$ similarly, gives

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 d\left[\frac{1}{n}(-\cos(nx))\right],$$

$$= \frac{x^2}{\pi} \cdot \frac{1}{n}\left(-\cos(nx)\right)\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{n}\left(-\cos(nx)\right) \cdot 2x\,dx,$$

$$= \frac{-\pi^2}{\pi n}(-1)^n - \frac{-\pi^2}{\pi n}(-1)^n + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{2x}{n} d\left(\frac{1}{n}\sin(nx)\right),$$

$$= 0 + \frac{2x}{\pi n^2}\sin(nx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{n^2}\sin(nx) \cdot 2dx,$$

$$= 0 + 0 + \frac{2}{n^2}\cos(nx)\Big|_{-\pi}^{\pi},$$

$$= \frac{2}{n^2}(-1)^n - \frac{2}{n^2}(-1)^n,$$

$$= 0.$$

Alternatively, $b_n = 0$ using the fact that $f(x)$ is an even function. Evaluating a_0 for $f(x) = x^2$ gives

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{-\pi^3}{3}\right] = \frac{2}{3}\pi^2.$$

Thus, the Fourier series for x^2 is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx).$$

Putting $x = \pi$ gives

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2},$$

$$= \frac{\pi^2}{3} + 4\zeta(2).$$

Therefore $\frac{2}{3}\pi^2 = 4\zeta(2)$ and so $\zeta(2) = \frac{\pi^2}{6}$. $\square$

THEOREM 13. $\zeta(2)$ is irrational.

Proof. Suppose $\zeta(2)$ is rational then there exist integers a and b such that $\zeta(2) = \frac{a}{b}$.

From theorem 12, $\zeta(2) = \frac{\pi^2}{6}$ which means that

$$\frac{a}{b} = \frac{\pi^2}{6},$$

which implies that $\pi^2 = \frac{6a}{b}$ which is rational which implies a contradiction and therefore

$\zeta(2)$ is irrational. $\square$

II. Apery's Proof of the Irrationality of $\zeta(3)$ using Van Der Poorten's Paper

2.1. Introduction.

Using Van der Poorten's paper, A Proof that Euler Missed ... Apéry's Proof of the Irrationality of $\zeta(3)$, this chapter presents a formal proof of the irrationality of $\zeta(3)$. An attempt is made at explaining some of the mysteries of the proof. The proof defines integers X_n, Y_n and a function $f(n)$ which converges to 0 as $n \to \infty$. It is then proved that given $\varepsilon > 0$ an N can be found such that

$$0 < X_n\zeta(3) - Y_n < f(n) = \frac{4 \cdot 27^n}{28^n} < \varepsilon \text{ for } n > N,$$

and from Theorem 6 in the first chapter this proves the irrationality of $\zeta(3)$.

2.2 The Formal Proof of the Irrationality of $\zeta(3)$.

DEFINITION. Using the notation $lcm\{1,2,3,\ldots,n\}$ to mean the lowest common multiple of $\{1, 2, 3, 4, 5, \ldots, n\}$, then $d_n = lcm\{1,2,3,\ldots,n\}$.

LEMMA 14. Using the definition of d_n, above, $d_n = \prod_{\substack{p \le n \\ p \text{ prime}}} p^{\lfloor \ln n / \ln p \rfloor}$.

Proof. Letting p be a prime number then for some real number, x, $n = p^x$ implies $\ln n = x \ln p$ which implies $x = \ln n / \ln p$. Thus $[x] = [\ln n / \ln p]$ will be the highest integer power of p for which $p^{[x]} < n$.

Given a set of positive integers $A_1, A_2, A_3, \ldots, A_n$ where $A_i = p_1^{y_{i1}} p_2^{y_{i2}} \ldots p_j^{y_{ij}} \ldots p_\alpha^{y_{i\alpha}}$, $i = 1 \ldots n$, p_j being a prime, $j = 1, \ldots, \alpha$, $p_\alpha > A_i$, for all A_i, $i = 1 \ldots n$, (note that y_j will be 0 for some of the p_j), then

$$lcm\{ A_1, A_2, A_3, \ldots, A_n \} = p_1^{m_1} p_2^{m_2} \ldots p_j^{m_j} \ldots p_\alpha^{m_\alpha}$$

where $m_j = \max\{y_{ij}, i = 1..n\}$. In the special case where $A_i = i$, $m_j = \lfloor \ln n / \ln p_j \rfloor$.

DEFINITION. The number of primes less than or equal n is defined to be $\pi(n)$.

LEMMA 15. The following is true

$$\prod_{\substack{p \le n \\ \text{p prime}}} p^{\ln n / \ln p} = n^{\pi(n)}.$$

Proof. Letting $p^{\ln n / \ln p} = y$, then $\frac{\ln n}{\ln p} \ln p = \ln y$ which implies $n = y$. Thus

$$\prod_{\substack{p \le n \\ p \text{ a prime}}} p^{\ln n / \ln p} = \text{the product of } \pi(n)\ n\text{'s or } n^{\pi(n)}. \qquad \square$$

LEMMA 16. For sufficiently large n,

$$n^{\pi(n)} < 3^n,$$

where n is a positive integer.

Proof. From the prime number theorem, given $\varepsilon > 0$ there exists a positive integer N such that for all $n > N$

$$\pi(n) < \frac{n}{\log n} + \frac{\varepsilon n}{\log n}.$$

Choose $\varepsilon = \log 3 - 1$ then there exists a positive integer N' such that

$$\begin{aligned} \pi(n) &< \frac{n}{\log n} + \frac{n \log 3}{\log n} - \frac{n}{\log n}, \quad \text{for all } n > N' \\ &= \frac{n \log 3}{\log n}. \end{aligned}$$

Thus for these n

$$n^{\pi(n)} < n^{\frac{n \log 3}{\log n}}.$$

Letting $X = n^{\frac{n \log 3}{\log n}}$, then $\log X = \frac{n \log 3}{\log n} \log n$ which implies $\log X = \log 3^n$

which implies $X = 3^n$.

Thus

$$n^{\pi(n)} < 3^n \quad \text{for sufficiently large } n. \qquad \square$$

COROLLARY 17. Given an integer, n, $d_n < 3^n$.

Proof. From Lemmas 14 and 15, $d_n = \prod_{\substack{p \le n \\ p \text{ a prime}}} p^{[\ln n / \ln p]} < \prod_{\substack{p \le n \\ p \text{ a prime}}} p^{\ln n / \ln p} = n^{\pi(n)}$.

Therefore $d_n < n^{\pi(n)}$ and thus from Lemma 16, $d_n < 3^n$ □

THEOREM 18. Letting $\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$, then $\zeta(3)$ is irrational.

Proof. Let $a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k}$, where $c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}$. Letting

$d_n = lcm\{1,2,3,\ldots,n\}$ it is shown in Corollary 20 that d_n^3 divides the denominator of a_n.

Next it is proved in Lemma 25 that a_n satisfies the recurrence relation

(2.2.1) $$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5) u_{n-1}, \; n \ge 2.$$

Letting $b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ then clearly b_n is an integer. It is then shown in Lemma 26 that b_n satisfies the same recurrence relation (2.2.1). Using the fact that both a_n and b_n satisfy the recurrence relation (2.2.1) the following is proved using Lemma 27 and 30

$$\left|\zeta(3) - \frac{a_n}{b_n}\right| = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}},$$

and then Lemma 32 proves the irrationality of $\zeta(3)$. The proof of Lemma 32 uses the fact that d_n^3 divides the denominator of a_n. □

LEMMA 19. Using the previous definitions of $c_{n,k}$ and d_n, if

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}\ ,\ k \le n \text{ and } d_n = lcm\{1,2,3,\ldots,n\} \text{ then } 2c_{n,k}\binom{n+k}{k} \text{ is a}$$

rational number whose denominator expressed in lowest terms divides $d_n^{\,3}$.

Proof. First it is shown that

$$\binom{n+k}{k} \Big/ \binom{n+m}{m} = \binom{n+k}{k-m} \binom{k}{m}. \tag{2.2.2}$$

Writing the left hand side in full produces

$$\frac{(n-k)!m!n!}{k!n!(n-m)!} = \frac{(n+k)!m!}{k!(n+m)!} = \frac{(n+k)!m!(k-m)!}{(k-m)!(n+m)!k!} = \frac{(n-k)!}{(k-m)!(n-m)!} \Big/ \frac{k!}{m!(k-m)!}$$
$$= \binom{n+k}{k-m} \Big/ \binom{k}{m}.$$

The number of times any given prime p divides the denominator is checked to see if that is less than the number of times p divides $d_n^{\,3}$. Let $ord_p x$ mean the highest exponent of p that divides x then, for $m \le n$,

$$ord_p \binom{n}{m} = ord_p \left[\frac{n(n+1)\cdots(n-m+1)}{m!} \right] \le ord_p \left[\frac{n!}{m!} \right] = ord_p(n!) - ord_p(m!)$$

$$\le ord_p(d_n) - ord_p m = \left[\frac{\log n}{\log p} \right] - ord_p m,$$

where $[m]$ denotes the greatest integer part of m. Next, it is checked that the denominator of each term in the expression for $2c_{n,k}\binom{n+k}{k}$ divides $d_n^{\,3}$. The terms of

$2\binom{n+k}{k}\sum_{m=1}^{n}\frac{1}{m^3}$ do not need to be checked. The following do need to be checked: the terms

of $2\binom{n+k}{k}\sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}$. The denominators of these terms are

$$m^3\binom{n}{m}\binom{n+m}{m}\binom{n+k}{k}\cdot 2$$

and using (2.2.2) this becomes

$$m^3\binom{n}{m}\binom{k}{m}\binom{n+k}{k-m}\cdot 2.$$

Thus,

$$ord_p\left[m^3\binom{n}{m}\binom{k}{m}\binom{n+k}{k-m}\cdot 2\right]$$

$$\leq 3ord_p m - \left\{\left[\frac{\log n}{\log p}\right] - ord_p m\right\} - \left\{\left[\frac{\log k}{\log p}\right] - ord_p m\right\} - ord_p\binom{n+k}{k-m} - ord_p 2.$$

$$\leq \left[\frac{\log n}{\log p}\right] + \left[\frac{\log k}{\log p}\right] + ord_p m.$$

As $m \leq k \leq n$ then

$ord_p m \leq ord_p d_m \leq ord_p d_k \leq ord_p d_n$ and therefore

$$\left[\frac{\log n}{\log p}\right] + \left[\frac{\log k}{\log p}\right] + ord_p m \leq \left[\frac{\log n}{\log p}\right] + \left[\frac{\log k}{\log p}\right] + ord_p d_n.$$

Now, from Lemmas 14 and 15, $ord_p d_x = \left[\frac{\log x}{\log p}\right]$ and thus

$$\left[\frac{\log n}{\log p}\right] + \left[\frac{\log k}{\log p}\right] + ord_p d_n \leq 3\left[\frac{\log n}{\log p}\right] = ord_p d_n^3.$$ □

COROLLARY 20. Using the previous definitions for a_n and d_n, the denominator of a_n divides d_n^3.

Proof. The definition of a_n is that

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k}.$$

From Lemma 19 the denominator of $2c_{n,k}\binom{n+k}{k}$ divides d_n^3. Therefore so does the denominator of a_n. $\square$

LEMMA 21. The equation

$$n^3 u_n + (n-1)^3 u_{n-2} = \left(34n^3 - 51n^2 + 27n - 5\right)u_{n-1},\ n > 2,$$

is equivalent to

$$(n+1)^3 u_n - \left(34n^3 + 51n^2 + 27n + 5\right)u_n + n^3 u_{n-1} = 0,\ n > 1.$$

Proof. Letting $n = n+1$ in the first equation, above, gives

$$(n+1)^3 u_{n+1} + n^3 u_{n-1} = \left[34(n+1)^3 - 51(n+1)^2 + 27(n+1) - 5\right]u_n,\ \ n > 1.$$

This gives

$$(n+1)^3 u_{n+1} + n^3 u_{n-1} = \left[34n^3 + 102n^2 + 102n + 34 - 51n^2 - 102n - 51 + 27n + 27 - 5\right]u_n.$$

Simplifying and rearranging produces

$$(n+1)^3 u_{n+1} - \left[34n^3 + 51n^2 + 27n + 5\right]u_n + n^3 u_{n-1} = 0,\ \ n > 1. \quad \square$$

LEMMA 22. If $B_{n,k} = 4(2n+1)\left[k(2k+1)-(2n+1)^2\right]\binom{n}{k}^2\binom{n+k}{k}^2$ then

$$B_{n,k} - B_{n,k-1} =$$

$$(n+1)^3\binom{n+1}{k}^2\binom{n+1+k}{k}^2 - \left(34n^3+51n^2+27n+5\right)\binom{n}{k}^2\binom{n+k}{k}^2 + n^3\binom{n-1}{k}^2\binom{n-1+k}{k}^2.$$

Proof. From the definitions of combinations the following are true:

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k} \tag{2.2.3}$$

$$\binom{n}{k} = \frac{n-k+1}{n+1}\binom{n+1}{k} \tag{2.2.4}$$

$$\binom{n+k}{k} = \frac{n+k}{n}\binom{n+k-1}{k} \tag{2.2.5}$$

$$\binom{n+k}{k} = \frac{n+1}{n+k+1}\binom{n+k+1}{k} \tag{2.2.6}$$

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1} \tag{2.2.7}$$

$$\binom{n+k}{k} = \frac{n+k}{k}\binom{n+k-1}{k-1} \tag{2.2.8}$$

The first identity will be proved only.

Thus

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k!(n-k)(n-k-1)!} = \frac{n}{n-k}\binom{n-1}{k}$$

and the first identity is proved.

Returning to the main identity that is to be proved the left hand side of the identity is worked on first. Using the definition of $B_{n,k}$ produces

$$B_{n,k} - B_{n,k-1} = 4(2n+1)\left[k(2k+1)-(2n+1)^2\right]\binom{n}{k}^2\binom{n+k}{k}^2$$

$$-4(2n+1)\left[(k-1)(2k-1)-(2n+1)^2\right]\binom{n}{k-1}^2\binom{n+k-1}{k-1}^2$$

Using (2.2.7) and (2.2.8) this becomes

$$\binom{n}{k}^2\binom{n+k}{k}^2(8n+4)\Big\{\left[2k^2+k-4n^2-4n-1\right]$$

$$-\left[2k^2-3k+1-4n^2-4n-1\right]\frac{k^2}{(n-k+1)^2}\frac{k^2}{(n+k)^2}\Big\}$$

Next, the right hand side of the identity we are trying to prove is worked on and thus

$$(n+1)^3\binom{n+1}{k}^2\binom{n+1+k}{k}^2-\left(34n^3+51n^2+27n+5\right)\binom{n}{k}^2\binom{n+k}{k}^2+n^3\binom{n-1}{k}^2\binom{n-1+k}{k}^2$$

Using (2.2.3), (2.2.4), (2.2.5) and (2.2.6) this becomes

$$(n+1)^3\frac{(n+1)^2}{(n-k+1)^2}\frac{(n+k+1)^2}{(n+1)^2}\binom{n}{k}^2\binom{n+k}{k}^2-\left(34n^3+51n^2+27n+5\right)\binom{n}{k}^2\binom{n+k}{k}^2$$

$$+n^3\frac{(n-k)^2}{n^2}\frac{n^2}{(n+k)^2}\binom{n}{k}^2\binom{n+k}{k}^2$$

$$=\frac{\binom{n}{k}^2\binom{n+k}{k}^2}{(n-k+1)^2(n+k)^2}\Big\{(n+1)^3(n+k+1)^2(n+k)^2$$

$$-\left(34n^3+51n^2+27n+5\right)(n+k)^2(n-k+1)^2+n^3(n-k)^2(n-k+1)^2\Big\}$$

The main identity to be proved now reduces to

$$(8n+4)\Big\{\left[2k^2+k-4n^2-4n-1\right](n-k+1)^2(n+k)^2-\left[2k^2-3k-4n^2-4n\right]k^4\Big\}$$

$$= (n+1)^3 (n+k+1)^2 (n+k)^2 - \left(34n^3 + 51n^2 + 27n + 5\right)(n+k)^2 (n-k+1)^2$$

$$+ n^3 (n-k)^2 (n-k+1)^2 .$$

This polynomial identity can be proved to be true by expanding both sides either by hand or more easily by using Mathematica.

Looking at some of the coefficients produces a convincing argument that this is correct.

	Left hand side	Right hand side
n^7	-32	$1-34-1=-32$
k^2	$4(2-2)=0$	0
k^3	$4(1-4+3)=0$	0
k^4	$4(-1-2+2)=-4$	$1-5=-4$
constant	0	0
n	0	0
n^2	-4	$1-5=-4$
n^3	$-16-8-8=-32$	$3+2-27-10=-32$
n^4	$-16-32-4-32-16=-100$	$1+3-6-4-32-16-16-32=-100$

$\square$

LEMMA 23. Using the previous definition of $c_{n,k}$, that is

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \text{ then } c_{n,k} - c_{n-1,k} = \frac{(-1)^k k!^2 (n-k-1)!}{n^2 (n+k)!} .$$

Proof. Using the definition of $c_{n,k}$ gives the following

$$c_{n,k} - c_{n-1,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}} - \sum_{m=1}^{n-1} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n-1}{m}\binom{n+m-1}{m}}$$

$$= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{\frac{2m^3 n!(n+m)!}{m!(n-m)!m!n!}} - \sum_{m=1}^{k} \frac{(-1)^{m-1}}{\frac{2m^3(n-1)!(n-1+m)!}{m!(n-1-m)!m!(n-1)!}}$$

$$= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^m(-1)(m-1)!^2(n-m)!}{(n+m)!2m} + \sum_{m=1}^{k} \frac{(-1)^m(m-1)!^2(n-1-m)!}{2m(n-1+m)!}$$

$$= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^m(m-1)!^2\{(n-1-m)!(n+m)! - (n-m)!(n-1+m)!\}}{2m(n+m)!(n-1+m)!}$$

$$= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^m(m-1)!^2(n-1+m)!(n-1-m)!\{n+m-(n-m)\}}{2m(n+m)!(n-1+m)!}$$

(2.2.9)
$$= \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^m(m-1)!^2(n-1-m)!}{(n+m)!}.$$

The fraction $\frac{(m-1)!^2(n-1-m)!(-1)^m}{(n+m)!}$ can be split up into two fractions with the purpose of telescoping.

Thus

$$\frac{(-1)^m(m-1)!^2(n-1-m)!}{(n+m)!} = \frac{(-1)^m(m-1)!^2(n-1-m)!\left[m^2+n^2-m^2\right]}{n^2(n+m)!}$$

$$= \frac{(-1)^m(m-1)!^2(n-1-m)!\left[m^2-(-1)(n-m)(n+m)\right]}{n^2(n+m)!}$$

$$= \frac{(-1)^m m!^2(n-1-m)! \; -(-1)^{m-1}(m-1)!^2(n-m)!(n+m)}{n^2(n+m)!}$$

$$= \frac{(-1)^m m!^2(n-1-m)!}{n^2(n+m)!} - \frac{(-1)^{m-1}(m-1)!^2(n-m)!}{n^2(n+m-1)!}$$

Thus (2.2.9) becomes

$$(2.2.10) \qquad = \frac{1}{n^3} + \sum_{m=1}^{k} \frac{(-1)^m m!^2(n-1-m)!}{n^2(n+m)!} - \frac{(-1)^{m-1}(m-1)!^2(n-m)!}{n^2(n+m-1)!}$$

The only difference between the first fraction and the second in the summation is that m in the first has been replaced with $m-1$ in the second and therefore if these two fractions are summed from 1 to k they will telescope giving

$$\frac{(-1)^k k!^2(n-k-1)!}{n^2(n+k)!} - \frac{(n-1)!}{n^2 \cdot n!} = \frac{(-1)^k k!^2(n-k-1)!}{n^2(n+k)!} - \frac{1}{n^3}$$

Substituting in (2.2.10) gives

$$c_{n,k} - c_{n-1,k} = \frac{(-1)^k k!^2(n-k-1)!}{n^2(n+k)!} \qquad \square$$

LEMMA 24. If $b_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$, then from earlier,

$B_{n,k} = 4(2n-1)\left(k(2k+1)-(2n+1)^2\right)b_{n,k}$. Using this and the previous definition of $c_{n,k}$,

that is, $c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}$ then

(2.2.11) $\left(B_{n,k}-B_{n,k-1}\right)c_{n,k}+(n+1)^3 b_{n-1,k}\left(c_{n-1,k}-c_{n,k}\right)-n^3 b_{n-1,k}\left(c_{n,k}-c_{n-1,k}\right)=A_{n,k}-A_{n,k-1}$

where $A_{n,k}=B_{n,k}c_{n,k}+\dfrac{5(2n+1)(-1)^{k-1}k}{n(n+1)}\binom{n}{k}\binom{n+k}{k}$.

Proof. The left side of the identity (2.2.11) is expanded term by term. The first term becomes

$$\left(B_{n,k}-B_{n,k-1}\right)c_{n,k}=B_{n,k}c_{n,k}-B_{n,k-1}c_{n,k}$$

$$=B_{n,k}c_{n,k}-B_{n,k-1}\left[c_{n,k-1}+\frac{(-1)^{k-1}}{2k^3\binom{n}{k}\binom{n+k}{k}}\right]$$

$$=B_{n,k}c_{n,k}-B_{n,k-1}c_{n,k-1}$$

(2.2.12) $$-4(2n+1)\left[(k-1)(2k-1)-(2n+1)^2\right]\binom{n}{k-1}^2\binom{n+k-1}{k-1}^2\cdot\frac{(-1)^{k-1}}{2k^3\binom{n}{k}\binom{n+k}{k}}.$$

Now, $-4(2n+1)\left[(k-1)(2k-1)-(2n+1)^2\right]\binom{n}{k-1}^2\binom{n+k-1}{k-1}^2\cdot\dfrac{(-1)^{k-1}}{2k^3\binom{n}{k}\binom{n+k}{k}}$

$$=-4(2n+1)\left[2k^2-3k+1-4n^2-4n-1\right]\binom{n}{k-1}^2\binom{n+k-1}{k-1}^2\cdot\frac{(-1)^{k-1}}{2k^3\binom{n}{k}\binom{n+k}{k}}$$

$=$

$$(-1)^{k-1}(-2)(2n+1)\left[2k^2-3k-4n^2-4n\right]\frac{k!(n-k)!k!n!n!^2(n+k-1)!^2}{k^3n!(n+k)!(k-1)!^2(n-k+1)!^2(k-1)!^2n!^2}$$

$=$

$$(-1)^{k-1}(-2)(2n+1)\left[2k^2-3k-4n^2-4n\right]\frac{(n+k-1)!}{k(n+k)(k-1)!^2(n-k+1)(n-k+1)!}$$

Finally, after substituting in (2.2.12) the first term becomes

$$B_{n,k}c_{n,k}-B_{n,k-1}c_{n,k-1}$$

$$(-1)^{k-1}(-2)(2n+1)\left[2k^2-3k-4n^2-4n\right]\frac{(n+k-1)!}{k(n+k)(k-1)!^2(n-k+1)(n-k+1)!}.$$

Using Lemma 23 the second term of the identity (2.2.11) becomes

$$(n+1)^3\binom{n+1}{k}^2\binom{n+k+1}{k}^2\cdot\frac{(-1)^k k!^2(n-k)!}{(n+1)^2(n+k+1)!}$$

$$=\frac{(-1)^k(n+1)(n+1)!(n+k+1)!^2k!^2(n-k)!}{k!^2(n+1-k)!^2k!^2(n+1)!^2(n+k+1)!}$$

$$=\frac{(-1)^k(n+1)(n+k+1)!}{(n+1-k)!(n+1-k)k!^2}.$$

Using Lemma 23 the third term of the identity (2.2.11) becomes

$$-n^3\binom{n-1}{k}^2\binom{n+k-1}{k}^2\cdot\frac{(-1)^k k!^2(n-k-1)!}{n^2(n+k)!}$$

$$=\frac{(-1)^k(-n)(n-1)!^2(n+k-1)!^2k!^2(n-k-1)!}{k!^2(n-1-k)!^2k!^2(n-1)!^2(n+k)!}$$

$$=\frac{(-1)^k(-n)(n+k-1)!}{(n-1-k)!(n+k)k!^2}.$$

Putting all three terms of the left hand side of (2.2.11) together this becomes

$$B_{n,k}c_{n,k}-B_{n,k-1}c_{n,k-1}=\frac{(-1)^{k-1}(2)(2n+1)\left[2k^2-3k-4n^2-4n\right](n+k-1)!}{k(n+k)(k-1)!^2(n-k+1)(n-k+1)!}$$

$$+\frac{(-1)^k(n+1)(n+k+1)!}{(n+1-k)!(n+1-k)k!^2}-\frac{(-1)^k n(n+k-1)!}{(n-1-k)!(n+k)k!^2}.$$

Taking out common factors from the last three terms this becomes

$$B_{n,k}c_{n,k}-B_{n,k-1}c_{n,k-1}-\frac{(-1)^k(n+1)(n+k-1)!}{(n+1-k)!(n+1-k)k!^2}$$

$$\{2(2n+1)\left[2k^2-3k-4n^2-4n\right]k+(n+k)(n+1)(n+k+1)(n+k)$$

$$-n(n+1-k)(n+1-k)(n-k)\}.$$

Simplifying the inside of the braces using Mathematica or by hand this expression is

$$10nk^3-10nk^2-10n^3k-15n^2k+5k^3-5k^2-5nk$$

$$=5k\left[2nk^2-2nk-2n^3-3n^2+k^2-k-n\right]$$

$$=5k(2n+1)\left(k^2-k-n^2-n\right)$$

$$=5k(2n+1)(n+k)(n-k+1)(-1).$$

The left hand side of (2.2.11) now becomes

$$B_{n,k}c_{n,k}-B_{n,k-1}c_{n,k-1}-\frac{(-1)^k(n+k-1)!}{(n+1-k)!(n+1-k)k!^2(n+k)}(-1)(5k)(2n+1)(n+k)(n-k+1)$$

(2.2.13) $$=B_{n,k}c_{n,k}-B_{n,k-1}c_{n,k-1}-\frac{(-1)^{k-1}(5k)(2n+1)(n+k-1)!}{k!^2(n+1-k)!}.$$

Looking at the right hand side of (2.2.11) the following is produced

$$B_{n,k}c_{n,k}+\frac{(-1)^{k-1}(5k)(2n+1)}{n(n+1)}\binom{n}{k}\binom{n+k}{k}-B_{n,k-1}c_{n,k-1}+\frac{(-1)^{k-2}5(k-1)(2n+1)}{n(n+1)}\binom{n}{k-1}\binom{n+k-1}{k-1}$$

$$= B_{n,k}c_{n,k} - B_{n,k-1}c_{n,k-1} - \frac{(-1)^{k-1}5(2n+1)}{n(n+1)}\left[k\binom{n}{k}\binom{n+k}{k} + (k-1)\binom{n}{k-1}\binom{n+k-1}{k-1}\right]$$

$$= B_{n,k}c_{n,k} - B_{n,k-1}c_{n,k-1} - \frac{(-1)^{k-1}5(2n+1)}{n(n+1)}\left[\frac{kn!(n+k)!}{k!(n-k)!k!n!} + \frac{(k-1)n!(n+k-1)!}{(k-1)!(n-k+1)!(k-1)!n!}\right]$$

$$= B_{n,k}c_{n,k} - B_{n,k-1}c_{n,k-1} - \frac{(-1)^{k-1}5(2n+1)}{n(n+1)}\left[\frac{k(n+k)!}{k!^2(n-k)!} + \frac{(k-1)(n+k-1)!}{(k-1)!^2(n-k+1)!}\right]$$

$$= B_{n,k}c_{n,k} - B_{n,k-1}c_{n,k-1} - \frac{(-1)^{k-1}5(2n+1)(n+k-1)!}{n(n+1)k!^2(n+1-k)!}\left[(n+1-k)k(n+k) + k^2(k-1)\right]$$

$$= B_{n,k}c_{n,k} - B_{n,k-1}c_{n,k-1} - \frac{(-1)^{k-1}5(2n+1)(n+k-1)!}{n(n+1)k!^2(n+1-k)!}\left[n^2k + kn\right]$$

$$= B_{n,k}c_{n,k} - B_{n,k-1}c_{n,k-1} - \frac{(-1)^{k-1}5(2n+1)(n+k-1)!kn(n+1)}{n(n+1)k!^2(n+1-k)!}$$

$$= B_{n,k}c_{n,k} - B_{n,k-1}c_{n,k-1} - \frac{(-1)^{k-1}5k(2n+1)(n+k-1)!}{k!^2(n+1-k)!}$$

LEMMA 25. Using the previous definitions of a_n and $c_{n,k}$, if $a_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 c_{n,k}$,

where $c_{n,k} = \sum_{m=1}^{n}\frac{1}{m^3} + \sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}$ then a_n satisfies the recurrence relation

$$n^3u_n + (n-1)^3u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1},\ n \geq 2.$$

Proof. From Lemma 21

$$n^3 u_n + (n-1)^3 u_{n-2} = \left(34n^3 - 51n^2 + 27n - 5\right) u_{n-1},\ n \ge 2$$

is equivalent to

$$(n+1)^3 u_{n+1} - \left(34n^3 + 51n^2 + 27n + 5\right) u_n + n^3 u_{n-1} = 0,\ n \ge 1.$$

Substituting a_n for u_n in the above identity produces

(2.2.14) $$(n+1)^3 \sum_{k=0}^{n+1} b_{n+1,k} c_{n+1,k} - P(n) \sum_{k=0}^{n} b_{n,k} c_{n,k} + n^3 \sum_{k=0}^{n-1} b_{n-1,k} c_{n-1,k} = 0,\ n \ge 1,$$

where $b_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$ and $P(n) = 34n^3 + 51n^2 + 27n + 5$.

Using the convention that $\binom{n}{r} = 0$ for $r > n$, (2.2.14) can be rewritten as

(2.2.15) $$\sum_{k=0}^{n+1} \left\{ (n+1)^3 b_{n+1,k} c_{n+1,k} - P(n) b_{n,k} c_{n,k} + n^3 b_{n-1,k} c_{n-1,k} \right\} = 0,\ n \ge 1.$$

Equation (2.2.15) will now be proved, which will prove the lemma.

From Lemma 22

$$B_{n,k} - B_{n,k-1} = (n+1)^3 b_{n+1,k} - P(n) b_{n,k} + n^3 b_{n-1,k},$$

where $B_{n,k} = 4(2n+1)\left[k(2k+1) - (2n+1)^2\right] b_{n,k}$, and therefore

$$-P(n) b_{n,k} c_{n,k} = \left(B_{n,k} - B_{n,k-1}\right) c_{n,k} - (n+1)^3 b_{n+1,k} c_{n,k} - n^3 b_{n-1,k} c_{n,k}.$$

Substituting in (2.2.15) produces

$$\sum_{k=0}^{n+1} \left\{ \left(B_{n,k} - B_{n,k-1}\right) c_{n,k} + (n+1)^3 b_{n+1,k} \left(c_{n+1,k} - c_{n,k}\right) - n^3 b_{n-1,k} \left(c_{n,k} - c_{n-1,k}\right) \right\} = 0,\ n \ge 1.$$

From Lemma 24 this becomes

$$\sum_{k=0}^{n+1}\left\{A_{n,k}-A_{n,k-1}\right\}=0,$$

where $A_{n,k}=B_{n,k}c_{n,k}+\dfrac{5(2n+1)(-1)^{k-1}k}{n(n+1)}\binom{n}{k}\binom{n+k}{k}$. Now, $\sum_{k=0}^{n+1}\left\{A_{n,k}-A_{n,k-1}\right\}=A_{n,n+1}-A_{n,-1}$.

Using the definition above $A_{n,n+1}=B_{n,n+1}c_{n,n+1}+\dfrac{5(2n+1)(-1)^{n}(n+1)}{n(n+1)}\binom{n}{n+1}\binom{2n+1}{n+1}$. The first term in this expression has, as a factor,

$$B_{n,n+1}=4(2n+1)\left[(n+1)(2n+3)-(2n+1)^2\right]\binom{n}{n+1}^2\binom{2n+1}{n+1}^2$$

which is zero as $\binom{n}{r}=0$ for $r>n$. The second term in the expression has the factor

$\binom{n}{n-1}$ which is also zero as $\binom{n}{r}=0$ for $r>n$. Therefore $\sum_{k=0}^{n+1}\left\{A_{n,k}-A_{n,k-1}\right\}=0$.

LEMMA 26. Letting $b_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}b_{n,k}$, then b_n satisfies the recurrence relation

$$n^3u_n+(n-1)^3u_{n-2}=\left(34n^3-51n^2+27n-5\right)u_{n-1},\ n\geq 2.$$

Proof. From earlier, $B_{n,k}=4(2n+1)\left(k(2k+1)-(2n+1)^2\right)\binom{n}{k}^2\binom{n+k}{k}^2$. From Lemma 22

$$B_{n,k}-B_{n,k-1}=(n+1)^3\binom{n+1}{k}^2\binom{n+1+k}{k}^2-(34n^3+51n^2+27n+5)\binom{n}{k}^2\binom{n+k}{k}^2+n^3\binom{n-1}{k}^2\binom{n-1+k}{k}^2$$

Let $G_{n,k}$ equal the expression on the right of this equation then

$$B_{n,k} = G_{n,k} + B_{n,k-1} = G_{n,k} + G_{n,k-1} + B_{n,k-2} = \dots .$$

Continuing in this way and using $B_{n,r} = 0$ for $r<0$ the following is derived

$$B_{n,k} = \sum_{r=0}^{k} G_{n,r} .$$

Thus, $$B_{n,n-1} = \sum_{r=0}^{n-1} G_{n,r} = (n+1)^3 \left[\binom{n+1}{n+1}^2 \binom{2n+2}{n+1}^2 + \binom{n+1}{n}^2 \binom{2n+1}{n}^2 + \dots + \binom{n+1}{0}^2 \binom{n+1}{0}^2 \right]$$

$$-(34n^3+51n^2+27n+5)\left[\binom{n}{n}^2 \binom{2n}{n}^2 + \binom{n}{n-1}^2 \binom{2n-1}{n}^2 + \dots + \binom{n}{0}^2 \binom{n}{n}^2 \right]$$

$$+n^3\left[\binom{n-1}{n-1}^2 \binom{2n-2}{n-1}^2 + \binom{n-1}{n-2}^2 \binom{2n-3}{n-2}^2 + \dots + \binom{n-1}{0}^2 \binom{n-1}{0}^2 \right],$$

$$= (n+1)^3 b_{n+1} - (34n^3 + 51n^2 + 27n + 5) b_n + n^3 b_{n-1} .$$

Now, $B_{n,r} = 0$ for $r > n$ so

$$0 = (n+1)^3 b_{n+1} - (34n^3 + 51n^2 + 27n + 5) b_n + n^3 b_{n-1},$$

and thus by Lemma 21, b_n satisfies the recurrence relation.

LEMMA 27. Given that a_n and b_n satisfy the recurrence relation

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5) u_{n-1}, \; n \geq 2$$

then

$$a_n b_{n-1} - a_{n-1} b_n = \frac{1}{n^3}(a_1 b_0 - a_0 b_1) .$$

For a_n and b_n as defined previously, $a_1b_0 - a_0b_1 = 6$.

Proof. Letting $P(n-1) = 34n^3 - 51n^2 + 27n - 5$ and substituting a_n and b_n in the above recurrence relation we get

$$\begin{cases} n^3a_n + (n-1)^3 a_{n-2} = P(n-1)a_{n-1} \\ n^3b_n + (n-1)^3 b_{n-2} = P(n-1)b_{n-1} \end{cases},$$

which gives

$$\begin{cases} n^3a_n - P(n-1)a_{n-1} + (n-1)^3 a_{n-2} = 0 \\ n^3b_n - P(n-1)b_{n-1} + (n-1)^3 b_{n-2} = 0 \end{cases}.$$

Multiplying the first equation by b_{n-1}, the second by a_{n-1} and subtracting gives

$$n^3\left(a_nb_{n-1} - a_{n-1}b_n\right) = (n-1)^3\left(a_{n-1}b_{n-2} - a_{n-2}b_{n-1}\right).$$

Therefore

$$\begin{aligned} a_nb_{n-1} - a_{n-1}b_n &= \frac{(n-1)^3}{n^3}\left(a_{n-1}b_{n-2} - a_{n-2}b_{n-1}\right) \\ &= \frac{(n-1)^3}{n^3}\cdot\frac{(n-2)^3}{(n-1)^3}\left(a_{n-2}b_{n-3} - a_{n-3}b_{n-2}\right) \\ &= \frac{(n-1)^3}{n^3}\cdot\frac{(n-2)^3}{(n-1)^3}\cdot\frac{(n-3)^3}{(n-2)^3}\left(a_{n-3}b_{n-4} - a_{n-4}b_{n-3}\right) \\ &= \frac{(n-(n-1))^3}{n^3}\left(a_{n-(n-1)}b_{n-n} - a_{n-n}b_{n-(n-1)}\right) \\ &= \frac{1}{n^3}\left[a_1b_0 - a_0b_1\right]. \end{aligned}$$

From before by simple calculation it can be seen that $a_1 b_0 - a_0 b_1 = 6 \times 1 - 0 \times 5 = 6$ and therefore

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}. \qquad \square$$

LEMMA 28. If $c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}$, $k \le n$, then

$c_{n,k} \to \zeta(3)$ as $n \to \infty$, uniformly in k.

Proof. This lemma is proved directly as follows

$$(2.2.16) \qquad |c_{n,k} - \zeta(3)| = \left| -\sum_{m=n+1}^{\infty} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right| \le \left| \sum_{m=n+1}^{\infty} \frac{1}{m^3} \right| + \left| \sum_{m=1}^{k} \frac{1}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right|.$$

Now, since $\sum_{m=1}^{\infty} \frac{1}{m^3}$ is convergent, it follows that given $\varepsilon' > 0$, for all $n > N$,

$\left| \sum_{m=n+1}^{\infty} \frac{1}{m^3} \right| < \varepsilon'$. Therefore

$$(2.2.17) \qquad \left| \sum_{m=n+1}^{\infty} \frac{1}{m^3} \right| < 2\varepsilon' \text{ for all } n > N_1.$$

It is necessary to find the smallest value of $2m^3 \binom{n}{m}\binom{n+m}{m}$ for $m \le k \le n$.

Case (i) $m \le \frac{n}{2}$, $m \ge 1$.

Using the fact that $\binom{n}{m}$ increases from $m = 1$ then the lowest value of $2m^3\binom{n}{m}$ occurs when $m = 1$, this value being $2n$. For $m \le \frac{n+m}{2}$ the smallest value of $\binom{n+m}{m}$ occurs when $m = 1$ Therefore the smallest value of $2m^3\binom{n}{m}\binom{n+m}{m}$ for $m \le \frac{n}{2}$ is

$$2\binom{n}{1}\binom{n+1}{1} = 2n(n+1).$$

Case (ii) $m \ge \frac{n}{2}$, $m \ge 1$.

The smallest value of $\binom{n}{m}$ occurs when $m = n$ this value being 1. The smallest value of $\binom{n+m}{m}$ occurs when $m = \left\lceil \frac{n}{2} \right\rceil$ as $\binom{n+m}{m}$ increases for $\left\lceil \frac{n}{2} \right\rceil \le m \le n$. The smallest value of $2m^3$ occurs when $m = \left\lceil \frac{n}{2} \right\rceil$. Thus the smallest value of

$$2m^3\binom{n}{m}\binom{n+m}{m} \ge 2\left\lceil \frac{n}{2} \right\rceil^3 \cdot 1 \cdot \binom{n+\left\lceil \frac{n}{2} \right\rceil}{\left\lceil \frac{n}{2} \right\rceil} \ge \frac{2n^3}{8}\binom{n+\left\lceil \frac{n}{2} \right\rceil}{\left\lceil \frac{n}{2} \right\rceil}.$$

For $n \ge 2$ $\binom{n+\left\lceil \frac{n}{2} \right\rceil}{\left\lceil \frac{n}{2} \right\rceil} \ge \binom{n+1}{1}$ and for $n \ge 3$, $\frac{n^3}{8} > n$. Thus, for $n \ge 3$ the smallest value of

$$2m^3\binom{n}{m}\binom{n+m}{m} \ge 2n(n+1).$$

Thus

(2.2.18) $$\left|\sum_{m=1}^{k}\frac{1}{2m^3\binom{n}{m}\binom{n+m}{m}}\right| \le k\cdot\frac{1}{2n(n+1)} \le \frac{n}{2n(n+1)} = \frac{1}{2(n+1)} \le \frac{1}{2n} < \varepsilon'$$

for all $n > \frac{1}{2\varepsilon'}$.

Let $N_2 = \frac{1}{2\varepsilon'}$ and let $N = \max\{N_1, N_2\}$, then, for $n > N$ it follows that, combining (2.2.16), (2.2.17) and (2.2.18),

$$\left|c_{n,k} - \zeta(3)\right| < 2\varepsilon' + \varepsilon'.$$

Letting $\varepsilon = 3\varepsilon'$ produces

$$\left|c_{n,k} - \zeta(3)\right| < \varepsilon$$

which is the required result. Note that the convergence is independent of *k* so the convergence is uniform. □

LEMMA 29. If $x_{n,k}$ and $y_{n,k}$ are real numbers and $y_{n,k}$ converges uniformly in n to a limit L then

$\dfrac{\sum_{k=0}^{n} x_{n,k} y_{n,k}}{\sum_{k=0}^{n} x_{n,k}}$ converges uniformly in n to the same limit L.

Proof. As $y_{n,k}$ converges uniformly in n to a limit L then, given $\varepsilon > 0$ there exists N such that for all $n > N$ and for all k

$$|y_{n,k} - L| < \varepsilon .$$

Thus for this set of n

$$|y_{n,k} - y_{n,0}| < 2\varepsilon$$

and therefore

$$y_{n,0} - 2\varepsilon < y_{n,k} < y_{n,0} + 2\varepsilon$$

Thus

$$\frac{\sum_{k=0}^{n} x_{n,k} y_{n,k}}{\sum_{k=0}^{n} x_{n,k}} < \frac{\sum_{k=0}^{n} x_{n,k}\left(y_{n,0} + 2\varepsilon\right)}{\sum_{k=0}^{n} x_{n,k}} = \frac{\left(y_{n,0} + 2\varepsilon\right)\sum_{k=0}^{n} x_{n,k}}{\sum_{k=0}^{n} x_{n,k}} = y_{n,0} + 2\varepsilon ,$$

and similarly $\frac{\sum_{k=0}^{n} x_{n,k} y_{n,k}}{\sum_{k=0}^{n} x_{n,k}} > y_{n,0} - 2\varepsilon$. Putting these two statements together produces

$$y_{n,0} - 2\varepsilon < \frac{\sum_{k=0}^{n} x_{n,k} y_{n,k}}{\sum_{k=0}^{n} x_{n,k}} < y_{n,0} + 2\varepsilon \qquad (2.2.19)$$

Now, there exists N' such that $|y_{n,0} - L| < \varepsilon$ for all $n > N'$. Thus

$$L - \varepsilon < y_{n,0} < L + \varepsilon .$$

Substituting in (2.2.19) gives

$$L - 2\varepsilon - \varepsilon < \frac{\sum_{k=0}^{n} x_{n,k} y_{n,k}}{\sum_{k=0}^{n} x_{n,k}} < L + 2\varepsilon + \varepsilon$$

for $n > \max\{N, N'\}$. Thus using $\varepsilon' = 3\varepsilon$ the result follows. $\square$

LEMMA 30. If $a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}$ then

$$\zeta(3) - \frac{a_n}{b_n} = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}$$

Proof. From lemma 27 $a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}$ implies

$$\frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}} = \frac{6}{n^3 b_n b_{n-1}} \tag{2.2.20}$$

Let $\zeta(3) - \frac{a_n}{b_n} = x_n$ then

$$x_n - x_{n-1} = \left[\zeta(3) - \frac{a_n}{b_n}\right] - \left[\zeta(3) - \frac{a_{n-1}}{b_{n-1}}\right] = \frac{a_{n-1}}{b_{n-1}} - \frac{a_n}{b_n} = \frac{6}{(n-1)^3 b_{n-1} b_n}.$$

and therefore

$$\zeta(3) - \frac{a_n}{b_n} = \frac{6}{(n+1)^3 b_{n-1} b_n} + x_{n-1}.$$

Using a similar argument it follows that

$$x_{n-1} - x_{n-2} = \frac{6}{(n+2)^3 b_{n-2} b_{n-1}}.$$

Repeating this process m times

$$\zeta(3)-\frac{a_n}{b_n}=\frac{6}{(n-1)^3 b_{n-1}b_n}+\frac{6}{(n-2)^3 b_{n-2}b_{n-1}}+\ldots+\frac{6}{(n-m)^3 b_{n-m}b_{n-m-1}}+x_{n-m}.$$

Now $\frac{a_n}{b_n}\to\zeta(3)$ (from Lemmas 28 and 29) which means that $\lim_{n\to\infty}x_n=0$ and therefore

$$\zeta(3)-\frac{a_n}{b_n}=\sum_{k=n+1}^{\infty}\frac{6}{k^3 b_k b_{k-1}}$$

From (2.2.20) and the fact that b_n is positive $\frac{a_n}{b_n}$ is increasing and therefore

$$\zeta(3)-\frac{a_n}{b_n}=\zeta(3)-\frac{a_n}{b_n}$$

LEMMA 31. Given $b_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$ then $b_n>\frac{28^n}{n^3}$ for $n>7$.

Proof. This is a proof by induction. For $n=8$.

$$b_8=\sum_{k=0}^{8}\binom{8}{k}^2\binom{8+k}{k}^2$$

$$=\binom{8}{0}^2\binom{8}{0}^2+\binom{8}{1}^2\binom{9}{1}^2+\binom{8}{2}^2\binom{10}{2}^2+\binom{8}{3}^2\binom{11}{3}^2+\binom{8}{4}^2\binom{12}{4}^2+\binom{8}{5}^2\binom{13}{5}^2+\binom{8}{6}^2\binom{14}{6}^2+\binom{8}{7}^2\binom{15}{7}^2+\binom{8}{8}^2\binom{16}{8}^2$$

≈ 16367912430 (using a calculator).

Now,

$$\frac{28^8}{8^3}\approx 737894528.$$

Thus $b_n > \frac{28^n}{n^3}$ for $n = 8$.

Again, using a calculator it is easily shown that $b_n > \frac{28^n}{n^3}$ for $n = 9$.

Now, it is assumed that $b_n > \frac{28^n}{n^3}$ for $n = j-1$ and for $n = j-2$.

Using the recurrence relation

$$n^3 u_n + (n-1)^3 u_{n-2} = \left(34n^3 - 51n^2 + 27n - 5\right) u_{n-1},\ n \geq 2,$$

that b_n satisfies, by Lemma 26, and substituting j for n and b for u, dividing by j^3, and then rearranging it follows that

$$b_j - \left(34 - 51j^{-1} + 27j^{-2} - 5j^{-3}\right) b_{j-1} + \left(1 - 3j^{-1} + 3j^{-2} - j^{-3}\right) b_{j-2} = 0,\ j \geq 2.$$

[Aside. An approximate solution to this recursion could be found from the solution of $b_j - 34b_{j-1} + b_{j-2} = 0$, which is : r^j, where $r^2 - 34r + 1 = 0$ which yields solutions $r = \frac{34 \pm \sqrt{34^2 - 4}}{2}$. However a lower bound is needed on the solution.]

Therefore

$$b_j = b_{j-1}\left(34 - 51j^{-1} + 27j^{-2} - 5j^{-3}\right) - b_{j-2}\left(1 - 3j^{-1} + 3j^{-2} - j^{-3}\right),\ j \geq 2.$$

Using the induction hypotheses and eliminating positive terms it follows that

$$b_j > \frac{28 \cdot 28^{j-1}}{(j-1)^3} + \frac{6 \cdot 28^{j-1}}{(j-1)^3} - \frac{51}{j} \cdot \frac{28^{j-1}}{(j-1)^3} - \frac{5}{j^3} \cdot \frac{28^{j-1}}{(j-1)^3}$$

$$-\frac{28^{j-2}}{(j-2)^3} - \frac{3}{j^2} \cdot \frac{28^{j-2}}{(j-2)^3} - \frac{1}{j^3} \cdot \frac{28^{j-2}}{(j-2)^3} \tag{2.2.21}$$

For $j > 9$

$$\frac{51}{j}\cdot\frac{28^{j-1}}{(j-1)^3} < (5.2)\cdot\frac{28^{j-1}}{(j-1)^3},$$

$$\frac{5}{j^3}\cdot\frac{28^{j-1}}{(j-1)^3} < (.006)\cdot\frac{28^{j-1}}{(j-1)^3},$$

$$\frac{28^{j-2}}{(j-2)^3} = \frac{1\cdot 28^{j-1}}{28\cdot(j-1)^3}\left(\frac{j-1}{j-2}\right)^3 < \frac{1}{28}\cdot\frac{28^{j-1}}{(j-1)^3}\left(\frac{8}{7}\right)^3,$$

as $\left(\frac{j-1}{j-2}\right)^3$ is decreasing as j increases. Thus

$$\frac{28^{j-2}}{(j-2)^3} < \frac{1}{28}\cdot\frac{28^{j-1}}{(j-1)^3}\left(\frac{8}{7}\right)^3 < (.06)\cdot\frac{28^{j-1}}{(j-1)^3}.$$

Similarly,

$$\frac{3}{j^2}\cdot\frac{28^{j-2}}{(j-2)^3} < (.04)\cdot\frac{28^{j-1}}{(j-1)^3}$$

and

$$\frac{1}{j^3}\cdot\frac{28^{j-2}}{(j-2)^3} < (.002)\cdot\frac{28^{j-1}}{(j-1)^3}.$$

Thus (2.2.21) now becomes

$$b_j > \frac{28^j}{(j-1)^3} + \frac{6\cdot 28^{j-1}}{(j-1)^3} - (5.2 + .006 + .06 + .04 + .002)\cdot\frac{28^{j-1}}{(j-1)^3} > \frac{28^j}{(j-1)^3} > \frac{28^j}{j^3}.$$

Thus, the lemma has been proved for $j \geq 8$. $\square$

LEMMA 32. Given $0<\zeta(3)-\frac{a_n}{b_n}=\sum_{k=n+1}^{\infty}\frac{6}{k^3b_kb_{k-1}}$ then $\zeta(3)$ is irrational.

Proof. From Lemma 30

$$0<\zeta(3)-\frac{a_n}{b_n}=\sum_{k=n+1}^{\infty}\frac{6}{k^3b_kb_{k-1}}$$

$$<\frac{1}{b_n^2}\sum_{k=n+1}^{\infty}\frac{6}{k^3}$$

$$<\frac{1}{b_n^2}T,\ \text{where } T \text{ is the constant } 6\zeta(3).$$

Multiplying by b_n it follows that

$$0<b_n\zeta(3)-a_n<\frac{T}{b_n} \tag{2.2.22}$$

It is known that a_n is a fraction whose denominator divides d_n^3 where d_n is the L.C.M. of the first n integers. The following substitutions are made:

$$a_n=\frac{p_n}{q_n}\text{ and } r_nq_n=d_n^3 \text{ where } p_n,\ q_n,\ r_n \text{ are all integers.}$$

Thus (2.2.22) now becomes

$$0<b_n\zeta(3)-\frac{p_nr_n}{d_n^3}<\frac{T}{b_n}.$$

Multiplying by d_n^3 gives

$$0<d_n^3b_n\zeta(3)-p_nr_n<\frac{Td_n^3}{b_n}.$$

From Corollary 17 $d_n^3<27^n$ and d_n^3 is an integer so the inequality now becomes

$$0 < X_n\zeta(3) - Y_n < \frac{T27^n}{b_n}, \text{ where } X_n \text{ and } Y_n \text{ are integers.}$$

From Lemma 31 $b_n > \frac{28^n}{n^3}$, for $n > 2$. Thus for $n > 2$

$$0 < X_n\zeta(3) - Y_n < \frac{T27^n n^3}{28^n}$$

which for large enough n gives

$$0 < X_n\zeta(3) - Y_n < \varepsilon .$$

By Theorem 6 this proves $\zeta(3)$ is irrational. □

Note that the stronger condition $\zeta(3) - \frac{a_n}{b_n} > 0$ is not needed. If it is just known that $\zeta(3) - \frac{a_n}{b_n} > 0$ the lemma would be proved.

2.3. Unraveling Some of the Mysteries

There are several mysteries to Apéry's proof. Firstly, where does $c_{n,k}$ come from? Secondly, where does a_n and b_n come from? Thirdly, where did that recurrence relation come from? In this section an attempt is made at providing some answers.

Here are two lemmas which will help with $c_{n,k}$.

LEMMA 33. The following identity holds

$$\sum_{k=1}^{K} \frac{a_1 a_2 a_3 \cdots a_{k-1}}{(x+a_1)(x+a_2)\cdots(x+a_k)} = \frac{1}{x} - \frac{a_1 a_2 a_3 \cdots a_K}{x(x+a_1)(x+a_2)\cdots(x+a_K)}$$

Proof. Let

$$A_k = \frac{a_1 a_2 a_3 \cdots a_k}{x(x+a_1)(x+a_2)\cdots(x+a_k)},$$

then

$$A_{k-1} - A_k = \frac{a_1 a_2 a_3 \cdots a_{k-1}}{x(x+a_1)(x+a_2)\cdots(x+a_{k-1})} - \frac{a_1 a_2 a_3 \cdots a_k}{x(x+a_1)(x+a_2)\cdots(x+a_k)},$$

$$= \frac{(x+a_k)(a_1 a_2 a_3 \cdots a_{k-1}) - a_1 a_2 a_3 \cdots a_k}{x(x+a_1)(x+a_2)\cdots(x+a_k)},$$

$$= \frac{(a_1 a_2 a_3 \cdots a_{k-1})}{(x+a_1)(x+a_2)\cdots(x+a_k)}.$$

Thus

$$\sum_{k=1}^{K} \frac{(a_1 a_2 a_3 \cdots a_{k-1})}{(x+a_1)(x+a_2)\cdots(x+a_k)} = \sum_{k=1}^{K} (A_{k-1} - A_k),$$

$$= (A_0 - A_1) + (A_1 - A_2) + (A_2 - A_3) + \ldots + (A_{K-1} - A_K),$$

$$= A_0 - A_K$$

$$= \frac{1}{x} - \frac{a_1 a_2 a_3 \cdots a_K}{x(x+a_1)(x+a_2)\cdots(x+a_K)}. \qquad \square$$

LEMMA 34. The following is an alternative series for $\zeta(3)$

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Proof. First, the substitutions $x=n^2$, $a_k=-k^2$ are made in

$$\frac{a_1a_2a_3\cdots a_k}{x(x+a_1)(x+a_2)\cdots(x+a_k)}$$

and taking $k=n-1$ gives

$$\frac{(-1)^{n-1}(n-1)!^2}{n^2(n^2-1^2)(n^2-2^2)\cdots\left(n^2-(n-1)^2\right)}$$

(2.3.1) $$=\frac{(-1)^{n-1}2n^2(n-1)!^2}{2n^2n^2(n-1)(n+1)(n-2)(n+2)\cdots(1)(2n-1)}.$$

Rearranging, the denominator becomes

$$2n^4(n-1)(n-2)(n-3)\cdots(1)(2n-1)(2n-2)\cdots(n+1)$$

$$=2n^3(2n-1)(2n-2)\cdots(n+1)n(n-1)(n-2)\cdots 1$$

$$=n^2(2n)!.$$

Thus the expression in (2.3.1) is

$$\frac{(-1)^{n-1}2n^2(n-1)!^2}{n^2(2n)!}$$

$$=\frac{2(-1)^{n-1}n!^2}{n^2(2n)!}$$

$$=\frac{2(-1)^{n-1}}{n^2\binom{2n}{n}}.$$

Using this result and again using $x = n^2$, $a_k = -k^2$ and $K = n-1$ then the identity

$$\sum_{k=1}^{K} \frac{a_1 a_2 a_3 \cdots a_{k-1}}{(x+a_1)(x+a_2)\cdots(x+a_k)} = \frac{1}{x} - \frac{a_1 a_2 a_3 \cdots a_K}{x(x+a_1)(x+a_2)\cdots(x+a_K)}$$

in Lemma 33 becomes

$$(2.3.2) \qquad \sum_{k=1}^{n-1} \frac{(-1)^{k-1}(k-1)!^2}{(n^2-1^2)(n^2-2^2)\cdots(n^2-k^2)} = \frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2\binom{2n}{n}}.$$

Let $\varepsilon_{n,k} = \frac{1}{2}\frac{k!^2(n-k)!}{k^3(n+k)!}$ then

$$(-1)^k n(\varepsilon_{n,k} - \varepsilon_{n-1,k}) = \frac{(-1)^k nk!^2}{2k^3}\left[\frac{(n-k)!}{(n+k)!} - \frac{(n-1-k)!}{(n-1+k)!}\right]$$

$$= \frac{(-1)^k nk!^2 (n-1-k)!}{2k^3 (n+k)!}\left[(n-k)-(n+k)\right]$$

$$= \frac{(-1)^k nk!^2 (n-1-k)!}{2k^3 (n+k)!}[-2k]$$

$$= \frac{(-1)^{k-1}(k-1)!^2}{1} \times$$

$$\frac{n(n-k-1)(n-k-2)\cdots 1}{(n+k)(n+k-1)\cdots(n+1)n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 1},$$

$$= \frac{(-1)^{k-1}(k-1)!^2}{\left[(n+k)(n-k)\right]\left[(n+k-1)(n-k+1)\right]\cdots\left[(n+1)(n-1)\right]},$$

$$= \frac{(-1)^{k-1}(k-1)!^2}{(n^2-1^2)(n^2-2^2)\cdots(n^2-k^2)},$$

and therefore $(-1)^k n(\varepsilon_{n,k} - \varepsilon_{n-1,k}) = \dfrac{(-1)^{k-1}(k-1)!^2}{(n^2-1^2)(n^2-2^2)\cdots(n^2-k^2)}$.

Substituting in (2.3.2) gives

$$n\sum_{k=1}^{n-1}(-1)^k(\varepsilon_{n,k}-\varepsilon_{n-1,k}) = \frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2\binom{2n}{n}}.$$

and thus

$$\sum_{k=1}^{n-1}(-1)^k(\varepsilon_{n,k}-\varepsilon_{n-1,k}) = \frac{1}{n^3} - \frac{2(-1)^{n-1}}{n^3\binom{2n}{n}}$$

and

(2.3.3) $$\sum_{n=1}^{N}\sum_{k=1}^{n-1}(-1)^k(\varepsilon_{n,k}-\varepsilon_{n-1,k}) = \sum_{n=1}^{N}\frac{1}{n^3} - \sum_{n=1}^{N}\frac{2(-1)^{n-1}}{n^3\binom{2n}{n}}$$

Now, $\sum_{n=1}^{N}\sum_{k=1}^{n-1}(-1)^k(\varepsilon_{n,k}-\varepsilon_{n-1,k})$ can be rewritten as

$$\sum_{1\le n\le N}\ \sum_{1\le k\le n-1}(-1)^k(\varepsilon_{n,k}-\varepsilon_{n-1,k})$$

$$= \sum_{1\le k\le N-1}\ \sum_{k+1\le n\le N}(-1)^k(\varepsilon_{n,k}-\varepsilon_{n-1,k})$$

(2.3.4) $$= \sum_{1\le k\le N-1}(-1)^k \sum_{k+1\le n\le N}(\varepsilon_{n,k}-\varepsilon_{n-1,k}).$$

Writing in full the inner sum becomes

$$(\varepsilon_{k+1,k}-\varepsilon_{k,k}) + (\varepsilon_{k+2,k}-\varepsilon_{k+1,k}) + \ldots + (\varepsilon_{N,k}-\varepsilon_{N-1,k})$$

$$= \varepsilon_{N,k} - \varepsilon_{k,k}.$$

Thus (2.3.4) becomes

$$= \sum_{1 \le k \le N-1} (-1)^k \left(\varepsilon_{Nk} - \varepsilon_{kk}\right)$$

which is

$$= \sum_{k=1}^{N-1} \frac{(-1)^k k!^2 (N-k)!}{2k^3 (N+k)!} - \sum_{k=1}^{N-1} \frac{(-1)^k k!^2}{2k^3 (2k)!}$$

$$= \sum_{k=1}^{N-1} \frac{(-1)^k k! k! (N-k)! N!}{2k^3 N! (N+k)!} + \frac{1}{2} \sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}$$

$$= \frac{1}{2} \sum_{k=1}^{N-1} \frac{(-1)^k}{k^3 \binom{N+k}{k} \binom{N}{k}} - \frac{1}{2} \sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}$$

Substituting in (2.3.3) gives

$$\frac{1}{2} \sum_{k=1}^{N-1} \frac{(-1)^k}{k^3 \binom{N+k}{k} \binom{N}{k}} + \frac{1}{2} \sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} = \sum_{n=1}^{N} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{2(-1)^{n-1}}{n^3 \binom{2n}{n}} \tag{2.3.5}$$

Extending the summations on the left to include the Nth terms does not alter the right side of the equation because the Nth term of the first sum equals

$$\frac{1}{2} \frac{(-1)^N}{N^3 \binom{2N}{N} \binom{N}{N}} = \frac{1}{2} \frac{(-1)^N}{N^3 \binom{2N}{N}}$$

and the Nth term of the second sum equals

$$\frac{1}{2} \frac{(-1)^{N-1}}{N^3 \binom{2N}{N}} = -\frac{1}{2} \frac{(-1)^N}{N^3 \binom{2N}{N}}.$$

Thus (2.3.5) becomes

$$\frac{1}{2}\sum_{k=1}^{N}\frac{(-1)^{k}}{k^{3}\binom{N+k}{k}\binom{N}{k}}+\frac{1}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k^{3}\binom{2k}{k}}=\sum_{n=1}^{N}\frac{1}{n^{3}}-\sum_{n=1}^{N}\frac{2(-1)^{n-1}}{n^{3}\binom{2n}{n}}.$$

Rearranging, it follows that

$$\sum_{n=1}^{N}\frac{1}{n^{3}}-\frac{1}{2}\sum_{k=1}^{N}\frac{(-1)^{k}}{k^{3}\binom{N+k}{k}\binom{N}{k}}=\frac{5}{2}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^{3}\binom{2n}{n}},$$

which becomes

$$\sum_{n=1}^{N}\frac{1}{n^{3}}+\frac{1}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k^{3}\binom{N+k}{k}\binom{N}{k}}=\frac{5}{2}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^{3}\binom{2n}{n}}. \qquad (2.3.6)$$

Now, it is easily seen that

$$\binom{N+k}{k}\binom{N}{k}>N^{2}, \qquad 1\le k\le N,$$

so the absolute value of each term in the second sum in (2.3.6) is less than $\frac{1}{N^{2}}$.

Thus

$$\left|\frac{1}{2}\sum_{k=1}^{N-1}\frac{(-1)^{k}}{k^{3}\binom{N+k}{k}\binom{N}{k}}\right|<\frac{1}{2}\sum_{k=1}^{N}\frac{1}{N^{2}}=\frac{N}{2N^{2}}=\frac{1}{2N}\to 0 \text{ as } N\to\infty.$$

Therefore this series converges absolutely. Taking the limit as $N\to\infty$ in equation (2.3.6) produces

$$\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{3}\binom{2n}{n}}. \qquad \square$$

It can be seen that the left hand side of (2.3.6) with k replaced with m, n with m and N with n is $c_{n,n}$ from Lemma 19.

Unfortunately $c_{n,k}$ is not a sufficiently fast approximate to prove the irrationality of $\zeta(3)$ by itself, whatever is done with k. It has too large a denominator relative to its closeness to $\zeta(3)$. The convergence has to be "accelerated". Van der Poorten describes Apéry's process whereby he comes up with a a rational expression that will do the trick. He says to consider two infinite triangular arrays (defined for $1 \le k \le n$) with entries $d_{n,k}^{(0)} = c_{n,k}\binom{n+k}{k}$ and $f_{n,k}^{(0)} = \binom{n+k}{k}$, respectively. Each array undergoes a sequence of transformations as follows:

Firstly, $d_{n,k}^{(0)} = c_{n,k}\binom{n+k}{k}$ is transformed as follows

$$d_{n,k}^{(0)} \to d_{n,n-k}^{(0)} = d_{n,k}^{(1)} = c_{n,n-k}\binom{2n-k}{n-k} = c_{n,n-k}\binom{2n-k}{n}$$

$$d_{n,k}^{(1)} \to \binom{n}{k} d_{n,n-k}^{(1)} = d_{n,k}^{(2)} = c_{n,n-k}\binom{n}{k}\binom{2n-k}{n}$$

$$d_{n,k}^{(2)} \to \sum_{k'=0}^{k}\binom{k}{k'} d_{n,k'}^{(2)} = d_{n,k}^{(3)} = \sum_{k'=0}^{k}\binom{k}{k'}\binom{n}{k'}\binom{2n-k'}{n} c_{n,n-k'}$$

$$d_{n,k}^{(3)} \to \binom{n}{k}\sum_{k'=0}^{k}\binom{k}{k'} d_{n,k'}^{(2)} = d_{n,k}^{(3)} = \sum_{k'=0}^{k}\binom{k}{k'}\binom{n}{k'}\binom{n}{k}\binom{2n-k'}{n} c_{n,n-k'}$$

$$d_{n,k}^{(4)} \to \sum_{k'=0}^{k}\binom{k}{k'} d_{n,k'}^{(4)} = d_{n,k}^{(5)} = \sum_{k'=0}^{k}\binom{k}{k'}\sum_{k''=0}^{k'}\binom{k'}{k''}\binom{n}{k''}\binom{n}{k'}\binom{2n-k''}{n} c_{n,n-k''}$$

$$= \sum_{k'=0}^{k} \sum_{k''=0}^{k'} \binom{k}{k'}\binom{k'}{k''}\binom{n}{k''}\binom{n}{k'}\binom{2n-k''}{n} c_{n,n-k''}$$

Now $f_{n,k}^{(0)} = \binom{n+k}{k}$ is transformed:

$$f_{n,k}^{(0)} = \binom{n+k}{k} \to \binom{2n-k}{n} = f_{n,k}^{(1)}$$

$$f_{n,k}^{(1)} \to \binom{n}{k}\binom{2n-k}{n} = f_{n,k}^{(2)}$$

$$f_{n,k}^{(2)} \to \sum_{k_1=0}^{k} \binom{k}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n} = f_{n,k}^{(3)}$$

$$f_{n,k}^{(3)} \to \sum_{k_1=0}^{k} \binom{k}{k_1}\binom{n}{k_1}\binom{n}{k}\binom{2n-k_1}{n} = f_{n,k}^{(4)}$$

$$f_{n,k}^{(4)} \to \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \binom{k}{k_2}\binom{k_2}{k_1}\binom{n}{k_1}\binom{n}{k_2}\binom{2n-k_1}{n} = f_{n,k}^{(5)}$$

Clearly the two expressions $\sum_{k'=0}^{k} \sum_{k''=0}^{k'} \binom{k}{k'}\binom{k'}{k''}\binom{n}{k''}\binom{n}{k'}\binom{2n-k''}{n}$ and $\sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \binom{k}{k_2}\binom{k_2}{k_1}\binom{n}{k_1}\binom{n}{k_2}\binom{2n-k_1}{n}$ in the 2 arrays $d_{n,k}^{(5)}$ and $f_{n,k}^{(5)}$ are identical. Apéry defines a "diagonal" of each of these arrays to be a set of entries $\{n, k(n)\}$. He then defines a "quotient" of the arrays to be the set of quotients of corresponding elements of two "diagonals" of 2 arrays $d_{n,k}^{(v)}$ and $f_{n,k}^{(v)}$. Using Lemmas 28 and 29 it is clear that all possible "quotients" converge to $\zeta(3)$. If the obvious diagonal, $\{n, n\}$, is taken, then the expression $\sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \binom{k}{k_2}\binom{k_2}{k_1}\binom{n}{k_1}\binom{n}{k_2}\binom{2n-k_1}{n}$ changes to

$$\sum_{k_2=0}^{n}\sum_{k_1=0}^{k_2}\binom{n}{k_2}\binom{k_2}{k_1}\binom{n}{k_1}\binom{n}{k_2}\binom{2n-k_1}{n}=\sum_{k_2=0}^{n}\sum_{k_1=0}^{k_2}\binom{k_2}{k_1}\binom{n}{k_1}\binom{n}{k_2}^2\binom{2n-k_1}{n}.$$

Simplifying the notation of this expression and rearranging produces

$$\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{2n-l}{n}.$$

The following 2 lemmas show that this expression is b_n.

LEMMA 35. The following identity holds

$$\sum_{r=k}^{n}\binom{n}{r}^2\binom{r}{k}=\binom{2n-k}{n}\binom{n}{k}.$$

Proof. Letting $n = n'$, $r = m'$, $k = p'$ in the standard combinatorial identity

$$\binom{n'}{m'}\binom{m'}{p'}=\binom{n'}{p'}\binom{n'-p'}{m'-p'}$$

produces

(2.3.7)
$$\sum_{r=k}^{n}\binom{n}{r}^2\binom{r}{k}=\sum_{r=k}^{n}\binom{n}{r}\binom{n}{k}\binom{n-k}{r-k}=\binom{n}{k}\sum_{r=k}^{n}\binom{n}{r}\binom{n-k}{r-k}.$$

Now Vandermonde's convolution formula [Riordan, page 8] is used:

$$\binom{n'}{m'}=\sum_{k'=0}\binom{n'-p'}{m'-k'}\binom{p'}{k'}.$$

Substitutions can be made for any of these variables as long as n', m', p', k' are positive integers, $n' > m'$ and $p' > k'$. The expression $n' - p'$ will be greater than $m' - k'$ because $n' > m'$ and $p' > k'$.

The substitutions $r-k=k'$ and $2n-k=n'$ are made to get

(2.3.8) $$\binom{2n-k}{m'}=\sum_{r=k}^{k'=p'}\binom{2n-k-p'}{m'-r+k}\binom{p'}{r-k}$$

The maximum value of k' is p' so $r-k<p'$ so $r<p'+k=n \Rightarrow p'=n-k$.

Substituting in (2.3.8) gives

(2.3.9) $$\binom{2n-k}{m'}=\sum_{r=k}^{k'=p'}\binom{2n-k-n+k}{m'-r+k}\binom{n-k}{r-k}$$

Let $m'=2n-k-n=n-k$. This substitution is well defined because

$n'=2n-k>n-k=m'$.

Thus (2.3.9) become

(2.3.10) $$\binom{2n-k}{n-k}=\sum_{r=k}^{n}\binom{n}{n-r}\binom{n-k}{r-k}.$$

Now, $\binom{2n-k}{n-k}=\binom{2n-k}{2n-k-n+k}=\binom{2n-k}{n}$ and $\binom{n}{n-r}=\binom{n}{r}$ so (2.3.10) becomes

$$\binom{2n-k}{n}=\sum_{r=k}^{n}\binom{n}{r}\binom{n-k}{r-k}$$

Substituting in (2.3.7) gives

$$\sum_{r=k}^{n}\binom{n}{r}^2\binom{r}{k}=\binom{n}{k}\sum_{r=k}^{n}\binom{n}{r}\binom{n-k}{r-k},$$

$$=\binom{n}{k}\binom{2n-k}{n}$$

LEMMA 36. The following identity holds

$$\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{2n-l}{n}=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2n-k}{n}^2.$$

Proof. Writing the left hand side of the above in full to get

$$\binom{n}{0}^2\binom{n}{0}\binom{0}{0}\binom{2n}{n}+\binom{n}{1}^2\binom{n}{0}\binom{1}{0}\binom{2n}{n}+\binom{n}{1}^2\binom{n}{1}\binom{1}{1}\binom{2n-1}{n}+$$

$$\binom{n}{2}^2\binom{n}{0}\binom{2}{0}\binom{2n}{n}+\binom{n}{2}^2\binom{n}{1}\binom{2}{1}\binom{2n-1}{n}+\binom{n}{2}^2\binom{n}{2}\binom{2}{2}\binom{2n-2}{n}+$$

$$\binom{n}{3}^2\binom{n}{0}\binom{3}{0}\binom{2n}{n}+\binom{n}{3}^2\binom{n}{1}\binom{3}{1}\binom{2n-1}{n}+\binom{n}{3}^2\binom{n}{2}\binom{3}{2}\binom{2n-2}{n}+\binom{n}{3}^2\binom{n}{3}\binom{3}{3}\binom{2n-3}{n}+$$

$$\ldots+$$

$$\binom{n}{n}^2\binom{n}{0}\binom{n}{0}\binom{2n}{n}+\binom{n}{n}^2\binom{n}{1}\binom{n}{1}\binom{2n-1}{n}+\ldots+\binom{n}{n}^2\binom{n}{n}\binom{n}{n}\binom{n}{n}$$

$$=\binom{n}{0}\binom{2n}{n}\left[\binom{n}{0}^2\binom{0}{0}+\binom{n}{1}^2\binom{1}{0}+\binom{n}{2}^2\binom{2}{0}+\ldots+\binom{n}{n}^2\binom{n}{0}\right]+$$

$$\binom{n}{1}\binom{2n-1}{n}\left[\binom{n}{1}^2\binom{1}{1}+\binom{n}{2}^2\binom{2}{1}+\binom{n}{3}^2\binom{3}{1}+\ldots+\binom{n}{n}^2\binom{n}{1}\right]+$$

$$\ldots+$$

$$\binom{n}{n}\binom{n}{n}\left[\binom{n}{n}^2\binom{n}{n}\right]$$

(2.3.11) $$=\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n}\sum_{r=k}^{n}\binom{n}{r}^2\binom{r}{k}$$

Now, from Lemma 35, $\sum_{r=k}^{n}\binom{n}{r}^2\binom{r}{k}=\binom{2n-k}{n}\binom{n}{k}$. So (2.3.11) becomes

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2n-k}{n}^2$$

Replacing k with $n-k'$ in $\sum_{k=0}^{n}\binom{n}{k}^2\binom{2n-k}{n}^2$ gives

$$\sum_{n-k'=0}^{k'=0}\binom{n}{n-k'}^2\binom{n+k'}{n}^2=\sum_{k'=0}^{n}\binom{n}{k'}^2\binom{n+k'}{k'}^2=b_n.$$

As for the recurrence relation the only thing that will be stated about it is that if

$P(n)=34n^3+51n^2+27n+5$ then $P(n-1)=-P(-n)$ which turns out to be useful.

III. Detailed Description of Beukers' paper

A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$

3.1. Introduction

In this chapter Beukers' paper is examined in detail. The proofs involve many double and triple integrals, the shape of which were motivated by Apéry's proof. Some of the mystery of where these integrals came from can be removed by the formulas

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \zeta(2) \text{ and } -\frac{1}{2}\int_0^1\int_0^1 \frac{\log xy}{1-xy}\,dx\,dy = \zeta(3).$$

The crux of the proofs are very similar to the Apéry proof because it is attempted by devious means, using the aforementioned integrals, to end up with

$$0<\left|A_n+B_n\zeta(2)\right|<\left\{\frac{p}{q}\right\}^n \text{ or } 0<\left|A_n+B_n\zeta(3)\right|<\left\{\frac{p}{q}\right\}^n$$

for all sufficiently large n where A_n, B_n are integers and $\frac{p}{q}$ is a rational number less than 1. As n increases $\left\{\frac{p}{q}\right\}^n$ gets arbitrarily close to 0. As discussed in Chapter I it is not possible to get arbitrarily close to an integer if you take an integer multiple of a rational number. It will either "hit" an integer exactly or it will be at least a distance of $\frac{1}{b}$ from any integer if the rational number in lowest terms is $\frac{a}{b}$. The respective inequalities prove the irrationality of $\zeta(2)$ or $\zeta(3)$.

Beukers' paper could be divided into 3 main sections. The first section contains a lemma ("Lemma 1") which connects 4 integrals to $\zeta(2)$ and $\zeta(3)$. The second section proves the irrationality of $\zeta(2)$ and the third section proves the irrationality of $\zeta(3)$. The first 2 sections of this thesis deal with "Lemma 1", the first section dealing with the interchange of the integration and differentiation processes which is used and the second section describing the rest of "Lemma 1" in detail. Section 3 of this thesis describes in detail Beukers' "Theorem 1", that $\zeta(2)$ is irrational. Section 4 of this thesis describes in detail Beukers' "Theorem 2", that $\zeta(3)$ is irrational.

3.2. Proving the Validity of Beukers' Interchange of the Integration and Differentiation Processes With the Improper Integral

$$\int_0^1\int_0^1 \frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\,dxdy.$$

LEMMA 37. Differentiating $\dfrac{x^{r+\sigma}y^{s+\sigma}}{1-xy}$ with respect to σ gives

$$\frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right] = \frac{x^{r+\sigma}y^{s+\sigma}\ln xy}{1-xy}.$$

Proof. Using the fact that $\dfrac{\partial}{\partial\sigma}a^{\sigma} = a^{\sigma}\ln a$

$$\frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right] = \frac{\partial}{\partial\sigma}\left[\frac{x^{r}y^{s}(xy)^{\sigma}}{1-xy}\right]$$

$$= \frac{x^{r}y^{s}\ln xy(xy)^{\sigma}}{1-xy}$$

$$= \frac{x^{r+\sigma}y^{s+\sigma}\ln xy}{1-xy}.$$

LEMMA 38. Let $I(\sigma) = \int_0^1\int_0^1 \dfrac{x^{r+\sigma}y^{s+\sigma}}{1-xy}dxdy$, where $\sigma \geq 0$ and r and s are non-negative integers. Then $I(\sigma) = \sum_{k=0}^{\infty}\dfrac{1}{(k+r+\sigma+1)(k+s+\sigma+1)}$, which is a convergent series.

Proof. Substituting the geometric series

$$(1-xy)^{-1} = \sum_{k=0}^{\infty}(xy)^{k}\text{, for } |xy| < 1$$

in $I(\sigma)$ and writing in full produces

$$I(\sigma) = \lim_{\delta_2 \to 0^+} \int_0^{1-\delta_2} \left[\lim_{\delta_1 \to 0^+} \int_0^{1-\delta_1} \sum_{k=0}^{\infty} x^{r-\sigma} y^{s-\sigma} (xy)^k dx \right] dy.$$

Because geometric series are uniformly convergent within their radius of convergence the series can be integrated term by term to obtain

$$I(\sigma) = \lim_{\delta_2 \to 0^+} \int_0^{1-\delta_2} \left[\lim_{\delta_1 \to 0} \sum_{k=0}^{\infty} \int_0^{1-\delta_1} x^{r-\sigma} y^{s-\sigma} (xy)^k dx \right] dy$$

$$= \lim_{\delta_2 \to 0^+} \int_0^{1-\delta_2} \left[\lim_{\delta_1 \to 0^+} \left(\sum_{k=0}^{\infty} \frac{x^{r+\sigma+k+1}}{r+\sigma+k+1} y^{s-\sigma+k} \Bigg|_0^{1-\delta_1} \right) \right] dy$$

$$= \lim_{\delta_2 \to 0^+} \int_0^{1-\delta_2} \left[\lim_{\delta_1 \to 0^+} \sum_{k=0}^{\infty} \frac{(1-\delta_1)^{r+\sigma+k+1}}{r+\sigma+k+1} y^{s+\sigma+k} \right] dy.$$

Now,

$$\frac{(1-\delta_1)^{r+\sigma+k+1}}{r+\sigma+k+1} y^{s+\sigma+k} \le y^{s+\sigma} \left((1-\delta_1) y \right)^k$$

and thus using the series above

$$\sum_{k=0}^{\infty} \frac{(1-\delta_1)^{r+\sigma+k+1}}{r+\sigma+k+1} y^{s+\sigma+k} \le y^{s+\sigma} \sum_{k=0}^{\infty} \left((1-\delta_1) y \right)^k .$$

From above

$$0 < (1-\delta_1) y \le (1-\delta_1)(1-\delta_2) < 1,$$

and thus $y^{s-\sigma} \sum_{k=0}^{\infty} \left((1-\delta_1) y \right)^k$ is a convergent geometric series.

By the Weierstrass M-test $\sum_{k=0}^{\infty}\frac{(1-\delta_1)^{r+\sigma+k+1}}{r+\sigma+k+1}y^{s+\sigma+k}$ is uniformly convergent and therefore the limiting process and the infinite summation process can be interchanged.

Thus, $$I(\sigma) = \lim_{\delta_2\to 0^+}\int_0^{1-\delta_2}\left[\sum_{k=0}^{\infty}\lim_{\delta_1\to 0^+}\frac{(1-\delta_1)^{r+\sigma+k+1}}{r+\sigma+k+1}y^{s+\sigma+k}\right]dy$$

$$= \lim_{\delta_2\to 0^+}\int_0^{1-\delta_2}\left[\sum_{k=0}^{\infty}\frac{1}{r+\sigma+k+1}y^{s+\sigma+k}\right]dy.$$

Again, $\sum_{k=0}^{\infty}\frac{1}{r+\sigma+k+1}y^{s+\sigma+k}$ is a power series which is dominated by the geometric series $y^{s+\sigma}\sum_{k=0}^{\infty}y^k = \frac{y^{s+\sigma}}{1-y}$ if $|y|<1$. For the integral above $y < 1-\delta_2 < 1$. Thus by uniform convergence integration can be done term by term to obtain

$$I(\sigma) = \lim_{\delta_2\to 0^+}\sum_{k=0}^{\infty}\int_0^{1-\delta_2}\frac{1}{r+\sigma+k+1}y^{s+\sigma+k}dy$$

$$= \lim_{\delta_2\to 0^+}\sum_{k=0}^{\infty}\frac{(1-\delta_2)^{s+\sigma+k+1}}{(r+\sigma+k+1)(s+\sigma+k+1)}.$$

As before,

$$\frac{(1-\delta_2)^{s+\sigma+k+1}}{(r+\sigma+k+1)(s+\sigma+k+1)} < (1-\delta_2)^k.$$

and the series $\sum_{k=0}^{\infty}(1-\delta_2)^k = \frac{1}{\delta_2}$ and therefore $\sum_{k=0}^{\infty}\frac{(1-\delta_2)^{s+\sigma+k+1}}{(r+\sigma+k+1)(s+\sigma+k+1)}$ is uniformly convergent by the Weierstrass M-test. Thus the limiting process and the summation process can be interchanged.

Thus,

$$I(\sigma) = \sum_{k=0}^{\infty} \lim_{\delta_2 \to 0} \frac{(1-\delta_2)^{s+\sigma+k+1}}{(r+\sigma+k+1)(s+\sigma+k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(r+\sigma+k+1)(s+\sigma+k+1)} .$$

$\square$

THEOREM 39. For $\sigma \geq 0$,

$$\text{(3.2.1)} \qquad \frac{d}{d\sigma}\left[\int_0^1\int_0^1 \frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\,dxdy\right] = \int_0^1\int_0^1 \frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right]dxdy.$$

From Lemma 37,

$$\frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right] = \frac{x^{r+\sigma}y^{s+\sigma}\ln xy}{1-xy}$$

and this is discontinuous in the square $0 \leq x, y \leq 1$ when $y = 0$ or $x = 0$ and when $x = y = 1$. The function $\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}$ is discontinuous when $x = y = 1$. Thus the common domain of these 2 functions in (x, y, σ) is $(0,1)\times(0,1)\times(0,\infty)$ and these two integrals are improper. To emphasize this the upper limits will be written as 1^- instead of 1. Similarly the lower limits will be written as 0^+ instead of 0 when applicable. So rewriting (3.2.1) more precisely it follows that

$$\text{(3.2.2)} \qquad \frac{d}{d\sigma}\left[\int_{0^+}^{1^-}\int_{0^+}^{1^-} \frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\,dxdy\right] = \int_{0^+}^{1^-}\int_{0^+}^{1^-} \frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right].$$

Note that

$$\frac{d}{d\sigma}\left[\int_{0^+}^{1^-}\int_{0^+}^{1^-}\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}dxdy\right] = \frac{\partial}{\partial\sigma}\left[\int_{0^+}^{1^-}\int_{0^+}^{1^-}\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}dxdy\right]$$

From Mattuck, pages 392-394 it is clear that in order to prove (3.2.2) three conditions need to be satisfied. First some definitions.

Let $G(x,y,\sigma) = \frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}$, $H(x,y,\sigma) = \frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right]$ where (x,y,σ) is an element of the set $(0,1)\times(0,1)\times(0,\infty)$ for both functions, $I(\sigma)$ defined as in lemma 38,

and $J(\sigma) = \int_{0^+}^{1^-}\int_{0^+}^{1^-}\frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right]dxdy$, where σ is an element of $(0,\infty)$.

Then the 3 conditions are:

(i) $G(x,y,\sigma)$ and $H(x,y,\sigma)$ have to be continuous;

(ii) $I(\sigma)$ has to be convergent for each σ - in other words the integral has to exist for each σ;

(iii) $J(\sigma)$ has to be uniformly convergent for each σ.

The proofs of these are as follows:

(i) This is trivial.

(ii) See lemma 39.

(iii) It will be shown that the improper integral defining $J(\sigma)$ is uniformly convergent for all σ.

From lemma 37,

$$\frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right]=\frac{x^{r+\sigma}y^{s+\sigma}\ln xy}{1-xy}.$$

This function is negative in $(0,1)\times(0,1)\times(0,\infty)$ and therefore if it is proved that the improper double integral defining σ is bounded then the function will converge uniformly in σ by the Weierstrass M-test.

Now, for $0<x,y<1$,

$$\left|\frac{x^{r+\sigma}y^{s+\sigma}\ln xy}{1-xy}\right|\le\frac{|\ln xy|}{1-xy}\le\begin{cases}\dfrac{\ln 2}{1-xy} & \text{if } \dfrac{1}{2}<xy\\ -2\ln xy & \text{if } xy\le\dfrac{1}{2}\end{cases}\qquad(3.2.3)$$

Next the double integral defining $J(\sigma)$ is split up in the following way:

$$\int_0^1\int_0^1\frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right]dxdy=\underset{xy\le\frac{1}{2}}{\int_0^1\int_0^1}\frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right]dxdy+\underset{\frac{1}{2}<xy}{\int_0^1\int_0^1}\frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\right]dxdy$$

$$=I_1+I_2.$$

From the estimate (3.2.3) above

$$|I_2|\le\lim_{\delta\to 0^+}\int_0^{1-\delta}\int_0^{1-\delta}\frac{\ln 2}{1-xy}dxdy$$

$$=\ln 2\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}$$

$$=\zeta(2)\ln 2$$

by Lemma 39 with $r=s=\sigma=0$.

Using the estimate (3.2.3) from above, again, it follows that

$$|I_1| \le \lim_{\delta_1 \to 0^+} \lim_{\delta_2 \to 0^+} \int_{\delta_1}^{1}\int_{\delta_2}^{1} -2\ln xy\,dx\,dy$$

$$= -2 \lim_{\delta_1 \to 0^+} \lim_{\delta_2 \to 0^+} \int_{\delta_1}^{1}\int_{\delta_2}^{1} (\ln x + \ln y)\,dx\,dy$$

$$= -4 \lim_{\delta_1 \to 0^+} \lim_{\delta_2 \to 0^+} \int_{\delta_1}^{1}\left[\int_{\delta_2}^{1} \ln x\,dx\right] dy \quad \text{(by symmetry)}$$

$$= -4 \lim_{\delta_2 \to 0^+} \int_{\delta_2}^{1} \ln x\,dx$$

$$= -4 \lim_{\delta_2 \to 0^+} \left[x\ln x - x\right]_{\delta_2}^{1}$$

$$= -4 \lim_{\delta_2 \to 0^+} \left[x\ln x - x\right]_{\delta_2}^{1}$$

$$= -4\left(-1 - \lim_{\delta_2 \to 0^+} \delta_2 \ln \delta_2\right)$$

$$= 4,$$

since by L'Hospital's rule $\lim_{\delta_2 \to 0^+} \delta_2 \ln \delta_2 = 0$. The conclusion is that the improper integral defining $J(\sigma)$ is bounded, and therefore uniformly convergent to $J(\sigma)$.

This also proves Theorem 39. □

3.3. Proving "Lemma 1" on Beukers' Paper.

LEMMA 40. Let r and s be integers, $r > s$, and let $\int_0^1\int_0^1 -\frac{x^r y^s}{1-xy}\,dxdy$ be equal to a rational number whose denominator is a divisor of d_r^2, then, for $s > r$, $\int_0^1\int_0^1 -\frac{x^r y^s}{1-xy}\,dxdy$ is a rational number whose denominator is a divisor of d_s^2.

Proof. The expression $\frac{x^r y^s}{1-xy}$ is continuous for $0 \le x, y < 1$. Therefore the iterated integral

$$\int_0^1\int_0^1 \frac{x^r y^s}{1-xy}\,dxdy = \int_0^1\int_0^1 \frac{x^r y^s}{1-xy}\,dydx = \int_0^1\int_0^1 \frac{y^s x^r}{1-yx}\,dydx$$

So y has been interchanged with x in the original integral $\int_0^1\int_0^1 \frac{x^r y^s}{1-xy}\,dxdy$ so if $s > r$ then this integral is a rational number whose denominator is a divisor of d_s^2 □

LEMMA 41. Differentiating $\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy}\,dxdy$ with respect to σ gives

$\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{s+\sigma} \ln xy}{1-xy}\,dxdy$.

Proof. Because of Theorem 39,

$$\frac{\partial}{\partial\sigma}\left[\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy}\,dxdy\right] = \int_0^1\int_0^1 \frac{\partial}{\partial\sigma}\left[\frac{x^{r+\sigma} y^{s+\sigma}}{1-xy}\right]dxdy$$

$$= \int_0^1\int_0^1 \frac{x^{r-\sigma}}{1-xy} y^{s-\sigma} \ln y\,dxdy + \int_0^1\int_0^1 \frac{x^{r-\sigma}\ln x}{1-xy} y^{s-\sigma}\,dxdy$$

$$= \int_0^1\int_0^1 \frac{x^{r-\sigma} y^{s-\sigma} \ln xy}{1-xy}\,dxdy \qquad \square$$

LEMMA 42. The following is true:

$$\sum_{0}^{\infty} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)} = \frac{1}{r-s}\left\{\frac{1}{s+1+\sigma} + \dots + \frac{1}{r+\sigma}\right\}.$$

Proof. The left-hand side becomes $\sum_{0}^{\infty}\left\{\frac{\frac{1}{-r+s}}{k+r+\sigma+1} + \frac{\frac{1}{-s+r}}{k+s+\sigma+1}\right\}$ by partial fractions.

$$= \sum_{0}^{\infty} \frac{1}{r-s}\left[\frac{1}{k+s+\sigma+1} - \frac{1}{k+s+\sigma+1}\right]$$

Letting $r - s = t$ produces

$$= \frac{1}{r-s}\left[\frac{1}{s+\sigma+1} - \frac{1}{r+\sigma+1}\right.$$

$$+ \frac{1}{s+\sigma+2} - \frac{1}{r+\sigma+2}$$

$$+ \frac{1}{s+\sigma+3} - \frac{1}{r+\sigma+3}$$

$$\cdot \qquad \cdot$$

$$\cdot \qquad \cdot$$

$$+\frac{1}{s+\sigma+t-1}-\frac{1}{r+\sigma+t-1}$$

$$+\frac{1}{r+\sigma}-\frac{1}{r+\sigma+t}$$

$$+\frac{1}{r+\sigma+1}-\frac{1}{r+\sigma+t+1}$$

$$\cdot \qquad \cdot$$

$$\cdot \qquad \cdot]$$

$$=\frac{1}{r-s}\left\{\frac{1}{s+1+\sigma}+\ldots+\frac{1}{r+\sigma}\right\}.$$

LEMMA 43. Using the definition of d_n, above, $d_n^2 = lcm\{1^2, 2^2, \ldots, n^2\}$.

Proof. Since $lcm\{a,b\} = \prod_{p_i} p_i^{\max(a_i,b_i)}$ where $a = \prod p_i^{a_i}$ and $b = \prod p_i^{b_i}$ and $a_i, b_i \geq 0$,

then $lcm\{a^2,b^2\} = \prod_{p_i} p_i^{2\max(a_i,b_i)} = [lcm\{a,b\}]^2$. Similarly it can be seen that

$lcm\{1^2,2^2,\ldots,n^2\} = \prod_{p_i} p_i^{2\max(1_i,2_i,\ldots k_i,\ldots n_i)}$ where $k = \prod_{p_i} p_i^{k_i}$, $k = 1..n$, $k_i \geq 0$. Clearly

$\prod_{p_i} p_i^{2\max(1_i,2_i,\ldots k_i,\ldots n_i)} = [lcm\{1,2,\ldots,n\}]^2 = d_n^2$. □

COROLLARY 44. Using the definition of d_n, above, $d_n^3 = lcm\{1^3,2^3,\ldots,n^3\}$

Proof. See Lemma 43, above. □

THEOREM 45. Let r and s be non-negative integers.

(a) $\int_0^1\int_0^1 \frac{x^r y^s}{1-xy}dxdy$ is a rational number whose denominator is a divisor of d_r^2 if $r > s$.

Note: $\int_0^1\int_0^1 \frac{x^r y^s}{1-xy}dxdy$ is a rational number whose denominator is a divisor of d_s^2 if

$s > r$ (see Lemma 40).

(b) $\int_0^1\int_0^1 -\frac{\log xy}{1-xy} x^r y^s dxdy$ is a rational number whose denominator is a divisor of d_r^3 if

$r > s$. Note: $\int_0^1\int_0^1 -\frac{\log xy}{1-xy}x^r y^s dxdy$ is a rational number whose denominator is

a divisor of d_s^3 if $s > r$ (see Lemma 40).

(c) $\int_0^1\int_0^1 \frac{x^r y^r}{1-xy}dxdy = \zeta(2) - \frac{1}{1^2} - \frac{1}{2^2} - \ldots - \frac{1}{r^2}$ if $r = s$.

(d) $\int_0^1\int_0^1 -\frac{\log xy}{1-xy}x^r y^r dxdy = 2\left\{\zeta(3) - \frac{1}{1^3} - \frac{1}{2^3} - \ldots - \frac{1}{r^3}\right\}$ if $r = s$.

Remark. In case $r = 0$ the following sums, $1^{-2} + 2^{-2} + \ldots + r^{-2}$ and $1^{-3} + 2^{-3} + \ldots + r^{-3}$, vanish.

Proof. Throughout this proof $r \geq s$ unless otherwise stated. Let σ be any non-negative number. Consider the integral

$$I(\sigma) = \int_0^1\int_0^1 \frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}dxdy .$$

Developing $(1-xy)^{-1}$ into a geometric series and then integrating term by term the resulting power series it follows that

(3.3.1)
$$I(\sigma) = \sum_{k=0}^{\infty}\frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)} .$$

For details of the derivation of (3.3.1) please see lemma 38. Using Lemma 42 this becomes

(3.3.2)
$$\frac{1}{r-s}\left\{\frac{1}{s+1+\sigma} + \ldots + \frac{1}{r+\sigma}\right\},$$

and putting $\sigma = 0$ this becomes

$$I(\sigma) = \frac{1}{r-s}\left\{\frac{1}{s+1} + \frac{1}{s+2} + \dots + \frac{1}{r}\right\}.$$

Using lemma 43 this equals a fraction whose denominator divides the lcm of

$\{1^2, 2^2, \dots, r^2\} = d_r^2$. So part (a) of our theorem is proved.

Returning to (3.3.2) it follows that

$$\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy}\,dxdy = \frac{1}{r-s}\left\{\frac{1}{s+1+\sigma} + \dots + \frac{1}{r+\sigma}\right\}.$$

Using Lemma 41 both sides are differentiated with respect to σ to give

$$\int_0^1\int_0^1 \frac{\ln xy}{1-xy} x^{r+\sigma} y^{s+\sigma}\,dxdy = \frac{1}{r-s}\left[-(s+1+\sigma)^{-2} + (-1)(s+2+\sigma)^{-2} + \dots + (-1)(r+\sigma)^{-2}\right].$$

Putting $\sigma = 0$ and multiplying both sides of the equation by -1 gives

$$\int_0^1\int_0^1 -\frac{\ln xy}{1-xy} x^r y^s\,dxdy = \frac{1}{r-s}\left[(s+1)^{-2} + (s+2)^{-2} + \dots + (r)^{-2}\right].$$

This is a rational number whose denominator divides d_r^3 (see corollary 44). So part (b) of the theorem is proved.

Assume $r = s$ then by (3.3.1)

$$\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy}\,dxdy = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)^2} \tag{3.3.3}$$

Substituting $\sigma = 0$ produces

$$\int_0^1\int_0^1 \frac{x^r y^r}{1-xy}dxdy = \frac{1}{(r+1)^2}+\frac{1}{(r+2)^2}+\dots$$

$$= \zeta(2)-\frac{1}{1^2}-\frac{1}{2^2}-\dots-\frac{1}{r^2}.$$

This proves part (c).

Equation (3.3.3) is now differentiated with respect to σ. By lemma 41 this gives

$\int_0^1\int_0^1 \frac{\ln xy}{1-xy}x^{r+\sigma}y^{r+\sigma}dxdy$ on the left side of the equation. Now, $\sum_{k=0}^{\infty}\frac{1}{(k+r+\sigma+1)^2}$ is a

uniformly convergent series in σ by the Weierstrass M-test as

$\sum_{k=0}^{\infty}\frac{1}{(k+r+\sigma+1)^2} < \sum_{k=0}^{\infty}\frac{1}{k^2}$ for all $\sigma \geq 0$ and $\sum_{k=0}^{\infty}\frac{1}{k^2}$ is convergent. Thus the series can be

differentiated term by term giving $\sum_{k=0}^{\infty}\frac{-2}{(k+r+\sigma+1)^3}$. Thus

$$\int_0^1\int_0^1 \frac{\log xy}{1-xy}x^{r+\sigma}y^{r+\sigma}dxdy = \sum_{k=0}^{\infty}\frac{-2}{(k+r+\sigma+1)^3}.$$

Putting $\sigma = 0$ gives

$$\int_0^1\int_0^1 \frac{\log xy}{1-xy}x^r y^r dxdy = \sum_{k=0}^{\infty}\frac{-2}{(k+r+1)^3},$$

$$= -2\left\{\frac{1}{(r+1)^3}+\frac{1}{(r+2)^3}+\dots\right\},$$

$$= -2\left\{\zeta(3)-\left(\frac{1}{1^3}+\frac{1}{2^3}+\dots+\frac{1}{r^3}\right)\right\}.$$

Therefore

$$\int_0^1\int_0^1 -\frac{\log xy}{1-xy}x^r y^r dxdy = 2\left\{\zeta(3)-\left(\frac{1}{1^3}+\frac{1}{2^3}+\ldots+\frac{1}{r^3}\right)\right\}.$$

This proves part (d). □

3.4. Proving the Irrationality of $\zeta(2)$.

LEMMA 46. The following holds

$$\int_0^1\int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}dxdy = \left[A_n + B_n\zeta(2)\right]d_n^{-2},$$

where $P_n(x)=\frac{1}{n!}\left(\frac{d}{dx}\right)^n x^n(1-x)^n$, A_n, B_n are integers, and d_n is the lowest common multiple of the first n integers.

Proof. First it is claimed that

(3.4.1) $$P_n(x)=\sum_{r=0}^{n}\binom{n}{r}^2(-1)^r x^r(1-x)^{n-r}$$

The proof of this is as follows. First, it is clear that $\frac{1}{n!}\left\{\frac{d}{dx}\right\}^i x^n(1-x)^n$, $i\le n$, has $i+1$ terms of the form $x^{n-j}(1-x)^{n-i+j}$, $0\le j\le i$. This is because the lowest exponent for either x or $(1-x)$ will be $n-i$ and the sum of the two exponents will be $2n-i$ so the exponents for x and $(1-x)$ will be as follows:

$$n-i,n;\quad n-i+1,n-1;\quad n-i+2,n-2;\quad \ldots\quad ;\quad n-i+i,n-i=n,n-i.$$

Clearly there are $i+1$ terms.

Then it can be shown that

$$\left\{\frac{d}{dx}\right\}^i x^n(1-x)^n = \sum_{r=0}^{i}(-1)^r\binom{i}{r}\binom{n}{i-r}(i-r)!\binom{n}{r}r!\,x^{n-i+r}(1-x)^{n-r} \tag{3.4.2}$$

Equation (3.4.2) shall be proved by induction on i. For $i = 1$

$$\left\{\frac{d}{dx}\right\} x^n(1-x)^n = nx^{n-1}(1-x)^n + nx^n(-1)(1-x)^{n-1}.$$

The first term in the summation on the right side of (3.4.2), i.e. $r = 0$, is

$$\binom{1}{0}\binom{n}{1}(1)!\binom{n}{0}0!\,x^{n-1}(1-x)^n = nx^{n-1}(1-x)^n.$$

The second term in the summation on the right side of (3.4.2), i.e. $r = 1$, is

$$(-1)\binom{1}{1}\binom{n}{0}0!\binom{n}{1}1!\,x^n(1-x)^{n-1} = (-1)nx^n(1-x)^{n-1}.$$

So (3.4.2) is true for $i = 1$. It shall be assumed to be true for $i = k$, i.e.

$$\left\{\frac{d}{dx}\right\}^k x^n(1-x)^n = \sum_{r=0}^{k}(-1)^r\binom{k}{r}\binom{n}{k-r}(k-r)!\binom{n}{r}r!\,x^{n-k+r}(1-x)^{n-r}$$

$$= \sum_{r=0}^{k} a_r x^{n-k+r}(1-x)^{n-r}, \text{ where } a_r = (-1)^r\binom{k}{r}\binom{n}{k-r}(k-r)!\binom{n}{r}r!. \tag{3.4.3}$$

Now it is necessary to prove that $\left\{\frac{d}{dx}\right\}^{k+1} x^n(1-x)^n$ satisfies (3.4.2) with $i = k+1$.

The r^{th} term of $\left\{\frac{d}{dx}\right\}^{k+1} x^n(1-x)^n$ will be of the form $b_r x^{n-k-1+r}(1-x)^{n-r}$. This term comes from differentiating the r^{th} and $(r-1)^{th}$ terms in the expansion of $\left\{\frac{d}{dx}\right\}^k x^n(1-x)^n$. The $(r-1)^{th}$ term is $a_{r-1}x^{n-k+r-1}(1-x)^{n-r+1}$. The r^{th} term is given in (3.4.3). Differentiating these 2 terms and finding the 2 out of the 4 terms which will give the r^{th} term of $\left\{\frac{d}{dx}\right\}^{k+1} x^n(1-x)^n$ produces

$$b_r x^{n-k-1+r}(1-x)^{n-r} = (n-k+r)a_r x^{n-k+r-1}(1-x)^{n-r} + (n-r+1)a_{r-1}x^{n-k+r-1}(1-x)^{n-r}(-1).$$

Thus $b_r = (n-k+r)a_r - (n-r+1)a_{r-1}$.

Now it is necessary to show that

$$(n-k+r)(-1)^r\binom{k}{r}\binom{n}{k-r}(k-r)!\binom{n}{r}r! - (n-r+1)(-1)^{r-1}\binom{k}{r-1}\binom{n}{k+1-r}(k-r+1)!\binom{n}{r-1}(r-1)!$$
$$= (-1)^r\binom{k+1}{r}\binom{n}{k+1-r}(k-r+1)!\binom{n}{r}r!$$

Working on the left side it follows that

$$(-1)^r\binom{n}{r}r!\binom{n}{k+1-r}(k+1-r)!\left[\frac{(k+1-r)(n-k+r)}{(n-k+r)(k+1-r)}\binom{k}{r}+\frac{(n-r+1)r}{(n-r+1)r}\binom{k}{r-1}\right]$$

$$= (-1)^r\binom{n}{r}r!\binom{n}{k+1-r}(k+1-r)!\left[\binom{k}{r}+\binom{k}{r-1}\right].$$

Using the identity $\binom{k}{r}+\binom{k}{r-1}=\binom{k+1}{r}$ this simplifies to $(-1)^r\binom{n}{r}r!\binom{n}{k+1-r}(k+1-r)!\binom{k+1}{r}$.

From the fact that there are $k+2$ terms in $\left\{\frac{d}{dx}\right\}^{k+1} x^n(1-x)^n$ and the lowest exponent of x or $(1-x)$ is $n-k-1$ (3.4.2) is proved. Thus

$$\left\{\frac{d}{dx}\right\}^i x^n(1-x)^n = \sum_{r=0}^{i}\left[(-1)^r\binom{i}{r}\binom{n}{i-r}(i-r)!\binom{n}{r}r!\,x^{n-i+r}(1-x)^{n-r}\right].$$

Substituting n for i:

$$\left\{\frac{d}{dx}\right\}^n x^n(1-x)^n = \sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{n}{n-r}(n-r)!\binom{n}{r}r!\,x^r(1-x)^{n-r}$$

$$= \sum_{r=0}^{n}(-1)^r\binom{n}{r}^2 n!\,x^r(1-x)^{n-r}.$$

Thus

$$P_n(x) = \frac{1}{n!}\left\{\frac{d}{dx}\right\}^n x^n(1-x)^n = \sum_{r=0}^{n}\binom{n}{r}^2(-1)^r x^r(1-x)^{n-r}$$

and (3.4.1) is proved. Thus $P_n(x)$ is a polynomial in x of degree n with integer coefficients. The fact that they are integers is crucial to making the rest of the proof work.

For simplicity the expression $P_n(x)$ is written as $\sum_{i=0}^{n}\alpha_i x^i$.

The next stage of the proof is to work on the original integral $\int_0^1\int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}dxdy$.

Writing $(1-y)^n$ as $\sum_{j=0}^{n}\beta_j y^j$ produces

$$(1-y)^n P_n(x) = \sum_{0\le i,j\le n} \gamma(i,j)x^i y^j.$$

Substituting this expression in $\int_0^1\int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}dxdy$ gives

$$\int_0^1\int_0^1 \frac{\sum_{0\le i,j\le n} \gamma(i,j)x^i y^j}{1-xy}dxdy.$$

Splitting this integral up into $(n+1)^2$ integrals and using Theorem 45 parts (a) and (c) this integral becomes

$$\sum_{0\le i,j\le n} \gamma(i,j)\,q(i,j)+\gamma(0,0)\left[\zeta(2)\right]+\sum_{i=1}^{n}\gamma(i,i)\left[\zeta(2)-\sum_{j=1}^{i}\frac{1}{j^2}\right]$$

where $q(i,j)$ is a rational number whose denominator is a divisor of d_i^2 or d_j^2 depending on whether $i > j$ or $j > i$, respectively. Rewriting produces

(3.4.4) $$\sum_{0\le i,j\le n,\ i\ne j,\ t=\max\{i,j\}} \frac{\gamma(i,j)k(i,j)}{d_t^2} + \sum_{i=0}^{n}\gamma(i,i)\zeta(2) - \sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{j^2}$$ where

$k(i,j)$ are integers. From earlier lemmas *lcm* $\{d_1^2, d_2^2, \ldots, d_n^2\}$ is d_n^2 and

lcm $\{1^2, 2^2, \ldots, n^2\}$ is d_n^2. So the lowest common denominator of the fractions in (3.4.4)

is d_n^2. So now (3.4.4) becomes

$$\frac{1}{d_n^2}\left[E_n + \zeta(2)B_n + F_n\right]$$

where E_n, B_n, F_n are integers. Letting $E_n + F_n = A_n$ it follows that

$$\frac{1}{d_n^2}\left[A_n + B_n\zeta(2)\right]$$

and the lemma is proved. □

LEMMA 47. The integral

$$\mathrm{I} = \int_0^1\int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}\,dx\,dy = (-1)^n \int_0^1\int_0^1 \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^{n+1}}\,dx\,dy$$

where $n!P_n(x) = \left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$.

Proof. Integrating by parts $\mathrm{I} = \int_0^1\int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}\,dx\,dy$ it follows that

$$\int_0^1\int_0^1 \frac{(1-y)^n}{1-xy}\,d\left[\frac{1}{n!}\left\{\frac{d}{dx}\right\}^{n-1} x^n(1-x)^n\right]dy,$$

$$= \int_0^1 \left\{ \frac{(1-y)^n P_n(x)}{1-xy} \frac{1}{n!} \left\{ \frac{d}{dx} \right\}^{n-1} x^n (1-x)^n \Bigg|_0^1 \right\} dy$$

$$- \int_0^1 \int_0^1 \frac{1}{n!} \left\{ \frac{d}{dx} \right\}^{n-1} x^n (1-x)^n \frac{(1-xy)\cdot 0 - (1-y)^n(-y)}{(1-xy)^2} dx dy.$$

Differentiating $x^n(1-x)^n$, $n-1$ times, will always produce an expression with x and $1-x$ as factors and consequently the first integral in the above expression will equal 0. Therefore

$$I = 0 - \int_0^1 \int_0^1 \frac{1}{n!} \left\{ \frac{d}{dx} \right\}^{n-1} \left(x^n (1-x)^n \frac{(1-y)^n y}{(1-xy)^2} \right) dx dy.$$

Integrating by parts in a similar way this gives

$$I = 0 + \int_0^1 \int_0^1 \frac{1}{n!} \left\{ \frac{d}{dx} \right\}^{n-2} x^n (1-x)^n \left[\frac{(1-xy)^2 \cdot 0 - (1-y)^n (y) \cdot 2(1-xy)(-y)}{(1-xy)^4} \right] dx dy,$$

since the first term in the integration by parts process evaluates to 0 again as now $x^n(1-x)^n$ has been differentiated $n-2$ times. Thus

$$I = \int_0^1 \int_0^1 \frac{1}{n!} \left\{ \frac{d}{dx} \right\}^{n-2} x^n (1-x)^n \left[\frac{2y^2(1-y)^n}{(1-xy)^3} \right] dx dy.$$

This integration by parts process is repeated a further $n-1$ times and each time the first integral will evaluate to 0, and the end result is

$$(-1)^n \int_0^1 \int_0^1 \frac{1}{n!} \frac{x^n (1-x)^n \cdot 2 \cdot 3 \cdots n (1-y)^n y^n}{(1-xy)^{n-1}} dx dy.$$

$$= (-1)^n \int_0^1 \int_0^1 \frac{x^n (1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}} dx dy.$$ □

LEMMA 48. The following inequality holds

$$\frac{y(1-y)x(1-x)}{1-xy} \le \left(\frac{\sqrt{5}-1}{2}\right)^5 \text{ for } 0 \le x, y \le 1.$$

Proof. Calculus is used to find the maximum of the function above in $0 \le x, y \le 1$.

Let $g = \dfrac{y(1-y)x(1-x)}{1-xy} = \dfrac{(y-y^2)(x-x^2)}{1-xy} = (y-y^2)\left[\dfrac{x-x^2}{1-xy}\right]$

then

$$\frac{\partial g}{\partial x} = (y-y^2)\left[\frac{(1-2x)(1-xy)-(x-x^2)(-y)}{(1-xy)^2}\right]$$

$$= \frac{(y-y^2)(1-2x+x^2y)}{(1-xy)^2}$$

The maximum occurs when $\dfrac{\partial g}{\partial x} = 0$ so either $y = 1$ or $y = 0$ or

(3.4.5) $\qquad 1-2x+x^2y = 0$ which implies $y = \dfrac{-1+2x}{x^2}, x \ne 0$

But the first 2 possibilities produce minimums as $g \ge 0$ so the only critical point that needs to be considered is

$$y = \frac{-1+2x}{x^2}, x \ne 0.$$

As x and y are symmetric, from $\dfrac{\partial g}{\partial y}$ the following is derived

$$1-2y+y^2x = 0.$$

Substituting from (3.4.5) produces

$$0 = 1-2\left(\frac{-1+2x}{x^2}\right)+\left(\frac{-1+2x}{x^2}\right)^2 x$$

$$= 1 + \frac{2-4x}{x^2} + \left(\frac{1+4x^2-4x}{x^4}\right)^2 x$$

$$= x^3 + 2x - 4x^2 + 1 + 4x^2 - 4x$$

$$= x^3 - 2x + 1$$

$$= (x-1)(x^2 + x - 1).$$

Since $x = 1$ is a minimum value, the local maximum must come from the solutions of

$x^2 + x - 1 = 0$.

Thus

$$x = \frac{-1 \pm \sqrt{1-4(-1)}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}.$$

As $\frac{-1-\sqrt{5}}{2}$ is not in the domain it can be seen that $x = \frac{-1+\sqrt{5}}{2}$ is the only solution and this must be the absolute maximum because the boundary of the region is the absolute minimum.

Because of symmetry at this maximum $y = x$.

Substituting in g it follows that

$$g = \frac{x(1-x)x(1-x)}{1-x^2}$$

$$= \frac{x^2(1-x)^2}{1-x^2}$$

$$= \frac{x^2(1-x)}{1+x}.$$

Substituting $x = \frac{-1+\sqrt{5}}{2}$ in this expression it follows that

$$g = \frac{\left(\frac{-1+\sqrt{5}}{2}\right)^2\left(1-\frac{-1+\sqrt{5}}{2}\right)}{1+\left(\frac{-1+\sqrt{5}}{2}\right)}$$

$$= \frac{\left(\frac{-1+\sqrt{5}}{2}\right)^2\left(\frac{3-\sqrt{5}}{2}\right)}{\frac{1+\sqrt{5}}{2}}$$

(3.4.6) $$= \frac{\left(\frac{-1+\sqrt{5}}{2}\right)^2\left(3-\sqrt{5}\right)\left(\sqrt{5}-1\right)}{\left(1+\sqrt{5}\right)\left(\sqrt{5}-1\right)}$$

Now, $$\left(\frac{-1+\sqrt{5}}{2}\right)^2 = \frac{1}{4}+\frac{5}{4}-\frac{\sqrt{5}}{2} = \frac{3}{2}-\frac{\sqrt{5}}{2},$$

and thus $$3-\sqrt{5} = 2\cdot\left(\frac{-1+\sqrt{5}}{2}\right)^2.$$

Substituting in (3.4.6) gives

$$= \frac{\left(\frac{-1+\sqrt{5}}{2}\right)^4 2\left(\frac{\sqrt{5}-1}{2}\right)\cdot 2}{\left(1+\sqrt{5}\right)\left(\sqrt{5}-1\right)}$$

$$= \left(\frac{\sqrt{5}-1}{2}\right)^5. \qquad \square$$

LEMMA 49. It is true that, for sufficiently large n

$$9^n\left\{\frac{\sqrt{5}-1}{2}\right\}^{5n}\varsigma(2)<\left\{\frac{5}{6}\right\}^n.$$

Proof The left-hand side is equal to $\left[9\left\{\frac{\sqrt{5}-1}{2}\right\}^5\sqrt[n]{\varsigma(2)}\right]^n$.

Now, $\left(\sqrt{5}-1\right)^5 = 25\sqrt{5} + 5(25)(-1) + 10(5\sqrt{5})(-1)^2 + 10(5)(-1)^3 + 5\sqrt{5}(-1)^4 + (-1)^5$

$= 25\sqrt{5} - 125 + 50\sqrt{5} - 50 + 5\sqrt{5} - 1$

$= 80\sqrt{5} - 176.$

So the left-hand side now becomes

$$\left[9\left\{\frac{80\sqrt{5}-176}{32}\right\}\sqrt[n]{\varsigma(2)}\right]^n.$$

As $n \to \infty$, $\sqrt[n]{\varsigma(2)} \to 1$ as it is known that $\sum_1^\infty \frac{1}{n^2}$ is convergent and greater than 1.

It is sufficient to show that $9\left\{\frac{80\sqrt{5}-176}{32}\right\} < \frac{5}{6}$. Using a calculator this is clearly shown to be true. The inequality is more elegantly proved as follows:

$91125 < 91204$, or in other words $\left(135\sqrt{5}\right)^2 < 302^2$ which implies $135\sqrt{5} < 302$.

Subtracting 297 from both sides gives $135\sqrt{5} - 297 < 302 - 297$ or $135\sqrt{5} - 297 < 5$.

Dividing by 6 gives $\frac{135\sqrt{5}-297}{6} < \frac{5}{6}$ and factoring the left gives $\frac{27\left(5\sqrt{5}-11\right)}{6} < \frac{5}{6}$

which simplifies to $\frac{9\left(\sqrt{5}-11\right)}{2} < \frac{5}{6}$ which finally implies $\frac{9\left(80\sqrt{5}-176\right)}{32} < \frac{5}{6}$. $\square$

THEOREM 50. $\zeta(2)$ is irrational.

Proof. For a positive integer n consider the integral

$$T = \int_0^1\int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}\,dx\,dy,$$

where $P_n(x)$ is the Legendre-type polynomial given by $n!\,P_n(x) = \left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$.

From lemma 46, T is equal to $\left[A_n + B_n\zeta(2)\right]d_n^{-2}$ for some integers A_n and B_n. Doing an n-fold partial integration on T it follows that

$$T = (-1)^n \int_0^1\int_0^1 \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^{n+1}}\,dx\,dy. \tag{3.4.7}$$

Lemma 47 describes the details of doing this. From lemma 48 it follows that

$$\frac{y(1-y)x(1-x)}{1-xy} \le \left\{\frac{\sqrt{5}-1}{2}\right\}^5 \text{ for all } 0 \le x \le 1,\ 0 \le y \le 1.$$

Substituting in (3.4.7) produces

$$|T| \le \int_0^1\int_0^1 \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \frac{1}{1-xy}\,dx\,dy$$

$$= \left[\frac{\sqrt{5}-1}{2}\right]^{5n} \int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy$$

$$=\left[\frac{\sqrt{5}-1}{2}\right]^{5n}\zeta(2),$$

using Theorem 45, part (c) with $r=0$. Now, $T=\int_0^1\int_0^1\frac{(1-y)^n P_n(x)}{1-xy}dxdy$ is positive for $0\le x\le 1,\ 0\le y\le 1$. This is because the function $\frac{(1-y)^n P_n(x)}{1-xy}$ cannot be negative as none of its components can be: $(1-y)^n\ge 0$; $1-xy\ge 0$; $P_n(x)\ge 0$ because this is a sum of terms each one containing $1-x$ and/or x as a factor and no other factors involving x. And the function is continuous and positive at some point for example at $\left(\frac{1}{2},\frac{1}{2}\right)$. So $T>0$. Using the results we have found it follows that

$$0<\left|A_n+B_n\zeta(2)\right|d_n^{-2}\le\left\{\frac{\sqrt{5}-1}{2}\right\}^{5n}\zeta(2),$$

which implies

(3.4.8) $$0<\left|A_n+B_n\zeta(2)\right|\le d_n^2\left\{\frac{\sqrt{5}-1}{2}\right\}^{5n}\zeta(2).$$

The less than or equal symbol in the preceding is actually a strict inequality as $\left\{\frac{\sqrt{5}-1}{2}\right\}^5$ is a local maximum for the function $\frac{y(1-y)x(1-x)}{1-xy}$ and as this function is continuous and/or constant in the interior of the square $0\le x\le 1,\ 0\le y\le 1$ then there must be a measurable set inside this square where the function is less than $\left\{\frac{\sqrt{5}-1}{2}\right\}^5$.

Therefore the integral must be strictly less than $\left\{\frac{\sqrt{5}-1}{2}\right\}^{5n}\zeta(2)$. However, this is not necessary for the proof; it is just put in for interest. From (3.4.8), Corollary 17 (in Chapter 2) and Lemma 49 it follows that given $\varepsilon > 0$,

$$0 < |A_n - B_n\zeta(2)| \le d_n^2\left\{\frac{\sqrt{5}-1}{2}\right\}^{5n}\zeta(2) < 9^n\left\{\frac{\sqrt{5}-1}{2}\right\}^{5n}\zeta(2) < \left\{\frac{5}{6}\right\}^n < \varepsilon$$

for sufficiently large n.

By Theorem 6 from Chapter 1 this proves that $\zeta(2)$ is irrational. □

3.4. Proving the irrationality of $\zeta(3)$.

LEMMA 51. The following holds

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy}P_n(x)P_n(y)\,dxdy = \left[A_n + B_n\zeta(3)\right]d_n^{-3}\quad .\ A_n, B_n \text{ are integers.}$$

Proof. From (3.4.1) $P_n(x)$ and $P_n(y)$ are polynomials in x and y, respectively, with integer coefficients. Let $P_n(x) = \sum_{i=0}^{n} a_i x^i$ and let $P_n(y) = \sum_{i=0}^{n} a_i y^i$ then $P_n(x)P_n(y) = \sum_{0 \le i,j \le n} \gamma(i,j) x^i y^j$ where $\gamma(i,j)$ are integers. Substituting in

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy}P_n(x)P_n(y)\,dxdy$$

it follows that

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy}\left(\sum_{0\le i,j\le n} \gamma(i,j)x^i y^j\right)dxdy.$$

Splitting up into several integrals and using Theorem 45 (parts b and d) this then becomes

$$\sum_{0\le i,j\le n} \gamma(i,j)q(i,j)+$$

(3.4.9) $$2\gamma_{00}\{\zeta(3)\}+2\gamma_{11}\left\{\zeta(3)-\frac{1}{1^3}\right\}+\ldots+2\gamma_{nn}\left\{\zeta(3)-\frac{1}{1^3}-\frac{1}{2^3}-\frac{1}{3^3}-\ldots-\frac{1}{n^3}\right\}$$

Now, $q(i,j)$ is a divisor of d_i^3 or d_j^3 depending on whether $i>j$ or $j>i$, respectively.

Then (3.4.9) can be written as

$$\sum_{0\le i,j\le n,\ i\ne j,\ t=\max\{i,j\}} \frac{\gamma(i,j)k(i,j)}{d_t^3}+$$

$$2\sum_{t=0}^{n}\gamma_{tt}\zeta(3)-2\left[\left(\frac{1}{1^3}\right)+\left(\frac{1}{1^3}+\frac{1}{2^3}\right)+\ldots+\left(\frac{1}{1^3}+\ldots+\frac{1}{n^3}\right)\right].$$

Using similar arguments to the ones used in Lemma 46 it follows that

$lcm\{1^3,2^3,\ldots,n^3,d_1^3,d_2^3,\ldots,d_n^3\}=d_n^3$. Equation (3.4.9) now becomes

$$\frac{1}{d_n^3}\left[C_n+\zeta(3)B_n+D_n\right],\ \text{for some integers } C_n, B_n, D_n.$$

Simplifying it follows that

$$\frac{1}{d_n^3}\left[A_n+\zeta(3)B_n\right],\ \text{for some integer } A_n.$$ □

LEMMA 52. It is true that $\int_0^1 \frac{1}{1-(1-xy)z}dz = -\frac{1}{1-xy}\ln xy$.

Proof. The left-hand side is

$$= -\frac{1}{1-xy}\ln\left[1-(1-xy)z\right]\Big|_0^1$$

$$= -\frac{1}{1-xy}\ln\left[1-(1-xy)\right]+\frac{1}{1-xy}\ln 1$$

$$= -\frac{1}{1-xy}\ln xy$$

LEMMA 53. The following holds

$$I = \int_0^1 \frac{P_n(x)P_n(y)}{1-(1-xy)z}dx = \int_0^1 \frac{(xyz)^n(1-x)^n P_n(y)}{\left[1-(1-xy)z\right]^{n+1}}dx$$

where

$$P_n(x)=\frac{1}{n!}\left(\frac{d}{dx}\right)^n x^n(1-x)^n$$

Proof. (Integration by parts)

$$I = \int_0^1 \frac{P_n(y)}{1-(1-xy)z}d\left[\frac{1}{n!}\left(\frac{d}{dx}\right)^{n-1} x^n(1-x)^n\right].$$

Integrating by parts gives:

$$\frac{P_n(y)}{1-(1-xy)z}\frac{1}{n!}\left(\frac{d}{dx}\right)^{n-1} x^n(1-x)^n\Big|_0^1$$

$$-\int_0^1 \frac{1}{n!}\left(\frac{d}{dx}\right)^{n-1} x^n(1-x)^n P_n(y)(-1)(1-(1-xy)z)^{-2}\,yz\,dx.$$

Now, $n-r$ differentiations of $x^n(1-x)^n$ will produce an expression where every term will have a common factor of $x^r(1-x)^r$. Using this fact and integrating by parts again in the above integral gives

$$0+\int_0^1 \frac{P_n(y)yz}{(1-(1-xy)z)^2}\,d\left[\frac{1}{n!}\left(\frac{d}{dx}\right)^{n-2} x^n(1-x)^n\right]$$

$$=\frac{P_n(y)yz}{(1-(1-xy)z)^2}\frac{1}{n!}\left(\frac{d}{dx}\right)^{n-2} x^n(1-x)^n\Big|_0^1$$

$$-\int_0^1 \frac{1}{n!}\left(\frac{d}{dx}\right)^{n-2}\left[x^n(1-x)^n\right]P_n(y)yz(-2)(1-(1-xy)z)^{-3}\,yz\,dx$$

$$=0+\int_0^1 \frac{P_n(y)y^2z^2}{(1-(1-xy)z)^3}\frac{2}{n!}\left(\frac{d}{dx}\right)^{n-2} x^n(1-x)^n$$

$$=0+\int_0^1 \frac{P_n(y)y^3z^3}{(1-(1-xy)z)^4}\frac{2\cdot 3}{n!}\left(\frac{d}{dx}\right)^{n-3} x^n(1-x)^n.$$

After $n-3$ more partial integrations it follows that

$$\int_0^1 \frac{P_n(y)y^nz^n}{(1-(1-xy)z)^{n+1}}\frac{2\cdot 3\cdot 4\cdots n}{n!}x^n(1-x)^n\,dx$$

$$=\int_0^1 \frac{(xyz)^n(1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}}\,dx.$$

LEMMA 54. If $I = \int \frac{(xyz)^n (1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}} dxdydz$ where $\int$ denotes the triple integral

$\int_0^1\int_0^1\int_0^1$, then

$$I = \int \frac{(1-x)^n (1-w)^n P_n(y)}{1-(1-xy)w} dxdydw, \text{ where } w = \frac{1-z}{1-(1-xy)z}.$$

Proof. Differentiating w with respect to z produces

$$\frac{dw}{dz} = \frac{(1-(1-xy)z)(-1)+(1-z)(1-xy)}{(1-(1-xy)z)^2}$$

$$= \frac{(1-z+xyz)(-1)+1-xy-z+xyz}{(1-(1-xy)z)^2}$$

$$= \frac{-xy}{(1-(1-xy)z)^2},$$

and hence

$$dz = \frac{(1-(1-xy)z)^2}{-xy} dw.$$

If $w = \frac{1-z}{1-(1-xy)z}$ then $w - wz(1-xy) = 1-z$,

$$-wz(1-xy)+z = -w+1,$$

$$z(1-w(1-xy)) = 1-w,$$

$$z = \frac{1-w}{1-w(1-xy)},$$

and hence

$$I = \int \frac{x^n y^n (1-w)^n (1-x)^n P_n(y)(1-(1-xy)z)^2}{(1-w(1-xy))^n (1-(1-xy)z)^{n-1}(-xy)} dxdydw$$

$$= -\int \frac{x^{n-1} y^{n-1} (1-w)^n (1-x)^n P_n(y)}{(1-w(1-xy))^n (1-(1-xy)z)^{n-1}} dxdydw .$$

Now, $1-(1-xy)z = 1 - \frac{(1-xy)(1-w)}{(1-w)(1-xy)}$

$$= \frac{1 - w + xyw - (1 - xy + xyw - w)}{1-w(1-xy)}$$

$$= \frac{xy}{1-w(1-xy)},$$

and, I becomes

$$= -\int \frac{x^{n-1} y^{n-1} (1-w)^n (1-x)^n P_n(y)(1-w(1-xy))^{n-1}}{(1-w(1-xy))^n (xy)^{n-1}} dxdydw$$

$$= -\int \frac{(1-w)^n (1-x)^n P_n(y)}{(1-w(1-xy))} dxdydw .$$

LEMMA 55. If $I = \int \frac{(1-x)^n (1-w)^n P_n(y)}{1-(1-xy)w} dy$ where $\int$ stands for the triple integration $\int_0^1\int_0^1\int_0^1$ then

$$I = \int \frac{x^n (1-x)^n y^n (1-y)^n w^n (1-w)^n}{(1-(1-xy)w)^{n+1}} dy .$$

Proof. Rearranging the above integral it follows that

$$I = \int \frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\frac{1}{n!}d\left[\left(\frac{d}{dy}\right)^{n-1} y^n(1-y)^n\right].$$

Integrating by parts gives

$$\int_0^1\int_0^1\left(\frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\frac{1}{n!}\left(\frac{d}{dy}\right)^{n-1} y^n(1-y)^n\Big|_0^1 -\right.$$
$$\left.\int_0^1\left(\frac{d}{dy}\right)^{n-1} y^n(1-y)^n\frac{d}{dy}\left(\frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\frac{1}{n!}\right)\right)dxdw.$$

Performing the differentiation $\left(\frac{d}{dy}\right)^{n-1} y^n(1-y)^n$ will produce an expression where every term will have common factors of y and $(1-y)$. Using this in the above triple integral gives

$$\int_0^1\int_0^1\left(0-\int_0^1\left(\frac{d}{dy}\right)^{n-1} y^n(1-y)^n\frac{d}{dy}\left(\frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\frac{1}{n!}\right)dy\right)dxdw$$

$$= \int_0^1\int_0^1\int_0^1\left(\frac{d}{dy}\right)^{n-1} y^n(1-y)^n\frac{(1-x)^n(1-w)^n}{(1-(1-xy)w)^2}\frac{1}{n!}xwdydxdw.$$

Integrating by parts again it follows that

$$\int\frac{(1-x)^n(1-w)^n}{(1-(1-xy)w)^2}\frac{1}{n!}d\left[\left(\frac{d}{dy}\right)^{n-2} y^n(1-y)^n\right]$$

$$= -\int\frac{(1-x)^n(1-w)^n}{(1-(1-xy)w)^3}\frac{-2}{n!}x^2w^2\left(\frac{d}{dy}\right)^{n-2} y^n(1-y)^n.$$

Integrating by parts again it follows that

$$\int \frac{(1-x)^n(1-w)^n}{(1-(1-xy)w)^3} x^2 w^2 \frac{2}{n!} d\left[\left(\frac{d}{dy}\right)^{n-3} y^n(1-y)^n\right]$$

$$-\int \frac{(1-x)^n(1-w)^n}{(1-(1-xy)w)^4} \frac{2(-3)}{n!} x^3 w^3 \left(\frac{d}{dy}\right)^{n-3} y^n(1-y)^n.$$

Continuing this integration by parts $n-3$ times the result is

$$\int \frac{(1-x)^n(1-w)^n}{(1-(1-xy)w)^{n+1}} x^n w^n y^n(1-y)^n$$

$$= \int \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{(1-(1-xy)w)^{n+1}} dy.$$

LEMMA 56. The function

$$f(w,x,y) = \frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \leq \left(\sqrt{2}-1\right)^4 \text{ for all } \quad 0 \leq x,y,w \leq 1.$$

Proof. The relative maximums of $f(w,x,y)$ occur when $\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ so it is necessary to find each of these partial derivatives. Rewriting $f(w,x,y)$ it follows that

$$\left(x-x^2\right)\left(y-y^2\right)\left(w-w^2\right)\left(1-w+xyw\right)^{-1}.$$

Differentiating with respect to x it follows that

$$\frac{\partial f}{\partial x} = \left(y-y^2\right)\left(w-w^2\right)\left[(1-2x)(1-w+xyw)^{-1} + \left(x-x^2\right)(-1)(1-w+xyw)^{-2} wy\right].$$

Putting $\frac{\partial f}{\partial x} = 0$ and simplifying gives

$$0 = (1-2x) + \frac{(x^2-x)wy}{1-w+xyw}, \quad y-y^2 \neq 0 \text{ and } w-w^2 \neq 0.$$

Therefore

$$0 = 1 - w + xyw - 2x + 2xw - 2x^2yw + x^2wy - xwy$$

(3.4.10) $$= 1 - w - 2x + 2xw - x^2yw.$$

As x and y are symmetric then from $\frac{\partial f}{\partial y} = 0$ it follows that

(3.4.11) $$0 = 1 - w - 2y + 2yw - y^2xw, \quad x - x^2 \neq 0 \text{ and } w - w^2 \neq 0.$$

Now,

$$\frac{\partial f}{\partial w} = (x-x^2)(y-y^2)\left[(1-2w)(1-w+xyw)^{-1} + (w-w^2)(-1)(1-w+xyw)^{-2}(xy-1)\right].$$

Putting $\frac{\partial f}{\partial w} = 0$ and simplifying gives

$$0 = (1-2w) + \frac{(w^2-w)(xy-1)}{1-w+xyw}, \quad x-x^2 \neq 0 \text{ and } y-y^2 \neq 0.$$

and therefore

$$0 = 1 - w + xyw - 2w + 2w^2 - 2xyw^2 + w^2xy - w^2 - wxy + w$$

$$= 1 - 2w + w^2 - xyw^2.$$

Rearranging gives

$$x = \frac{w^2 - 2w + 1}{yw^2}$$

$$= \left(\frac{w-1}{w}\right)^2 \frac{1}{y}.$$

Substituting in (3.4.11) gives

$$0 = 1 - w - 2y + 2yw - \frac{y(w-1)^2}{w},$$

and therefore

$$0 = w - w^2 - 2yw + 2yw^2 - \left[yw^2 + y - 2wy\right]$$

$$= w - w^2 + yw^2 - y.$$

Thus,

$$y = \frac{w^2 - w}{w^2 - 1}$$

$$= \frac{w(w-1)}{(w+1)(w-1)}$$

(3.4.12)
$$= \frac{w}{1+w}.$$

Because of the symmetry of x and y

$$x = \frac{w}{1+w}.$$

Substituting in (3.4.10) gives

$$0 = 1 - w - \frac{2w}{1+w} + \frac{2w^2}{1+w} - \left(\frac{w}{1+w}\right)^2\left(\frac{w}{1+w}\right)w,$$

and therefore

$$0 = (1+w)^3 - w(1+w)^3 - 2w(1+w)^2 + 2w^2(1+w)^2 - w^4$$

$$= 1 + 3w + 3w^2 + w^3 - w - 3w^2 - 3w^3 - w^4 - 2w - 2w^3 - 4w^2 + 2w^2 + 2w^4 + 4w^3 - w^4$$

$$= 1 - 2w^2.$$

Thus,

$$w = \pm\sqrt{\frac{1}{2}},$$

but $0 \le w \le 1$, so $w = \dfrac{1}{\sqrt{2}}$ or $\dfrac{\sqrt{2}}{2}$.

From (3.4.12) it follows that

$$x = \frac{\frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2} + 1}$$

$$= \sqrt{2} - 1.$$

Similarly,

$$y = \sqrt{2} - 1.$$

Now, inspecting $f(w, x, y)$ it is clear that it is a non-negative valued, continuous function for $0 \le x, y, w \le 1$ and it is 0 along the boundary of the region $0 \le x, y, w \le 1$. From further inspection the function $f(w, x, y)$ is greater than 0 at $\left(\sqrt{2} - 1, \sqrt{2} - 1, \frac{1}{\sqrt{2}}\right)$ so this point on the curve must be a maximum for if it were a minimum or a point of inflexion another critical point would be required the domain of which must be internal to the region $0 \le x, y, w \le 1$ and the only other possibilities for critical points for $0 \le x, y, w \le 1$ occur when $w - w^2 = y - y^2 = x - x^2 = 0$, but these possibilities occur when (w, x, y) is on the boundary $0 \le x, y, w \le 1$.

Now the function value $f\left(\sqrt{2}-1, \sqrt{2}-1, \frac{1}{\sqrt{2}}\right)$ is calculated. To get the required result it is useful to write $1-x$ and $1-y$ in the following way:

$$1-x = 1-\left(\sqrt{2}-1\right) = -\sqrt{2}+2 = \left(\sqrt{2}-1\right)\sqrt{2}.$$

Also note that $1-w = 1-\frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$.

So now using $\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w}$ for $f(w,x,y)$ the value of the function at this critical point is

$$\frac{\left(\sqrt{2}-1\right)^2\left(\sqrt{2}\right)^2\left(\sqrt{2}-1\right)^2\left(\frac{1}{\sqrt{2}}\right)^2\left(\sqrt{2}-1\right)}{1-\left(1-\left(\sqrt{2}-1\right)^2\right)\frac{\sqrt{2}}{2}} = \frac{\left(\sqrt{2}-1\right)^5}{1-\left(1-2-1+2\sqrt{2}\right)\frac{\sqrt{2}}{2}}$$

$$= \frac{2\left(\sqrt{2}-1\right)^5}{2-\left(2\sqrt{2}-2\right)\sqrt{2}}$$

$$= \frac{2\left(\sqrt{2}-1\right)^5}{2-4+2\sqrt{2}} = \frac{2\left(\sqrt{2}-1\right)^5}{2\left(\sqrt{2}-1\right)} = \left(\sqrt{2}-1\right)^4 \qquad \square$$

LEMMA 57. If $I = \int \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{\left(1-(1-xy)w\right)^{n+1}}\,dx\,dy\,dw$ where $\int$ denotes the triple integration $\int_0^1\int_0^1\int_0^1$ then $I > 0$.

Proof. Clearly for $0 \le x, y, w \le 1$, $I \ge 0$. Putting $x = y = w = \frac{1}{2}$ into

$$\frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{(1-(1-xy)w)^{n+1}}$$

gives a value which is not zero. Because the function is continuous for $0 \le x, y, w \le 1$ there must be a set of values of the function, of measure greater than 0, where all the values are greater than 0.

Therefore $I \neq 0$ so $I > 0$. □

LEMMA 58. The following holds:

$$2\zeta(3)27^n\left(\sqrt{2}-1\right)^{4n} < \left(\frac{4}{5}\right)^n \text{ for sufficiently large } n.$$

Proof. The left-hand side of the above inequality is

$$\left[\sqrt[n]{2}\sqrt[n]{\zeta(3)}\cdot 27\left(\sqrt{2}-1\right)^4\right]^n.$$

So it is just necessary to show that there exists n such that

$$\sqrt[n]{2}\sqrt[n]{\zeta(3)}\cdot 27\left(\sqrt{2}-1\right)^4 < \frac{4}{5}.$$

This will give the required result as the left side of this inequality is positive.

Rearranging it can be seen that n is required such that

$$\frac{1}{n}\ln\left(2\zeta(3)\right) < \ln\left[\frac{4}{5\cdot 27\cdot\left(\sqrt{2}-1\right)^4}\right].$$

Now, if

$$\ln\left[\frac{4}{5\cdot 27\cdot\left(\sqrt{2}-1\right)^4}\right] > 0$$

then

$$n > \ln\left(2\zeta(3)\right) \div \ln\left[\frac{4}{5\cdot 27\cdot\left(\sqrt{2}-1\right)^4}\right]$$

and the proof is complete as $2\zeta(3) > 1$. So it is necessary to show that

$$\frac{4}{5\cdot 27\cdot\left(\sqrt{2}-1\right)^4} > 1.$$

This can be proved on a calculator, of course, but more elegantly it is proved as follows.

First the expression $\left(\sqrt{2}-1\right)^4$ is rewritten:

$$\left(\sqrt{2}-1\right)^4 = \left(\sqrt{2}\right)^4 + 4\left(\sqrt{2}\right)^3(-1) + 6\left(\sqrt{2}\right)^2(-1)^2 + 4\left(\sqrt{2}\right)(-1)^3 + (-1)^4$$

$$= 4 - 8\sqrt{2} + 12 - 4\sqrt{2} + 1$$

$$= 17 - 12\sqrt{2}.$$

Now,

$$5248800 > 5248681$$

which on taking the square root of both sides gives

$$1620\sqrt{2} > 2291.$$

Rearranging it follows that

$$2295-1620\sqrt{2}<4$$

and then factoring and dividing by 5 produces

$$27\left(17-12\sqrt{2}\right)<\frac{4}{5}$$

which is the same as

$$27\left(\sqrt{2}-1\right)^4<\frac{4}{5}$$

or

$$\frac{4}{5\cdot 27\cdot\left(\sqrt{2}-1\right)^4}>1.$$

THEOREM 59. The number $\zeta(3)$ is irrational.

Proof. Consider the integral

$$I=\int_0^1\int_0^1\frac{-\ln xy}{1-xy}P_n(x)P_n(y)\,dx\,dy$$

where $P_n(x)=\frac{1}{n!}\left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$. From Lemma 51, $I=\left[A_n+B_n\zeta(3)\right]d_n^{-3}$ for some integers A_n and B_n. From Lemma 52, which states that

$$\int_0^1\frac{1}{1-(1-xy)z}dz = -\frac{1}{1-xy}\ln xy$$

I can be rewritten as $\int \frac{P_n(x)P_n(y)}{1-(1-xy)z}dxdydz$ where $\int$ denotes the triple integration

$\int\int\int$. After an n-fold partial integration with respect to x our integral changes into

$$\int \frac{(xyz)^n(1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}}dxdydz.$$

The details of this are given in Lemma 53. A substitution is now made

$$w = \frac{1-z}{1-(1-xy)z},$$

which gives

$$\int \frac{(1-x)^n(1-w)^n P_n(y)}{1-(1-xy)w}dxdydw.$$

The details of this are given in Lemma 54. After another n-fold partial integration with respect to y it follows that (see Lemma 55 for details)

$$\int \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{(1-(1-xy)w)^{n+1}}dxdydw. \tag{3.5.5}$$

From Lemma 56 it can be seen that the maximum of

$$x(1-x)y(1-y)w(1-w)(1-(1-xy)w)^{-1}$$

occurs for $x = y$ and then

$$x(1-x)y(1-y)w(1-w)(1-(1-xy)w)^{-1} \le \left(\sqrt{2}-1\right)^4$$

for all $0 \le x,y,w \le 1$. As $\frac{1}{1-(1-xy)w}$ is positive for $0 \le x,y,w \le 1$ then I is bounded above by

$$\left(\sqrt{2}-1\right)^{4n}\int\frac{1}{1-(1-xy)w}dxdydw.$$

As $\dfrac{1}{1-(1-xy)w}$ is continuous for $0<x,y,w<1$, the order of iteration of the integral can be changed at will. Note that it is not necessary to consider continuity at $x=0, w=1$ or at $y=0, w=1$ as the integral is undefined at these points. So then

$$I\le\left(\sqrt{2}-1\right)^{4n}\int\frac{1}{1-(1-xy)w}dwdxdy$$

$$=\left(\sqrt{2}-1\right)^{4n}\int_0^1\int_0^1\left[\int_0^1\frac{1}{1-(1-xy)w}dw\right]dxdy,$$

$$=\left(\sqrt{2}-1\right)^{4n}\int_0^1\int_0^1\left[\frac{-\log xy}{1-xy}\right]dxdy.$$

Using Theorem 45 part (d) it follows that

$$I\le 2\left(\sqrt{2}-1\right)^{4n}\zeta(3).$$

Using (3.5.5) in Lemma 57 it can be seen that $I>0$. So now the following holds

$$0<\left[A_n+B_n\zeta(3)\right]d_n^{-3}=I\le 2\left(\sqrt{2}-1\right)^{4n}\zeta(3).$$

Rearranging, using Corollary 17 (in Chapter 2) and then Lemma 58 it follows that

$$0<\left[A_n+B_n\zeta(3)\right]\le 2\zeta(3)d_n^{3}\left(\sqrt{2}-1\right)^{4n}<2\zeta(3)27^n\left(\sqrt{2}-1\right)^{4n}<\left(\frac{4}{5}\right)^n.$$

So, finally it can be seen that, for sufficiently large n,

$$0 < \left| A_n - B_n \zeta(3) \right| < \left(\frac{4}{5} \right)^n$$

By Theorem 6 this proves the irrationality of $\zeta(3)$. □

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