1989

Total coloring of graphs

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Order Number 1337813

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San Jose State University, 1989

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Total Coloring of Graphs

A Thesis

Presented to

The Faculty of the Department of Mathematics

and Computer Science

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

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May, 1989

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Abstract

A total coloring of a graph G is an assignment of colors to both the vertices and edges of G such that adjacent or incident vertices and edges of G are not colored with the same color. The total coloring conjecture is presented which states that to total color any graph, the number of colors needed is equal to the maximum degree of the graph plus two colors. The total coloring conjecture is shown to hold for graphs of small degree, complete graphs, complete bipartite graphs, complete balanced r-partite graphs, complete r-partite graphs, and 2-degenerate graphs. Furthermore, the complete balanced r-partite graphs are classified according to how many colors are needed to complete a total coloring. Outerplanar graphs with a few exceptions are shown to be total colorable using one more color than the maximum degree of the graph.

Acknowledgements

I would like to thank my supervisor, John Mitchem, for giving me invaluable direction and encouragement. I would also like to thank Susan Cabaniss and Edward Schmeichel for helpful feedback on an earlier draft of this paper. Finally, a big thanks for being very supportive and understanding goes to Sally Nelson Huffman.

T. E. H.

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Graph Coloring

1.1 Introduction

Brooks published what has become his most well-known paper in 1941 in which he proved that the vertices of most graphs can be properly colored with at most the maximum degree of the graph colors. In 1964, Vizing published what is also a famous result, a proof that the edges of a graph can be properly colored with at most the maximum degree of the graph plus one colors. In the following year, Behzad (1965) conjectured that to properly color both the vertices and edges of a graph, the number of colors needed is equal to the maximum degree of the graph plus two colors. This conjecture has sometimes been known as the Behzad conjecture. Since Vizing is known to have indepently made the same conjecture (1968), it is also called the total coloring conjecture.

The main purpose of this thesis is to investigate the total coloring conjecture. Later in this chapter, two proofs are shown of weaker forms of the total coloring conjecture, in which the bound on the number of colors needed is not as low as in the total coloring conjecture. In chapter two, the total coloring conjecture is shown to hold for graphs of small degree and graphs of large degree. The third chapter contains proofs that the total coloring conjecture holds for several types of graphs, among them the complete graphs and the complete balanced, r-partite graphs. The main result of the fourth and final chapter is that outerplanar graphs, with a few exceptions, can be properly total colored using only the maximum degree of the graph plus one colors.

Before going on to those subjects, the definitions and notation that will be used are described.

1.2 Definitions and Notation

A graph G = (V(G), E(G)) is a finite nonempty set V(G) of elements called vertices or points, together with a set E(G) of two element subsets of V(G) called edges. Each edge must consist of distinct vertices (no loops), and at most one edge joins a pair of vertices (no multiple edges). In a multigraph, multiple edges are allowed between distinct vertices, but again no loops are allowed. For the most part, this thesis concerns graphs, but multigraphs will be referred to on occasion.

If the context is clear we will denote V(G) or E(G) by V or E respectively. The number of vertices in V(G) is called the order of G, and the number of edges in E(G) is called the size of G. They will also be referred to by |V(G)| and |E(G)| respectively. If a graph G has order p and size q, then we say that G is a (p,q) graph.

Two vertices that are connected by an edge are said to be adjacent. If e = vw is an edge in a graph G, then each of v and w is an end point of e. Two edges which have a common vertex as an end point are also called adjacent. If an edge e has a vertex v as an end point, then we say that e is incident with v. The term independent is used to describe two vertices that are not adjacent, two edges that are not adjacent, or a vertex v and an edge e where e is not incident with v. A set of vertices, edges, or vertices and edges is independent if each pair of elements in the set is independent.

The number of edges incident with a vertex v is called the degree of v and is denoted by d(v) or by $d_G(v)$ to emphasize that the degree is being measured in the graph G. The maximum degree of a vertex in a graph G is denoted by D(G). A graph in which every vertex has degree r is called an r-regular graph. An r-factor of a graph G is an r-regular subgraph of G which contains every vertex of G.

A path is a sequence of vertices $v_1, v_2, \ldots, v_n$ in a graph G such that v_i is adjacent to v_{i+1} for $1 \leq i \leq n - 1$. The length of P is equal to n. Two vertices u and v in a graph G are a distance x apart if the shortest uv path in G has length x. A cycle is a sequence of vertices $v_1, v_2, \ldots, v_n$ in a graph G such that v_i is adjacent to v_{i+1} for $1 \leq i \leq n - 1$ and v_n is adjacent to v_1. The cycle is odd if n is odd and it is even if n is even. A chord of a cycle $C = v_1, v_2, \ldots, v_n$ in G is an edge connecting vertices of C that is not itself in C.

A graph is disconnected if it contains vertices u and v such that there is no path containing u and v. In a connected graph G, if the removal of any set of n - 1 vertices leaves a graph that is still connected and is not a single vertex, then we say that G is n-connected. A cutpoint is a vertex in a graph whose removal creates a disconnected graph. A graph with no cutpoints is called a block. A block of a graph is a subgraph that is a block and is maximal with respect to this property. An endblock is a block of a graph containing exactly one cutpoint of the graph.

A coloring or vertex coloring of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices are assigned the same color. Similarly, an edge coloring of a graph G is an assignment of colors to the edges of G such that no two adjacent edges are assigned the same color. A total coloring of a graph G is an assignment of colors to both the vertices and edges of G such that no two adjacent vertices are assigned the same color, no two adjacent edges are assigned the same color, and no vertex and incident edge are assigned the same color. A coloring using n colors is called an n-coloring. An n-edge coloring and an n-total coloring are defined similarly.

Each of these types of graph colorings are sometimes called proper to emphasize that the assignment of colors meets these definitions. The minimum number of colors that are necessary and sufficient to vertex color, edge color, and total color a graph are called the chromatic number, edge chromatic number, and total chromatic number respectively. These are denoted by $X(G)$, $X'(G)$, and $X''(G)$ respectively.

The colors that are assigned will be usually be denoted by the counting numbers, 1, 2,..., n. We use the notation c(v) and c(vw) to indicate the color that has been assigned to vertex v and edge vw respectively.

A color class of a proper vertex coloring for a graph G is a set of vertices of G that are assigned the same color. The color classes of a graph G have the properties that each color class is an independent set and each vertex in G is contained in exactly one color class. A color class for an edge coloring or total coloring is defined similarly.

In this paper, a set of vertices, edges, or vertices and edges will sometimes be called a color class even when it has not been established that the set meets the definition of a color class. This will only be done when the proof or demonstration that the set is a color class is forthcoming. This slight misuse of the term should not be confusing--in fact it is done to indicate a desired property of the set.

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their end points. Such a drawing of a planar graph G is called an embedding of G in the plane.

The complete graph of order n, written K_n, is the graph containing n vertices with edges connecting each pair of vertices in the graph.

A subset M of E(G) is called a matching in the graph G if no two edges in M are adjacent.

1.3 Vertex Coloring

This section provides an introduction to ideas related to graph coloring, as well as background material for the subject of total coloring. Vertex coloring is probably the most popular area of the different types of graph coloring, and there is a large amount of literature in this area. Most introductory textbooks in graph theory will contain a section which is an overview of results in vertex coloring (Behzad, Chartrand, and Lesniak-Foster, 1979, chap. 11; Bondy and Murty, 1976, chap. 8; Gould, 1988, chap. 8).

One very famous problem related to vertex coloring is the Four Color Map Problem which was solved in 1976 (Appel and Haken, 1977a, 1977b).

Theorem 1.1. Every planar graph can be vertex colored with four colors.

This theorem has a history dating from at least from 1852 when a student of Augustus DeMorgan's named Frederick Guthrie asked DeMorgan to consider the problem which became known as the Four Color Map Problem. DeMorgan communicated the problem to other mathematicians who were not able to solve it. A well known attempt at a proof was given by Alfred Bay Kempe in 1879. His proof used color interchanges, an idea that has been applied to other coloring problems. The proof stood until 1890 when John Heawood found that it was in error. Heawood modified Kempe's argument to prove that every planar graph is 5-colorable. In the period through 1976 the four color problem was worked on by many mathematicians, but no proof resulted.

In 1976, Apple and Haken with computer help from Koch published their proof of the four color theorem. Appel and Haken's proof is very long, and it involved reducing the possible structures of a planar graph to a finite number of cases and checking each case with a computer. Given the history of the problem, it is perhaps understandable that a proof that is so long and so different from the normal proof in mathematics was required to solve the problem. The four color theorem certainly illustrates the fact that in graph theory, a problem may be very easy to state and very difficult to solve.

A vertex coloring problem that is more closedly related to the total coloring conjecture is to find good bounds on the chromatic number of a graph G in terms of D(G), the maximum degree of the graph. It is easy to show the following.

Theorem 1.2. $X(G) \leq D(G) + 1$ for any graph G.

Proof. The counting numbers will be used to represent the colors. The colors are assigned to the vertices of G in any order according to the following rule. The vertex v is given the lowest color that is not already assigned to an adjacent vertex of v. Since each vertex is adjacent to at most D(G) other vertices, no more than D(G) + 1 colors will be required. This completes the proof.

A stronger and more famous result due to Brooks (1941) states that any graph G other than an odd cycle or a complete graph can be colored with no more than D(G) colors. There have been several proofs of Brooks' Theorem published (Melnikov and Vizing, 1969; Ponstein, 1969; Tverberg, 1983), most of them using a recoloring strategy similar to what Kempe used in his attempt to prove the Four Color Theorem. There have been several improvements to

Brooks' Theorem which we will not discuss further (Beutelspacher and Hering, 1983; Borodin and Kostochka, 1977; Catlin, 1978; Durandhar, 1982). We give a proof of Brooks' Theorem that is due to Lovász (1975). Lovász credits Ponstein (1969) with previously using a related idea.

<u>**Theorem 1.3.**</u> If G is a connected graph that is not a complete (D(G) + 1)-graph or an odd cycle, then G is D(G)-colorable.

<u>Proof.</u> Let G be a (p,q) graph and let n = D(G). The theorem is true trivially if n = 0. If n = 1, then G is K_2 and the theorem doesn't apply. If G is an even cycle or a path, then the colors 1 and 2 can be used on successive vertices to complete a proper coloring. We let G be a connected graph with D(G) ≥ 3 for the remainder of the proof.

Assume that G contains two vertices a and b that are a distance two apart such that G - a - b is connected. Note that if G is 3-connected, then this will always be true since G is not a complete graph. Since G - a - b is connected, we can arrange the vertices of G in order $v_1, v_2, \ldots, v_{p-2}$ such that v_1 is adjacent to a and b and each vertex v_i, $2 \leq i \leq p - 2$ is adjacent to a vertex that appears earlier in the list.

Since a and b are not adjacent, they can both be assigned color 1. The vertices $v_{p-2}, v_{p-1}, \ldots, v_2$ are in turn assigned a color from the set {1, 2,..., n}. This is always possible because each vertex is adjacent to less than n vertices that have been previously colored. This may not be true for v_1, but it is adjacent to a and b which both are assigned color 1, so there is a color available for v_1. Thus the theorem is true as long as the assumption that vertices a and b exist is true.

As noted above, a and b will exist if G (not complete) is 3-connected graph. We assume that G is 2-connected but not 3-connected. Some vertex v must exist which is not adjacent to all of the other vertices and which has a degree of three or more. If G - v is 2-connected, then let a be vertex v and let b be any point that is a distance 2 from v. Otherwise, G - v must have a cutpoint.

Since G - v has a cutpoint and G is 2-connected, G - v must contain two endblocks A and B such that there exists a vertex a contained in A and a vertex b contained in B that are each adjacent to v. These vertices would then be nonadjacent and separated by a distance of two in G such that G - a - b is connected. The existence of the vertices a and b is demonstrated and the proof is complete if G is 2-connected.

Finally, we consider the case where G has a cutpoint v. In this case the vertices a and b may not exist so that G - a - b is connected. Let G_1, G_2,..., G_p be the components of G - v. The subgraphs induced by $V(G_i) \cup \{v\}$ can each be colored with D(G) colors for $1 \leq i \leq p$. Then the colorings of these subgraphs can be combined into a coloring of G by permuting or renaming the colors on some subgraphs if necessary. Thus the theorem will be true for this case if it is true for the other cases. This completes the proof of Brooks' Theorem.

1.4 Edge Coloring

There is a large amount of literature about edge colorings of graphs. Many textbooks about graph theory contain an introduction to the subject (Behzad, Chartrand, and Lesniak-Foster, 1979, chap. 11; Bondy and Murty, 1976, chap. 6; Gould, 1988, chap. 8). We will concentrate on Vizing's Theorem (1964) which provides a very good bound on the edge chromatic number of a graph. It states that X'(G) is either D(G) or D(G) + 1 for any graph. The lower bound is quite easy to see. Some vertex v in G has D(G) incident edges. These edges must be assigned D(G) different colors, so at least D(G) colors are required to edge color G.

There are several proofs of Vizing's Theorem. The one that follows is due to Ehrenfeucht, Faber, and Kierstead (1984). See also "Other Interesting Coloring Problems" by Mitchem (1987).

Lemma 1.4. Let G be a graph with D(G) = k. If G contains a vertex v of degree k which is adjacent to at most one vertex of degree k, and if X'(G-v) $\leq$ k, then X'(G) = k.

Proof. Let v be a vertex of degree k, with v_1, v_2,..., v_k being the vertices that are adjacent to v, and let v_1 have degree equal to or greater than the degrees of v_2, v_3,..., or v_k. Furthermore, let G - v be edge colored with the k colors 1, 2,..., k. The general strategy will be to extend this k-edge coloring of G - v to a k-edge coloring of G by coloring the edges incident with v one at a time.

Since v is adjacent to at most one vertex of degree k, it follows that at v_1 there is at least one color that is not assigned to an incident edge, call it c_1', and at v_i, $2 \le i \le k$ there are at least two colors that are not assigned to incident edges, call them c_i' and c_i''. Call these the missing colors at each vertex and let S be the collection of missing colors at each vertex where duplications are allowed. Let u be the vertex adjacent to v with one missing color, so initially $u = v_1$. Consider three cases.

Case 1. Some color, call it c_i', occurs exactly once in the collection S of missing colors on the vertices $v_1, v_2, \ldots, v_k$.

In this case color edge vv_i with c_i'. Remove c_i' and c_i'' from the collection S of missing colors. If $v_i = v_1$, then relabel one of the remaining v_j with u and remove one of c_j' or c_j'' from S.

Case 2. The color c_i' that is missing at u occurs exactly once as a c_j' or c_j'' for a different vertex v_j.

In this case color edge vu with c_i', remove both occurances of c_i' from the collection S of missing colors, and relabel v_j with u.

Case 3. All other cases. In this case the color $a = c_i'$ (the missing color at u) occurs at least three times in the collection S of missing colors, and all other colors appear at least twice. Then there must exist some color b which is not in S and has not been previously used on any edge vv_i. This can be seen as follows.

Say that there are j edges incident with v that remain to be colored. Since color a occurs at least three times in S, and all other colors occur at least twice, there are $j - 1$ or less different colors in the collection S of missing colors. The graph $G - v$ has been k colored and $j \le k$, so there is at least one color b where $1 \le b \le k$, where b is not in S, and where b has not been previously used on any edge vv_i.

Now the colors a and b are interchanged along a longest path P starting at u consisting of only these colors. This will always be possible while preserving the proper edge coloring. Then b becomes the missing color at u, and we are back to either Case 1 or Case 2.

We have succeeded in coloring one more edge that is incident to v in G (Cases 1 or 2) or else setting up a situation where one more edge incident to v can be colored (Case 3). By repeating this proceedure, a k-edge coloring of G is completed. This completes the proof of the lemma.

Theorem 1.5. If G is any graph, then $X'(G) \leq D(G) + 1$.

Proof by induction on D(G). If $D(G) = 1$, then the theorem is clearly true. Assume that the theorem is true for any graph G with $D(G) < k$, and let G be a graph with $D(G) = k$.

Let M be a maximal independent set of edges in G. Then in G - M, no two vertices of degree k are adjacent because otherwise M is not maximal. Remove the vertices of degree k from G - M to form the graph H. In the graph H, $D(H) \leq k - 1$, so by the inductive hypothesis, $X'(H) \leq k$. Lemma 1.4 is now used repeatedly to extend the k-edge coloring to the graph G - M. Finally, the edges of M are colored with color k + 1, completing the edge coloring of G with k + 1 colors. By induction the theorem is proven.

1.5 Total Coloring

Conjecture 1.6 (Behzad, 1965; Vizing, 1968). For every graph G, $D(G) + 1 \leq X''(G) \leq D(G) + 2$.

The bound of $D(G) + 2$ cannot be improved, for there are graphs G where $D(G) + 2$ colors are required for total coloring. The smallest example is K_2 which requires three colors while $D(K2) = 1$.

The lower bound for the total chromatic number is easy to see. For a vertex v in G where $d(v) = D(G)$, one color must be assigned to v itself and D(G) colors must be assigned to the edges incident with v.

The strong similarity between the total coloring conjecture and Vizing's theorem for edge coloring is apparent. At first glance, it might seem somewhat amazing that this conjecture could be true. The required conditions for total coloring would seem to be much more difficult to meet than the conditions for edge coloring.

At the same time by Brooks' Theorem it follows that most graphs can be vertex colored with no more than D(G) colors. This bound for vertex coloring is not good for many graphs--in fact for a star graph, which is one vertex adjacent to k other vertices of degree one, D(G) is equal to $|V(G)| - 1$, and X(G) is just two. So perhaps the total coloring conjecture is true.

A beginning student of total graph coloring might start by constructing total colorings for small graphs to see if they can be colored with no more than $D(G) + 2$ colors. This would be a reasonable way to get used to the ideas related to total coloring and it could have an additional consequence. In the event that a graph G is discovered that cannot be total colored with D(G) + 2 colors, the total coloring conjecture would be disproved! Such a counterexample has not been found in the twenty-three years since the conjecture was made.

The opposite strategy is to attempt to prove that the total coloring conjecture is true, or to attempt to prove that it is true for some graphs. This is what is done in chapters two and three.

There are other ways to work on this problem. One is to attempt to classify graphs as being either type 1 (requires D(G) + 1 colors for total coloring) or type 2 (requires D(G) + 2 colors for total coloring). A restatement of the total coloring conjecture is to say that all graphs are either type 1 or type 2. In chapter four it is shown that outerplanar graphs with a few exceptions are type 1. There are other results related to total coloring described in survey papers (Behzad, 1971; Chetwynd 1988).

We first consider the problem of bounding the total chromatic number in the general case. The goal here is to derive the lowest possible bound for the total chromatic number of any graph, and of course if that bound is shown to be D(G) + 2, then the total coloring conjecture will be proven. We begin with some lesser results.

Theorem 1.7. If G is a graph, then $X''(G) \leq 2D(G) + 1$.

Proof. If G is a complete graph or a cycle then D(G) + 2 colors will suffice to total color G. These results will be presented later as Theorems 3.1 and 4.4. Otherwise, G can be vertex colored with D(G) colors by Brooks' Theorem and edge colored with D(G) + 1 colors by Vizing's Theorem. Combining these colorings yields a total coloring with 2D(G) + 1 colors or less.

A stronger result was proven by Hind (1988) and is stated in the next theorem.

Theorem 1.3. If G is a graph, then $X''(G) \le \lfloor (3/2)\, D(G) \rfloor + 2$.

Proof. If $G = K_n$ or $G = C_{2k+1}$, then assume for now that the result holds (see Theorems 3.1 and 4.4). Assume that $G \ne K_n$ for any n and $G \ne C_{2k+1}$ for any k. We know that by Brooks' Theorem $X(G) \le D(G)$, and that by Vizing's Theorem, $X'(G) \le D(G) + 1$. Color the vertices of G with colors 1, 2,..., D(G) and color the edges of G with colors D(G)+1, D(G)+2,..., 2D(G)+1.

Let $j = (1/2)(D(G) + 1) + 1$ if D(G) is odd, and $j = (1/2)\, D(G) + 2$ if D(G) is even. Define G_1 to be the graph with vertex set V(G) and an edge set containing all of the edges in E(G) that have been assigned colors from the set {D(G)+1, D(G)+2,..., D(G)+j}. Similarly graph G_2 has vertex set V(G) and edge set containing all of the edges in E(G) that have been assigned colors from the set {D(G)+j+1, D(G)+j+2,..., 2D(G)+1}.

The colors for the vertices and for the edges in set G_1 are unchanged. Consider the graph G_2. Because of the way the graph G_2 was defined, we have $D(G_2) \le D(G) + 1 - j$. The number on the right hand side is equal to the largest integer less than or equal to $(D(G) - 1) / 2$, which we will call k.

Reassign a color to each edge in G_2 from the set {1, 2,..., D(G)} to create a proper total coloring of G_2 with at most D(G) colors. To see that this is possible, assume that some edges in G_2 have already been recolored, and assume that we are about to assign a new color to edge e = vw. The number of edges adjacent to e in G_2 that have already been recolored could be as high as $D(G_2) - 1$. Two different colors are assigned to v and w, so therefore the number of colors that is available for assignment to e is at least $D(G) - (D(G_2) - 1 + 2)$. This number equals $D(G) - k + 1$, which is always at least one, so there is a color available to color e.

The graph G has been total colored with $\lfloor (3 / 2) D(G) + 2 \rfloor$ or less colors. The theorem is proven.

Kostochka (1977a) has proven the related and slightly stronger result that $X''(G) \leq \lfloor (3 / 2) D(G) \rfloor$ for multigraphs G having $D(G) \geq 6$ (with a few exceptions). Theorem 1.8 is notable, however, because the proof is so short and because it uses vertex and edge colorings to create a total coloring. In the same paper (Hind, 1988), there is a proof of an even stronger result:

Theorem 1.9. For any multigraph G, $X''(G) \leq X'(G) + 2\lceil \sqrt{X(G)} \rceil$.

In the proof, Hind uses some ideas similar to those used in Theorem 1.8. Specifically, he uses an edge coloring of a graph to create a total coloring of a different graph. If D(G) is large, then this bound will be much smaller than (3 / 2) D(G) because of the square root term. This bound is the best one known to date for graphs in which D(G) is large.

Unfortunately, Hind does not believe that the proof technique that he uses can be used to prove much stronger results.

Specifically, he says that it is unlikely that his technique can be used to obtain an inequality of the form $X''(G) \leq X'(G) + c$, where c is a constant.

Another bound on the total chromatic number has been recently published by Yap (1988). We list the following without a proof.

Theorem 1.10. For any graph G, $X''(G) \leq (7 / 5)\, D(G) + 3$.

Chapter 2

Total Coloring of Graphs of Small Degree and Graphs of Large Degree

It seems to be very difficult to prove that the total coloring conjecture is true for all graphs, so a reasonable strategy is to attempt to prove that it is true for certain types of graphs. This section will focus on some of the progress that has been made in total coloring two types of graphs, the graphs of small degree and the graphs of large degree.

In studying the graphs of small degree, the goal is to prove that the total coloring conjecture holds for all graphs of maximum degree k for as large a value of k as possible. The graphs with maximum degree two are the cycles and the paths. If a graph G is a path, then the colors 1, 2, and 3 can be assigned to successive vertices and edges of G to complete the total coloring with three colors. The total coloring conjecture does hold for the cycles as well, and this will be proven as Theorem 4.4.

The graphs of maximum degree three and those of maximum degree four are more difficult. Proofs for these cases will be presented in this chapter. Unfortunately, the proofs that have been published for these do not seem to generalize or extend to graphs of higher degree. To date no mathematician has published a proof that the total coloring conjecture is true in general for graphs of maximum degree five or for any higher degree. For the graphs of high degree, which are those graphs where the maximum degree is close to the order of the graph, the total coloring conjecture has also been proven to hold. After looking at the graphs of small degree, several theorems will be presented that concern graphs of high degree.

2.1 Total Coloring Graphs With Maximum Degree Three

First we will present a well known theorem by König (1936, chap. 10) that will be useful when discussing the graphs of small degree.

Theorem 2.1. Any multigraph G is a subgraph of a regular multigraph of degree D(G).

Proof. Let G be a multigraph with order |V(G)| and maximum degree D(G). If G is D(G)-regular then we are done, so we assume otherwise. If |V(G)| and D(G) are both odd, then we form a new multigraph G' by adding a vertex v to G and an edge between v and any vertex in G with degree less than D(G). Otherwise, let multigraph G' be a copy of multigraph G.

The first theorem of graph theory says that the sum of the degrees of the vertices of any graph is an even number, and this theorem holds for multigraphs as well. Using this theorem, we note that the sum of the degrees of the vertices of G', call it S, is an even number. We also note that |V(G')| * D(G') is an even number because at least one of these quantities is not odd.

We form a new multigraph G" as follows. Add (|V(G')| * D(G)) - S vertices to G' (an even number), and connect each new vertex to a vertex of G' so that the vertices in G' all have degree D(G). The newly added vertices all have degree one, and they are now paired. For each pair of vertices we add more vertices and edges to form a copy of $K_{D(G)+1}$ - e, where the paired vertices are the ones that are not connected by an edge in the copy of $K_{D(G)+1}$ - e.

In the new graph G", every vertex has a degree of D(G) so it is D(G)-regular. The multigraph G is a submultigraph of G", so the theorem is proven.

Observe that if G is a graph (rather than a multigraph) then no multiple edges have been introduced into the graph G" in the proof of Theorem 2.1.

We now consider the total chromatic number of an arbitrary graph of maximum degree three. Using Theorem 2.1, it is clear that if the total coloring conjecture holds for arbitrary 3-regular graphs (cubic graphs), then it holds for any graph of maximum degree three. This can be seen as follows. Given an arbitrary graph G of maximum degree three, construct a cubic graph G' that contains G as a subgraph by Theorem 2.1. Next, G' is total colored with five colors (assume that this is possible for now), so that the graph G is also total colored with five colors. Therefore, to prove that the total coloring conjecture holds for graphs having D(G) = 3, it suffices to prove it for cubic graphs.

Two different proofs that the total coloring conjecture holds for graphs of maximum degree three were published in 1971. One proof by Rosenfeld uses induction on the order of G. In the other proof by Vijayaditya a cubic graph is shown to be decomposable into a union of cycles and matchings which can be colored and then reassembled into the original graph.

A third proof has been found by Yap (in press-a). His proof is significantly different than the first two. It is a description of an algorithm that will construct a total coloring of an arbitrary cubic graph. All three proofs seem to be similar in length and complexity. The proof by Rosenfeld is presented here because it is a good illustration of using induction to prove a result about total coloring.

In Rosenfeld's proof he assumes inductively that the theorem will hold for multigraphs although he explicitly states that the theorem is for graphs rather than multigraphs. If this assumption is not made, then there are more cases to consider and some of the cases become more complicated. The following is a correction of Rosenfeld's proof.

Theorem 2.2. Any graph G with $D(G) \leq 3$ can be total colored with five colors.

Proof by induction on p, the number of vertices in G. If G is disconnected, then the total chromatic number of G will be the maximum of the total chromatic numbers of each component of G, so we assume that G is connected.

If G has a maximum degree of zero, one, or two, then it is clear that five colors suffice to total color G (in fact four colors are enough to total color these cases). Thus the theorem is true if $p \leq 3$. We assume that $p \geq 4$ and $D(G) = 3$.

If $p = 4$, then G is one of the graphs shown in Figure 2.1. These graphs are shown along with a proper total coloring so the theorem holds for $p = 4$.

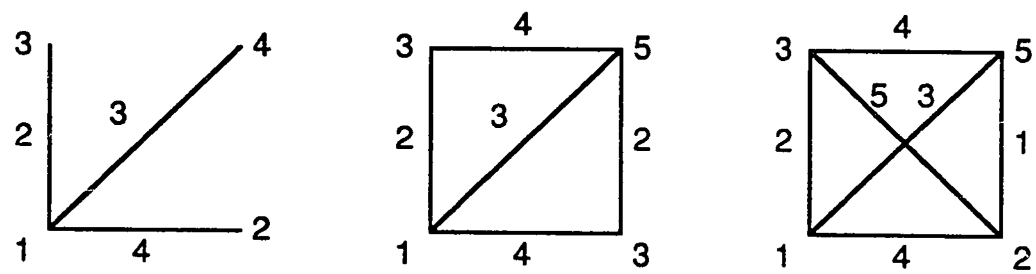

Figure 2.1. Total Coloring of Graphs of Order 4.

Assume that the theorem is true for $1 \leq p \leq k - 1$ and let G be a graph of order k. If G has one or more cutpoints, then consider the blocks of G. Each of these blocks can be total colored with five colors by the induction hypothesis since each block has order less than p. The total colorings for these blocks can be used to form a total coloring of G by beginning with the total coloring of one block, and extending it to the other blocks one at a time in the following way.

Say that a subgraph G' of the graph G has been total colored, and that a block B shares a cutpoint with G'. The colors used in the total coloring of B are permuted or renamed if necessary so that the color assigned to v in G' is assigned to v in B. The colors assigned to the edges in B incident with v are permuted or renamed if necessary so that a different color is assigned to each edge incident to v. In this way, the total coloring is extended from G' to G' U B. By repeating this process a proper total coloring of G is constructed and the theorem is true if G has a cutpoint.

Assume for the remainder of the proof that G has no cutpoint. Since $D(G) \leq 3$ it follows that each vertex in G has degree two or three.

Recall that a <u>chord</u> of a cycle $C = v_1, v_2, \ldots, v_n$ in G is an edge connecting vertices of C that is not itself in C. A cycle $C = v_1, v_2, \ldots, v_n$ in G is called an <u>admissable</u> cycle if it is chordless, and, if g_i is a vertex in G - C that is adjacent to vertex v_i, then $g_i \neq g_j$ if $i \neq j$. Assume for now that G contains an admissable cycle $C = v_1, v_2, \ldots, v_n$.

Remove vertices $v_1, v_2, \ldots, v_n$ and their incident edges from G to form a new graph G'. By the induction hypothesis, G' can be total colored with five colors. There are now two cases to consider.

In case 1, there are three conditions which are true. First, for each i, $1 \leq i \leq n$ there exists an edge $v_i g_i$ where v_i is contained in the cycle and g_i is not. Second, we have $c(g_i) \neq c(g_{i+1})$ for all i, $1 \leq i \leq n$. Third, if the edges incident to g_i are labelled with $e_{i,1}$, $e_{i,2}$, and $e_{i,3}$ where $e_{i,3} = g_i v_i$, then the set of colors $\{c(e_{i,j}) \mid 1 \leq i \leq n, 1 \leq j \leq 2\}$

contains two or fewer colors. In some cases g_i may have degree just two, in which case there is no edge $e_{i,2}$. We will describe the total coloring of G assuming that each g_i has degree three because the case where g_i has degree two is less constrained (there is one more color to choose from when assigning a color to edge v_ig_i).

Without loss of generality we assume that $c(g_1) = 1$, $c(g_2) = 4$, and that $\{c(e_{i,j}) \mid 1 \le i \le n, 1 \le j \le 2\} = \{2, 3\}$. Now we make the following color assignments. Let $c(v_1) = 4$, $c(v_2) = 3$, $c(v_1v_2) = 1$, and $c(v_2v_3) = 2$. Let $c(v_i) = c(g_{i+1})$ and let $c(v_iv_{i+1}) = 2$ or 3, alternating, for all i, $3 \le i \le n$. This partial total coloring is clearly proper.

The edges which are not yet colored are the edges v_ig_i for $1 \le i \le n$. For each of these edges there is a color available from the set $\{1, 2, 3, 4, 5\} \setminus \{c(e_{i,1}), c(e_{i,2}), c(g_i), c(v_i), c(v_{i-1}v_i), c(v_i,v_{i+1})\}$ because of the way in which colors have previously been assigned. This completes the total coloring of G for G belonging to case 1.

The case 2 consists of the graphs which do not meet the conditions of case 1. If there is some vertex v_i on the cycle C with degree two, then the vertices are now relabelled as follows. Relabel v_i with v_1, and relabel the vertices on the cycle so that for each i, $1 \le i \le n$, vertex v_{i+1} is adjacent to vertex v_i. The vertices g_i are also relabelled so that v_i is adjacent to g_i for each vertex v_i that has degree three.

If every vertex v_i on the cycle C has degree three, then we locate vertices g_i and g_{i+1} where $\{c(e_{i,1}), c(e_{i,2})\} \neq \{c(e_{i+1,1}), c(e_{i+1,2})\}$ or $c(g_i) = c(g_{i+1})$. The vertex g_i is relabelled as g_1, and all of the other vertices g_i and v_i are relabelled so that the vertices of the cycle C are labelled sequentially and each v_i is adjacent to g_i. Without loss of generality, we assume that $c(g_1) = 1$, $c(e_{1,1}) = 2$, and $c(e_{1,2}) = 3$.

Let G_2 be the graph induced by $V(G') \cup \{v_1, v_2\}$. The goal is to assign colors so that $c(v_2) \neq 3$. If vertex v_2 is of degree two, then this is easily done as follows. Let $c(v_1) = 2$, $c(v_1v_2) = 1$, and $c(v_2) = 4$. For now no color will be assigned to edge v_1g_1 if such an edge exists.

If vertex v_2 has degree three, then we must be more careful. The section of G_2 that is of interest is shown in Figure 2.2. In Figure 2.2 we have also assumed that vertex v_1 has degree three since this will be more difficult to color than if v_1 has degree two.

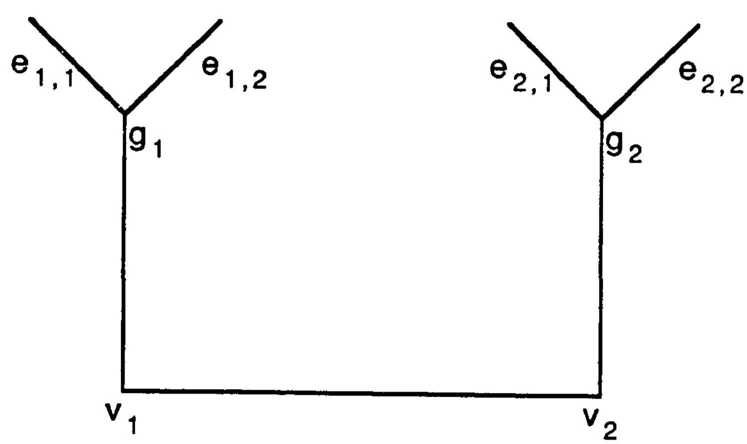

Figure 2.2. Section of graph G_2.

The set S is defined to be the set of colors {1, 2, 3, 4, 5}. This will be useful for describing the coloring assignments. Now let $c(v_1) = 2$ and $c(v_1v_2) = 1$. If none of $c(g_2)$, $c(e_{2,1})$, $c(e_{2,2})$ is 3, then we assign $c(v_2g_2) = 3$ and $c(v_2)$ is assigned some color from the set S \ {1, 2, 3, $c(g_2)$}. If $c(g_2) = 3$, then $c(v_2g_2)$ is assigned some color from the set S \ {1, $c(g_2)$, $c(e_{1,1})$, $c(e_{1,2})$}, and $c(v_2)$ is assigned some color from the set S \ {1,2, 3, $c(v_2g_2)$}.

Lastly, we consider the situation when one of $e_{2,1}$ or $e_{2,2}$ is colored 3. Assume that $c(e_{2,1}) = 3$. If $c(e_{1,2}) = 2$, then $c(g_2) = 1$ because this is a case 2 graph. We assign $c(v_2g_2) = 4$ and $c(v_2) = 5$. If $c(e_{1,2}) \neq 2$ and $c(g_2) = 2$, then $c(v_2g_2)$ is assigned some color from the set S \ {1, 2, 3, $c(e_{1,2})$} and $c(v_2)$ is assigned some color from the set S \{1, 2, 3, $c(v_2g_2)$}. If $c(e_{1,2}) \neq 2$ and $c(g_2) \neq 2$, then we assign $c(v_2g_2) = 2$ and $c(v_2)$ is assigned some color from the set S \ {1, 2, 3, $c(g_2)$}.

For now a color will not be assigned to edge v_1g_1 if such an edge exists. Observe that all other edges and vertices in graph G_2 are properly colored and that $c(v_2) \neq 3$. The fact that $c(v_2) \neq 3$ will become important when we attempt to complete the total coloring for the graph G.

Let G_i be the graph induced by the union of $V(G_{i-1})$ and vertex v_i for all i, $3 \leq i \leq n$. Our goal is to extend the 5-total coloring of graph G_{i-1} to graph G_i. If both vertices v_{i-1} and v_i have degree three, then proceed as follows. There is a choice of two colors available for edge $v_{i-1}v_i$. Similarly, there is a choice of two colors available for edge v_ig_i. Color these two edges with different colors, and then assign a color to vertex v_i other than $c(v_{i-1})$, $c(v_{i-1}v_i)$, $c(g_i)$, or $c(v_ig_i)$. This extends the 5-total coloring of G_{i-1} to G_i. If either vertex v_{i-1} or v_i has degree two, then there will be more colors to choose from when assigning a color to the uncolored vertices and egdes in G_i. So the 5-total coloring of G_{i-1} can be extended to G_i in this case as well.

The total coloring of graphs G_i through G_n can be completed in this way using 5 colors. After graph G_n is total colored, we add edge v_nv_1 to form graph G. The edge v_nv_1 must now be assigned a color. If vertex v_1 has degree three, then edge v_1g_1 must also be assigned a color to complete the total coloring of G.

Consider the situation where vertex v_n has degree three. First, assume that $c(v_n) \neq 2$. If $\{c(v_n), c(v_ng_n), c(v_{n-1},v_n)\} \neq \{3, 4, 5\}$, then edge v_1v_n is assigned some color from the set $S \setminus \{1, 2, c(v_n), c(v_ng_n), c(v_{n-1}v_n)\}$. If v_1 has degree three, then edge v_1g_1 is assigned some color from the set $S \setminus \{1, 2, 3, c(v_nv_1)\}$. If $\{c(v_n), c(v_ng_n), c(v_{n-1},v_n)\} = \{3, 4, 5\}$ and $c(v_n) \neq 3$, then we reassign $c(v_1) = 3$, and we assign $c(v_1v_n) = 2$. If v_1 has degree three, then we assign $c(v_ig_i) = 4$. If $\{c(v_n), c(v_ng_n), c(v_{n-1},v_n)\} = \{3, 4, 5\}$ and $c(v_n) = 3$, then vertex v_1 is reassigned some color from the set $S \setminus \{1, 2, 3, c(v_2)\}$, and we assign $c(v_nv_1) = 2$. If v_1 has degree three, then edge v_1g_1 is assigned some color from the set $S \setminus \{1, 2, 3, c(v_1)\}$. This completes the total coloring with five colors in the situation when $c(v_n) \neq 2$.

Now assume that $c(v_n) = 2$. If $c(v_ng_n) = 3$ or $c(v_{n-1}v_n) = 3$ then we reassign $c(v_1) = 3$, and edge v_nv_1 is assigned some color from the set $S \setminus \{1, 2, 3, c(v_ng_n), c(v_{n-1}v_n)\}$. If v_1 has degree three, then edge v_1g_1 is assigned some color from the set $S \setminus \{1, 2, 3, c(v_nv_1)\}$. Finally, if $c(v_ng_n) \neq 3$ and $c(v_{n-1}v_n) \neq 3$, then vertex v_1 is reassigned some color from the set $S \setminus \{1, 2, 3, c(v_2)\}$, and we assign $c(v_nv_1) = 3$. If v_1 has degree three, then edge v_1g_1 is assigned some color from the set $S \setminus \{1, 2, 3, c(v_1)\}$. This completes the total coloring with five colors in the situation when $c(v_n) = 2$.

If vertex v_n has degree two, then the 5-total coloring can be extended from G_n to G in a similar way. In this case there will be one more color to choose from when assigning a color to edge v_nv_1. If vertex v_1 has degree three, then there will be a color available to assign to edge v_1g_1 by the same argument as was used above.

The total coloring of G has been completed with five colors in the case where G contains an admissable cycle. It remains to show that G can be 5-total colored even if it does not contain an admissable cycle. There are several cases to consider.

The definition of an admissable cycle implies that if the shortest cycle in G has a length of five or more, then the shortest cycle is also an admissable cycle. Therefore, if G does not contain an admissable cycle, then it must contain a nonadmissable 3-cycle or 4-cycle. An example of these with labellings for the vertices is shown in Figure 2.3.

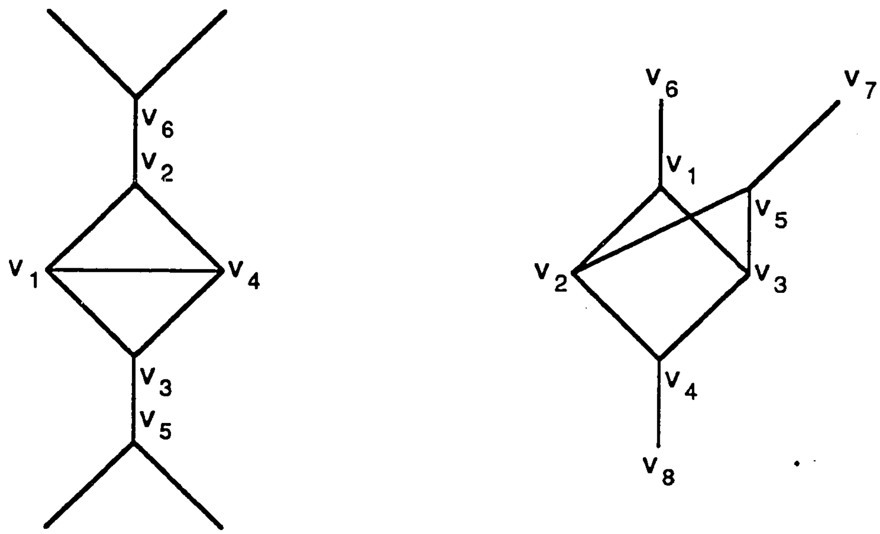

Figure 2.3. Nonadmissable 3-cycle and 4-cycle.

If G contains a nonadmissable 3-cycle, then we will proceed as follows. If v_5 and v_6 are the same vertex, then we note that this vertex cannot be adjacent to any other vertex because G has no cutpoints. The graph G must then be the graph shown in Figure 2.4 along with a proper total coloring with five colors.

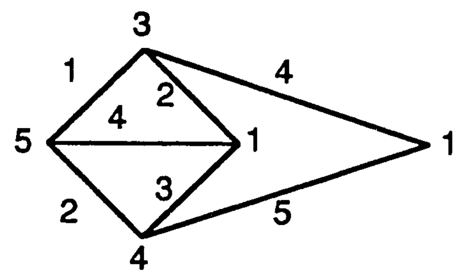

Figure 2.4. Total Coloring of G where $v_5 = v_6$.

If v_5 and v_6 are distinct vertices and nonadjacent, then we proceed as follows. Form the graph G' by removing vertices v_1, v_2, v_3, and v_4, and connecting the other edges that are incident with v_2 and v_3 in G to each other, forming a single edge e. By the induction hypothesis, we total color G' with five colors. We can assume without loss of generality that c(e) = 4, and the two endpoints of e are colored with 2 and 3. This situation is shown in the first diagram of Figure 2.5.

Now in the original graph G it is possible to complete the total coloring by keeping the color assignments from graph G', and making the additional color assignments as shown in Figure 2.5. Thus the total coloring of G with five colors is done.

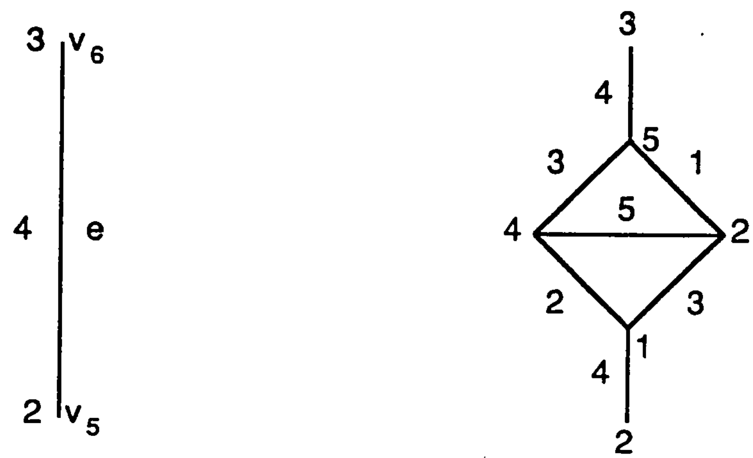

Figure 2.5. Construction of Total Coloring When v_5 and v_6 are distinct and nonadjacent.

If v_5 and v_6 are distinct vertices and adjacent, then a similar strategy will work. Form the graph G' by removing vertices v_1 and v_4, and connecting vertices v_2 and v_3 in G to each other. By the induction hypothesis, G' is total colored with five colors. We can assume without loss of generality that $c(v_5v_6) = 1$ and $c(v_5v_3) = 4$. This situation is shown in the first diagram of Figure 2.6.

Now in the original graph G it is possible to complete the total coloring by keeping the color assignments from graph G' and making additional color assignments. If $c(v_2v_6) = 4$, then we assume without loss of generality that v_6 is assigned color 2 or 3. The total coloring is completed in the same way as in the previous case, shown in the second diagram of Figure 2.5 with $c(v_6) = 3$. If $c(v_2v_6)$

$\neq 4$, then we can assume without loss of generality that $c(v_2v_6) = 3$. Since vertices v_5 and v_6 cannot be assigned color 1, the total coloring is completed by making the additional color assignments shown in the second diagram of Figure 2.6. Thus the total coloring of G with five colors is done.

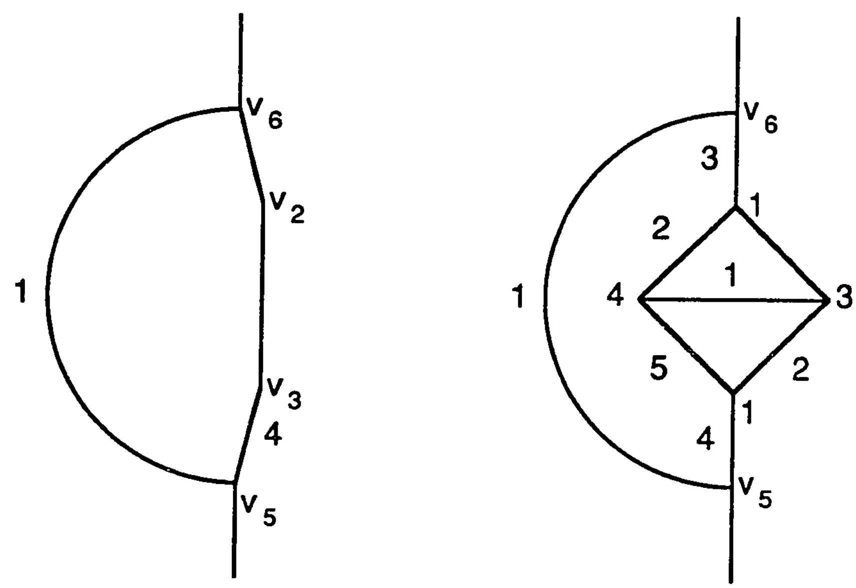

Figure 2.6. Construction of Total Coloring When v_5 and v_6 are distinct and adjacent.

A similar strategy will work in the case where G contains a nonadmissable 4-cycle. First, if vertices v_6, v_7, and v_8 are the same vertex, then G is the graph shown in Figure 2.7. A total coloring of G with five colors is shown.

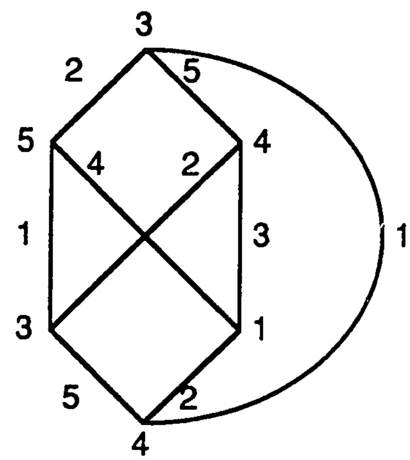

Figure 2.7. Total Coloring When $v_6 = v_7 = v_8$.

Second, if exactly two of the vertices v_6, v_7, and v_8 are the same vertex, then we proceed as follows. By symmetry we assume that $v_6 = v_7$ and refer to this vertex as v_6. If v_8 is adjacent to v_6, then v_8 cannot be adjacent to any other vertex because G does not have a cutpoint. The graph G is therefore the graph shown in Figure 2.8 along with a proper total coloring using five colors.

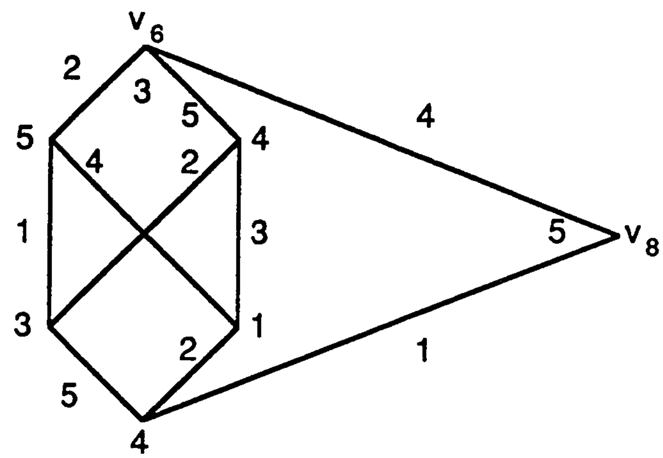

Figure 2.8. Total Coloring When v_8 is Adjacent to v_6.

If v_8 is not adjacent to v_6, then we remove vertices v_1, v_2, v_3, v_4, v_5, and v_6, v_7 and add a new edge e to form a graph G' which is total colored with five colors by the induction hypothesis. The six vertices and their incident edges are added back and assigned colors to complete the total coloring. This construction is shown in Figure 2.9.

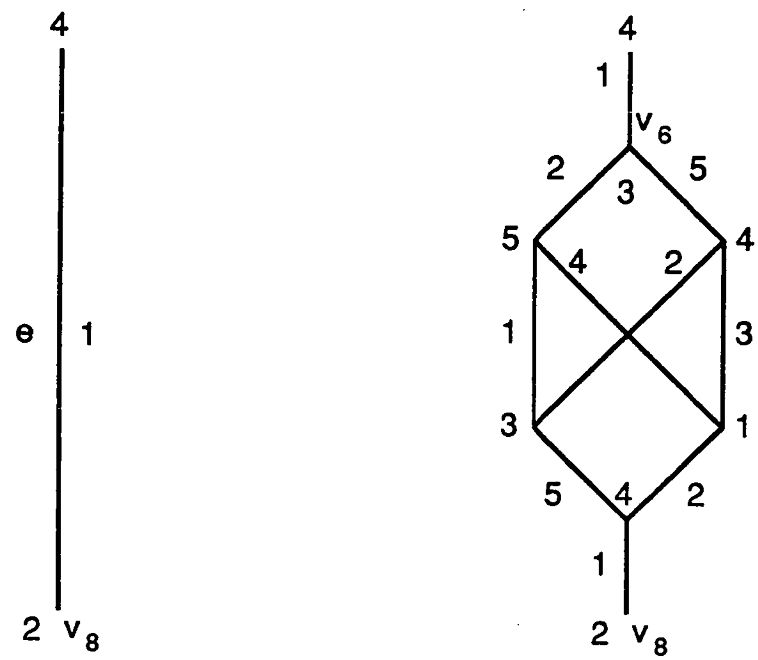

Figure 2.9. Total Coloring When v_8 is not adjacent to v_6.

In the case where vertices v_6, v_7, and v_8 are distinct, we remove vertices v_1, v_2, v_3, v_4, and v_5 and add a new vertex v' to form a graph G' which is total colored with five colors by the induction hypothesis. The five vertices and their incident edges are added back and assigned colors to complete the total coloring. This construction is shown in Figure 2.10.

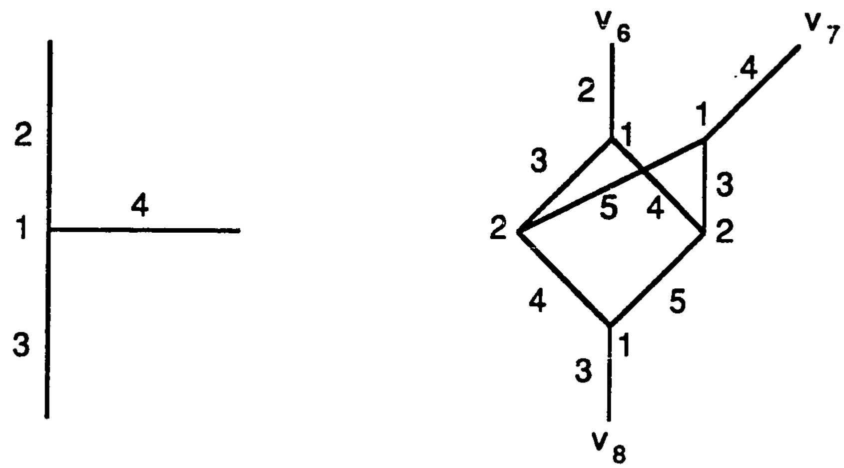

Figure 2.10. Total Coloring When v_8, v_6, and v_7 are distinct.

All possibilities for graph G have been considered. These possibilities are that G contains an admissable cycle, and that G contains a nonadmissable 3-cycle or 4-cycle. In all cases the graph G has been total colored with five colors. By induction, then theorem is proven.

2.2 Total Coloring Graphs With Maximum Degree Four

Kostochka (1977b) has published a proof that any multigraph with D(G) = 4 can be total colored with six colors. Thus the total coloring conjecture is true for the case where D(G) = 4. In terms of maximum degree this is the best result known to date, for no mathematician has published a proof that the conjecture holds in general for D(G) equal to five or more.

Kostochka's proof is a proof by construction. Using just three pages he describes a method of assigning colors to the vertices and edges of any graph with D(G) = 4 such that the assignment is a total coloring using six colors or less. This description of Kostochka's paper will follow his method of proof fairly closely, beginning with some preliminary lemmas.

Lemma 2.3. If $X''(G) \leq 6$ for any 4-regular multigraph G, then $X''(H) \leq 6$ for any multigraph H with $D(H) \leq 4$.

Proof. Let H be an arbitrary multigraph with maximum degree four. Assume for now that $X''(G) \leq 6$ for any 4-regular multigraph is true. If H is a submultigraph of some 4-regular multigraph G, then the proof is done, for by total coloring G with six colors a total coloring of H with six colors is completed also. By Theorem 2.1, H is a submultigraph of some 4-regular multigraph G, so the lemma is proven.

Lemma 2.4. Let c be a proper coloring of the vertices of any cycle $C_n = v_1, v_2, \ldots, v_n$. Let M be a set of four colors. Then there exists a coloring f of the edges of C_n in colors from M such that c U f is a total coloring of C_n.

Note that the colors in M can be different from the colors used to color the vertices of C_n, although it would appear to be more difficult to color the edges if the sets of colors were the same.

<u>Proof.</u> If only two colors are used to color the vertices of C_n, then C_n must be an even cycle. There are at least two colors in M not used to color the vertices. Each of these can be assigned to alternating edges of C_n to complete the total coloring.

If k colors are used to color the vertices of C_n where $k \geq 3$, then there must be three consecutive vertices v_1, v_2, v_3 in C_n such that $c(v_1) \neq c(v_3)$. This can be seen as follows.

Assume that there are not three consecutive vertices where $c(v_1) \neq c(v_3)$. Let $c(v_1) = 1$ and $c(v_2) = 2$. Then for $c(v_i) = c(v_{i+2})$ to hold for all i, the colors one and two must alternate along the consecutive vertices of C_n. But this is contrary to the assumption that $k \geq 3$. Therefore three consecutive vertices v_1, v_2, and v_3 must exist with $c(v_1) \neq c(v_3)$.

If $c(v_1)$ is contained in M, then we color edges v_2v_3 with $c(v_1)$. If $c(v_1)$ is not contained in M, then we color edge v_2v_3 with any color in M other than $c(v_2)$ or $c(v_3)$. Beginning with edge v_3v_4 we now color each edge in succession with an available color from M other than that assigned to the two endpoints or the previous edge. Since $|M| = 4$, there is always a color available. Finally, the edge v_1v_2 can be colored from the set $M \setminus \{c(v_1), c(v_2), c(v_nv_1), c(v_2v_3)\}$. This set must be nonempty because of the choice of color for edge v_2v_3. This completes the proof of Lemma 2.4.

These lemmas are now used in the proof of the following theorem, which is the main result of Kostochka's paper.

Theorem 2.5. Any multigraph G with D(G) = 4 can be total colored with six colors.

Proof. By applying Lemma 2.3, it will be sufficient to complete a coloring of an arbitrary 4-regular multigraph using six colors with the knowledge that once this is completed, the result will immediately hold for any multigraph with maximum degree less than or equal to four. In the case where G is K_5, the complete graph on five vertices, five colors will suffice to total color G (this will be shown in Theorem 3.1), so we need only show that any arbitrary 4-regular multigraph different than K_5 can be total colored in six colors.

Let G be an arbitrary 4-regular multigraph that is not K_5. By Brooks' Theorem, the vertices of G can be colored with four colors. We color the vertices of G with the colors 1, 2, 3, and 4.

The edges of G are first divided into two sets, and then each set is considered separately. This is done by using a theorem due to Petersen (1891) which states that if G is a 2k-regular graph (k is an integer), then the edges of G can be partitioned into k 2-factors. Note that for a set of edges to be a 2-factor, the set must be composed only of cycles. It may be just one cycle, or it may be many cycles. The cycles may contain an even or an odd number of edges. In any case, by applying Petersen's theorem to G, we can partition the edges into two 2-factors F_1 and F_2.

Using Lemma 2.4, color the edges of F_1 with the colors 1, 2, 3, and 4.

It remains to color the edges in the second 2-factor, F_2. To accomplish this we will present some definitions, remarks, and an algorithm. We must be careful to color the odd cycles in F_2, for the colors five and six will not be enough alone to complete this task. After these steps are done the proof will be complete.

Let $A(v_i)$ denote the colors used to color the two edges in F_1 that are incident with v_i.

A vertex v_1 contained in an odd cycle of F_2 will be called a <u>t-vertex</u> if there exists a vertex v_2 such that v_1v_2 is contained in F_2 and $|c(v_2) \cup A(v_1) \cup A(v_2)| \leq 3$.

<u>Remark 1.</u> Consider any edge v_1v_2 contained in F_2. If $|A(v_1) \cup A(v_2)| = 3$, then at least one of the vertices v_1 and v_2 must be a t-vertex. Vertices must be different colors, so one of these can be a fourth color, but the other must be one of the first three colors. Hence either $|c(v_1) \cup A(v_1) \cup A(v_2)| = 3$ or $|c(v_2) \cup A(v_1) \cup A(v_2)| = 3$. In other words, v_1 or v_2 is a t-vertex.

<u>Remark 2.</u> Consider any edges v_1v_2 contained in F_2. If $|A(v_1) \cup A(v_2)| = 2$, then both vertices v_1 and v_2 are t-vertices. The color for v_1 (or the color for v_2) together with the two colors used in $A(v_1) \cup A(v_2)$ gives three colors. In other words both v_1 and v_2 are t-vertices.

<u>Remark 3.</u> Each odd cycle $C = (v_1, v_2, \ldots, v_{2k+1})$ in F_2 contains at least two t-vertices. This can be seen as follows.

Suppose this is not true. Then C must have zero or one t-vertices. For C to have zero t-vertices, we must have the situation where $|A(v_i) \cup A(v_{i+1})| = 4$ for each i, $1 \leq 1 \leq 2k+1$. Assume that $A(v_1) = \{1, 2\}$. Then we must have

$$A(v_1) = \{1,2\}$$
$$A(v_2) = \{3,4\}$$
$$A(v_3) = \{1,2\}$$

.

.

$$A(v_{2k}) = \{3,4\}$$
$$A(v_{2k+1}) = \{1,2\}$$

But then $|A(v_{2k+1}) \cup A(v_1)| \leq 3$, a contradiction because by Remark 1 C would then have a t-vertex. So C has at least one t-vertex.

Since C is an odd circuit, there are two consecutive vertices v_1 and v_2 such that $|A(v_1) \cup A(v_2)| \leq 3$. This can easily be seen by using an argument similar to the one just given to show that C has at least one t-vertex.

If $|A(v_1) \cup A(v_2)| = 2$ for some vertices v_1 and v_2 in C, then we have 2-t-vertices by Remark 2 and we are done. Otherwise we know that $|A(v_1) \cup A(v_2)| = 3$ for some vertices v_1 and v_2 in C. By Remark 1 we have that either v_1 or v_2 is a t-vertex. Let us assume that v_1 is the t-vertex.

We can rename the colors on the edges of G such that $A(v_1) = \{1,2\}$ and $A(v_2) = \{1,3\}$. Furthermore, we can rename the colors on the vertices of G such that $c(v_1) = 4$. This is true because if $c(v_1) =$

3, then v_2 would also be a t-vertex and we would be done (it can't be color 1 or 2 because $A(v_1) = \{1,2\}$).

Now since v_1 is the only t-vertex, by Remark 1 we have $A(v_i) = \{1,3\}$ for i even, and $A(v_i) = \{2,4\}$ for i odd, $i \geq 3$.

Then $|A(v_{2k+1}) \cup A(v_1) \cup c(v_1)| = |\{1, 2, 4\}| = 3$. So v_{2k+1} is a second t-vertex, a contradiction. Thus Remark 3 is true--each odd cycle in F_2 must contain two t-vertices.

In the next stage of the proof we will show that we can select from the set of t-vertices in G a subset T with two properties. First, each odd cycle of F_2 contains exactly one vertex of T. Second, the subgraph of G induced by the vertices of T contains no odd cycles. In the final construction of the total coloring, these two properties will become very important. They will allow us to color the edges of F_2 using the colors five, six, and the colors one through four on certain carefully chosen edges (adjacent to the vertices in T).

More on that later--for now we will describe an algorithm that will generate the set T, and then show that the set T that is generated has the desired properties.

0 T = the empty set

1 IF an odd cycle of F_2 exists containing no vertices of T
THEN x = any t-vertex in the cycle
ELSE STOP--the algorithm is done
call the vertex x a t_0-vertex.
label the cycle of F_1 containing x with C.

orient C so x is considered to be the initial vertex.

2 FOR each vertex z in succession around the cycle C.

IF (z is a t-vertex) and

(z is in an odd cycle of F_2 that contains no vertices from T and no t_0-vertices)

THEN z is called a t_0-vertex

ELSE z is called usual.

3 IF (at least one vertex of C is called usual) or

(C is an even cycle)

THEN put all of the t_0-vertices in T

GOTO 1

ELSE put all of the t_0-vertices in T except v, the vertex following the initial vertex x in the cycle.

let x = the other t-vertex in the odd cycle of F_2 that contains v (it must exist by Remark 3).

call x a t_0-vertex.

orient the F_1 cycle C that contains v.

GOTO 2.

The algorithm is finite because the graph is finite so there are just a finite number of odd cycles in F_2 (and a finite number of t-vertices). Each odd cycle of F_2 contains at least one vertex of T because the algorithm will not stop if an odd cycle of F_2 has no t-vertices in T. In step 1 the t-vertex will be picked before the algorithm will stop.

Each odd cycle of F_2 contains at most one vertex of T because of the following reasons. In step 2, there is a check to prevent two t-vertices from the same odd cycle from being selected. If one t-vertex in an odd cycle is rejected in step 3 (the vertex called v), then the second t-vertex in the odd cycle of F_2 starts the process for the next iteration. In this way all t-vertices in the same odd cycle of F_2 cannot be rejected by different iterations of step 3.

The subgraph induced by T contains no odd cycles which can be seen as follows. Because each odd cycle of F_2 contains only one vertex of T, all of the edges in the subgraph induced by T will be in F_1. If, on a given iteration of the algorithm, all vertices in an odd cycle are t_0-vertices, then one vertex is rejected from being added to T (the vertex called v), preventing an odd cycle from being present in the subgraph.

We are now in a position to complete the construction of the total coloring by applying the algorithm. Recall that using colors 1, 2, 3, and 4 we have already colored all of the vertices in G, and also all of the edges in F_1. We have also determined which vertices in G are in the subset T with the properties listed above. We will now color the edges in F_2.

The edges in the even cycles of F_2 can be colored using colors five and six on alternate edges. For the odd cycles of F_2, we recolor the vertices of T in colors five and six so that a proper coloring is obtained. Because the subgraph generated by T contains no odd cycles, it is possible to do this using only colors five and six.

The edges of the odd cycles in F_2 are colored as follows. For each odd cycle C in F_2, label the vertex contained in T with v_1. Label the other vertices in C consecutively so $C = v_1, v_2, \ldots, v_{2k+1}$, where v_2 is the vertex given in the definition of the t-vertex. Assign color five to v_1, then color edge $v_{2k+1}v_1$ and all edges $v_{2s}v_{2s+1}$, $1 \leq s \leq k$, with color five also. The edges $v_{2s-1}v_{2s}$, $2 \leq s \leq k$, are colored with color six. Finally, the edge v_1v_2 can be colored with some color from {1, 2, 3, 4} because v_1 is a t-vertex and by definition $|A(v_1) \cup A(v_2) \cup c(v_2)| \leq 3$. Because of the preparation done to guarantee that one t-vertex exists in each odd cycle of F_2, there will always be a color available to complete the coloring of the odd cycles.

All vertices and edges in G have been colored using six colors or less. Accepting for now that the total coloring conjecture holds for cycles and paths, it follows from Theorems 2.2 and 2.5 that the total coloring conjecture holds for all graphs with maximum degree four or less. For the graphs of small degree, this is the strongest statement that can be made today because proofs for the graphs with higher degree have not been found.

2.3 Total Coloring Graphs of Large Degree

We now turn our attention to the graphs of high degree. In this area there have been advances published recently by several authors. All of the following theorems have been published since 1985.

Theorem 2.6. (Bollobás and Harris, 1985). If G is a graph with $D(G) \geq 3919$, then $X''(G) \leq (11 / 6)\, D(G)$.

Theorem 2.7. (Chetwynd and Hilton, submitted). If G is a regular graph of odd order and D(G) ≥ (19 / 21) |V(G)| then X"(G) ≤ D(G) + 2.

A characterization of which of these graphs are type 1 and which graphs are type 2 is also given for graphs meeting these conditions in Chetwynd and Hilton's proof.

Theorem 2.8. (Chetwynd and Hilton, submitted). If G is a regular graph of odd order and D(G) ≥ (6 / 7) |V(G)|, then X"(G) ≤ D(G) + 2.

Theorem 2.9. (Chetwynd and Hilton, submitted). If G is a regular graph and D(G) ≥ (3 / 4) |V(G)| then X"(G) ≤ D(G) + 3.

Theorem 2.10. (Chetwynd and Hilton, submitted). If G is a graph with D(G) ≥ (3 / 4) |V(G)| and |V(G)| is even, then X"(G) ≤ D(G) + 2.

Theorem 2.11. (Yap, in press-b). If G is a graph with D(G) ≥ |V(G)| - 4, then X"(G) ≤ D(G) + 2.

The proofs for these theorems are somewhat different than the proofs we have considered so far. In the proofs by Chetwynd and Hilton an edge coloring of the graph being studied is used to create a total coloring for the graph. The fact that the graphs have so many edges (D(G) close to the order of the graph) is of course an important fact.

Hilton notes that the restrictions required on the maximal degree of G are not the best possible. In other words, it may be possible to establish weaker conditions on the maximum degree of the graph which will lead to the same conclusions. He also points out that it is not clear that the proof techniques used are good enough to give the best possible results.

Hilton makes use of an interesting but simple idea that relates to edge coloring. A graph G is called overfull if $\lceil |E(G)| / D(G) \rceil > (|V(G)| / 2)$. If a graph is overfull, then it cannot be edge colored with D(G) colors, because no color class can contain more than $|V(G)| / 2$ edges. A similar idea is used later in Chapter 3 when studying the total coloring of complete balanced, r-partite graphs.

Chapter 3
Total Coloring of Complete Graphs and Complete Balanced k-partite Graphs

A different approach to the total coloring conjecture is to study the total chromatic number for some broad classes of graphs. An ultimate goal would be to extend the classes of graphs to include all graphs. We will investigate this approach to the total coloring conjecture by studying complete graphs, complete balanced r-partite graphs, and complete r-partite graphs.

An r-partite graph is one whose vertices can be partitioned into r subsets so that no edge has both endpoints in the same subset. A complete r-partite graph is one that is r-partite and in which each vertex is joined to all of the other vertices which are not in the same partite set. A complete balanced r-partite graph is a complete r-partite graph with the same number of vertices in each partite set.

3.1 Total Coloring of Complete Graphs and Complete Bipartite Graphs

In 1967, Behzad, Chartrand, and Cooper published some results about coloring complete graphs and complete bipartite graphs. Their results included vertex coloring and edge coloring as well as total coloring. We will present their results as Theorems 3.1 and 3.2 because they will be used to prove later theorems, and also because their proofs use some interesting arguments.

<u>Theorem 3.1.</u>

(i) $X(K_n) =$ n

(ii) $X'(K_n) =$ n for n odd, $n \geq 3$

n - 1 for n even

(iii) $X''(K_n) =$ n for n odd

n + 1 for n even

<u>Proof.</u> (i) Since each vertex is adjacent to each other vertex it is clear that n colors are necessary, and of course no more than n colors are needed to color n vertices. This result has been well known for a long time. Behzad, Chartrand, and Cooper included it for completeness (they were not the first to prove it).

(ii) First consider K_n, $n \geq 3$ and odd. No independent set of edges can contain more than (n - 1) / 2 edges because this is the biggest integer smaller than half of the number of vertices. Since there are n(n - 1) / 2 edges in K_n, it follows that $X'(K_n) \geq n$. Label the vertices of K_n 1, 2,..., n. For p = 1, 2,..., n let S_p be the set of edges joining p - q and p + q where q = 1, 2,..., (n - 1) / 2 and p - q and p + q are taken modulo n. Each S_p is an independent set of edges, and there are n(n - 1) / 2 edges in the union of the sets S_p. Therefore this is a proper edge coloring, and $X'(K_n) \leq n$. Therefore $X'(K_n) = n$ for n odd, $n \geq 3$.

For n even, K_n is the union of n - 1 1-factors (König, 1936, p. 157). First observe that that $X'(K_2) = 1$, clearly. Let $n \geq 4$. Label the vertices 1, 2,..., n and let K_{n-1} be the graph formed by deleting n and all edges incident with n. Then for p = 1, 2,..., n-1 and S_p as before, we let D_p be the union of S_p and edge (pn). The sets D_p are each independent, each edge is in exactly one set D_p, and together they partition K_n. Therefore, $X'(K_n) = n - 1$ for n even.

(iii) Each vertex v has n - 1 incident edges. The colors for v and each incident edge must be different, so $X''(g) \geq (n - 1) + 1 = n$. Label the vertices 1, 2,..., n and define the sets S_p, p = 1, 2,..., n as in (ii). Now let S_p' be the union of S_p and the vertex p. S_p' is an independent set of vertices and edges and every vertex and edge is in exactly one Sp'. Therefore $X''(K_n) = n$ for n odd.

Now consider the complete graph K_n, where n is even. There are n vertices and n(n -1) / 2 edges, so a total of n(n + 1) / 2 elements. Each color class can contain at most one vertex because any two vertices are adjacent. Therefore no color class can contain more than n / 2 elements. It follows that $X''(K_n) \geq n + 1$ for n even. To the graph K_n we add a vertex labelled n + 1 along with edges (i, n+1) for i = 1, 2,..., n. This forms K_{n+1} which is total colorable with n + 1 colors since n + 1 is odd. Since K_n is a subgraph of K_{n+1} we have $X''(K_n) \leq X''(K_{n+1}) = n + 1$. Therefore, $X''(K_n) = n + 1$ for n even. This completes the proof of Theorem 3.1.

Theorem 3.2. If G = $K_{m,n}$ is a complete bipartite graph, then G is type 1 if $m \neq n$ and G is type 2 if $m = n$.

Proof. We know that $X''(G) \geq D(G) + 1$ for all graphs. Let G = $K_{m,n}$ be a complete bipartite graph with partite sets $S_1 = \{u_1, u_2,..., u_m\}$ and $S_2 = \{v_1, v_2,..., v_n\}$, and suppose that $m \leq n$. Let $A_p = \{(u_i, v_{i+p}) \mid i = 1, 2,...,m\}$ for p = 1, 2, ..., n, the subscripts being taken modulo n. The edges of $K_{m,n}$ are partitioned into the n independent sets $A_1, A_2,..., A_n$.

Assume that m < n. For p = 1, 2,...,n, define $B_p = A_p \cup \{v_p\}$, and define $B_{n+1} = S_1$. The sets B_p, p = 1, 2,..., n+1 are independent sets of vertices and edges which partition the vertices and edges of $K_{m,n}$. Therefore, $X''(K_{m,n}) \leq n + 1$, so that $X''(K_{m,n}) = n + 1$. In other words, G is type 1.

If m = n, we note that no independent set of vertices and edges in $K_{n,n}$ can contain more than n elements. The total number of vertices and edges in $K_{n,n}$ is $n^2 + 2n$, so at least n + 2 independent sets will be needed to complete the total coloring. If we define A_p as before for p = 1, 2,..., n and define $A_{n+1} = S_1$, and $A_{n+2} = S_2$, then we have a partitioning of the vertices and edges of $K_{n,n}$ into n + 2 independent sets. Therefore $X''(K_{n,n}) = n + 2$, and $K_{n,n}$ is type 2.

3.2 Total Coloring of Complete Balanced r-partite Graphs

The following result is due to Rosenfeld (1971), and it will be useful for proving later theorems.

Theorem 3.3. If G is a complete balanced r-partite graph, then $X''(G) \leq D(G) + 2$. Furthermore, such a total coloring exists using r colors to color the vertices of G.

Proof. Let G have r partite sets and n vertices per partite set. Label the vertices $x_{i,j}$, where $1 \leq i \leq r$ and $1 \leq j \leq n$. For each j, $1 \leq j \leq n$, let H_j be the complete subgraph induced by vertices $x_{i,j}$, $1 \leq i \leq r$, and let H be the union of all subgraphs H_j. If r is odd, then each subgraph H_j can be total colored with r colors by Theorem 3.1, and there will be r colors used to color the vertices. The color classes for total coloring these complete subgraphs can be combined into color classes for H, so that by using $D(H) + 1 = r$ colors, all vertices in G and all edges in G with endpoints whose second coordinate is the same are colored. The subgraph G' obtained from G by removing the edges that have already been colored satisfies $D(G') = (r - 1)(n - 1)$, so by Vizing's Theorem, $X'(G') \leq (r - 1)(n - 1) + 1$. Also $D(G) = D(H) + D(G')$ so it follows that $X''(G) \leq X''(H) + X'(G') = D(H) + 1 + D(G') + 1 = D(G) + 2$. Hence the theorem is true if r is odd.

To prove that the theorem is true in general, we use induction on r, the number of partite sets in G. If $r = 1$, then one color suffices to total color G because G has no edges. If $r = 2$, then $X''(G) \leq D(G) + 2$ by Theorem 3.2. As shown in Theorem 3.2, a total coloring exists with two colors used to color the vertices.

Assume that the theorem is true for $r < k$, and let G be a complete balanced k-partite graph. If k is odd, then the theorem is true as was previously shown, so assume that k is even. If $n = 1$, then G is a complete graph and the theorem is true by Theorem 3.2, so assume that $n \geq 2$.

Let $k = 2m$, and define G_1 to be the complete balanced m-partite graph induced by vertices $x_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Define G_2 to be the isomorphic graph induced by vertices $x_{i,j}$, $m + 1 \leq i \leq k$, $1 \leq j \leq n$. By the inductive hypothesis, $X''(G_1) \leq D(G_1) + 2$ and $X''(G_2) \leq D(G_2) + 2$, and each of G_1 and G_2 can be total colored so that m colors are used to color the vertices. Now since $n \geq 2$, we have $D(G_1) + 2 = D(G_2) + 2 \geq 2m$ which means that $n(m - 1) + 2 \geq k$. Therefore, it follows that in the graph $G_1 \cup G_2$ the vertices of G_2 can be colored with a subset of the colors used on the edges of G_1 and vise versa, so that $n(m - 1) + 2$ colors suffice to total color $G_1 \cup G_2$.

The graph obtained from G by removing the edges that have been colored is a complete nm by nm bipartite graph. This graph is edge colorable with nm colors by Theorem 3.2. We have $X''(G) \leq X''(G_1 \cup G_2) + X'(K_{nm,nm}) = n(m - 1) + 2 + nm = 2nm - n + 2 = n(2m - 1) + 2 = D(G) + 2$. By induction the theorem is proven.

The results from Behzad, Chartrand, and Cooper, and from Rosenfeld can be used directly to prove results about other types of graphs. The balanced, complete r-partite graphs contain many complete subgraphs, so one might guess that the results from Behzad, Chartrand, and Cooper can be used or extended to total color these graphs. In fact this is what Bermond has done in his paper published in 1974. Bermond was able to classify each complete balanced r-partite graphs as being either type 1 or type 2.

__Theorem 3.4__ Let G be a complete, balanced r-partite graph with n vertices in each partite set.

Case i. If n is even and r is odd, then G is type 1.

Case ii. If n is even and r is even ($r \neq 2$), then G is type 1.

If n is even and r is even ($r = 2$), then G is type 2.

Case iii. If n is odd and r is odd, then G is type 1.

Case iv. If n is odd and r is even, then G is type 2.

__Proof.__

Case i. If n is even and r is odd, then G is type 1.

To show that these graphs are type 1, the color classes of a total coloring using D(G) + 1 colors will be described in detail The color classes will be shown to be independent, every vertex and edge will be colored, and no edge or vertex will be colored twice, so that a proper total coloring will be completed. As a check we will show that the total number of vertices and edges in the color classes is the same as the total number in the graph.

In G, there are r partite sets each containing n vertices so there are a total of rn vertices. Each of the vertices is adjacent to the $(r - 1)(n) = rn - n$ vertices in the other partite sets so there are $(rn)(rn - n)(1 / 2) = (r^2n^2 - rn^2)(1 / 2)$ edges in G.

The degree of each vertex is $(r - 1)(n) = rn - n$ so we must show that $rn - n + 1$ colors suffice to total color G.

In order to do this we introduce the following notation. For each i, $1 \leq i \leq r$ we let $x_{i,j}, 1 \leq j \leq n$ be the jth vertex in the ith partite set.

There will be two different types of color classes in coloring G. For the first type we consider the sets $S_1, S_2, \ldots, S_n$ where each S_j consists of all vertices with second subscript j. The graph induced by each S_j is the complete graph on r vertices. By Theorem 3.1, r colors suffice to total color this graph and each color class will contain one vertex and r edges.

In the graph G, since the sets S_1 through S_n are disjoint, we can combine the color classes for the sets into r color classes for G. In this way, we color all rn vertices in G, and $(n)(r)(r - 1)(1 / 2) = (r^2n - rn)(1 / 2)$ edges of G.

The second type of color class will contain only edges. We let $1 \leq a < b \leq n$ and consider the subgraph H with vertex set $S_a \cup S_b$ and edge set consisting of all edges joining vertices with different second coordinates. This subgraph is $K_{r,r}$ with one perfect matching removed. By Theorem 3.2 the edges of $K_{m,m}$ can be colored with m colors. Thus the edges of the subgraph H can be colored with r - 1 colors and each color class contains r edges.

Observe that the complete graph K_n can be edge colored with n - 1 colors. This is again from Theorem 3.2. Each color class can be viewed as a partition of {1, 2,..., n} into a family of n / 2 disjoint two element subsets. Since we have an edge coloring with n -1 color classes it follows that there exist n - 1 different families, each with n / 2 disjoint two element subsets of {1, 2,..., n}. Furthermore each member is paired with each other member exactly once in the various families. It follows that the collections {S_1, S_2,..., S_n} can be partitoned into a family of n / 2 disjoint pairs S_a, S_b ($1 \leq a < b \leq n$) and there exist n -1 such families such that each S_a is paired exactly once with each other S_b.

Now for each pair S_a, S_b, we consider the subgraph H_{ab} of G with vertex set $S_a \cup S_b$ and edge set consisting of all edges joining vertices with different second coordinates. As noted above, the graph H_{ab} is $K_{r,r}$ with one perfect matching removed, its edges can be colored with r - 1 colors, and each color class contains r edges.

For each family of s disjoint pairs S_a, S_b, we can combine one color class from each subgraph H_{ab} to form one color class for G. In this way, by using r - 1 colors, we can color all of the edges between the paired sets of vertices given by the family of n / 2 pairs S_a, S_b. Each color class will contain (n / 2)(r) edges.

Since there are (n - 1) families of s disjoint pairs, and for each of these there are r -1 color classes, there are in total $(r - 1)(n - 1) = rn - r - n + 1$ color classes of this type.

The results relating to the second type of color class are now summarized. There are n - 1 color classes in an edge coloring of K_n (each corresponds to one family of s disjoint pairs S_a, S_b ($1 \leq a < b \leq n$)). There are r - 1 color classes in G associated with each family of s disjoint pairs. Thus, there are $(r - 1)(n - 1) = rn - r - n + 1$ total color classes in G. There are r edges in each subgraph Hab corresponding to one pair S_a, S_b. There are $n / 2$ pairs of S_a, S_b in each family of s disjoint pairs. Thus, there are $(n / 2)(r) = rn / 2$ total edges in one color classes in G. In all color classes of second type, there are a total of $(rn - r - n + 1)(rn / 2) = (r^2n^2 - r^2n - rn^2 + rn)(1 / 2)$ edges.

We have finished the description of the two types of color classes. The results are summarized now. There are r color classes of the first type, and a total of rn vertices and $(r^2n - rn)(1 / 2)$ edges are colored by these color classes. There are $rn - r - n + 1$ color classes of the second type, and a total of zero vertices and $(r^2n^2 - r^2n - rn^2 + rn)(1 / 2)$ edges are colored by these color classes. In all there are $rn - n + 1$ color classes which color rn vertices and $(r^2n^2 - rn^2) / 2$ edges of G.

These totals agree with the total number of vertices and edges in G and no vertex of edge has been colored twice. Therefore, we have completed a proper total coloring of G using one more color that the degree of G, and G is a type 1 graph. This completes the proof for this case.

Case ii. If n is even and r is even ($r \neq 2$), then G is type 1.

If n is even and r is even ($r = 2$), then G is type 2.

This case requires the most complicated proof. Hence instead of giving all of the details, we will outline the main ideas.

If $r = 2$, then G is type 2 by Theorem 3.2. The strategy for proving this case when $r \neq 2$ is similar to Case i, that being to describe D(G) + 1 sets of vertices and edges and show that they are color classes of a proper total coloring of G. A general overview of the work to be done will be given first, followed by a detailed description.

Once again, let r be the number of partite sets, and n be the number of vertices in each partite set, and let $x_{i,j}$ refer to the vertex in the ith row and jth column, $1 \leq i \leq r$, $1 \leq j \leq n$. In the previous case i where r was odd and n was even, recall that there were two types of color classes. The first was based on total coloring the complete subgraphs that are induced by each column of vertices. We referred to these as the first type of color class. For the second type of color class the columns of vertices were paired, and the color classes were made up of edges between the paired columns.

In this case with r even and n even, the exact same construction will easily yield a total coloring with D(G) + 2 colors. The reason that D(G) + 2 colors are used rather than D(G) + 1 is due to the first type of color class. Previously, the subgraphs K_r on the columns of vertices could be total colored with r colors because r was odd, now r + 1 colors are required because r is even. This is by Theorem 3.1.

This situation is not hopeless, however, for there is one consequence of this construction that can be used to our advantage. In the total coloring of K_r, r = 2k, there is one color class with k edges and r color classes with k - 1 edges and one vertex. For each of the color classes which contains a vertex there is another vertex which is independent of the color class. That is a vertex which is not incident with any edge that is contained in the color class and is not itself in the color class. We call a vertex with these properties spare with respect to the color class.

Now in forming the color classes of the first type, the color classes of the subgraphs K_r on each column of vertices are combined into a color class of the graph G. In one color class of the first type there are k edges for each in column of vertices. In r color classes of the first type there are k - 1 edges and one vertex for each column of vertices. For each of these r color classes there is one spare vertex in each column of vertices. This is the key point. If edges between pairs of these spare vertices can be added to r of the first type of color classes in such a way that this will color a perfect matching of G, then one less color class of the second type will be needed. Thus the total coloring would be completed with D(G) + 1 colors rather than D(G) + 2, completing the proof.

Bermond has shown that it is possible to carry out this strategy to construct a total coloring of G. The details of the proof consist of formulas describing which vertices and edges are in each color class.

Case iii. If n is odd and r is odd, then G is type 1.

For this case, we will follow the proof given by Bermond quite closely. This method of proof is very constructive and straightforward--D(G) + 1 color classes will be constructed and described to complete a total coloring of the graph G.

Recall that the color classes of a total coloring are defined to be sets of vertices and edges of the graph G such that each set is independent, and such that every vertex or edge in G is in exactly one color class. The graph, $K_{5,5,5}$, is used as an example to illustrate the pattern of total coloring that is being described.

Once again, let r be the number of partite sets, and n be the number of vertices in each partite set, and let $x_{i,j}$ refer to the vertex in the ith row and jth column, $1 \leq i \leq r$, $1 \leq j \leq n$.

To help describe the total coloring of G it will be useful to describe some edge sets contained in two other graphs, K_r and K_n.

By Theorem 3.1, the complete graph on r vertices (r odd) can be total colored with r colors. Using this theorem, we form in K_r the r edge sets R_i $(1 \leq i \leq r)$, each containing (r -1) / 2 edges, and we form in K_n the n edge sets N_j $(1 \leq j \leq n)$, each containing (n - 1) / 2 edges. These edge sets will be enumerated for a small example as an illustration. The method used to construct these edge sets is the same as that used in the proof of Theorem 3.1.

If G is $K_{5,5,5}$, then we have the following. R_i are edge sets in a complete graph on 3 vertices labelled v_1, v_2, and v_3, and N_j are edges sets in the complete graph on 5 vertices with vertices labelled u_1, u_2, u_3, u_4, and u_5.

$R_1 = \{(v_2v_3)\}$, $R_2 = \{(v_1v_3)\}$, $R_3 = \{(v_1v_2)\}$

$N_1 = \{(u_2u_5), (u_3u_4)\}$, $N_2 = \{(u_1u_3), (u_4u_5)\}$, $N_3 = \{(u_2u_4), (u_1u_5)\}$

$N_4 = \{(u_3u_5), (u_1u_2)\}$, $N_5 = \{(u_1u_4), (u_2u_3)\}$

These edge sets will now be used to define the total coloring of G. In the following discussion sets of vertices and edges of G will be described, and will be shown to be color classes of G.

There will be n(r-1) color classes of one type, and one color class of a second type. The first type of color class will be denoted by $C_{i,j}$, $1 \leq i \leq r - 1$, $1 \leq j \leq n$. The color class $C_{i,j}$ will contain one vertex, $x_{i,j}$. For each edge v_av_c in set R_i, the edge $x_{a,j}x_{c,j}$ is included in $C_{i,j}$. Since none of the edges is incident with each other or the vertex $x_{i,j}$, the elements thusfar in $C_{i,j}$ are independent.

To complete the description of $C_{i,j}$ consider each edge u_bu_d contained in edge set N_j, and each value of a, $1 \leq a \leq r$. The edge $x_{a,b}x_{a+i,d}$ is included in $C_{i,j}$, where a + i is taken modulo r. The values of b and d are never equal to j in this construction, so each of these edges must be independent of the vertex and edges described in the previous paragraph. Also, since the set N_j is an independent set of edges in K_n, and since the values of a and a + i are each used just once for a given edge u_bu_d in K_n, we know that none of these edges have a common end point. Thus the color class $C_{i,j}$ is an independent set.

The second type of color class, of which there is just one, contains the vertices $x_{r,a}$ $(1 \leq a \leq n)$, and the edges $x_{aj}x_{cj}$ $(1 \leq j \leq n)$, for each edge $v_a v_c$ contained in set R_j. This color class is clearly an independent set.

It remains to show that each vertex and edge in G is contained in one of the color classes that have been described. For the vertices of G, this is easy to see. The vertices $x_{r,i}$ $(1 \leq i \leq n)$ are in the one color class of the second type. Each of the other vertices $x_{i,j}$ is in color class $C_{i,j}$.

Consider edge $e = x_{a,b}x_{c,d}$, where $a \leq r$, $c \leq r$, $b \leq n$, $d \leq n$, and $a \neq c$. If $b = d$, then e is in the color class $C_{i,b}$ if $v_a v_c$ is contained in set R_i in the edge coloring of K_r, $i \neq r$. If $v_a v_c$ is contained in R_r, then e is in the one color class of the second type. If $b \neq d$, then we note that $u_b u_d$ is contained in just one set N_j in the edge coloring of K_n. Also $c = a + i$ modulo r for one and only one i, $1 <= i <= r$. Therefore, the edge $x_{a,b}x_{c,d}$ will be contained in exactly one color class $C_{i,j}$, the one where $u_b u_d$ is contained in set N_j and $c = a + i$. We have shown that each edge and vertex is contained in one color class and that each color class defined is an independent set of vertices and edges. Therefore we have described a proper total coloring of G.

To illustrate the method of construction, the color classes of the example graph $K_{5,5,5}$ will be shown in Figures 3.1 through 3.11.

The color classes shown in Figures 3.1 through 3.10 are of the first type, and the color class shown in Figure 3.11 is of the second type. As argued above, each of the 15 vertices and 75 edges of this graph is in one color class and also that each color class is an independent set, so that the graph has in fact been total colored with D(G) + 1 = 11 colors, and this is a type one graph.

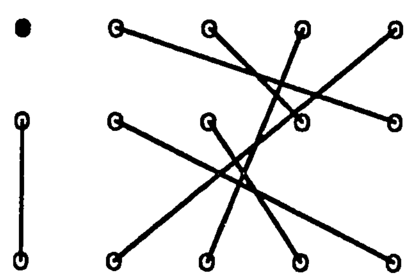

Figure 3.1. The color class C_{11} contains vertex x_{11} and edges $x_{21}x_{31}$, $x_{12}x_{25}$, $x_{31}x_{24}$, $x_{22}x_{35}$, $x_{23}x_{34}$, $x_{32}x_{15}$, $x_{33}x_{14}$.

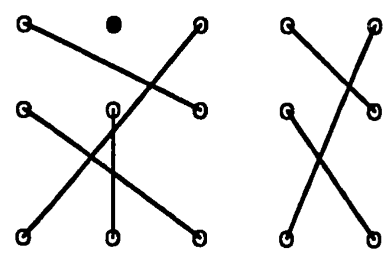

Figure 3.2. The color class C_{12} contains vertex x_{12} and edges $x_{22}x_{32}$, $x_{11}x_{23}$, $x_{14}x_{25}$, $x_{21}x_{33}$, $x_{24}x_{35}$, $x_{31}x_{13}$, $x_{34}x_{15}$.

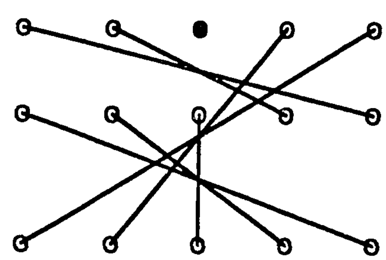

Figure 3.3. The color class C_{13} contains vertex x_{13} and edges $x_{23}x_{33}$, $x_{12}x_{24}$, $x_{11}x_{25}$, $x_{22}x_{34}$, $x_{21}x_{35}$, $x_{32}x_{14}$, $x_{31}x_{15}$.

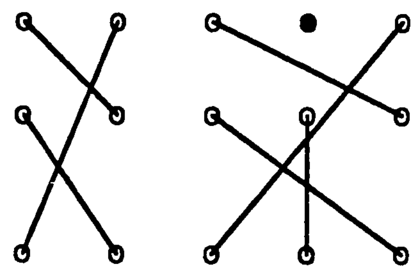

Figure 3.4. The color class C_{14} contains vertex x_{14} and edges $x_{24}x_{34}$, $x_{13}x_{25}$, $x_{11}x_{22}$, $x_{23}x_{35}$, $x_{21}x_{32}$, $x_{33}x_{15}$, $x_{31}x_{12}$.

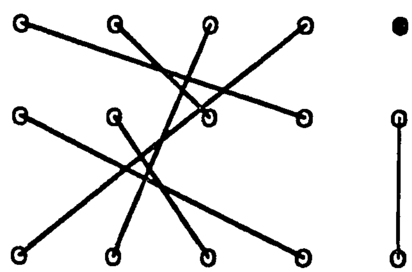

Figure 3.5. The color class C_{15} contains vertex x_{15} and edges $x_{25}x_{35}$, $x_{11}x_{24}$, $x_{12}x_{23}$, $x_{21}x_{34}$, $x_{22}x_{33}$, $x_{31}x_{14}$, $x_{32}x_{13}$.

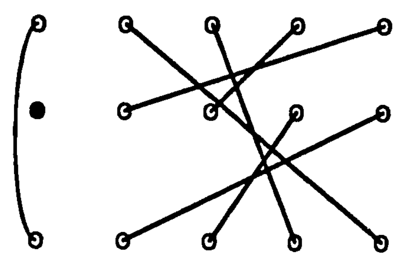

Figure 3.6. The color class C_{21} contains vertex x_{21} and edges $x_{11}x_{31}$, $x_{12}x_{35}$, $x_{13}x_{34}$, $x_{22}x_{15}$, $x_{23}x_{14}$, $x_{32}x_{25}$, $x_{33}x_{24}$.

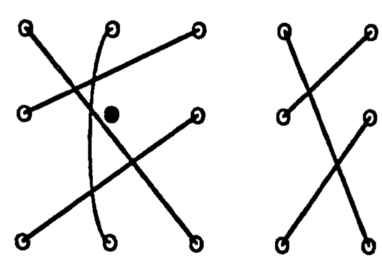

Figure 3.7. The color class C_{22} contains vertex x_{22} and edges $x_{12}x_{32}$, $x_{11}x_{33}$, $x_{14}x_{35}$, $x_{21}x_{13}$, $x_{24}x_{15}$, $x_{31}x_{23}$, $x_{34}x_{25}$.

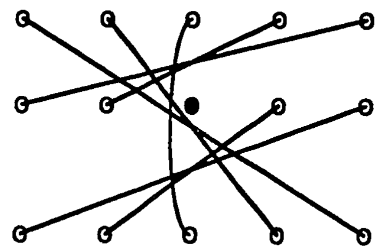

Figure 3.8. The color class C_{23} contains vertex x_{23} and edges $x_{13}x_{33}$, $x_{12}x_{34}$, $x_{11}x_{35}$, $x_{22}x_{14}$, $x_{21}x_{15}$, $x_{32}x_{24}$, $x_{31}x_{25}$.

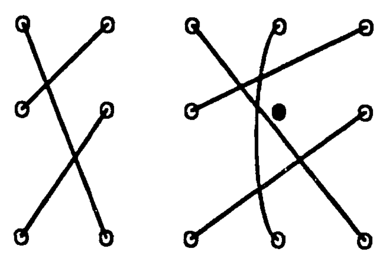

Figure 3.9. The color class C_{24} contains vertex x_{24} and edges $x_{14}x_{34}$, $x_{13}x_{35}$, $x_{11}x_{32}$, $x_{23}x_{15}$, $x_{21}x_{12}$, $x_{33}x_{25}$, $x_{31}x_{22}$.

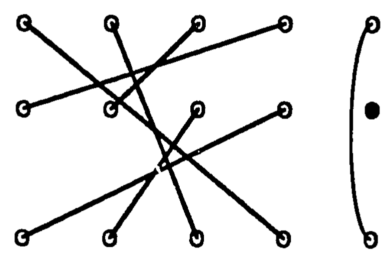

Figure 3.10. The color class C_{25} contains vertex x_{25} and edges $x_{15}x_{35}$, $x_{11}x_{34}$, $x_{12}x_{33}$, $x_{21}x_{14}$, $x_{22}x_{13}$, $x_{31}x_{24}$, $x_{32}x_{23}$.

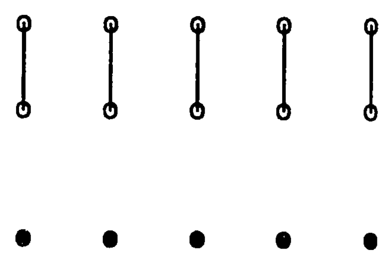

Figure 3.11. The last color class contains vertices x_{31}, x_{32}, x_{33}, x_{34}, x_{35} and edges $x_{11}x_{21}$, $x_{12}x_{22}$, $x_{13}x_{23}$, $x_{14}x_{24}$, $x_{15}x_{25}$.

Case iv. If n is odd and r is even, then G is type 2.

By Theorem 3.3 all complete, balanced r-partite graphs satisfy the total coloring conjecture, so they are either type 1 or type 2. We will assume that G is a type 1 graph and derive a contradiction.

In G, each vertex has maximum degree, call it D. Since G is type 1, D + 1 colors are used to total color G. For each vertex v there are D different colors on the edges incident to v, and a (D+1)th color on v itself.

Because each vertex is adjacent to all vertices in the other partite sets, the vertices present in each color class must all be in the same partite set. Each partite set contains an odd number of vertices, so there must be some color class C containing an odd number of vertices.

There are an even number of vertices in G, so there must be an odd number of vertices in G that are not in C. Therefore, there must be some vertex u not in C and not incident to an edge in C.

But this is a contradiction, because the D +1 elements at u (u itself plus the D edges incident to u) must be colored with the D+1 colors used to color G. Thus our assumption that G is type 1 was false, and G is a type 2 graph. This completes the proof of Theorem 3.4 for the final case.

Extensions to Theorem 3.3 (p. 49) and Theorem 3.4 (p. 51) could take several forms. One extension is to remove the restriction that G be balanced. This theorem is due Yap (1988).

Theorem 3.5. If G is a complete r-partite graph, the X"(G) ≤ D(G) + 2.

Proof. Let O_i denote the graph with i vertices and no edges, and let $G_1 + G_2$ denote the graph which contains all vertices and edges in G_1 and G_2 and also edges between each pair of vertices (g_1, g_2) where g_1 is contained in G_1 and g_2 is contained in G_2.

Now let $G = O_{p1} + O_{p2} + ... + O_{pr}$, where $p_1 \le p_2 \le ... \le p_r$. The graph G is a subgraph of $H = O_{p2} + O_{p2} + ... + O_{pr}$, and D(H) = D(G), so we may assume that $p_1 = p_2$. We now have only to consider three cases.

Case 1. $p_1 = p_2 = ... = p_r$.

By Theorem 3.3, G can be total colored with D(G) + 2 colors so we are done with this case.

Case 2. $p_1 = p_2 = ... = p_{2m+1} < p_{2m+2} \le ... \le p_r$.

The graph G' is formed by adding a vertex v, to G and adding an edge between v and every vertex not in the partite set O_{p2m+1}. Thus G' is the following graph.

$G' = O_{p1} + O_{p2} + ... + O_{p2m} + (O_{p2m+1} \cup \{v\}) + O_{p2m+2} + ... + O_{pr}$.

Let F be a 1-factor of $O_{p1} + O_{p2} + ... + O_{p2m}$. This 1-factor clearly exists because there are an even number partite sets in this graph, all of the same size. Let G" = G' - F.

By the way these graph have been defined, it is clear that D(G") = D(G). By Vizing's Theorem, X'(G") ≤ D(G") + 1. Let t = D(G") + 1, then edge color G" using the colors 1, 2,..., t. This edge coloring can now be extended to a total coloring of G using colors 1, 2,..., t, t+1 in the following way.

c(w) = t+1 if w is a vertex contained in O_{p2m+1}.

c(w) = c(vw) if w is any other vertex in G.

c(e) = t+1 if e is an edge contained in F.

c(e) = c(e) if e is any other edge in G.

It is clear that all vertices and edges in G have been colored using t+1 colors and that this total coloring is proper. Thus X"(G) ≤ D(G) + 2.

Case 3. $p_1 = p_2 = ... = p_{2m} < p_{2m+1} \leq ... \leq p_r$.

This case will be quite similar to Case 2. The graph G' is formed by adding a vertex v to G and adding an edge between v and every vertex not in the partite set O_{pr}. Thus G' is the following graph.

$G' = O_{p1} + O_{p2} + ... + O_{p2m} + O_{p2m+1} + O_{p2m+2} + ... + (O_{pr} \cup \{v\})$

Let F be a 1-factor of $O_{p1} + O_{p2} + ... + O_{p2m}$ and let G" = G' - F. It is clear that D(G") = D(G). By Vizing's Theorem, X'(G") ≤ D(G") + 1. Let t = D(G") + 1, then edge color G" using the colors 1, 2,..., t. This edge coloring can now be extended to a total coloring of G using colors 1, 2,..., t, t+1 in the following way.

c(w) = t+1 if w is a vertex contained in O_{pr}.

c(w) = c(vw) if w is any other vertex in G.

c(e) = t+1 if e is an edge contained in F.

c(e) = c(e) if e is any other edge in G.

All vertices and edges have been colored using t+1 colors and this total coloring is proper. Thus $X''(G) \leq D(G) + 2$.

Thus in all cases, the graph G has been total colored with D(G) + 2 colors. Therefore the theorem is true.

Note that if we remove the restriction that G is complete, then we would be back at the total coloring conjecture, for every graph G is an r-partite graph for some value of r.

Chapter 4
Total Coloring of Outerplanar Graphs and 2-Degenerate Graphs

In this chapter the total coloring of graphs which are outerplanar and which are 2-degenerate is considered. An outerplanar graph is a graph which is planar and which can be embedded in a plane in such a way that all vertices are incident with the same region of the plane (the outer region). The main result that will be shown is that certain trivial outerplanar graphs are type 2, and that all other outerplanar graphs are type 1.

All outerplanar graphs are also 2-degenerate graphs. A 2-degenerate graph G = (p,q) is a graph whose vertices can be labelled $v_1, v_2, \ldots, v_p$ such that

$$\deg_G(v_1) \leq 2$$

$$G_1 = G - v_1, \quad \deg_{G1}(v_2) \leq 2$$

....

$$G_{p-1} = G_{p-2} - v_{n-1}, \quad \deg_{Gp-1}(v_p) \leq 2.$$

We will show that all 2-degenerate graphs must be either type 1 or type 2, and that some 2-degenerate graphs are type 2. Thus the result for outerplanar graphs (that they are type 1 except for some trivial exceptions) does not extent to the broader family of 2-degenerate graphs.

4.1 Total Coloring Outerplanar Graphs

First, we will present some definitions that will be useful for this discussion.

Recall that a cut vertex is a vertex v in a connected graph G such that G - v is a disconnected graph, and that a block is a connected graph G containing no cut vertices.

Note that an outerplanar block with order at least three will always contain a hamiltonian cycle (a cycle containing all vertices of the graph exactly once). In an embedding of an outerplanar block in a plane, all of the edges of the hamiltonian cycle will be incident with the outer region of the plane.

An outer path is a path $P = v_1, v_2, \ldots, v_n$ contained in an outerplanar block where $n \geq 3$, vertices v_1 and v_n are adjacent vertices of degree 3 or more, and vertices v_i, $2 \leq i \leq n-1$ have degrees of 2. We say that the length of P is n - 1. The internal vertices of P are vertices v_2 through v_{n-1}.

A chord in an outerplanar block is an edge xy not on the hamiltonian cycle C. Its length is the length of a shortest x-y path on the hamiltonian cycle C.

Let v be a cutpoint of an outerplanar graph G. To split G at v means the following. Remove v from G. This disconnects G into subgraphs $G_1, G_2, \ldots, G_n$ because v was a cutpoint. Add vertex v_1 to G_1, and add vertex v_2 to the union of $G_2 \cup G_3 \cup \ldots \cup G_n$. Now add an edge between v_1 and each vertex of G_1 which was adjacent to v and add an edge between v_2 and each vertex of $G_2,\ldots,$ and G_n which was adjacent to v.

To <u>join</u> graphs G_1 and G_2 together we reverse the splitting process as follows. Remove vertex v_1 from G_1 and vertex v_2 from G_2. Add a new vertex v to the graph. Now add an edge between v and each vertex which was adjacent to vertex v_1 or v_2.

<u>Lemma 4.1</u> If G is an outerplanar block, and all outer paths are of length 2 and all endpoints of outer paths have the maximum degree in the graph, then D(G) = 3 or 4.

<u>Proof.</u> Let G be an outerplanar block with all outer paths of length 2 such that all outer paths have degree equal to D(G). Assume that $D(G) \geq 5$.

Let xy be a chord with length greater than 2 which is as small as possible (such exists because $D(G) \geq 5$). Let P: $x = v_0, v_1, v_2, \ldots, v_{t-1}, v_t = y$, $t \geq 3$, be a shortest xy path on the hamiltonian cycle C. For i = 1, 2,...,t, $\deg(v_i) \leq 4$ because if one of these vertices had degree five or more there would be a chord of length greater than two and less than that of chord xy.

It follows that if there exists a chord different than xy joining 2 vertices v_i, v_j, $0 \leq i < j \leq t$, then $j = i + 2$. Therefore $\deg(v_{i+1}) = 2$. If there exists no chord joining v_i, v_j, $0 \leq i < j \leq t$, then $\deg(v_i) = 2$ where $1 \leq i \leq t - 1$. In both cases there exists some vertex v_i having degree = 2 where $1 \leq i \leq t - 1$.

It follows that either P is an outer path of length greater than two, or some v_i, $0 \leq i \leq t$, is on an outer path of length 2 adjacent to a vertex of degree less than D(G). In the first case this contradicts one of the conditions of the lemma, and in the second case this contradicts the assumption that D(G) is 5 or more. Therefore our assumption that D(G) $\geq$ 5 is false, and the lemma is proven.

Lemma 4.2 In an outerplanar block G, if all end vertices of outer paths have degree four, and all outer paths have length two, then there must exist two vertices that are each on an outer path, are each of degree two, and are separated by a distance of two.

Proof. Let G be an outerplanar graph with all end vertices of outer paths having degree 4 and all outer paths having length 2. If there is an end vertex v of an outer path with both chords of length 2, then the lemma is true. Otherwise, we assume that there is no such vertex, and select v such that its larger chord vx is as small as possible. Let P: $v = v_1, v_2, \ldots, v_t = x$ be the shortest vx path on the hamiltonian cycle C. Now P must contain a vertex v_j, $1 < j < t$ of degree 2 and thus contains an outer path of length 2 with end vertices v_j and v_k, $1 \leq i \leq k \leq t$, at least one of which is not an end vertex of P. But then this vertex (the one that is not an end vertex of P) has degree 4 and both of its chords are smaller than chord vx, a contradiction. Therefore, there is an end vertex of an outer path with both chords of length 2, and the lemma is true.

Lemma 4.3 Let G be an outerplanar block with D(G) = 3. Let S $= e_1, v_1, e_0, v_p, e_{p+1}$ be a sequence of edges and incident vertices of G which have been 4 total colored such that $c(e_1) \neq c(e_{p+1})$ (see Figure 4.1). Let H be the graph formed by adding a path $P = v_1, e_2, v_2, \ldots, e_p,$

v_p to G, where all interior elements of P are not part of G. Let C be the cycle formed by P and edge e_0. Then for any matching M on C which includes e_0 we can extend the 4 total coloring to C such that if S' = e_i, v_i, e_{i+1}, v_{i+1}, e_{i+2} are consecutive edges and vertices on C and e_{i+1} is contained in M, then 4 colors are used on S' and $c(e_i) \neq c(e_{i+2})$.

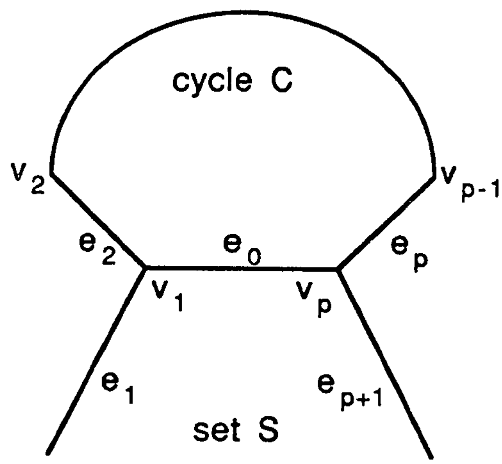

Figure 4.1. Illustration of the set S and the cycle C.

Proof. Without loss of generality assume that the colors 1 through 4 are used on S as shown in figure 4.2. Note that given these colors for the set S, the color 1 must be used on edge e_2, and the color 3 must be used on the edge e_p.

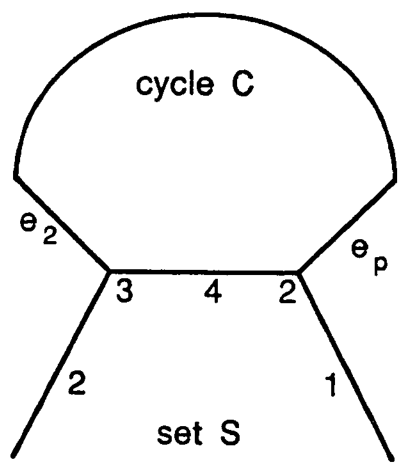

Figure 4.2. Colors assumed for the elements in S.

The cases where C contains 3, 4, 5, or 6 edges are shown along with a total coloring satisfying the conditions of the lemma in figure 4.3. All possible matchings M are shown in these cases.

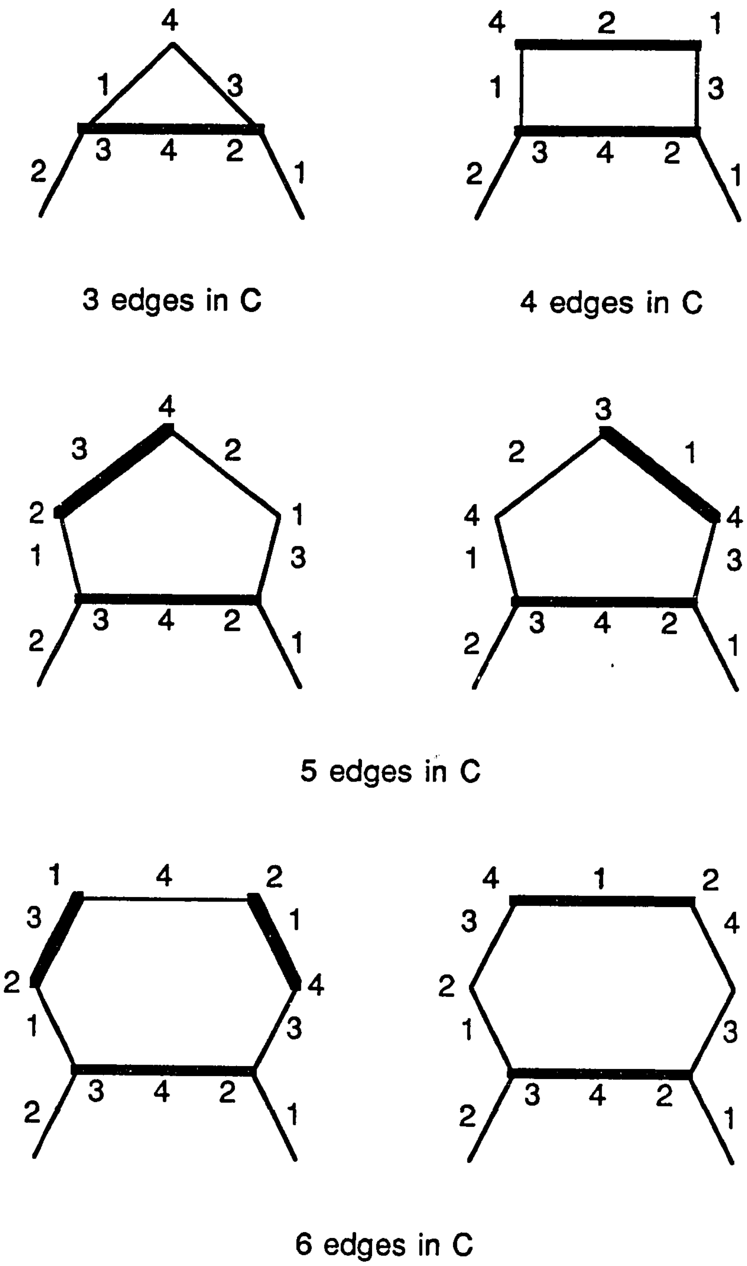

Figure 4.3. Total coloring when C contains 3, 4, 5, or 6 edges.

If C contains seven or more edges, then we will color C as follows. Once again we note that given the stated coloring of S, the color 1 must be assigned to edge e_2 and the color 3 must be assigned to the edge e_p.

Beginning with vertex v_2, we will use the colors 1, 2, 3, and 4 on successive elements (vertices and edges) unless an edge in the matching M is encountered, ie. $S' = \{e_{i-1}, v_i, e_i, v_{i+1}, e_{i+1}\}$ is contained in C and e_{i+2} is contained in M. In this case colors 1, 2, 3, and 4 will be used successively through v_{i+1}, e_{i+2} will be colored with $c(v_i)$, and the colors 1, 2, 3, and 4 will again be used in succession as we continue to color C. This process is shown in an example in figure 4.4.

Figure 4.4. Example of process used to color C.

It is clear that the colors assigned meet the conditions of the lemma. It remains to show that this procedure, or some adjustment of this procedure, can be used to complete the total coloring of C.

To show this we will illustrate all possible cases of coloring pattern (any of the colors 1 through 4 may be on vertex v_{p-1}) and also all possible cases of edges which are included in the matching M. These are shown in figures 4.5 through 4.8. In some cases the conditions of the lemma will be satisfied and we will be done with that case. In most cases the conditions of the lemma will not be satisfied and certain edges and vertices of C will be recolored. These recolorings are shown below the original coloring.

The cycle is shown back to the point where the colors on the vertices and edges are not affected by the recoloring. The cases where the last edge in the matching M is earlier than the fourth last edge are taken care of by figure 4.8 below because the inclusion of any earlier edges in the cycle C does not change the edges and vertices which must be recolored. This completes the proof of the lemma.

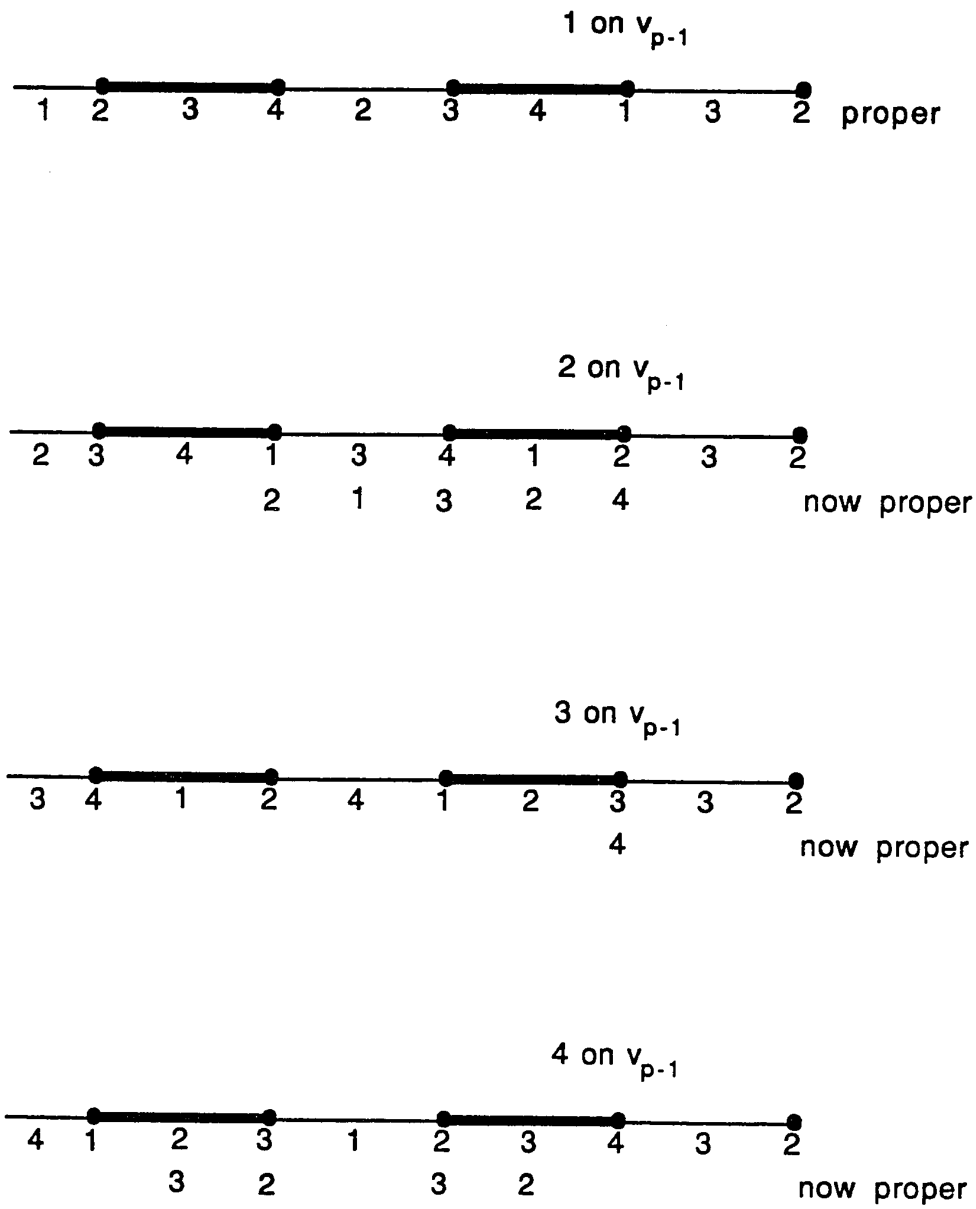

Figure 4.5. 2nd to last and 4th to last edges are in M.

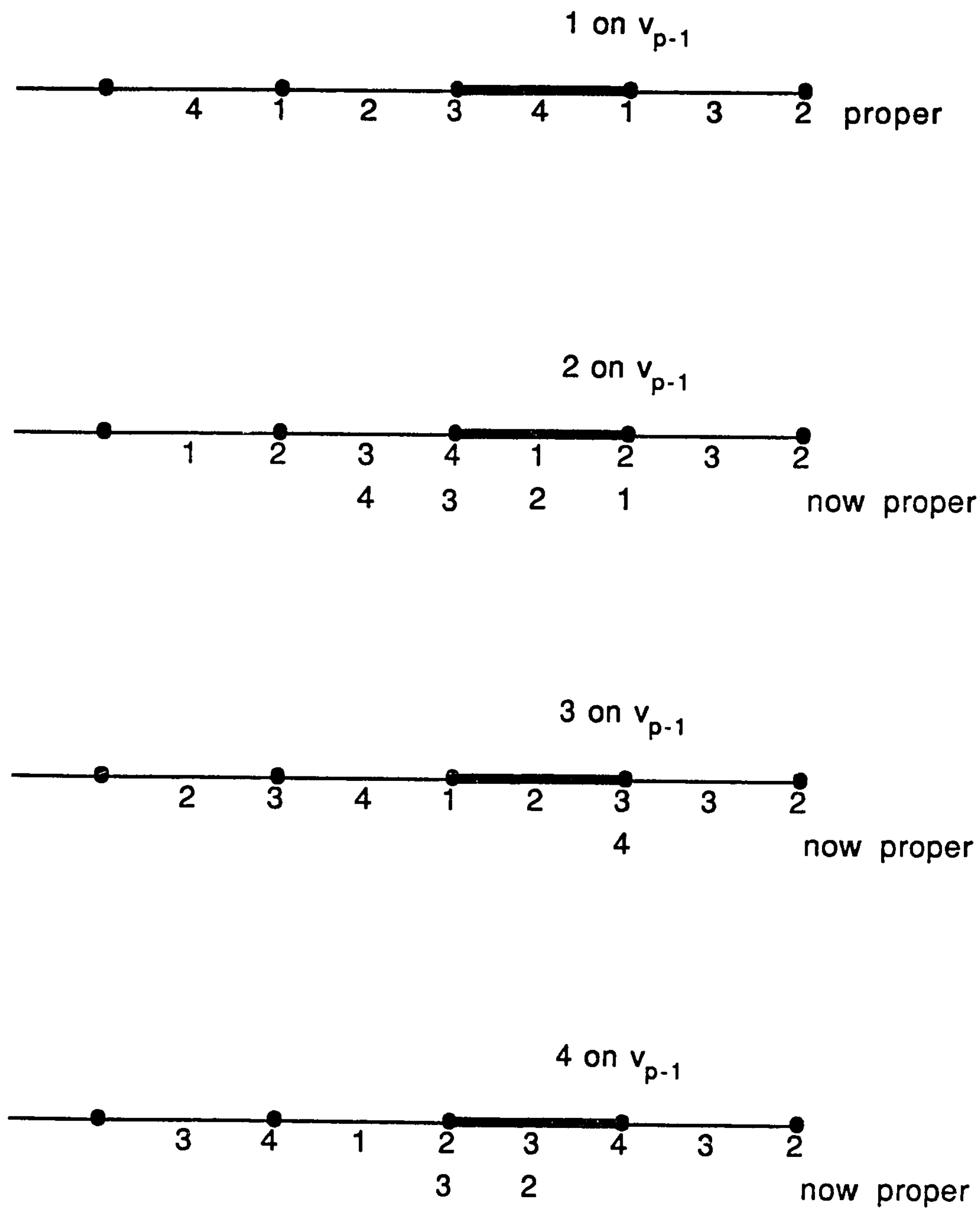

Figure 4.6. 2nd to last and 5th to last or earlier edges are in M.

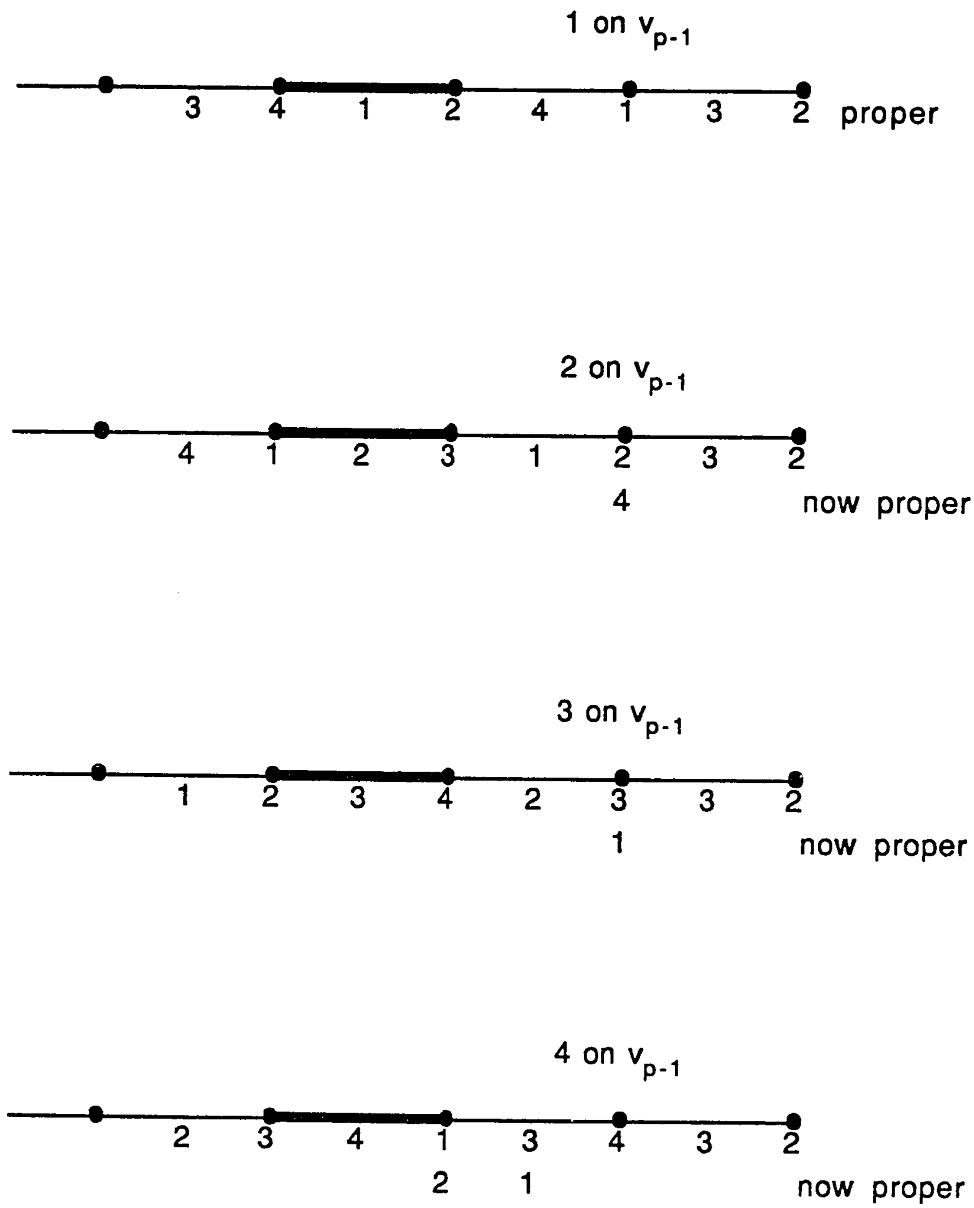

Figure 4.7. 3rd to last edge is in M.

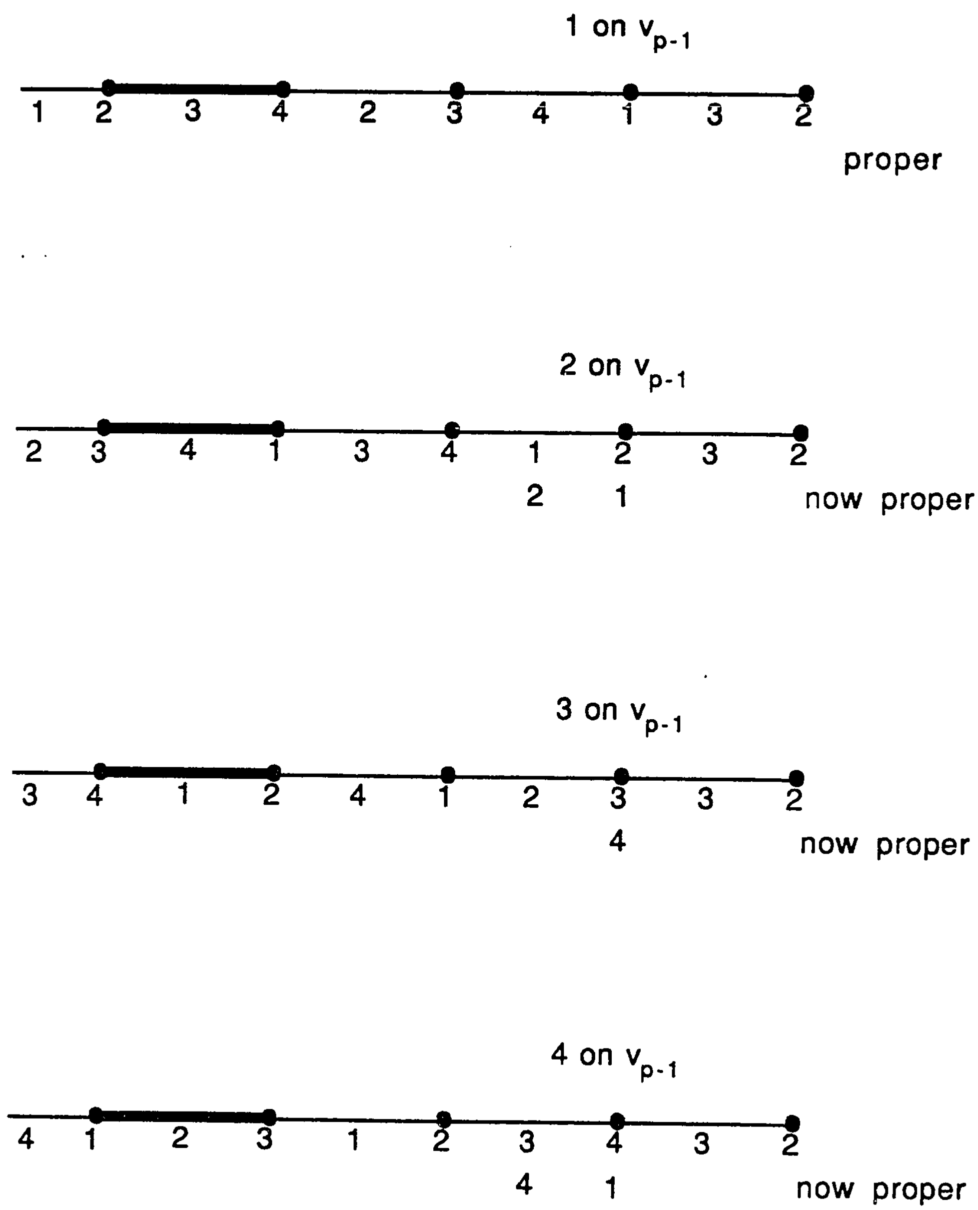

Figure 4.8. 4th to last or earlier edge is the last one in M.

Theorem 4.4 A cycle of length congruent to 0 (mod 3) is a type 1 graph, and all other cycles are type 2 graphs.

Proof. For a cycle of length congruent to 0 (mod 3), the colors 1, 2, and 3 can be used on successive elements (vertices and edges) to complete a total coloring. Thus these cycles are type 1 graphs.

We will show that the cycles of length congruent to 1 (mod 3) or 2 (mod 3) can be total colored with 4 colors but not with 3 colors, so that they are type 2 graphs. First, we show that 4 colors suffice to color these cycles.

Case i. The cycle has length 2k, $k \geq 2$. Colors 1 and 2 are used on consecutive vertices of the cycle, and colors 3 and 4 on consecutive edges of the cycle.

Case ii. The cycle has length 2k + 1, $k \geq 1$.

$c(v_i) = 1$, for $1 \leq i \leq 2k$ and i odd

$c(v_i) = 2$, for $1 < i \leq 2k$ and i even

$c(v_i,v_{i+1}) = 3$, for $1 \leq i \leq 2k$, and i odd

$c(v_i,v_{i+1}) = 4$, for $1 < i \leq 2k$, and i even

$c(v_{2k+1}) = 3$

$c(v_{2k+1},v_1) = 2$

Second, we show that a total coloring using 3 colors is not possible. Assume that a total coloring with 3 colors is possible. Let some vertex v_1 have color 1, and let an incident edge have color 2. Then the next vertex must have color 3, and the colors 1, 2, and 3 must be used on successive elements (edges and vertices) of the cycle.

If the cycle has length $3k + 1$, $k \geq 1$, then vertex v_{3k+1} will have color 1, violating the total coloring since v_{3k+1} is adjacent to v_1.

If the cycle has length $3k + 2$, $k \geq 1$ then edge $e = v_{3k+2}v_1$ will have color 1, violating the total coloring since e is incident to v_1.

In both cases, the assumption is false, so the cycle cannot be total colored with 3 colors. The cycles of length congruent to 1 (mod 3) or 2 (mod 3) are therefore of type 2. We will refer to these as the type 2 cycles.

<u>**Theorem 4.5**</u> If G is an outerplanar block with D(G) = 2 or 3, then for any matching M of the edges of G where each edge in M joins vertices of degree two, there exists a total coloring of G using four colors such that if $S = \{e_{i-1}, v_i, e_i, v_{i+1}, e_{i+1}\}$ is a sequence of edges and incident vertices of G with e_i contained in M, then all four colors are used to color S and $c(e_{i-1}) \neq c(e_{i+1})$.

<u>Proof</u> by induction on p, the number of vertices in G. For the case where p = 3 we will color G similar to the case shown in figure 4.3 with 3 edges in C. Label the vertices of G with v_1, v_2, v_3 and assume that edge v_2v_3 is in the matching M. Assign $c(v_2v_1) = 1$, $c(v_3) = 2$, $c(v_2) = c(v_1v_3) = 3$, and $c(v_1) = c(v_2v_3) = 4$ to complete the total coloring satisfying the conditions of the theorem.

Assume that the theorem is true for all p, $3 \leq p < k$. Let G be an outerplanar block of order k, and let M be any matching of the edges of G.

Case i. D(G) = 2. G is a cycle. Select one edge of the matching M and call it e_0, and call its incident vertices v_1 and v_p. Add vertices

v_0 and v_{p+1} to G and edges $e_1 = v_0v_1$ and $e_{p+1} = v_pv_{p+1}$. We assign $c(e_1) = 2$, $c(v_1) = 3$, $c(e_0) = 4$, $c(v_p) = 2$, and $c(e_{p+1}) = 1$. Now by lemma 4.3, the rest of G can be total colored with four colors so that the conditions concerning the coloring around the edges in the matching M are met. The vertices v_0 and v_{p+1} are now removed (along with edges e_1 and e_{p+1}) and we are left with the graph G and a total coloring of G that satisfies the conditions of Theorem 1.

Case ii. $D(G) = 3$. Choose an outer path $P = v_1, v_2,\ldots,v_p$ in G (there must be one since $D(G) = 3$) and remove it to form a new graph G'. By remove it we mean remove the interior vertices and any edges that are incident to the interior vertices. Let M' be the intersection of M and E(G') and let e_1 be the chord corresponding to P. G' is an outerplanar block and $M' \cup \{e_1\}$ is a matching in G' in which each edge joins degree two vertices in G'. By the inductive assumption G' can be 4 total colored such that the conditions of the theorem are satisfied. Now by Lemma 4.3 we can extend the 4 total coloring to G so that if $S = \{e_i, v_i, e_{i+1}, v_{i+1}, e_{i+2}\}$ is contained in P and e_{i+1} is contained in M, then all four colors are used to total color G and $c(e_i) \neq c(e_{i+2})$. By induction, the proof is complete.

Theorem 4.6 All outerplanar blocks with $D(G) = 3$ are type 1.

Proof. If G has two adjacent vertices u and v of degree two, then form a matching M such that $M = \{uv\}$. Then by Theorem 4.5, $X''(G) = 4$.

If G has no two adjacent vertices of degree two, then remove a vertex v of degree two whose adjacencies are v_i and v_{i+1}. Such a vertex must exist since G is outerplanar. Now v_i and v_{i+1} are adjacent with degree two in G - v. Let M = $\{v_iv_{i+1}\}$. By Theorem 4.5, G - v is 4-total colorable with all four colors used on S = $\{e_i, v_i, e_{i+1}, v_{i+1}, e_{i+2}\}$ and $c(e_i) \neq c(e_{i+2})$. Then the 4-total coloring can easily be extended to G.

In both cases G has been total colored with four colors, so it is a type 1 graph.

<u>**Theorem 4.7**</u> All outerplanar blocks with $D(G) \geq 4$ are type 1.

<u>Proof.</u> We use induction on p, the number of vertices in G. Figure 4.9 shows that the theorem is true for p = 5.

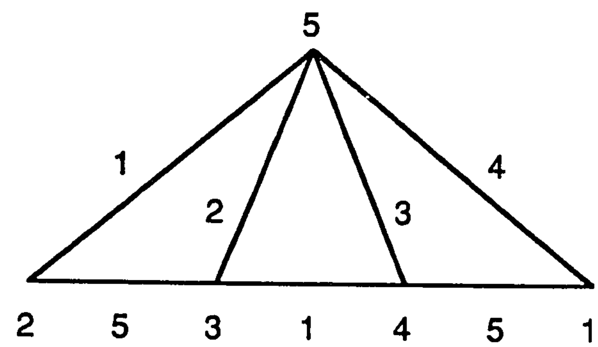

Figure 4.9. The graph where p = 5 and D(G) = 4 is type 1.

Assume that the theorem is true for all p , $5 \leq p \leq k$. This will be our inductive assumption. Let G be an outerplanar block with D(G)

≥ 4 containing $k + 1$ vertices. Since $D(G) \geq 4$, G must contain at least two outer paths. We now classify the graph G into one of two cases.

Case 1. All of the outer paths in G have length 2, and all of the end points of the outer paths have degree equal to the maximum degree of G.

Case 2. All other outerplanar blocks.

The case 2 graphs are handled first. If some outer path has length greater than 2, then select one such path. Otherwise, select an outer path where one or more end points has degree other than the maximum degree in G. Call this outer path P, and label the vertices of P so $P = v_1, v_2, \ldots, v_n$ $(n \geq 3)$. Remove the path from G to form the graph G' (by remove the path we mean remove the internal vertices of the path, and remove all edges incident to the internal vertices of the path).

G' can be total colored with $D(G) + 1$ colors by the induction hypothesis in all cases except where $D(G) = 4$ and $D(G') = 3$. In this case G' is 5-total colorable by Theorem 4.6. Color G' with $D(G) + 1$ colors, and consider the graph G. In graph G we know that $\deg(v_1) - 1$ edges incident to v_1 have been colored. The vertex v_1 has a different color. Since $\deg(v_1) \leq D(G)$, there must be at least one color available from the set $\{1, 2, \ldots, D(G)+1\}$ to color the edge of P incident to v_1. Similarly, some color is available to color the edge in P incident to v_n.

In the case where P has length two, at least one of the endpoints of P must have degree less than the maximum degree of G because of the criteria used to select P. For this reason, a second color will be available to color one of the end edges of P, and the two edges can be assigned different colors. The middle vertex of P can be assigned another color such as that on edge v_1v_n.

In the case where P has length 3 or more, the vertices and edges of P can be easily colored with colors from the set {1, 2, 3, 4, 5}.

The total coloring of G has been completed with D(G) + 1 colors, and the graph G is therefore of type 1.

The case 1 graphs are now considered. In this case, G is a graph where all outer paths have length 2, and all end vertices of outer paths have degree equal to the maximum degree of G.

By Lemma 4.1, the maximum degree of G must be either 3 or 4. Since this theorem concerns graphs where $D(G) \geq 4$, we know that $D(G) = 4$.

By Lemma 4.2 we know that there exists two vertices in G that are on different outer paths, are of degree 2, and are a distance 2 from each other. We refer to the first diagram of Figure 4.10 where these two vertices are labelled v_2 and v_7.

Select the outer path containing v_7 and remove it from G to form G'. G' is a type 1 graph by the inductive hypothesis. Color G' with five colors and define the following sets.

$S_1 = \{c(v_1v_3), c(v_2,v_3), c(v_3), c(v_3,v_4)\}$

$S_2 = \{c(v_3,v_4), c(v_4), c(v_4,v_5), c(v_4,v_6)\}$

If S_1 is not equal to S_2, then it follows that $\{a\} = \{1, 2, 3, 4, 5\} \setminus S_1$ is not equal to $\{b\} = \{1, 2, 3, 4, 5\} \setminus S_2$. We can assign color a to edge v_3v_7, color b to edge v_4v_7, and $c(v_3v_4)$ to vertex v_7 to complete the total color of G using 5 colors.

If S_1 is equal to S_2, then we must modify the already assigned colors to complete the total coloring of G. Without loss of generality we can assume that the following color assignments have been made: $c(v_4v_5) = 1$, $c(v_4v_6) = 2$, $c(v_4) = 3$, $c(v_3v_4) = 4$, $c(v_3) = 1$, $c(v_1v_3) = 3$, $c(v_2v_3) = 2$. These color assignments are shown in the second diagram of Figure 4.10.

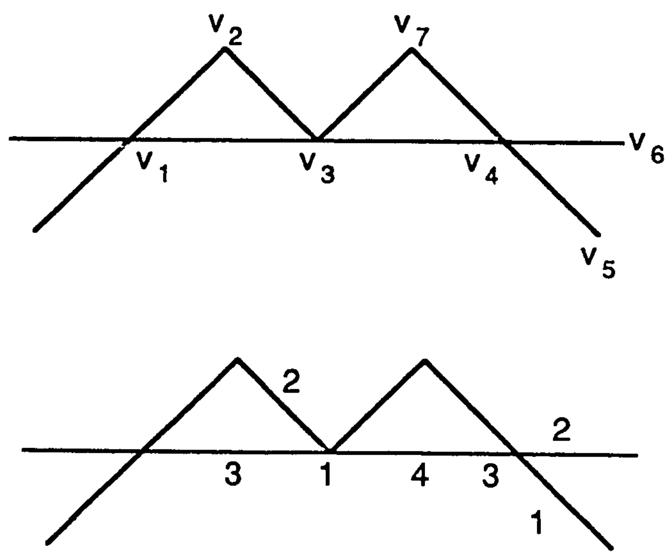

Figure 4.10. Labellings of vertices and edges and illustration of colors that are assumed.

If $c(v_2) = 5$, then assign $c(v_2) = 3$ and $c(v_3) = 5$. This is always possible since $c(v_1)$ cannot be 5 because it is adjacent to v_2. Otherwise if $c(v_1) = 5$, then assign $c(v_2v_3) = 5$. This is always possible because the edges adjacent to v_1 cannot be colored 5.

In the remaining case when $c(v_1) \neq 5$ and $c(v_2) \neq 5$, assign $c(v_3) = 5$. This is clearly possible given the assignments that we have made. The sets S_1 and S_2 are now not equal and the total coloring using 5 colors can be completed as above. The graph G has now been total colored using 5 colors. Since $D(G) = 4$, G is a type 1 graph.

In both case 1 and case 2, G has been shown to be a type 1 graph. By induction, the theorem is true.

Theorem 4.8 All connected outerplanar graphs G except K_2 and the type 2 cycles are type 1.

Proof. We use induction on i, the number of blocks in the graph G. If $i = 1$ (and the graph is not K_2 or a type 2 cycle) then the graph is an outerplanar block, and by Theorems 4.4, 4.6 and 4.7 this is a type 1 graph.

Assume that the theorem is true for all outerplanar graphs containing i or fewer blocks. Let G be an outerplanar graph having $i + 1$ blocks. Split G at one of the cutpoints v into subgraphs G_1 and G_2, each an outerplanar graph. Color each of the subgraphs in the minimal number of colors possible. This will be $D(G_i) + 2$ if the subgraph is a type 2 cycle or K_2. If the subgraph is an outerplanar block other than a type 2 cycle or K_2, then this will be $D(G_i) + 1$ by Theorems 4.6 and 4.7. Finally, this will be $D(G_i) + 1$ by the inductive assumption if the subgraph is an outerplanar graph that is not a block. We now consider three separate cases.

Case i. Both subgraphs are type 2 cycles. Each of the cycles is colored with colors 1, 2, 3, and 4. Rename the colors on one cycle so that color 1 is on the cut vertex of G and colors 2 and 3 are on the incident edges. Now rename the colors used on the second cycle so that color 1 is on the cut vertex, and colors 4 and 5 are on the incident edges. The graphs can now be joined back together, and we have completed the total coloring with 5 colors, so the graph is type 1.

Case ii. One subgraph, G_1, is a type 2 cycle and the other, G_2, is not. Rename the colors on G_1 so that v is colored 1 and its incident edges are colored 2 and 3. Similarly rename the colors of G_2 so that v is colored 1 and its its incident edges are colored 4, 5,...,t, where $t \leq D(G) + 1$. This is always possible because $Deg_{G2}(v) \leq D(G) - 2$. The graphs can now be joined back together yeilding a total coloring of G with $D(G) + 1$ colors.

Case iii. Neither G_1 nor G_2 is a type 2 cycle. Rename the colors on G_1 so that v is colored 1 and its incident edges are colored 2, 3,...,t+1, where $t = deg_{G1}(v)$. Rename the colors of G_2 so that v is colored 1 and its incident edges are colored t+2, t+3,..., $t+deg_{G2}(v)+1$. Since $t+deg_{G2}(v)+1 = deg_G(v)+1 \leq D(G) + 1$, this change is possible.

The two subgraphs can now be joined back together and a total coloring is completed with D(G) + 1 colors.

The graph G has been total colored with D(G) + 1 colors, so it is type 1. By induction, the theorem is true.

4.2 Total Coloring 2-Degenerate Graphs

We now turn our attention to a larger family of graphs, the 2-degenerate graphs.

It is easily seen that an outerplanar graph is a 2-degenerate graph because an outerplanar graph is either a cycle or it has outer paths. In either case it has a vertex of degree 2. By removing an outer path one forms another outerplanar graph (also having a vertex of degree 2 or smaller) which completes the demonstration that outerplanar graphs are 2-degenerate. As a first step it will be shown that the total coloring conjecture holds for 2-degenerate graphs.

Theorem 4.9 For any 2-degenerate graph G, X"(G) ≤ D(G) + 2, where D(G) is the maximum degree of G.

Proof by induction on p, the number of vertices in G. If p = 1, then X"(G) = 1 ≤ D(G) + 2 = 2, and the theorem is true. Assume that the theorem is true for all 2-degenerate graphs with k or fewer vertices, and let G be a 2-degenerate graph containing k + 1 vertices. Label the vertices in G with $v_1, v_2, \ldots, v_k, v_{k+1}$ such that

$deg_G(v_1) \leq 2$ and $deg_G(v_1) = d(G)$, the minimum degree of G

$G_1 = G - v_1$, $\quad deg_{G1}(v_2) \leq 2$,

$G_2 = G_1 - v_2$, $\quad deg_{G2}(v_3) \leq 2$,

...

$G_k = G_{k-1} - v_k, \quad deg_{Gk}(v_{k+1}) \leq 2$

This is possible because G is a 2-degenerate graph.

Remove vertex v_1 from G to form G_1. By the induction hypothesis, G_1 can be total colored with $D(G_1) + 2$ colors. Do this, and then consider graph G.

If $deg(v_1) = 1$, then v_1 is adjacent to exactly one vertex v_i, $2 \leq i \leq k + 1$. Vertex v_i has degree no greater than D(G) - 1 in graph G_1, so D(G) or fewer colors have been used to color v_i and its incident edges. Some color from the set {1, 2,..., D(G)+2} is therefore available to color edge v_1v_i. Since D(G) is at least 1, some color from the same set is available to color the vertex v_1 other than that used on vertex v_i or edge v_1v_i. This completes the total coloring of G using D(G) + 2 colors or less in the case where $deg(v_1) = 1$.

If $deg(v_1) = 2$, then v_1 is adjacent to two vertices v_i and v_j. By Theorem 4.4 all cycles can be total colored with D(G) + 2 colors, so we will assume that G some component of G has a maximum degree of 3 or more so that $D(G) + 2 \geq 5$.

The vertices v_i and v_j have degree no greater than D(G) - 1 in graph G_1, so D(G) or fewer colors have been used to color each vertex and its incident edges. Therefore, some color from the set {1, 2,..., D(G)+2} is available to color edge v_1v_i, and some other color from the same set is available to color edge v_1v_j. Since $D(G) + 2 \geq 5$, a color is available for vertex v_1 other than that used on vertices v_i, v_j, or edges v_1v_i, or v_1v_j. This completes the total coloring of G using D(G) + 2 colors or less for the case where $deg(v_1) = 2$.

In all cases, the graph G has been total colored using D(G) + 2 or less colors. By induction, the theorem is true.

Now that the 2-degenerate graphs have been shown to satisfy the total coloring conjecture, one might wonder if they are also type 1 graphs. This would be an extension of Theorem 4.8 from the outerplanar graphs to the 2-degenerate graphs.

Theorem 4.8 cannot be extended in this way, however. We show that there are in fact 2-degenerate graphs that are not type 1.

Consider a 2-degenerate, order 7 graph G with vertices having degrees 3, 3, 3, 3, 3, 3, 2. One graph that has this property is formed by taking $K_{3,3}$ and inserting a vertex into one edge. It is easy to see that such a graph is 2-degenerate because the removal of the degree 2 vertex will create a graph with 2 vertices having degree 2. The degree 2 vertices can be removed in succession, with the resulting graph at each stage having at least 2 vertices of degree 2 or less, hence the graph is 2-degenerate.

We will show that of the order 7 graphs with vertices of degree 3, 3, 3, 3, 3, 3, 2, there are exactly 2 graphs that can be total colored with 4 colors and are hence type 1. All other order 7 graphs with vertices of degree 3, 3, 3, 3, 3, 3, 2, are therefore not type 1, including the one described above which is a 2-degenerate graph.

In a graph G with vertices of degree 3, 3, 3, 3, 3, 3, 2, there are 7 vertices and 10 edges. To color these 17 elements with 4 colors, some color class C must have at least [17 / 4] = 5 elements. The 5 elements in this color class can be either vertices or edges. Each of the various combinations of vertices and edges will be considered individually.

If C contains 5 vertices and 0 edges, then the 2 vertices not in C can be incident to a total of 6 edges. The vertices in C are incident to at least 14 edges, a contradiction to proper total coloring since 2 vertices in C would have to be adjacent.

If C contains 4 vertices and 1 edge, then the 3 vertices not in C can be incident to a total of 9 edges. The vertices in C are incident to at least 11 edges, again a contradiction.

If C contains 3 vertices and 2 edges, then we cannot deduce a contradiction in the same way as the previous cases. We will consider this case later.

If C contains 2 vertices and 3 edges, then 8 vertices will be required and there are only 7 in G, a contradiction.

If C contains 1 vertex and 4 edges, then 9 vertices will be required and there are only 7 in G, a contradiction.

If C contains 0 vertices and 5 edges, then 10 vertices will be required and there are only 7 in G, again a contradiction.

We now consider the case where C contains 3 vertices and 2 edges. Label the 3 vertices in C with u_1, u_2, and u_3. Label the 4 vertices not in C with v_1, v_2, v_3, v_4, and assume that edges v_1v_2 and v_3v_4 are in C. Without loss of generality we can assume that u_1 is adjacent to v_1, v_2, and v_3. Now if u_2 is adjacent to v_1, v_2, and v_3, then u_3 can be adjacent only to v_4 without violating the constraints that D(G) = 3 and u_3 is not adjacent to u_1 or u_2. Hence u_2 cannot be adjacent to all of v_1, v_2, and v_3.

The case 1 where u_2 is adjacent to v_1, v_2, and v_4, is shown in figure 4.11. Three separate diagrams are shown. The first diagram shows the graph drawn in a way which follows from the description above. The second diagram shows the graph drawn embedded in the plane, and the third diagram shows a proper total coloring for the graph.

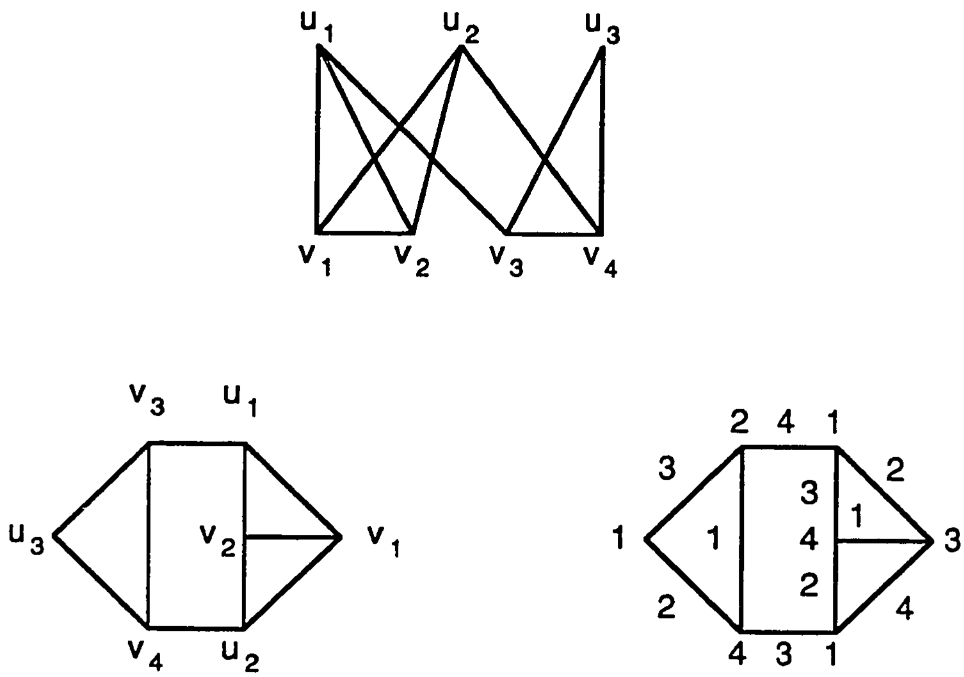

Figure 4.11. Case 1 graph drawn as described in the text, drawn as a planar graph, and drawn with assigned colors.

The case 2 where u_2 is adjacent to v_1, v_3, and v_4 is shown in Figure 4.12. Once again, three diagrams are shown, the first with the graph drawn which follows from the description, the second with the graph embedded in the plane, and the third showing a proper total coloring for the graph.

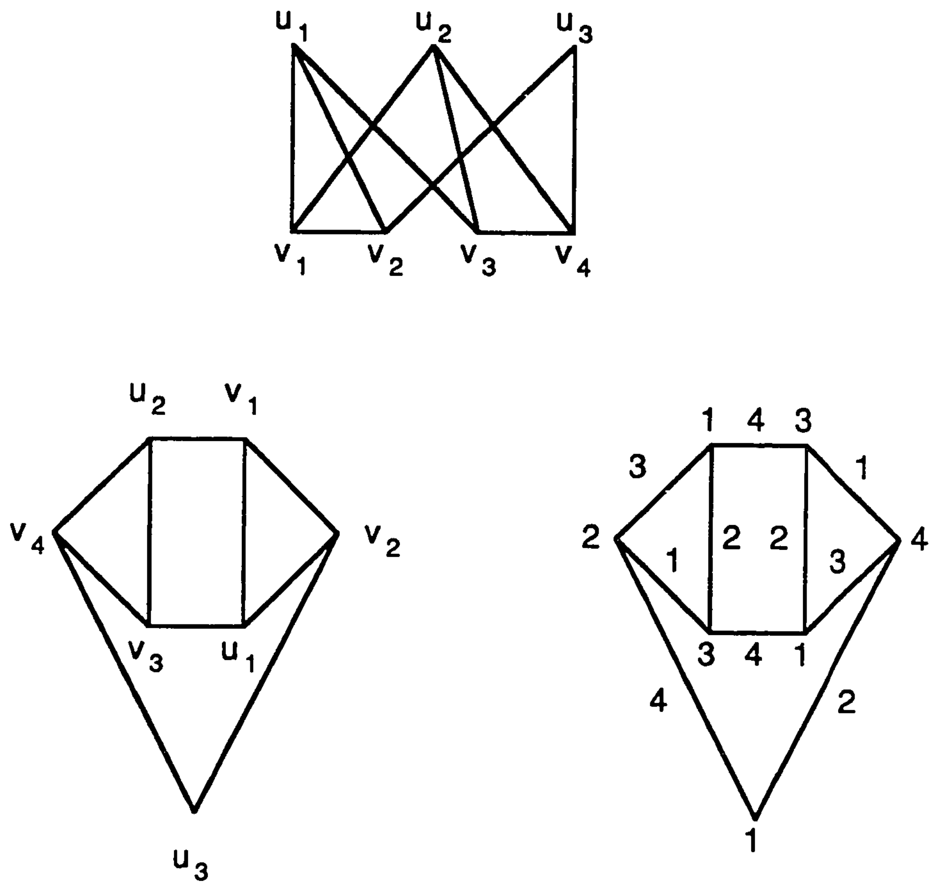

Figure 4.12. Case 2 graph drawn as described in the text, drawn as a planar graph, and drawn with assigned colors.

If $\deg(u_2) = 2$, then $\deg(u_3) = 3$. The graphs possible with these conditions are isomorphic to the graphs with $\deg(u_2) = 3$ and $\deg(u_3) = 2$, so they have already been considered.

If vertices u_1, u_2, and u_3 each have degree 3, then 9 edges must connect these to vertices v_1, v_2, v_3, and v_4. But with edges v_1v_2 and

v_3v_4 already present, the sum of the degrees of the v_i vertices is 13, a contradiction to the constraint that D(G) = 3. Hence all of u_1, u_2, and u_3 cannot have degree 3.

We have considered all order 7 graphs with vertices having degrees 3, 3, 3, 3, 3, 3, 2, and found that only two are type 1 graphs. The other graphs having vertices with these degrees are not type 1 graphs. One example is $K_{3,3}$ with a vertex inserted into one of its edges, which is a 2-degenerate graph.

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