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A mathematical model for retirement planning

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A MATHEMATICAL MODEL FOR RETIREMENT PLANNING

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ABSTRACT

A MATHEMATICAL MODEL FOR RETIREMENT PLANNING

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In this thesis, an integer-programming model for retirement planning is developed, where a set of investments of a person is known. Also known are the growth-rates of the investments, the length of planning horizon, and the rate of increase in cost of living after retirement. The objective is to find the time periods in which the investments should be liquidated and put in the bank to earn interest with which the person can cover his or her annual expenses. If the withdrawal schedules of investments are known, the model can be used to determine what investments an individual needs to have or how much cost of living has to be after retirement to satisfy his or her retirement needs. The developed model is simple and flexible enough to incorporate tax related and other changes. For example, a one time big expense, like a child's wedding or education incurred during the retirement period, can be included in this model.

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Chapter 1: Introduction

Retirement planning is a very important and complex topic in today's finance. There have been numerous researches on different variables of retirement planning. O'Neill (1999) discusses three variables of utmost importance. One of the most important variables is pre-retirement savings. This variable is tightly coupled with the financial goal of the retiree. Another important factor to consider is the assumed rate of return on the savings. If the savings is diversified, the planning becomes more complicated. And finally, life expectancy is another important factor that influences retirement planning. This chapter will show how the earlier assumptions have been challenged. Then different dimensions of problems in modern retirement planning will be discussed and finally a proposal will be presented that will explain how this solution fits into the big picture of the problem.

1.1. Challenging the earlier assumptions

The assumptions and methodologies of retirement planning have changed considerably in the last two decades. McCarthy (2002) noted that earlier advisors often followed a standard sequence of steps to calculate the required savings that a person needs for the retirement life. Those steps were dependent on a fixed set of assumptions. Tacchino and

Saltzman (1999) challenged the assumptions behind many plans. For example, they showed that "almost 50 percent of retired households continued to save and build assets eight years into their retirement." Stein's (1999) study indicated "for many people it might be possible to save a third, or more, of the capital required for retirement in the five years just prior to retirement." Tacchino and Saltzman (1999) also challenged the linear assumption of spending. In their study, they showed that "there is a gradual reduction in spending starting shortly after retirement." Basu (2005) proposes a model where he divides the planning horizon of 30 years into three age bands to reflect post retirement life style changes. Then he calculates the present value of the post retirement expenses. Bernicke (2005) runs a hypothetical retirement income projections comparing traditional retirement planning and reality retirement planning.

1.2. Problems in modern retirement planning

There are several facets of problems related to modern retirement planning. Firstly, some people value other aspects of their retirement life over their financial needs. For example, for some people leisure time could be an important factor. Boscaljon (2005) discusses a "threedimensional model illustrating the trade-off preferences of risk, return, and leisure time."

Secondly, determining the right portfolio of investments is a challenging problem. Everett and Anthony (1996) discuss a spreadsheet-based model to diversify the portfolio. DeMiguel and Uppal (2005) propose a nonlinear programming based model to determine the optimal portfolio policy with tax implication.

Thirdly, a problem arises while dealing with future uncertainty about different parameters of financial planning. Undoubtedly it is very hard to forecast the right inflation rate or rate of return on investments. However, Greninger et al (2000) conducted a Delphi study among 188 financial planners and educators and reported that the planners and educators agreed on 4% inflation rate, 8.5% rate of return on investment, and 70-89% of current income as the replacement ratio.

Finally, a schedule of withdrawal is a great concern for the retirement planners. Spitzer and Singh (2006) propose a model to determine the portfolio depletion schedule when the objective is to generate an equal income stream in the retirement period. Ragsdale et al (1994) specifically focus on tax-deferred accounts and develop an optimization model for scheduling withdrawals from a tax-deferred retirement account.

1.3. The proposal

In this thesis report, a deviation from the idea of Spitzer and Singh (2006) is made and the withdrawal problem is addressed from a different angle. An analytical model is designed in which the set of investments of a person is known. Also known are the growth rate of the investments and the rate of increase of cost of living after retirement. Then the model seeks to find the time periods in which the investments should be liquidated and put in the bank to earn simple interest with which the person can cover his or her spending. The objective of this model is to maximize the residue money at the end of the retirement period.

The model is based on mixed integer programming. A mixed integer programming is a model in which some of the variables are required to be integers (Winston, 2004). To the best of the author's knowledge, similar work has not been done yet.

1.4. Advantages

Some of the advantages of the model are it is simple, fast, and flexible. The model can be useful for financial planners. A simple computer program can be developed for the financial planners so that they can help their clients effectively in their retirement planning. Also, the simplicity of the model can appeal to individuals who want to do financial planning on their own computer.

1.5. Organization of the paper

In Chapter 2, the mathematical model of the retirement planning problem will be discussed. First, the problem definition will be presented. Next, the assumptions and notations will be stated. Finally, the problem will be formulated as a mixed integer-programming model. In Chapter 3, the model will be expanded to take care of non-recurring expenses and incomes that may take place during the retirement period. A discussion will follow showing how the model needs to be changed to incorporate tax implications and capital gains and losses. In Chapter 4, different scenarios of retirement planning will be discussed and the results derived from the model will be presented. In Chapter 5, some future works that can be carried out based on the proposed model will be discussed. In Appendix A, a brief overview of the LINGO solver that has been used for solving the model will be presented. The actual LINGO programs used to solve each of the scenarios described in Chapter 4 will be presented in Appendix B.

Chapter 2: Mathematical model

In this chapter, the mathematical model of the problem will be discussed. First, the problem statement will be defined. Then the assumptions and notations used for the model will be stated, and finally the constraints and objective will be developed in the form of mathematical equations.

2.1. Problem statement

Suppose that an early retirement is of an interest for an individual. The person wants to make sure that his/her post-retirement life remains financially peaceful and he/she does not have to borrow any money from anyone to cover the annual expenses. In the beginning of his/her career, the person made some fortune, which was invested in different types of assets. After retirement, the person wishes to liquidate the investments by following a particular schedule and put the money in a bank account from where he/she can earn interest perpetually. The problem at hand is to figure out the schedule of liquidation. That is, in which year and what investment should be liquidated so that for any year the total income is greater than the expenses of that year and also the total savings including the non-liquidated assets at the end of the planning horizon is maximized.

2.2. Assumptions and notations

i) Investment i (dollar amount denoted by I_i) grow by r_i% every year on average.

ii) All investments are liquidated fully. There is no scope of partial liquidation.

iii) Current annual expense is C. C' is the annual expense at the onset of retirement period. The assumption made here is that the annual expense grows by r_c% every year and r_c is same for both before and after retirement life.

iv) Bank pays r_b% interest rate for any deposited amount and for any period of savings.

v) Let T be the planning horizon i.e., number of years used for retirement planning.

vi) Let R denote the number of years remaining to retire. It is also assumed that prior to retirement, the person will not make any more investments.

vii) Investment k is liquidated at time period t_k years from R. If the current year is chosen as the base line whose value is 0, it means that I_1 dollar investment will be liquidated $R+t_1$ years from now. Similarly, I_2 dollar investment will be liquidated

$R+t_2$ years from now and so forth. The objective is to find the value of $t_1, t_2, ...t_n$.

viii) Investments are liquidated and put in the bank at the beginning of the year. Expense and Interest earning are recorded at the end of each year.

ix) Let D_t = dollars available in the bank at the beginning of time period t, $t = 1,2,.......T$.

x) L_{it} = 1 if investment i is liquidated at the beginning of time period t, $t = 1,2,.......T$

= 0 otherwise.

2.3. Developing constraints and objectives

In this section, the constraints and objective function of the problem are discussed.

2.3.1. Income-expense constraint for each year

Suppose it is year t. Let's find out how much money will be there at the bank at the beginning of the period t. The money at the bank at the beginning of the period t-1 has grown by $r_b\%$. So the money at the beginning of period t due to bank's interest = $(1+r_b)D_{t-1}$. (1)

If any investment is liquidated at period t, then that money will be added to the bank's beginning balance at period t. For investment k (with present value I_k), the dollar amount of the investment at the beginning of the retirement period is $I_k^{'} = I_k(1+r_k)^R$. At the beginning of period t of retirement life the dollar amount of that investment will be = $I_k(1+r_k)^{t-1}$. Since $L_{kt}=1$ if the k-th investment is liquidated at period t and 0 otherwise, the term $I_k^{'}(1+r_k)^{t-1}L_{kt}$ can be added over k (i.e., all investments) to find out how much money comes to the bank from liquidation of investments. Hence money generated by liquidating investments = $\sum_k I_k^{'}(1+r_k)^{t-1}L_{kt}$. (2)

Adding (1) and (2), the total money at the bank at the beginning of period t = $(1+r_b)D_{t-1} + \sum_k I_k^{'}(1+r_k)^{t-1}L_{kt}$. (3)

Since $C^{'}$ is the expense at the beginning of retirement period and r_c is the rate of increase of expenses each year, expense at year t

$$= C^{'}(1+r_c)^{t-1} \quad (4)$$

Since it is assumed that expenses are made at the beginning of the year, equation (4) can be subtracted from equation (3) to get the beginning balance of period t.

$$D_t = (1+r_b)D_{t-1} + \sum_k I_k^{'}(1+r\ )^{t-1}L_{kt} - C^{'}(1+r_c)^{t-1} \quad (5)$$

The constraint expressed by equation (5) will be there for all time periods $t=1,2,\ldots.T$.

2.3.2. Available balance constraint

For each year, it is required to have positive bank balance. So for time periods $t = 1,2,\ldots\ldots.T$, $D_t \geq 0$. (6)

2.3.3. Binary value constraint of L variables

In section 2.2 "assumption and notations," it has been discussed that

L_{it} = 1 if i-th investment is liquidated at period t

= 0 otherwise. (7)

2.3.4. Number of liquidation for an investment constraint

Since an investment can be liquidated, if at all done, only once in the whole retirement period, the sum of L_{it} over period t must be less than or equal to 1. It can be stated for all investments i

$$\sum_t L_{it} \leq 1 \quad (8)$$

2.3.5. Objective function

At the end of period T, or at the beginning of period *T+1*, the money at bank = $(1+r_b)D_T$. (9)

At the end of period T, or at the beginning of period *T+1*, the dollar value of the investment k = $I_k^{'}(1+r_k)^T$. If the investment k is liquidated in the retirement period, then the term $\sum L_{it}$ equals to 1. So $(1-\sum L_{it})$ = 1 means the investment k is not liquidated yet and 0 means the investment k is already liquidated. So, total value of non-liquidated investments at the end of period T

$$= \sum \quad I_k^{'}(1+r_k)^T \;\; (1-\sum L_{it}) \tag{10}$$

So, total money at hand at the end of period T

$$= (1+r_b)D_T \; + \sum \quad I_k^{'}(1+r_k)^T \;\; (1-\sum L_{it}) \tag{11}$$

The objective function is to maximize equation (11).

2.4. Summary

Let $I_i^{'} = I_i\,(1+r_i)^R$ and $C^{'} = C_i\,(1+r_c)^R$

Maximize $Z = (1+r_b)D_T + \sum_i I_i^{'}(1+r_i)^T(1-\sum_t L_{it})$ (12)

$$(1+r_b)D_{t-1} + \sum_{i=1}^{k} I_i^{'}(1+r_i)^{t-1}L_{it} - C^{'}(1+r_c)^{t-1} = D_t \quad \text{for } t=1,2,\ldots T \tag{13}$$

$$\sum_{t=1}^{T} L_{it} \leq 1 \qquad \text{for } i=1,2,\ldots.k \tag{14}$$

$$D_t \geq 0 \quad \text{for } t=1,2,\ldots T\text{-}1 \tag{15}$$

$$L_{it} = 0/1 \quad \text{for } i=1,2,\ldots.k \text{ and } t=1,2,\ldots.T \tag{16}$$

The objective function (12) maximizes the number of dollars available at the end of time period T including non-liquidated investments. Constraint set (13) evaluates cash available in the bank at the beginning of time period t = cash available at the bank at the beginning of time period $(t-1)x(1+r_b)$ + cash generated at time period t by liquidating investments – annual expenses for year $(t-1)$. Constraint set (14) ensures that each investment i gets liquidated in exactly one time period. Constraint set (15) ensures that at the beginning of each year t, nonnegative amount of dollars is available in the bank for t = 1, 2, ..., T after the expenses for the period have been deducted. Constraint set (16) is a binary condition on the investment liquidation variable for each i and t.

Chapter 3: Expansion of the model

In the previous chapter, the basic model of the retirement planning problem was developed. In this chapter, the model will be expanded to incorporate various features.

3.1. Fixed income and nonrecurring planned expenses

The model can be easily modified to take care of the fixed income and non-recurring planned expenses that may occur in the retirement life. For example, if social security benefits start at age 67 or a world tour is planned for the 60th birthday, then this model needs to be expanded. Fixed income may result from the maturity of a bond, maturity of a CD, ordinary dividends, other interest income and annuities, social security benefits, pensions, etc. Expenses that are normally large, are outside the scope of regular annual expenses, and are planned to take place in the future in a non-recurring fashion are termed as non-recurring planned expenses. Events such as children's education, marriage, charity, etc. fall into the category of non-recurring planned expenses.

Consider two variables f_t and e_t which denote the fixed income at year t and non-recurring expense at year t respectively. It is to be noted that here year t means t years from the onset of retirement life. In the left hand side of equation (13), f_t should be added to reflect the additional income earned in year t. Also e_t should be subtracted from the left hand side of equation (13) to take care of the additional expenses in year t. The modified model is presented below.

$$\text{Maximize} \quad Z = (1+r_b)D_T + \sum_i I_i'(1+r_i)^T (1-\sum_t L_{it}) \tag{17}$$

$$(1+r_b)D_{t-1} + f_t + \sum_{i=1}^{k} I_i'(1+r_i)^{t-1} L_{it} - e_t - C'(1+r_c)^{t-1} = D_t \quad \text{for } t = 1,2,.....T \tag{18}$$

$$\sum_{t=1}^{T} L_{it} \leq 1 \quad \text{for } i=1,2,....k \tag{19}$$

$$D_t \geq 0 \quad \text{for } t=1,2,...T\text{-}1 \tag{20}$$

$$L_{it} = 0/1 \quad \text{for } i=1,2,....k \text{ and } t=1,2,....T \tag{21}$$

3.2. Tax consideration

In this section, it will be discussed how the model can be expanded to take care of the tax situation. If the income is in the form of 401K payment or social security benefits or ordinary interest and dividend payment, then the terms f_t and $r_b D_{t-1}$ (interest income in year t for available dollars at the beginning of year t) need to be multiplied by a factor $(1-T_x)$ where T_x is the tax rate whose value depends on the annual income bracket.

On the other hand, if an income is made in the form of property selling or any other capital gain, then the following steps need to be executed to calculate the tax. First, net adjusted basis is calculated by subtracting the depreciation from original price and adding improvements, if any, that has been made on the property. Next, capital gain is calculated by taking out the net adjusted basis and cost of sale from the actual sale price. Last, the recaptured depreciation is calculated and federal and state capital tax rate on the capital gain are added to deduce the actual tax dollars. For example, if a property is purchased at price P and sold at S, then (excluding the depreciation and other items), the tax = $T_x(S-P)$ where T_x is the capital gain tax rate. So

net gain = $S - T_x(S-P) = S(1-T_x) + PT_x$. (22)

In this model, it is assumed that the investment k originally purchased at I_k dollars, gets sold at $I_k(1+r_k)^{R+t_k}$ dollars at year t_k of the retirement period. So from (22), net gain = $I_k(1+r_k)^{R+t_k}(1-T_x) + I_kT_x$. (23)

Therefore, in equation (12) and (13), wherever the term $I_k(1+r_k)^{R+t_k}$ is used it should be replaced by the net gain term $I_k(1+r_k)^{R+t_k}(1-T_x) + I_kT_x$ from equation (23). The modified model is presented below.

Maximize $Z = (1+r_b)D_T + \sum_i I_i'(1+r_i)^T(1-\sum_t L_{it})$ (24)

$$(1+r_b)D_{t-1} + f_t + \sum_{i=1}^{k}(I_k(1+r_k)^{R+t_k}(1-T_x) + I_kT_x)L_{it} - e_t - C'(1+r_c)^{t-1} = D_t \quad \text{for}$$

$t = 1,2,.....T$ (25)

$$\sum_{t=1}^{T} L_{it} \leq 1 \quad \text{for } i=1,2,....k \tag{26}$$

$$D_t \geq 0 \quad \text{for } t=1,2,...T\text{-}1 \tag{27}$$

$$L_{it} = 0/1 \quad \text{for } i=1,2,....k \text{ and } t=1,2,....T \tag{28}$$

Now it will be shown what happens if there is a capital loss. Consider a case where the purchase value of a property is P and selling value is S where $S \leq P$. If the tax rate is T_x, then the net cash flow = money had by selling the property + money saved by not paying capital gain tax. Since S is the selling value, money had by selling the property $= S$. So,

tax dollars saved= $T_x(P - S)$. (29)

So net cash flow = $S(1 - T_x) + PT_x$. (30)

The result derived from equation (30) for capital loss is identical to equation (22) for capital loss. Hence the model for capital loss remains same as capital gain.

3.3. Partial liquidation

As mentioned in assumption ii) of section 2.2, all investments considered so far were fully liquidated. There was no scope of partial liquidation. That is, it was not possible to withdraw a part of an investment and let another part of that investment grow at the original growth rate. But in investments like a 401K, a person can withdraw as much as is needed and leave the rest of the money to grow at the original rate. In this section, the original model is extended to take care of this case. First, the additional assumptions and notations used to extend the original model are listed.

i) $A_1, A_2, \ldots\ldots A_m$ denote the dollar value of the current investments that can be partially liquidated.

ii) g_j indicates the growth rate (in %) of investment j where $j = 1,2,3,\ldots.m$. Like r_i and r_c, it is assumed that g_j is same before and after the retirement.

iii) A_{jt} denotes the dollar value of the investment j at the beginning of period t. So assuming all the m investments are kept alive till the retirement period starts, it can be said that $A_{s1} = A_s(1+g_s)^R$ for $s = 1,2,\ldots.m$.

iv) x_{jt} denotes the dollar amount withdrawn from investment j at the beginning of period t.

The additional constraints and the modified objective function will be developed in the following sections.

3.3.1. Additional income constraint

Since x_{jt} dollars are withdrawn from jth investment at the beginning of period t and added to the total income, the total income increases by $\sum x_{jt}$.

Hence the constraint (13) is modified as

$$D_t = (1+r_b)D_{t-1} + \sum_k I_k^{'}(1+r_k)^{t-1} L_{kt} - C^{'}(1+r_c)^{t-1} + \sum_k x_{kt} \quad \text{for } t = 1,2,....T \tag{31}$$

3.3.2. Investment growth constraints

At the starting of the retirement period, the dollar value of all the m investments is calculated as $A_{s1} = A_s(1+g_s)^R$ for $s = 1,2,.....m$. Since the dollar value of the jth investment at beginning of period t is A_{jt} and x_{jt} dollars are withdrawn at period t, the remaining dollars for investment j= $A_{jt} - x_{jt}$. After one year, that is, at the beginning of the period t+1, dollar value of the investment j, $A_{(t+1)j} = (A_{jt} - x_{jt})(1+g_j)$. So for $t = 1,2,.....T$

$$A_{(t+1)j} = (A_{jt} - x_{jt})(1+g_j)\,. \tag{32}$$

3.3.3. Withdrawal of investment constraint

Since it is not possible to withdraw more money from an investment that is already present in that investment, for all j and t, $x_{jt} \le A_{jt}$. (33)

3.3.4. Modifying the objective value

The dollar value of jth investment at the end of period T or at the beginning of period $T+1 = A_{j(T+1)}$. (34)

Adding equation (34) over j, the total dollar value of investments not withdrawn yet = $\sum_j A_{j(T+1)}$. (35)

The value in equation (34) is added to equation (12) to modify the objective value. Here is the modified form of the objective function.

Maximize Z= =$(1+r_b)\ D_T + \Sigma_i\ I_i'\ (1+r_i)^T\ \ (1 - \Sigma_t\ L_{it}) + \sum_i A_{i(T+1)}$ (36)

The expanded model is summarized below.

$$\text{Maximize } Z = (1+r_b)D_T + \sum_i I_i'(1+r_i)^T(1-\sum_t L_{it}) + \sum_i A_{i(T+1)} \quad (37)$$

$$D_t = (1+r_b)D_{t-1} + \sum_k I_k'(1+r_k)^{t-1}L_{kt} - C'(1+r_c)^{t-1} + \sum_k x_{kt} \text{ for } t = 1,2,....T \quad (38)$$

$$A_{(t+1)j} = (A_{jt} - x_{jt})(1+g_j) \quad \text{for} \quad t = 1,2,.....T \quad (39)$$

$$A_{s1} = A_s(1+g_s)^R \quad \text{for } s = 1,2,.....m \quad (40)$$

$$x_{jt} \le A_{jt} \quad \text{for all } j \text{ and } t \quad (41)$$

$$\sum_{t=1}^{T} L_{it} \le 1 \quad \text{for } i=1,2,....k \quad (42)$$

$$D_t \ge 0 \quad \text{for } t=1,2,...T\text{-}1 \quad (43)$$

$$L_{it} = 0/1 \quad \text{for } i=1,2,....k \text{ and } t=1,2,....T \quad (44)$$

Chapter 4: Scenarios and results

In this chapter, six scenarios are presented and how the model works in different situations is demonstrated. Each scenario corresponds to a unique set of retirement planning parameters, which includes number of investments, the current dollar value of the investments, the growth rate of each investment, current annual expense, rate of increase of annual expense, length of planning horizon, tax rate, and the interest rate offered by the bank.

In scenario 1, a case is considered where the retirement parameters yield a feasible solution and hence no change needs to be done for any parameter.

In scenario 2, the parameters are chosen in such a way that makes the solution infeasible, and then five alternatives (by tuning one or more parameters) are discussed to attain a feasible solution set. In this paper, the parameters considered are the number of investments, bank's interest rate, length of planning horizon, current expense, and expense growth rate.

In scenario 3, the same set of parameters as presented in scenario 1 are considered along with some additional parameters like planned non-recurring expenses and fixed income stream as described in Chapter 3.

In scenario 4, it is shown how the model can be used to manage an original plan in case of change of parameters in retirement planning.

In scenario 5, the tax implications are considered. For simplicity, it is assumed that a single tax rate is present.

Finally in scenario 6, some partially liquidated investments are considered along with tax. In this scenario, the planning horizon is extended to see if it significantly changes the running time of the model. The data used in these scenarios are fictitious and have been intentionally kept simple to clearly demonstrate how the model works.

4.1. Scenario 1

This case assumes that an individual has 3 investments at present. The amounts and growth rates are shown in Table 4.1. This table is referred as investment data sheet.

Table 4.1: Investment data sheet for Scenario 1

Investment	Dollars (in 10^4)	Growth Rate (%)
1	10	15
2	13	10
3	10	13

The individual has no other sources of income in his/her retirement life. There are no planned non-recurring expenses in the retirement period. The person will retire after 10 years from now and the life expectancy is 30 years after retirement starts.

The bank is expected to pay a simple interest rate of 5%. Current annual expense is $20,000. It is estimated that every year the annual expense will rise by 12%. However, after retirement, rate of increase of annual expense is estimated to go down to 5%. It is estimated that at the onset of the retirement, the individual will have zero bank balance.

Table 4.2 shows the schedule of investment liquidation, total earning, and total expense for each year after retirement, which optimizes the objective function. It is to be noted that the objective is to maximize the total money left at hand at the end of the retirement planning horizon.

Table 4.2: Results for scenario 1 (figures in $10K)

Liquidated Money	Total Earning	Total Expenses	Available Dollars
33.72	33.72	6.21	27.51
0.00	0.00	6.52	22.36
0.00	0.00	6.85	16.63
0.00	0.00	7.19	10.27
0.00	0.00	7.55	3.23
62.54	62.54	7.93	58.01
0.00	0.00	8.32	52.59
0.00	0.00	8.74	46.48
0.00	0.00	9.18	39.62
0.00	0.00	9.64	31.97
0.00	0.00	10.12	23.45
0.00	0.00	10.62	13.99
0.00	0.00	11.16	3.54
248.91	248.91	11.71	240.92
0.00	0.00	12.30	240.66
0.00	0.00	12.91	239.78
0.00	0.00	13.56	238.21
0.00	0.00	14.24	235.89
0.00	0.00	14.95	232.73
0.00	0.00	15.70	228.67
0.00	0.00	16.48	223.62
0.00	0.00	17.31	217.50
0.00	0.00	18.17	210.20
0.00	0.00	19.08	201.63
0.00	0.00	20.03	191.68
0.00	0.00	21.04	180.23
0.00	0.00	22.09	167.16
0.00	0.00	23.19	152.32
0.00	0.00	24.35	135.59
0.00	0.00	25.57	116.80

The model suggests that the second investment should be liquidated at the beginning of retirement period. Then in year 6, the third investment and finally at year 14, the first investment should be liquidated. For the actual LINGO program and output, see Appendix B.1.

4.2. Scenario 2

Here all the data are same as of previous scenario except the dollar amount of the second investment is changed from 150K to 90K. Refer Table 4.3 for the investment data sheet.

Table 4.3: Investment data sheet for scenario 2

Investment	Dollars (in 10^4)	Growth Rate (%)
1	10	15
2	9	10
3	10	13

With this data, no feasible solution is found. It tells that with the current set of investments it is not possible to meet up the expenses in the retirement life the person is planning to incur. However, there are many other parameters in this model, which could be changed so that a feasible solution can be attained. For sake of brevity, five different alternatives are considered.

4.2.1. Alternative 1

The first alternative is to make one more investment. For example, if the person can make another investment of $90K at the growth rate of

8%, a feasible solution is reached. See Table 4.4 for the new investment data sheet.

Table 4.4: Investment data sheet for alternative 1

Investment	Dollars (in 10^4)	Growth Rate (%)
1	10	15
2	9	10
3	10	13
4	9	8

The liquidation schedule derived from the model is presented in Table 4.5. For the actual LINGO program, see Appendix B.2.1.

Table 4.5: Liquidation schedule for alternative 1

Year	Investment liquidated
1	4
4	2
8	3
17	1

Table 4.6 shows the schedule of investment liquidation, total earning, and total expense for each year after retirement.

Table 4.6: Results for scenario 2.1 (figures in $10K)

Liquidated Money	Total Earning	Total Expenses	Available Dollars
19.43	19.43	6.21	13.22
0.00	0.00	6.52	7.36
0.00	0.00	6.85	0.88
31.07	31.07	7.19	24.80
0.00	0.00	7.55	18.49
0.00	0.00	7.93	11.49
0.00	0.00	8.32	3.74
79.86	79.86	8.74	75.04
0.00	0.00	9.18	69.62
0.00	0.00	9.64	63.46
0.00	0.00	10.12	56.52
0.00	0.00	10.62	48.72
0.00	0.00	11.16	40.00
0.00	0.00	11.71	30.29
0.00	0.00	12.30	19.50
0.00	0.00	12.91	7.56
378.57	378.57	13.56	372.95
0.00	0.00	14.24	377.36
0.00	0.00	14.95	381.28
0.00	0.00	15.70	384.65
0.00	0.00	16.48	387.40
0.00	0.00	17.31	389.46
0.00	0.00	18.17	390.77
0.00	0.00	19.08	391.22
0.00	0.00	20.03	390.75
0.00	0.00	21.04	389.25
0.00	0.00	22.09	386.63
0.00	0.00	23.19	382.77
0.00	0.00	24.35	377.56
0.00	0.00	25.57	370.87

4.2.2. Alternative 2

Sometimes making an additional investment may not seem to be a viable option. In that case, the individual can try to reduce the rate of increase of expenses after retirement. In scenario 2, the rate was assumed to be 5%. If that can be lowered down to 3%, a feasible solution can be obtained.

The resultant liquidation schedule table is shown in Table 4.7. For the actual LINGO program and output, see Appendix B.2.2.

Table 4.7: Liquidation schedule for alternative 2

Year	Investment liquidated
1	2
4	3
12	1

Table 4.8 shows the schedule of investment liquidation, total earning, and total expense for each year after retirement.

Table 4.8: Results for scenario 2.2 (figures in $10K)

Liquidated Money	Total Earning	Total Expenses	Available Dollars
23.34	23.34	6.21	17.13
0.00	0.00	6.40	11.59
0.00	0.00	6.59	5.58
48.98	48.98	6.79	48.05
0.00	0.00	6.99	43.46
0.00	0.00	7.20	38.43
0.00	0.00	7.42	32.94
0.00	0.00	7.64	26.95
0.00	0.00	7.87	20.43
0.00	0.00	8.10	13.34
0.00	0.00	8.35	5.66
188.22	188.22	8.60	185.56
0.00	0.00	8.86	185.98
0.00	0.00	9.12	186.16
0.00	0.00	9.40	186.07
0.00	0.00	9.68	185.70
0.00	0.00	9.97	185.01
0.00	0.00	10.27	184.00
0.00	0.00	10.58	182.62
0.00	0.00	10.89	180.86
0.00	0.00	11.22	178.69
0.00	0.00	11.56	176.06
0.00	0.00	11.90	172.97
0.00	0.00	12.26	169.35
0.00	0.00	12.63	165.20
0.00	0.00	13.01	160.45
0.00	0.00	13.40	155.08
0.00	0.00	13.80	149.03
0.00	0.00	14.21	142.27
0.00	0.00	14.64	134.75

4.2.3. Alternative 3

Alternative 2 may not always be a viable option, that is, a person may not always control the rate of increase of annual expenses if the inflation rate is too high. In that case an alternative could be to work for few more years and thus reduce the retirement planning horizon. In this example, if the current planning horizon is reduced from 30 years to 25 years and increase the time left to retire from 10 years to 15 years, a feasible solution is reached.

The liquidation schedule for this alternative presented in Table 4.9 looks almost similar to the one described in alternative 2 except that investment 1 should be liquidated in year 11, that is, 1 year earlier than the case of alternative 2. For the actual LINGO program and output, see Appendix B.2.3.

Table 4.9: Liquidation schedule for alternative 3

Year	Investment liquidated
1	2
4	3
12	1

Table 4.10 shows the schedule of investment liquidation, total earning, and total expense for each year after retirement. Note that number of years in this case is 25.

Table 4.10: Results for scenario 2.3 (figures in $10K)

Liquidated Money	Total Earning	Total Expenses	Available Dollars
37.60	37.60	10.95	26.65
0.00	0.00	11.49	16.49
0.00	0.00	12.07	5.24
90.24	90.24	12.67	83.07
0.00	0.00	13.31	73.92
0.00	0.00	13.97	63.64
0.00	0.00	14.67	52.16
0.00	0.00	15.40	39.36
0.00	0.00	16.17	25.16
0.00	0.00	16.98	9.43
329.19	329.19	17.83	321.26
0.00	0.00	18.72	318.60
0.00	0.00	19.66	314.87
0.00	0.00	20.64	309.97
0.00	0.00	21.67	303.79
0.00	0.00	22.76	296.23
0.00	0.00	23.90	287.14
0.00	0.00	25.09	276.41
0.00	0.00	26.35	263.88
0.00	0.00	27.66	249.41
0.00	0.00	29.05	232.84
0.00	0.00	30.50	213.98
0.00	0.00	32.02	192.66
0.00	0.00	33.62	168.67
0.00	0.00	35.31	141.79

4.2.4. Alternative 4

Another alternative could be to find another bank that pays higher interest rate. For example, if the bank interest rate is increased from 5% to 6%, a feasible solution is achieved.

The liquidation schedule remains exactly the same as of alternative 3. For the actual LINGO program and output, see Appendix B.2.4. Table 4.11 shows the schedule of investment liquidation, total earning, and total expense for each year after retirement.

Table 4.11: Results for scenario 2.4 (figures in $10K)

Liquidated Money	Total Earning	Total Expenses	Available Dollars
37.60	37.60	10.95	26.65
0.00	0.00	11.49	16.75
0.00	0.00	12.07	5.69
90.24	90.24	12.67	83.60
0.00	0.00	13.31	75.31
0.00	0.00	13.97	65.86
0.00	0.00	14.67	55.14
0.00	0.00	15.40	43.04
0.00	0.00	16.17	29.45
0.00	0.00	16.98	14.24
329.19	329.19	17.83	326.45
0.00	0.00	18.72	327.31
0.00	0.00	19.66	327.29
0.00	0.00	20.64	326.28
0.00	0.00	21.67	324.19
0.00	0.00	22.76	320.88
0.00	0.00	23.90	316.24
0.00	0.00	25.09	310.12
0.00	0.00	26.35	302.38
0.00	0.00	27.66	292.86
0.00	0.00	29.05	281.39
0.00	0.00	30.50	267.77
0.00	0.00	32.02	251.82
0.00	0.00	33.62	233.30
0.00	0.00	35.31	211.99
0.00	0.00	37.07	187.64
0.00	0.00	38.92	159.97
0.00	0.00	40.87	128.70
0.00	0.00	42.91	93.51
0.00	0.00	45.06	54.06

4.2.5. Alternative 5

If either of the above alternatives seem infeasible from the retiree's point of view, another alternative could be to retrench the rate of increase of current expenses. For example, if the current rate of increase of expenses is brought down to 10% from 12%, a feasible solution is reached.

The liquidation schedule at Table 4.12 shows that the liquidation years of first and third investments have been postponed by one year. This is obvious because as the expense has gone down the person can take the leverage of liquidating an investment later and gain more money.

Table 4.12: Liquidation schedule for alternative 5

Year	Investment liquidated
1	2
5	3
15	1

Table 4.13 shows the schedule of investment liquidation, total earning, and total expense for each year after retirement.

Table 4.13: Results for scenario 2.5 (figures in $10K)

Liquidated Money	Total Earning	Total Expenses	Available Dollars
37.60	37.60	8.35	29.24
0.00	0.00	8.77	21.93
0.00	0.00	9.21	13.82
0.00	0.00	9.67	4.84
101.97	101.97	10.15	96.90
0.00	0.00	10.66	91.08
0.00	0.00	11.20	84.44
0.00	0.00	11.76	76.90
0.00	0.00	12.34	68.41
0.00	0.00	12.96	58.86
0.00	0.00	13.61	48.20
0.00	0.00	14.29	36.32
0.00	0.00	15.00	23.13
0.00	0.00	15.75	8.54
575.75	575.75	16.54	568.18
0.00	0.00	17.37	579.22
0.00	0.00	18.24	589.94
0.00	0.00	19.15	600.29
0.00	0.00	20.11	610.20
0.00	0.00	21.11	619.60
0.00	0.00	22.17	628.41
0.00	0.00	23.28	636.55
0.00	0.00	24.44	643.94
0.00	0.00	25.66	650.48
0.00	0.00	26.94	656.06
0.00	0.00	28.29	660.57
0.00	0.00	29.71	663.89
0.00	0.00	31.19	665.90
0.00	0.00	32.75	666.44
0.00	0.00	34.39	665.37

It is to be noted that there can be several other possible alternatives to get a feasible solution for scenario 2. In this paper, only five possibilities have been presented to show how the model can be easily programmed to deal with this kind of situation.

4.3. Scenario 3

In this scenario, it is assumed that the person has the following investment data sheet at present. See Table 4.14.

Table 4.14: Investment data sheet for scenario 3

Investment	Dollars (in 10^4)	Growth Rate (%)
1	10	15
2	9	10
3	10	13

It is also assumed that the person will retire after 15 years from now and the life expectancy is 30 years after retirement period starts. The bank is expected to pay a simple interest rate of 5%. Current annual expense is $20,000. It is estimated that every year the annual expense rises by 12%. However, after retirement, rate of increase of annual expense is estimated to go down to 5%. It is estimated that at the onset of the retirement, the individual will have zero bank balance.

The person is also expecting to receive $40,000 annually each year perpetually starting from year 18. In year 13, he/she is planning to incur an expense of $100,000.

The liquidation schedule for this case is presented below in Table 4.15. For the actual LINGO program and output, see Appendix B.3.

Table 4.15: Liquidation schedule for scenario 3

Year	[illegible]
1	2
4	3
11	1

Table 4.16 shows the schedule of investment liquidation, total earning, and total expense for each year after retirement.

Table 4.16: Results for scenario 3 (figures in $10K)

Liquidated Money	Other income	Total Earning	Planned Big Expense	Total Expenses	Available Dollars
37.60	0.00	37.60	0.00	10.95	26.65
0.00	0.00	0.00	0.00	11.49	16.49
0.00	0.00	0.00	0.00	12.07	5.24
90.24	0.00	90.24	0.00	12.67	83.07
0.00	0.00	0.00	0.00	13.31	73.92
0.00	0.00	0.00	0.00	13.97	63.64
0.00	0.00	0.00	0.00	14.67	52.16
0.00	0.00	0.00	0.00	15.40	39.36
0.00	0.00	0.00	0.00	16.17	25.16
0.00	0.00	0.00	0.00	16.98	9.43
329.19	0.00	329.19	0.00	17.83	321.26
0.00	0.00	0.00	0.00	18.72	318.60
0.00	0.00	0.00	10.00	29.66	304.87
0.00	0.00	0.00	0.00	20.64	299.47
0.00	0.00	0.00	0.00	21.67	292.77
0.00	0.00	0.00	0.00	22.76	284.65
0.00	0.00	0.00	0.00	23.90	274.99
0.00	4.00	4.00	0.00	25.09	267.64
0.00	4.00	4.00	0.00	26.35	258.68
0.00	4.00	4.00	0.00	27.66	247.95
0.00	4.00	4.00	0.00	29.05	235.30
0.00	4.00	4.00	0.00	30.50	220.57
0.00	4.00	4.00	0.00	32.02	203.58
0.00	4.00	4.00	0.00	33.62	184.13
0.00	4.00	4.00	0.00	35.31	162.03
0.00	4.00	4.00	0.00	37.07	137.06
0.00	4.00	4.00	0.00	38.92	108.99
0.00	4.00	4.00	0.00	40.87	77.57
0.00	4.00	4.00	0.00	42.91	42.53
0.00	4.00	4.00	0.00	45.06	3.60

4.4. Scenario 4

This scenario is different in nature from all others described previously. In previous scenarios it was shown how an individual's retirement life could be planned. Here it is shown how the plan can be managed if there is a change of parameters in the retirement life and it turns out that the original plan does not yield a feasible solution any more. To present the case, it is assumed that the retirement plan was originally made based on the data of the scenario 3. It is also assumed that everything was going fine till 10 years after retirement when there came a big recession. Investment growth rates plummeted from 15% to 9%. Planned non-recurring expense in the 13th year went up from \$100,000 to \$200,000. The rate of increase of expenses also went up to 10%.

Since the individual is now at the 10th year of his/her retirement period, two investments have already been liquidated (2 and 3). Only one investment (investment 1) is remaining to be liquidated. The available balance at the end of the 10th year serves as the beginning balance. The problem here is to design a new plan of liquidation to meet the objectives. While dealing with this kind of situation, it is necessary to adjust the expenses and the investments at the 10th year of the retirement period with time value of money. For the actual LINGO program and output, see Appendix B.4.

Table 4.17 shows the schedule of investment liquidation, total earning, and total expense for each year starting from year 10 of retirement till the end of the planning horizon.

Table 4.17: Results for scenario 4 (figures in $10K)

Liquidated Money	Other income	Total Earning	Planned Big Expense	Total Expenses	Available Dollars
329.19	0.00	329.19	0.00	17.83	320.79
0.00	0.00	0.00	0.00	19.43	333.43
0.00	0.00	0.00	20.00	41.18	325.59
0.00	0.00	0.00	0.00	23.09	335.06
0.00	0.00	0.00	0.00	25.17	343.40
0.00	0.00	0.00	0.00	27.43	350.31
0.00	0.00	0.00	0.00	29.90	355.44
0.00	4.00	4.00	0.00	32.59	362.38
0.00	4.00	4.00	0.00	35.53	367.10
0.00	4.00	4.00	0.00	38.72	369.08
0.00	4.00	4.00	0.00	42.21	367.78
0.00	4.00	4.00	0.00	46.01	362.55
0.00	4.00	4.00	0.00	50.15	352.65
0.00	4.00	4.00	0.00	54.66	337.25
0.00	4.00	4.00	0.00	59.58	315.40
0.00	4.00	4.00	0.00	64.95	285.99
0.00	4.00	4.00	0.00	70.79	247.80
0.00	4.00	4.00	0.00	77.16	199.42
0.00	4.00	4.00	0.00	84.11	139.25
0.00	4.00	4.00	0.00	91.68	65.50

4.5. Scenario 5

In this scenario, the tax effect on the model is considered. It is assumed that a person has the following investment data sheet at present. See Table 4.18.

Table 4.18: Investment data sheet for scenario 5

Investment	Dollars (in 10^4)	Growth Rate (%)
1	10	15
2	13	10
3	2	9
4	30	7
5	10	10
6	20	15

It is also assumed that the person will retire after 15 years from now and the life expectancy is 30 years after retirement period starts. The bank is expected to pay a simple interest rate of 5%. Current annual expense is $20,000. It is estimated that every year the annual expense rises by 12%. However, after retirement, the rate of increase of annual expense is estimated to go down to 10%. It is estimated that at the onset of the retirement, the individual will have zero bank balance. The single tax rate is 20%. There is no fixed income streams or planned non-recurring expenses in the retirement life.

The liquidation schedule for this case is presented below in Table 4.19. It is to be noted that investment 3 is not necessary to be liquidated at all. For the actual LINGO program and output, see Appendix B.5.

Table 4.19: Liquidation schedule for scenario 5

Year	Investment liquidated
1	4
6	2
10	5
13	6
27	1

Table 4.20 shows the schedule of investment liquidation, total earning, and total expense for each year after retirement.

Table 4.20: Results for scenario 5 (figures are in $10K)

Beginning Balance	Expenses	Earning	Available Dollars at next year
0.00	10.95	82.77	61.27
61.27	12.04	0.00	51.68
51.68	13.25	0.00	40.50
40.50	14.57	0.00	27.55
27.55	16.03	0.00	12.62
12.62	17.63	87.46	68.06
68.06	19.39	0.00	51.39
51.39	21.33	0.00	32.12
32.12	23.47	0.00	9.93
9.93	25.81	98.50	65.32
65.32	28.39	0.00	39.54
39.54	31.23	0.00	9.88
9.88	34.36	870.71	676.49
676.49	37.79	0.00	665.75
665.75	41.57	0.00	650.81
650.81	45.73	0.00	631.12
631.12	50.30	0.00	606.06
606.06	55.33	0.00	574.97
574.97	60.87	0.00	537.10
537.10	66.95	0.00	491.64
491.64	73.65	0.00	437.65
437.65	81.01	0.00	374.15
374.15	89.11	0.00	300.00
300.00	98.02	0.00	213.98
213.98	107.83	0.00	114.71
114.71	118.61	0.00	0.69
0.69	130.47	3080.43	2336.59
2336.59	143.52	0.00	2286.54
2286.54	157.87	0.00	2220.13
2220.13	173.66	0.00	2135.28

4.6. Scenario 6

In this scenario, the earlier assumptions that all investments need to be fully liquidated are relaxed. Here a mix of investments is considered, some of which need to be fully liquidated while others may be partially liquidated. The tax effect as discussed in scenario 4.5 is considered here.

It is assumed that a person has the following investment data sheet at present. See Table 4.21. These investments are fully liquidated.

Table 4.21: Investment data sheet for scenario 6

Investment	Dollars (in 10^4)	Growth Rate (%)
1	10	15
2	13	10
3	2	9
4	15	7

In addition to these investments, it is assumed that the person has the following investments at present (see Table 4.22) that can be partially liquidated.

Table 4.22: Partial investment data sheet for scenario 6

Investment	Dollars (in 10^4)	Growth Rate (%)
1	8	10
2	12	14
3	11	12

It is also assumed that the person will retire after 15 years from now and the life expectancy is 40 years after retirement period starts. The bank is expected to pay a simple interest rate of 5%. Current annual expense is $20,000. It is estimated that every year the annual expense rises by 12%. However, after retirement, the rate of increase of annual expense is estimated to go down to 10%. It is estimated that at the onset of the retirement, the individual will have zero bank balance.

The single tax rate is 25%. There is a fixed income stream of $40,000 starting from year 11 of the retirement period. Also there are two planned non-recurring expenses: $100,000 at year 16 and $250,000 at year 33.

The liquidation schedule for the fully liquidated investments for this case is presented below in Table 4.23.

Table 4.23: Liquidation schedule for scenario 6

Year	Investment liquidated
1	4
4	3
5	2
16	1

Table 4.24 shows the withdrawal schedule for partial liquidations. In Table 4.25, the total income schedule for the whole retirement period is presented. Table 4.26 shows the expense schedule. The last columns of Table 4.25 and 4.26 are pulled into Table 4.27, which summarizes the income-expense schedule. For the actual LINGO program and output see Appendix B.6.

Table 4.24: Withdrawal schedule for partial liquidation (figures in $10K)

Investment 1		Investment 2		Investment 3		Total withdrawal	
Amount	Withdrawal	Amount	Withdrawal	Amount	Withdrawal	Before tax	After tax
33.42	0.00	85.66	0.00	60.21	0.00	0.00	0.00
36.76	0.00	97.65	0.00	67.43	0.00	0.00	0.00
40.44	0.10	111.32	0.00	75.53	0.00	0.10	0.08
44.37	9.33	126.90	0.00	84.59	0.00	9.33	6.99
38.55	0.00	144.67	0.00	94.74	0.00	0.00	0.00
42.40	0.00	164.92	0.00	106.11	0.00	0.00	0.00
46.64	0.00	188.01	0.00	118.84	0.00	0.00	0.00
51.30	10.81	214.33	0.00	133.10	0.00	10.81	8.11
44.54	31.29	244.34	0.00	149.08	0.00	31.29	23.47
14.58	14.58	278.55	0.00	166.96	19.84	34.42	25.81
0.00	0.00	317.54	0.00	164.78	33.86	33.86	25.39
0.00	0.00	362.00	0.00	146.64	37.64	37.64	28.23
0.00	0.00	412.68	0.00	122.07	41.81	41.81	31.36
0.00	0.00	470.45	0.00	89.89	46.39	46.39	34.79
0.00	0.00	536.32	2.71	48.72	48.72	51.43	38.57
0.00	0.00	608.32	0.00	0.00	0.00	0.00	0.00
0.00	0.00	693.48	0.00	0.00	0.00	0.00	0.00
0.00	0.00	790.57	0.00	0.00	0.00	0.00	0.00
0.00	0.00	901.25	0.00	0.00	0.00	0.00	0.00
0.00	0.00	1027.43	0.00	0.00	0.00	0.00	0.00
0.00	0.00	1171.26	0.00	0.00	0.00	0.00	0.00
0.00	0.00	1335.24	0.00	0.00	0.00	0.00	0.00
0.00	0.00	1522.18	0.00	0.00	0.00	0.00	0.00
0.00	0.00	1735.28	24.19	0.00	0.00	24.19	18.14
0.00	0.00	1950.65	139.77	0.00	0.00	139.77	104.83
0.00	0.00	2064.40	154.15	0.00	0.00	154.15	115.61
0.00	0.00	2177.69	169.96	0.00	0.00	169.96	127.47
0.00	0.00	2288.81	187.36	0.00	0.00	187.36	140.52
0.00	0.00	2395.66	206.49	0.00	0.00	206.49	154.87
0.00	0.00	2495.65	227.54	0.00	0.00	227.54	170.66
0.00	0.00	2585.65	250.69	0.00	0.00	250.69	188.02
0.00	0.00	2661.85	276.16	0.00	0.00	276.16	207.12
0.00	0.00	2719.68	337.51	0.00	0.00	337.51	253.14
0.00	0.00	2715.67	335.00	0.00	0.00	335.00	251.25
0.00	0.00	2713.96	368.90	0.00	0.00	368.90	276.67
0.00	0.00	2673.37	406.19	0.00	0.00	406.19	304.64
0.00	0.00	2584.59	447.21	0.00	0.00	447.21	335.41
0.00	0.00	2436.62	492.33	0.00	0.00	492.33	369.25
0.00	0.00	2216.49	541.96	0.00	0.00	541.96	406.47
0.00	0.00	1908.97	596.56	0.00	0.00	596.56	447.42

Table 4.25: Total income schedule (figures in $10K)

Liquidated money	After tax	Withdrawal	Other income	After Tax	Total income
41.39	34.79	0.00	0.00	0.00	34.79
0.00		0.00	0.00	0.00	0.00
0.00		0.08	0.00	0.00	0.08
9.43	7.57	6.99	0.00	0.00	14.56
79.51	62.88	0.00	0.00	0.00	62.88
0.00		0.00	0.00	0.00	0.00
0.00		0.00	0.00	0.00	0.00
0.00		8.11	0.00	0.00	8.11
0.00		23.47	0.00	0.00	23.47
0.00		25.81	0.00	0.00	25.81
0.00		25.39	4.00	3.00	28.39
0.00		28.23	4.00	3.00	31.23
0.00		31.36	4.00	3.00	34.36
0.00		34.79	4.00	3.00	37.79
0.00		38.57	4.00	3.00	41.57
662.12	499.09	0.00	4.00	3.00	502.09
0.00		0.00	4.00	3.00	3.00
0.00		0.00	4.00	3.00	3.00
0.00		0.00	4.00	3.00	3.00
0.00		0.00	4.00	3.00	3.00
0.00		0.00	4.00	3.00	3.00
0.00		0.00	4.00	3.00	3.00
0.00		0.00	4.00	3.00	3.00
0.00		18.14	4.00	3.00	21.14
0.00		104.83	4.00	3.00	107.83
0.00		115.61	4.00	3.00	118.61
0.00		127.47	4.00	3.00	130.47
0.00		140.52	4.00	3.00	143.52
0.00		154.87	4.00	3.00	157.87
0.00		170.66	4.00	3.00	173.66
0.00		188.02	4.00	3.00	191.02
0.00		207.12	4.00	3.00	210.12
0.00		253.14	4.00	3.00	256.14
0.00		251.25	4.00	3.00	254.25
0.00		276.67	4.00	3.00	279.67
0.00		304.64	4.00	3.00	307.64
0.00		335.41	4.00	3.00	338.41
0.00		369.25	4.00	3.00	372.25
0.00		406.47	4.00	3.00	409.47
0.00		447.42	4.00	3.00	450.42

Table 4.26: Expense schedule (figures in $10K)

Regular Expenses	Other planned expenses	Total expenses
10.95	0.00	10.95
12.04	0.00	12.04
13.25	0.00	13.25
14.57	0.00	14.57
16.03	0.00	16.03
17.63	0.00	17.63
19.39	0.00	19.39
21.33	0.00	21.33
23.47	0.00	23.47
25.81	0.00	25.81
28.39	0.00	28.39
31.23	0.00	31.23
34.36	0.00	34.36
37.79	0.00	37.79
41.57	0.00	41.57
45.73	10.00	55.73
50.30	0.00	50.30
55.33	0.00	55.33
60.87	0.00	60.87
66.95	0.00	66.95
73.65	0.00	73.65
81.01	0.00	81.01
89.11	0.00	89.11
98.02	0.00	98.02
107.83	0.00	107.83
118.61	0.00	118.61
130.47	0.00	130.47
143.52	0.00	143.52
157.87	0.00	157.87
173.66	0.00	173.66
191.02	0.00	191.02
210.12	0.00	210.12
231.14	25.00	256.14
254.25	0.00	254.25
279.67	0.00	279.67
307.64	0.00	307.64
338.41	0.00	338.41
372.25	0.00	372.25
409.47	0.00	409.47
450.42	0.00	450.42

Table 4.27: Income-expense schedule for each year (figures in $10K)

Beg. Balance	Income	Expenses	Balance	Interest	After tax	Balance for next year
0.00	34.79	10.95	23.84	1.19	0.89	24.74
24.74	0.00	12.04	12.70	0.63	0.48	13.17
13.17	0.08	13.25	0.00	0.00	0.00	0.00
0.00	14.56	14.57	0.00	0.00	0.00	0.00
0.00	62.88	16.03	46.85	2.34	1.76	48.60
48.60	0.00	17.63	30.97	1.55	1.16	32.14
32.14	0.00	19.39	12.74	0.64	0.48	13.22
13.22	8.11	21.33	0.00	0.00	0.00	0.00
0.00	23.47	23.47	0.00	0.00	0.00	0.00
0.00	25.81	25.81	0.00	0.00	0.00	0.00
0.00	28.39	28.39	0.00	0.00	0.00	0.00
0.00	31.23	31.23	0.00	0.00	0.00	0.00
0.00	34.36	34.36	0.00	0.00	0.00	0.00
0.00	37.79	37.79	0.00	0.00	0.00	0.00
0.00	41.57	41.57	0.00	0.00	0.00	0.00
0.00	502.09	55.73	446.36	22.32	16.74	463.10
463.10	3.00	50.30	415.80	20.79	15.59	431.39
431.39	3.00	55.33	379.06	18.95	14.21	393.27
393.27	3.00	60.87	335.41	16.77	12.58	347.99
347.99	3.00	66.95	284.03	14.20	10.65	294.69
294.69	3.00	73.65	224.04	11.20	8.40	232.44
232.44	3.00	81.01	154.43	7.72	5.79	160.22
160.22	3.00	89.11	74.11	3.71	2.78	76.89
76.89	21.14	98.02	0.00	0.00	0.00	0.00
0.00	107.83	107.83	0.00	0.00	0.00	0.00
0.00	118.61	118.61	0.00	0.00	0.00	0.00
0.00	130.47	130.47	0.00	0.00	0.00	0.00
0.00	143.52	143.52	0.00	0.00	0.00	0.00
0.00	157.87	157.87	0.00	0.00	0.00	0.00
0.00	173.66	173.66	0.00	0.00	0.00	0.00
0.00	191.02	191.02	0.00	0.00	0.00	0.00
0.00	210.12	210.12	0.00	0.00	0.00	0.00
0.00	256.14	256.14	0.00	0.00	0.00	0.00
0.00	254.25	254.25	0.00	0.00	0.00	0.00
0.00	279.67	279.67	0.00	0.00	0.00	0.00
0.00	307.64	307.64	0.00	0.00	0.00	0.00
0.00	338.41	338.41	0.00	0.00	0.00	0.00
0.00	372.25	372.25	0.00	0.00	0.00	0.00
0.00	409.47	409.47	0.00	0.00	0.00	0.00
0.00	450.42	450.42	0.00	0.00	0.00	0.00

4.7. Conclusion

All of the above scenarios and some other scenarios (not listed here) have been run in a machine with the following configuration using LINGO 9.0.

Processor: Intel Celeron M processor 330

Memory: 512MB

Hard Disk: 60GB

Operating System: Microsoft Windows XP

With this configuration, a very high performance of the LINGO programs was noticed. The upper limit of the run, which used almost 700,000 solver iterations for a practical size problem, finished below 2 minutes. The high performance of the model is ensured because all the constraints are linear in nature. Keeping this high performance in mind, it was decided not to develop any heuristic for the model.

Chapter 5: Future work

There is a great market opportunity for retirement planning software. Based on this model, an application that can cater the needs of the retirement planners can be developed. In this chapter, some of future work that can be done based on this model will be discussed. First, one of the existing popular software will be reviewed. Then how the model can be simulated and integrated with other applications will be briefly discussed, and finally the potential customers of the model will be discussed.

5.1. Existing retirement planning software

Currently there are many sophisticated software packages available in the market. An example is RetirementWorks® II which is available in three versions. According to Still River Retirement Planning Software Inc. (http://www.stillriverretire.com/SRRW2_products.asp), "One, the original, uses non-stochastic scenario testing. Another uses Monte Carlo analysis to test financial plans under randomized mortality and investment experience, as well as with randomized medical expense costs, long-term care requirements, and high inflation scenarios. The third combines Monte Carlo analysis with other stochastic modeling techniques to provide a faster computing model more suitable for Internet use."

According to Still River web site, the company considers four different strategies for liquidating assets at retirement. The fourth strategy called "Weighted Asset Analysis," considers five quantitative criteria by which assets are evaluated; expected after-tax financial performance during the retiree's lifetime; expected after-tax financial performance for heir; investment risk; liquidity and balance of asset ownership between partners. Each of the criteria is weighted and the solution is found out quantitatively.

The model described in this paper does not assign any specific weight to the investments. It is assumed that a person is indifferent about liquidation of investments and the two major goals are to maximize the money at hand at the end of the retirement period and to have annual earning always greater than or equal to annual expenses for any given year.

5.2. Simulation

As mentioned in the introduction section, the model presented in this paper is analytical in nature. It has been assumed that the rate of increase of annual expenses and the growth rate of all investments are constant. It has also been assumed that the bank's interest rate and tax rates to be constant. In the future, it is planned to extend the model by making interest rates, investment growth rates, and inflation rates random. It is also planned to use simulation technique to derive the results. Winston (2004) discusses the simulation techniques in Chapter 21 of the book "Operation Research, Applications and Algorithms."

5.3. Building application software

A custom application based on this model can be built for use by a financial advisor or an individual. The application can be written using a Windows development environment. According to LINDO Systems Inc, LINGO allows users to embed the solver in another application and through Dynamic Link Library (DLL) and Object Linking and Embedding (OLE) interfaces of LINGO; programmers can access all of the features and commands available interactively. The application should be simple, intuitive, and generic enough to address the needs of a wide variety of customers.

5.4. Potential customers of the software

Companies that design and develop custom retirement applications for individuals can benefit from this model. One example of such a company is Intuit. Intuit has a Quicken Personal series of products. These products help individuals to manage personal finances; the services offered by these suites of products range from simple checkbook entry to complex portfolio optimization. If this model can be integrated into their product line, they may find a new customer base.

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Appendix A: A brief overview of LINGO and its use in the model

LINGO is a comprehensive tool from LINDO Systems Inc. (http://www.lindo.com/products/lingo/lingom.html), designed to make building and solving linear, nonlinear, and integer optimization models faster, easier, and more efficient. LINGO provides a completely integrated package that includes a powerful language for expressing optimization models, a full featured environment for building and editing problems, and a set of fast built-in solvers. In this chapter, a brief overview of LINGO is provided along with the demonstration of how LINGO is used to build up the model.

A.1. Organization of the LINGO model

The LINGO model consists of the following three parts.

1. **Objective function**: It defines what the model should optimize. In this model, there is one objective function and that is defined in this section.
2. **Variables**: These are the elements in a model that can be changed to optimize the objective functions.
3. **Constraints**: These are the formulas that limit the value of the variables. The constraints of this model as described in Chapters 2 and 3 are defined in this section.

A.2. Using SETS in LINGO

LINGO allows users to group several same nature variables into one common class called a set. Sets may include attributes for each of its members. For example, in this model one set is defined as INVESTMENTS, which has two attributes; AMOUNT (the present dollar value) and RATE (rate of growth). The number of elements of the INVESTMENT set is determined by the NO_INVESTMENTS (number of investments) variable. Similarly another set YEAR has the following attributes; AVL_DOLLARS (available dollars), OTHER_INCOME (other income), BIG_EXPENSES (planned expenses), BEG_BALANCE (beginning balance of the year), TOTAL_EXPENSES (total expenses of the year) , TOTAL_EARNING (total earning of the year), and LIQUIDATED_MONEY (money liquidated in the year). The set YEAR will have as many elements as length of planning horizon. A set can be derived, that is, it can have other sets as its attribute. In this model, there is a set called KNAPSACK, which is a derived set because it has set attributes INVESTMENTS and YEAR as its members.

A.3. Set-looping functions

These functions operate over an entire set. Two of the most important functions used in this model are as follows.

1. **@FOR**: generates constraints over members of a set.
2. **@SUM**: sums expression over members of a set.

A.4. DATA section

The values of the variables are defined in this section. This section is defined after the SETS section. It starts with the tag DATA: and ends with the ENDDATA tag. There are two DATA sections in this model. The first DATA section will define the variables, which are used in the immediately following SETS section. Those variables are as follows.

1. **NO_INVESTMENTS**: Number of fully liquidated investments.
2. **PLAN_HZ**: Planning horizon or length of the retirement period.
3. **TIME_LEFT_TO_RETIRE**: Number of years left to retire.
4. **Rb**: Bank's interest rate.
5. **Rc**: Rate of increase of annual expenses after retirement.
6. **CUR_COST_INCR_RATE**: Rate of increase of annual expenses before retirement.
7. **BEG_BANK_BALANCE**: Bank balance at the onset of the retirement period.
8. **T**: Tax rate.
9. **NO_PL_INVESTMENTS**: Number of partially liquidated investments.

The other DATA section is placed at the end of the model and defines the values of the members of the sets defined in SETS section.

1. **RATE**: Growth rate of the fully liquidated investments.
2. **AMOUNT**: Present dollar value of the fully liquidated investments.
3. **PL_RATE**: Growth rate of the partially liquidated investments.
4. **PL_AMOUNT**: Present dollar value of the partially liquidated investments.
5. **OTHER_INCOME**: Fixed income at the retirement period.
6. **BIG_EXPENSES**: Planned non-recurring expenses at the retirement period.

A.5. Variable domain functions

All variables in LINGO are positive. The variable domain function used extensively in this model is @BIN, which indicates the variable is binary and can take values either 0 or 1. The L_{it} variables used in this model (the member variables of the KNAPSACK set) are binary.

A.6. Logical operators

LINGO provides a rich set of logical operators out of which #NE# (not equal to) and #EQ# have been used for this model.

Appendix B: LINGO programs and outputs of different scenarios

B.1. Scenario 1

```
DATA:
NO_INVESTMENTS          = 3;
PLAN_HZ                 = 30;
TIME_LEFT_TO_RETIRE     = 10;
Rb                      = 0.05;
Rc                      = 0.05;
CUR_COST_INCR_RATE      = 0.12;
CUR_EXPENSES            = 2;
BEG_BANK_BALANCE        = 0;
ENDDATA
SETS:
INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;
YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME, BIG_EXPENSES,
BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LIQUIDATED_MON
EY;
KNAPSACK (INVESTMENTS, YEAR): IS_LIQUID;

ENDSETS
```

```
MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));

Cprime =
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));
```

```
@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);


@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);


DATA:

RATE = 0.15  0.10  0.13;

AMOUNT = 10  13   10;

OTHER_INCOME = 0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  0 0 0 0 0   0 0 0 0

0    0 0 0 0 0 ;

BIG_EXPENSES =   0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  0 0 0 0 0   0 0 0 0

0    0 0 0 0 0 ;

ENDDATA
```

Solution:

Global optimal solution found.

Objective value:	122.6408
Extended solver steps:	12
Total solver iterations:	4469

B.2. Scenario 2

```
DATA:

NO_INVESTMENTS          = 3;
PLAN_HZ                 = 30;
TIME_LEFT_TO_RETIRE     = 10;
Rb                      = 0.05;
Rc                      = 0.05;
CUR_COST_INCR_RATE      = 0.12;
CUR_EXPENSES            = 2;
BEG_BANK_BALANCE        = 0;

ENDDATA

SETS:

INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;

YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,

BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,

LIQUIDATED_MONEY;

KNAPSACK (INVESTMENTS, YEAR): IS_LIQUID;

ENDSETS

MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -

@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *

(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));
```

```
Cprime=
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);

@FOR (YEAR(I)| I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I)| I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));

@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);

@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);
```

```
DATA:
RATE            = 0.15  0.10  0.13;
AMOUNT          = 10  9   10;
OTHER_INCOME    = 0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  0 0 0 0 0   0 0 0
0 0    0 0 0 0 0;
BIG_EXPENSES  = 0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  0 0 0 0 0   0 0 0 0
0    0 0 0 0 0;
ENDDATA
```

B.2.1. Scenario 2- Alternative 1

```
DATA:

NO_INVESTMENTS          = 4;

PLAN_HZ                 = 30;

TIME_LEFT_TO_RETIRE     = 10;

Rb                      = 0.05;

Rc                      = 0.05;

CUR_COST_INCR_RATE      = 0.12;

CUR_EXPENSES            = 2;

BEG_BANK_BALANCE        = 0;

ENDDATA

SETS:

INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;

YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,

BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI

QUIDATED_MONEY;

KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;

ENDSETS
```

```
MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));

Cprime=
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));
```

```
@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);

@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);

DATA:
RATE         = 0.15  0.10  0.13 0.08;
AMOUNT    = 10  9    10  9;
OTHER_INCOME = 0 0 0 0 0    0 0 0 0 0   0 0 0 0 0   0 0 0 0 0    0 0 0 0
0    0 0 0 0 0 ;
BIG_EXPENSES = 0 0 0 0 0    0 0 0 0 0   0 0 0 0 0   0 0 0 0 0    0 0 0 0 0
0 0 0 0 0 ;
ENDDATA
```

Solution:

Global optimal solution found.

Objective value: 389.4120

Extended solver steps: 1247

Total solver iterations: 37054

B.2.2. Scenario 2- Alternative 2

```
DATA:
NO_INVESTMENTS          = 3;
PLAN_HZ                 = 30;
TIME_LEFT_TO_RETIRE     = 10;
Rb                      = 0.05;
Rc                      = 0.03;
CUR_COST_INCR_RATE      = 0.12;
CUR_EXPENSES            = 2;
BEG_BANK_BALANCE        = 0;
ENDDATA
SETS:
INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;
YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,
BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI
QUIDATED_MONEY;
KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;
ENDSETS
MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));
```

```
Cprime=
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);
@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));
@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));
@FOR (KNAPSACK : @BIN(IS_LIQUID));
@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);
@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);

DATA:
RATE        = 0.15  0.10  0.13;
AMOUNT      = 10  9   10 ;
OTHER_INCOME = 0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  0 0 0 0 0   0 0 0 0
0    0 0 0 0 0 ;
```

BIG_EXPENSES = 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 ;

ENDDATA

Solution:

Global optimal solution found.

Objective value:	141.4838
Extended solver steps:	48
Total solver iterations:	8749

B.2.3. Scenario 2- Alternative 3

```
DATA:

NO_INVESTMENTS          = 3;

PLAN_HZ                 = 25;

TIME_LEFT_TO_RETIRE     = 15;

Rb                      = 0.05;

Rc                      = 0.05;

CUR_COST_INCR_RATE      = 0.12;

CUR_EXPENSES            = 2;

BEG_BANK_BALANCE        = 0;

ENDDATA


SETS:

INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;

YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,

BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI

QUIDATED_MONEY;

KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;

ENDSETS
```

```
MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));

Cprime=
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));
```

```
@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);

@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);

DATA:
RATE         = 0.15  0.10  0.13;
AMOUNT       = 10  9   10 ;
OTHER_INCOME = 0 0 0 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0
0 ;
BIG_EXPENSES = 0 0 0 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0 0;
ENDDATA
```

Solution:

Global optimal solution found.

Objective value: 148.8837

Extended solver steps: 20

Total solver iterations: 5777

B.2.4. Scenario 2- Alternative 4

```
DATA:
NO_INVESTMENTS          = 3;
PLAN_HZ                 = 30;
TIME_LEFT_TO_RETIRE     = 15;
Rb                      = 0.06;
Rc                      = 0.05;
CUR_COST_INCR_RATE      = 0.12;
CUR_EXPENSES            = 2;
BEG_BANK_BALANCE        = 0;
ENDDATA

SETS:
INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;
YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,
BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI
QUIDATED_MONEY;
KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;
ENDSETS
```

```
MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));

Cprime=
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));
```

```
@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);


@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);


DATA:
RATE        = 0.15  0.10  0.13;
AMOUNT      = 10  9   10 ;
OTHER_INCOME  = 0 0 0 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0
0   0 0 0 0 0;
BIG_EXPENSES =  0 0 0 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0 0
0 0 0 0 0;
ENDDATA
```

Solution:

Global optimal solution found.

Objective value: 57.30526

Extended solver steps: 6

Total solver iterations: 5955

B.2.5. Scenario 2- Alternative 5

```
DATA:
NO_INVESTMENTS         = 3;
PLAN_HZ                = 30;
TIME_LEFT_TO_RETIRE    = 15;
Rb                     = 0.05;
Rc                     = 0.05;
CUR_COST_INCR_RATE     = 0.10;
CUR_EXPENSES           = 2;
BEG_BANK_BALANCE       = 0;
ENDDATA

SETS:
INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;
YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,
BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI
QUIDATED_MONEY;
KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;
ENDSETS
```

```
MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));

Cprime=
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));
```

```
@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);

@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);

DATA:
RATE         = 0.15  0.10  0.13;
AMOUNT       = 10  9   10 ;
OTHER_INCOME = 0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  0 0 0 0 0   0 0 0 0
0   0 0 0 0 0;
BIG_EXPENSES = 0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  0 0 0 0 0   0 0 0 0 0
0 0 0 0 0;
ENDDATA
```

Solution:

Global optimal solution found.

Objective value:	698.6418
Extended solver steps:	102
Total solver iterations:	16900

B.3. Scenario 3

```
DATA:

NO_INVESTMENTS            = 3;

PLAN_HZ                   = 30;

TIME_LEFT_TO_RETIRE       = 15;

Rb                        = 0.05;

Rc                        = 0.05;

CUR_COST_INCR_RATE        = 0.12;

CUR_EXPENSES              = 2;

BEG_BANK_BALANCE          = 0;

ENDDATA


SETS:

INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;

YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,

BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI

QUIDATED_MONEY;

KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;

ENDSETS
```

```
MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));

Cprime=
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);
! Calculate the expenses at the starting year of retirement;

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));
```

```
@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);

@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);

DATA:
RATE            = 0.15  0.10  0.13;
AMOUNT          = 10  9   10 ;
OTHER_INCOME    = 0 0 0 0 0   0 0 0 0 0   0 0 0 0 0   0 0 4 4 4   4 4 4
4 4   4 4 4 4 4;
BIG_EXPENSES    = 0 0 0 0 0   0 0 0 0 0   0 0 10 0 0   0 0 0 0 0   0 0 0
0 0   0 0 0 0 0;
ENDDATA
```

Solution:

Global optimal solution found.

Objective value: 3.781538

Extended solver steps: 5

Total solver iterations: 4730

B.4. Scenario 4

```
DATA:

NO_INVESTMENTS          = 1;

PLAN_HZ                 = 20;

TIME_LEFT_TO_RETIRE     = 0;

Rb                      = 0.05;

Rc                      = 0.10;

CUR_COST_INCR_RATE      = 0.12;

CUR_EXPENSES            = 2;

BEG_BANK_BALANCE        = 9.43;

ENDDATA


SETS:

INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;

YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,

BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI

QUIDATED_MONEY;

KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;

ENDSETS
```

```
MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));

Cprime= 17.83 ;

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));

@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);
```

```
@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);

DATA:
RATE          = 0.09 ;
AMOUNT        = 329.19 ;
OTHER_INCOME  = 0 0 0 0 0   0 0 4 4 4   4 4 4 4 4   4 4 4 4 4;
BIG_EXPENSES  = 0 0 20 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0 0;
ENDDATA
```

The solution is infeasible. One alternative is to transfer the money in a high interest-yielding bank. For example, let's change the bank interest rate from 5% to 10%.

```
DATA:
NO_INVESTMENTS          = 1;
PLAN_HZ                 = 20;
TIME_LEFT_TO_RETIRE     = 0;
Rb                      = 0.10;
Rc                      = 0.09;
CUR_COST_INCR_RATE      = 0.12;
CUR_EXPENSES            = 2;
```

```
BEG_BANK_BALANCE          = 9.43;
ENDDATA

SETS:
INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;
YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,
BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI
QUIDATED_MONEY;
KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;
ENDSETS

MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));

Cprime= 17.83;
! Calculate the expenses at the starting year of retirement;

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
```

```
(1+Rb)*AVL_DOLLARS(I-1) + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) -
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1) * IS_LIQUID(J,I)) +
BEG_BANK_BALANCE + OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));
@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);
@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);

DATA:
RATE          = 0.09;
AMOUNT        = 329.19;
OTHER_INCOME  = 0 0 0 0 0   0 0 4 4 4   4 4 4 4 4   4 4 4 4 4;
BIG_EXPENSES    = 0 0 20 0 0   0 0 0 0 0   0 0 0 0 0   0 0 0 0 0;
ENDDATA
```

Solution:

Global optimal solution found.

Objective value:	72.05281
Extended solver steps:	0
Total solver iterations:	0

B.5. Scenario 5

```
DATA:
NO_INVESTMENTS          = 6;
PLAN_HZ                 = 30;
TIME_LEFT_TO_RETIRE     = 15;
Rb                      = 0.05;
Rc                      = 0.10;
CUR_COST_INCR_RATE      = 0.12;
CUR_EXPENSES            = 2;
BEG_BANK_BALANCE        = 0;
T                       = 0.20;
ENDDATA

SETS:
INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;
YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,
BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI
QUIDATED_MONEY;
KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;
ENDSETS
```

```
MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ));

Cprime=
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : (AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1)*(1-T)+ AMOUNT(J)*T)*
IS_LIQUID(J,I)) + (1+(1-T)*Rb)*AVL_DOLLARS(I-1) + (1-
T)*OTHER_INCOME(I)  - Cprime*(1+Rc)^(I-1) - BIG_EXPENSES(I) =
AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : (AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1)*(1-T)+AMOUNT(J)*T) *
IS_LIQUID(J,I)) + BEG_BANK_BALANCE + (1-T)*OTHER_INCOME(I)  -
Cprime*(1+Rc)^(I-1)- BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));
```

```
@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);


@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);


DATA:

RATE  = 0.15  0.10  0.09  0.07  0.10  0.15;

AMOUNT= 10  13   2  30  10  20;

OTHER_INCOME= 0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  0 0 0 0 0   0 0 0 0 0

0 0 0 0 0  ;

BIG_EXPENSES=   0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  0 0 0 0 0   0 0 0 0

0   0 0 0 0 0 ;

ENDDATA
```

Solution:

Global optimal solution found.

Objective value: 2338.701

Extended solver steps: 10087

Total solver iterations: 205180

B.6. Scenario 6

```
DATA:

    NO_INVESTMENTS             = 4;
    PLAN_HZ                    = 40;
    TIME_LEFT_TO_RETIRE        = 15;
    Rb                         = 0.05;
    Rc                         = 0.10;
    CUR_COST_INCR_RATE         = 0.12;
    BEG_BANK_BALANCE           = 0;
    T                          = 0.25;
    NO_PL_INVESTMENTS          = 3;

ENDDATA


SETS:

INVESTMENTS/1..NO_INVESTMENTS/: AMOUNT, RATE;

PL_INVESTMENTS/1..NO_PL_INVESTMENTS/: PL_AMOUNT, PL_RATE;

YEAR/1..PLAN_HZ/: AVL_DOLLARS, OTHER_INCOME,

BIG_EXPENSES,BEG_BALANCE,TOTAL_EXPENSES,TOTAL_EARNING,LI

QUIDATED_MONEY;

KNAPSACK (INVESTMENTS,YEAR): IS_LIQUID;
```

```
PL_BALANCE(PL_INVESTMENTS,YEAR):WD, AMT;
ENDSETS

MAX =(1+Rb)*AVL_DOLLARS(PLAN_HZ) +@SUM(INVESTMENTS(J): (1 -
@SUM(YEAR(I): IS_LIQUID(J,I))) * AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+PLAN_HZ)) +END_BAL;

Cprime=
CUR_EXPENSES*(1+CUR_COST_INCR_RATE)^(TIME_LEFT_TO_RETIRE);

@FOR (YEAR(I) | I #NE# 1:
@SUM (INVESTMENTS(J) : (AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1)*(1-T)+ AMOUNT(J)*T)*
IS_LIQUID(J,I)) + (1+(1-T)*Rb)*AVL_DOLLARS(I-1) + (1-
T)*OTHER_INCOME(I) + @SUM(PL_BALANCE(J,I): WD(J,I)) *(1-T) -
Cprime*(1+Rc)^(I-1) - BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (YEAR(I) | I #EQ# 1:
@SUM (INVESTMENTS(J) : (AMOUNT(J) *
(1+RATE(J))^(TIME_LEFT_TO_RETIRE+I-1)*(1-T)+AMOUNT(J)*T) *
IS_LIQUID(J,I)) + BEG_BANK_BALANCE + (1-T)*OTHER_INCOME(I) +
```

```
@SUM(PL_BALANCE(J,I): WD(J,I)) *(1-T)  - Cprime*(1+Rc)^(I-1)-
BIG_EXPENSES(I) = AVL_DOLLARS(I));

@FOR (KNAPSACK : @BIN(IS_LIQUID));

@FOR (PL_BALANCE : WD <= AMT );

@FOR (YEAR(I) | I #EQ# 1:
@FOR (PL_INVESTMENTS(J): AMT(J,I) = PL_AMOUNT(J)
*(1+PL_RATE(J))^(TIME_LEFT_TO_RETIRE)));

@FOR(YEAR(I) |I #NE# 1:
@FOR(PL_INVESTMENTS(J): AMT(J,I)= (AMT(J,I-1) - WD(J,I-1))*
(1+PL_RATE(J))));

END_BAL=@SUM(PL_BALANCE(J,I)|I #EQ# PLAN_HZ: (AMT(J,I) -
WD(J,I))*(1+PL_RATE(J)));

@FOR (INVESTMENTS(I):@SUM(YEAR(J):IS_LIQUID(I,J))<=1);

@FOR (YEAR(J): AVL_DOLLARS(J) >= 0);
```

```
DATA:
  RATE  = 0.15  0.10  0.09  0.07;
  AMOUNT= 10  13   2  15 ;
  PL_RATE=0.10  0.14  0.12;
  PL_AMOUNT= 8  12  11;
  OTHER_INCOME= 0 0 0 0 0   0 0 0 0 0  4 4 4 4 4  4 4 4 4 4   4 4 4
4 4   4 4 4 4 4   4 4 4 4 4   4 4 4 4 4;
  BIG_EXPENSES=  0 0 0 0 0   0 0 0 0 0  0 0 0 0 0  10 0 0 0 0 0   0 0
0 0 0   0 0 0 0 0   0 0 25 0 0   0 0 0 0 0;
ENDDATA
```

Solution:

Global optimal solution found.

Objective value:	1496.147
Extended solver steps:	225
Total solver iterations:	20171