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Investigation of repeated measures linear regression methodologies

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INVESTIGATION OF REPEATED MEASURES LINEAR REGRESSION METHODOLOGIES

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ABSTRACT

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Repeated measures regression is regression where the assumption of independent identically distributed observations is not met due to the fact that an observational unit has multiple readings of the outcome variable, thus standard methods of analysis are not valid. A substantial amount of research exists on repeated measures in the Analysis of Variance (ANOVA) setting; however, when the independent variables are not factor variables, ANOVA is not the appropriate tool for analysis. There is currently much controversy regarding regression methods in the repeated measures setting. At issue are topics such as parameter estimation, testing of parameters, and testing of models. We intend to examine currently used methodologies and investigate the properties of these various methods. Methodologies will include calculation of expected mean square values for appropriateness of statistical tests, as well as simulations in order to investigate the validity of the methods in situations where the truth is known.

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CHAPTER I - INTRODUCTION

Repeated measures designs come in several types: split-plot, change over, sources of variability, and longitudinal studies. Split plot designs may include experiments where an agricultural field is split into multiple plots. Each of the plots is treated with a different fertilizer, and crops are randomly assigned to subplots within each fertilized plot (Kotz & Johnson, 1988). Each fertilized plot will produce several types of crops and a measure of each will be collected, yielding repeated measures for each plot. Change over designs may be used when testing two types of drugs. First, half of the people are given drug A and the others are given drug B. Then drug A and drug B are switched and the experiment is rerun. This design is a repeated measures experiment because every person will have two measures, one for drug A and one for drug B (Kotz & Johnson, 1988). Source of variability studies may include taking several randomly selected items from a manufacturing process and allowing several people to test each one, possibly over several days (Kotz & Johnson, 1988). Longitudinal studies address situations such as the growth of chicks, where weights of each chick may be measured every few days.

The type of repeated measures experiment explored in this paper is classified as a special case of longitudinal study. Usually, in longitudinal designs, the observational units are sampled over an extended period of time. However, there exists a subset of longitudinal designs where time is not an independent variable (Ware, 1985). In this case, we will look at studies that are not broken into intervals of time but rather are categorized according to variables composed of concepts, items, or locations in space (Kotz & Johnson, 1988).

As an example, we may have some subjects in a cognitive study read multiple sentences (Lorch & Myers, 1990). We will collect time measurements for every sentence a person reads. In this experiment, each subject would have multiple measures, one per sentence, where the independent variable is the number associated with the sentences serial position in the set of sentences. Each sentence is an item and has a measurement associated with it; the time it takes to read it. This is not the typical type of longitudinal study because the entire experiment can be done in a short period of time.

Another such example includes a survey with several questions on it regarding the three branches of government (Kotz & Johnson, 1988). First, this design is classified as a longitudinal study but is not categorized by time. The repeated measures from each participant are obtained from questions on the survey. Second, the answers to the questions collected from each person are correlated and cannot be assumed to be independent. An anarchist would most likely view all three branches of government unfavorably, while someone else may support government and answer all questions more positively.

These types of longitudinal studies have been employed across many disciplines and have a plethora of practical applications. We found examples in the disciplines of psychology and cognitive development, such as the aforementioned experiment with subjects reading sentences (Lorch & Myers, 1990). Also, repeated measures designs have been used in survey analysis as in the government survey example (Kotz & Johnson, 1988). In the life sciences, people may be measured sporadically for kidney disease; a single reading could be affected by one particular meal so multiple readings are taken instead (Liu & Liang, 1992).

In many cases where single measurements are taken, the experiment could be redesigned to collect multiple readings on a subject. This can reduce the number of observational units needed when conducting a study and already has been used in many fields such as psychology, education, and medicine. The only instance where repeated measures is not possible is when the observational unit is altered or destroyed by the initial test, as in the example where a rope is tension-tested until it breaks. Repeated measurement experiments are common in most situations and fields of study. However, standard analysis will not suffice because the measurements are correlated. The standard errors, t statistics, and p values in most statistical tests are invalid when the measurements are not independent (Misangyi, LePine, Algina, & Goeddeke, 2006).

For further clarification, repeated measures can be broken down into two categories: replicate or duplicate observations (Montgomery, 1984). Replicate observations are defined as multiple responses taken at the same value of the independent variables. Different subjects are being used for every measurement being collected. The observations are assumed independent since distinct observational units are being tested. This scenario is discussed in many texts and analysis usually includes a goodness of fit (GOF) test. This test can be performed alongside ordinary least squares regression because some of the rows of the design matrix are identical. The GOF test evaluates whether a higher order model, such as a polynomial or a model with interaction terms, might fit the data better. Duplicate observations are repeated measures on the same observational unit; these measurements are not necessarily independent and may be correlated, thus defying one of the assumptions of ordinary least squares regression and ANOVA (Montgomery, 1984). However, duplicate observations may appear very similar to replicate observations since they too may have the same value of independent variables for multiple responses.

This paper will examine three types of models, all of which contain duplicate observations. The first model, discussed in Chapter II, will evaluate a special case where the same observational unit is tested at all values of a single independent variable. The multiple readings are usually correlated even though they are obtained from different values of the independent variable because they are acquired from the same observational unit (Crowder & Hand, 1990). This model is of interest because ANOVA tests can also be compared against several other regression methods. In Chapter III, a second model is constructed with repeated measurements taken on the same observational unit at the same independent variable value repeatedly. The third model, in Chapter IV, will be unstructured; the repeated measurements from an observational unit may be obtained from any set of values of the independent variable. Each of these chapters will contain examples, several methods of analysis, and results of simulations. A comprehensive discussion of results will be addressed in Chapter V.

CHAPTER II - MODEL 1: EVERY SUBJECT TESTED AT EVERY LEVEL

Say the efficacy of a drug needs to be tested and two methods of experimentation are available. First, an experimenter could collect observations on $N = I * J$ people, randomly assign I of them to one of J treatment groups, and give each treatment group a different dose of the drug. Under this design, exactly one reading per person is taken. Alternately, an experimenter could collect observations on I people and test them at each of the J levels of the factor. This second method is categorized as repeated measures because the same set of people is used in multiple tests. The repeated measures experimental design is beneficial as it eliminates the variation due to subjects, which can significantly reduce the mean square error, ultimately making detecting real differences between the treatment doses easier (Montgomery, 1984).

In Model 1, we will simulate an experiment with J treatments and I people. Each person is tested once per treatment, which helps minimize the amount of variation due to the subjects. This sort of study is used in many fields; one such example is in the field of psychology where several subjects were asked to read a set of sentences (Lorch & Myers, 1990). Each sentence was timed, so each subject provided multiple readings. In this particular model, we have I subjects with indices $i = 1, 2, \ldots, I,$ in our experiment with J repeated measures per person with indices $j = 1, 2, \ldots, J$. Each person is timed as they read J sentences, and a time is recorded for each sentence. For ease, we will let $N = I * J$ denote the total number of observations.

In Model 1, y_{ij} is the measurement of the time it took the i^{th} person to read the j^{th} sentence. In this example, $\mathbf{x}_1$ will be a dummy coded variable for which sentence is being read. But in other experiments, $\mathbf{x}_1$ could just as easily be a continuous variable, such as the dose of a drug taken for a study. Other independent variables may exist as well, such as the age of the person, gender, or IQ score, denoted $\mathbf{x}_2, \mathbf{x}_3 \ldots, \mathbf{x}_k$, however, only one independent variable is used in these simulations. $\boldsymbol{\beta}$ is a vector of the coefficients, and $\boldsymbol{\varepsilon}$ is the vector of errors. Let $\boldsymbol{\Sigma}$ be the covariance matrix of the errors. Our model will be of the form: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. In matrix notation we have:

$$\mathbf{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1J} \\ y_{21} \\ \vdots \\ y_{IJ} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{k1} \\ 1 & x_{12} & \cdots & x_{k2} \\ \vdots & \vdots & & \vdots \\ 1 & x_{1J} & \cdots & x_{kJ} \\ 1 & x_{11} & \cdots & x_{k1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{1J} & \cdots & x_{kJ} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \vdots \\ \varepsilon_{1J} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{IJ} \end{bmatrix} \tag{1-4}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{cov}(\varepsilon_{11},\varepsilon_{11}) & \mathrm{cov}(\varepsilon_{12},\varepsilon_{11}) & \ldots & \mathrm{cov}(\varepsilon_{1J},\varepsilon_{11}) & \mathrm{cov}(\varepsilon_{21},\varepsilon_{11}) & \ldots & \mathrm{cov}(\varepsilon_{IJ},\varepsilon_{11}) \\ \mathrm{cov}(\varepsilon_{11},\varepsilon_{12}) & \mathrm{cov}(\varepsilon_{12},\varepsilon_{12}) & \ldots & \mathrm{cov}(\varepsilon_{1J},\varepsilon_{12}) & \mathrm{cov}(\varepsilon_{21},\varepsilon_{12}) & \ldots & \mathrm{cov}(\varepsilon_{IJ},\varepsilon_{12}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \mathrm{cov}(\varepsilon_{11},\varepsilon_{1J}) & \mathrm{cov}(\varepsilon_{12},\varepsilon_{1J}) & \ldots & \mathrm{cov}(\varepsilon_{1J},\varepsilon_{1J}) & \mathrm{cov}(\varepsilon_{21},\varepsilon_{1J}) & \ldots & \mathrm{cov}(\varepsilon_{IJ},\varepsilon_{1J}) \\ \mathrm{cov}(\varepsilon_{11},\varepsilon_{21}) & \mathrm{cov}(\varepsilon_{12},\varepsilon_{21}) & \ldots & \mathrm{cov}(\varepsilon_{1J},\varepsilon_{21}) & \mathrm{cov}(\varepsilon_{21},\varepsilon_{21}) & \ldots & \mathrm{cov}(\varepsilon_{IJ},\varepsilon_{21}) \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(\varepsilon_{11},\varepsilon_{IJ}) & \mathrm{cov}(\varepsilon_{12},\varepsilon_{IJ}) & \ldots & \mathrm{cov}(\varepsilon_{1J},\varepsilon_{IJ}) & \mathrm{cov}(\varepsilon_{21},\varepsilon_{IJ}) & \ldots & \mathrm{cov}(\varepsilon_{IJ},\varepsilon_{IJ}) \end{bmatrix} \tag{5}$$

It is also noted that these matrices were created for a regression analysis. If RM ANOVA is utilized, which requires categorical variables, the **X** matrix must be adjusted. In this model, the **X** matrix used for RM ANOVA would have two categorical variables, which would be dummy coded with J - 1 levels for the treatments and I - 1 dummy coded variables for the subjects and result in rank(**X**) = (J - 1) + (I - 1) + 1.

Figure 2.1 shows how data from such a model would look if graphed, where each observational unit is given its own symbol. For example, if there was a study on a blood pressure drug the graph could look as following, where a patient (denoted +,×,○, or ♦) has three blood pressure readings taken, one at each dosage of a drug (maybe 1, 2, and 3mgs).

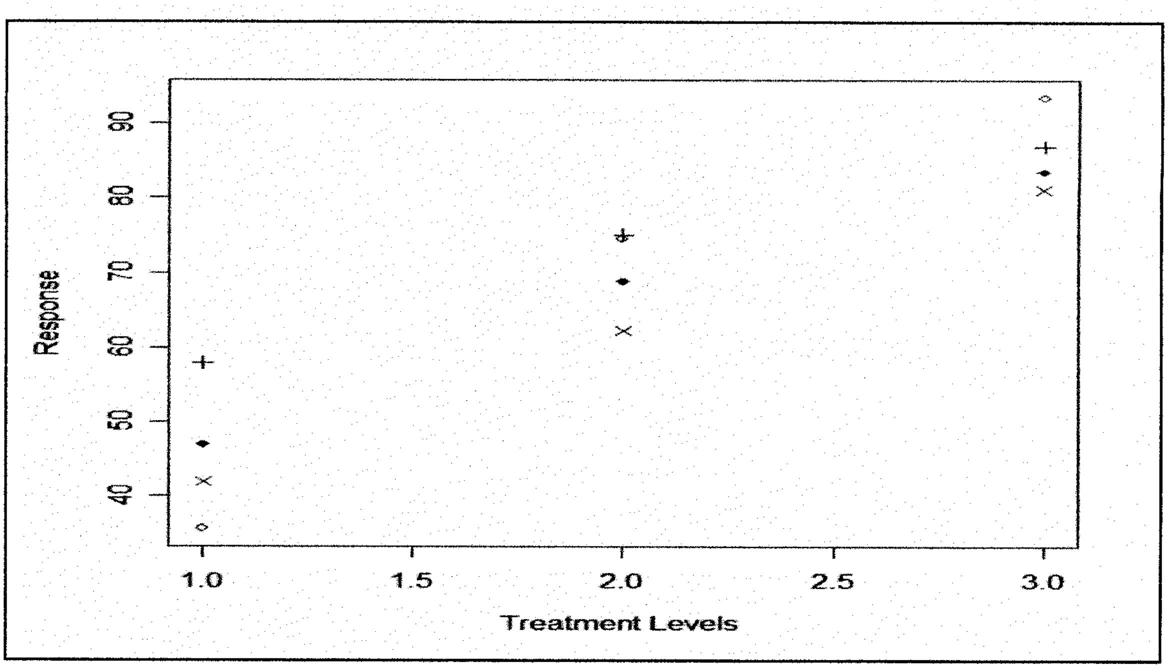

Figure 2.1. Graphical Representation of Model 1.

Methods

Crowder and Hand (1990) describe many inefficient methods of analyzing this experiment type, which are not included in our simulations. These methods include performing multiple t-tests on different groupings of the factors and observational units. This analysis tends to be invalid because the involvement of multiple hypothesis tests increase the chance of a Type I error (Bergh, 1995). Others suggest performing multiple regressions on these crossover design experiments, allotting one regression for each observational unit. How, then, do we compare these multiple regression functions? Some suggest performing an area under the curve analysis, but this examination only inspects one very specific feature of the regression curve. Similarly, other aspects like time to the peak, half-life of the curve, or distance back to the baseline are not accounted for (Crowder & Hand, 1990). Crowder and Hand (1990) suggest the best way to analyze data is to have one all-inclusive ANOVA or regression model, so as not to have multiple statistical tests.

Analysis with Repeated Measures ANOVA

For Model 1, each measurement on a subject is taken at each of the levels of treatment. Therefore, if, as before, an experimenter wanted to test a drug's effectiveness he or she could simply test each person at, for example, $J = 3$ dosage levels. All subjects must be tested at all treatment levels for repeated measures ANOVA (RM ANOVA) to be a valid test (Montgomery, 1985). The null hypothesis is that effectiveness is statistically the same at each treatment level (Misangyi et al., 2006). If the F value is large enough, then the null hypothesis will be rejected, and the treatments will be considered statistically distinct. The calculations for the F values are shown in Table 2.1. We are only interested in the second F value in Table 2.1, $MSTR / MSE$, which is used to test the null hypothesis that the treatments levels are the same. This analysis is identical to randomized complete block design with subjects as the blocks (Montgomery, 1985).

Table 2.1

ANOVA with one Within-Subject Factor

Source	*df*	*SS*	*MS*	*F*
Between Subject	I - 1	*SSBS*	*MSBS*	*MSBS / MSE*
Within Subject	$I\,(J-1)$	*SSWS*	*MSWS*	
Treatment	J - 1	*SSTR*	*MSTR*	*MSTR / MSE*
Error	$(J$ - 1$)(I-1)$	*SSE*	*MSE*	
Total	$IJ-1$	*SST*		

Note: Proofs of this table in Appendix E (Montgomery, 1985).

In a standard ANOVA setting, without repeated measures, all of the error terms are assumed independent and normally distributed, with equal variance. Also, in ANOVA, the independent variable is considered as a factor variable that contains levels. In RM ANOVA, since we break variation into two parts, one due to the observational units and one due to the factor levels, we also must modify our assumptions. Because the assumptions of independence and constant variance no longer necessarily hold, we must now change our assumption to one of sphericity, or circularity, which restricts the variances and correlations of the measurements (Bergh, 1995). Compound symmetry, a condition where all variances of the factor levels are equal and all covariances between each pair of factor levels are equal (O'Brien & Kaiser, 1985), can be examined in place of the less restrictive sphericity assumption because, if it does not hold, the sphericity assumption usually will not hold. First, the variance of each level of the factor must be identical so as not to violate our assumptions. For example, we may test children at ages

2, 4, 6, and 8; with sphericity it is assumed that the correlation between observations taken at ages 2 and 4, and the correlation between observations taken at ages 4 and 6 would be the same. Usually, it would not be the case that the correlation between observations taken at ages 2 and 8 is the same as the correlation between observations taken at ages 2 and 4, but compound symmetry requires this supposition (O'Brien & Kaiser, 1985). The sphericity assumption generally will be violated when individuals are being tested over a time range or more than two measurements are taken: two characteristics of most longitudinal designs (O'Brien & Kaiser, 1985). In this way, the sphericity assumption is almost always violated in the repeated measures situation making RM ANOVA troublesome (O'Brien & Kaiser, 1985) and may lead to an increase in Type I errors if the degrees of freedom are not adjusted (Misangyi et al., 2006).

In the previous sentence reading example, a possibility arises that some people may start to read each successive sentence at an accelerated pace as they become familiar with the topic; thus their rate of increase is not necessarily constant. This acceleration could cause the variance of the time to read each sentence to increase over the whole group and thus violate the sphericity assumption (Misangyi et al., 2006).

Some suggest performing a series of tests for sphericity. To do this, we define a special covariance matrix ($\mathbf{\Sigma_T}$) of the treatment levels ($\mathbf{T}_j$) in Model 1 such that:

$$\mathbf{\Sigma_T} = \begin{bmatrix} \text{var}(\mathbf{T}_1) & \text{cov}(\mathbf{T}_1,\mathbf{T}_2) & \cdots & \text{cov}(\mathbf{T}_1,\mathbf{T}_J) \\ \text{cov}(\mathbf{T}_2,\mathbf{T}_1) & \text{var}(\mathbf{T}_2) & & \text{cov}(\mathbf{T}_2,\mathbf{T}_J) \\ \vdots & & \ddots & \vdots \\ \text{cov}(\mathbf{T}_J,\mathbf{T}_1) & \text{cov}(\mathbf{T}_J,\mathbf{T}_2) & \cdots & \text{var}(\mathbf{T}_J) \end{bmatrix}. \qquad (6)$$

We can examine an estimate of this covariance matrix to verify the supposition of sphericity in our repeated measure situations. Compound symmetry is defined as constant variance and covariance of Σ_T (Baguley, 2004). Compound symmetry is a stricter form of sphericity, and therefore, if compound symmetry holds, the sphericity assumption is met. Unfortunately, sphericity can also be met by a much more general rule, so we must verify that the variances of the differences between each factor level $\mathbf{T}_v$ and $\mathbf{T}_m$ are equivalent (Baguley, 2004) for every set of *v* and *m*, using the formula:

$$\mathrm{var}(\mathbf{T}_v - \mathbf{T}_m) = \mathrm{var}(\mathbf{T}_v) + \mathrm{var}(\mathbf{T}_m) - 2\,\mathrm{cov}(\mathbf{T}_v, \mathbf{T}_m). \quad (7)$$

If the differences of the variances for all of the treatments are equivalent, sphericity will hold (Baguley, 2004). As long as the sphericity assumption is met, the *F* test does not need adjustment because it will not be biased.

An RM ANOVA test would suffice only if the sphericity condition were met (Cornell, Young, Seaman, & Kirk, 1992). When the sphericity assumption is violated, however, the *F* test will have a positive bias and we would be more likely to reject the null hypothesis when it is true (Misangyi et al., 2006). O'Brien and Kaiser (1985) note that in many cases, sphericity may fail when repeated observations are taken on a subject since those observations are correlated. In these cases, we are prone to assume the model is significant when, in reality, it is not (Misangyi et al., 2006). To avoid all of the problems with violations of sphericity, it is suggested that a multivariate analysis (MANOVA) be performed because these tests do not have an assumption about sphericity (Bergh, 1995). However, the MANOVA would be less powerful than the RM

ANOVA that relies on the sphericity assumption (Crowder & Hand, 1990). Model 1 is the only case in this paper where the sphericity assumption must hold.

Cornell et al. (1992) compared eight different types of tests for sphericity. The most commonly employed and discussed test for sphericity is the *W* test, also known as Maulchly's likelihood ratio test. This test has been shown to be rather futile because it does not work well with small sample sizes or in cases where Normality is in question (O'Brien & Kaiser, 1985). The *W* test also tends to be too conservative for light-tailed distributions and too liberal for heavy-tailed distributions (Crowder & Hand, 1990). Another test for sphericity is the *V* test, a locally best invariant test, which has been shown to be slightly superior to the *W* test (Cornell et al., 1992). The other possible tests are the *T* test, a ratio of the largest to smallest eigenvalues of the sample covariance matrix ($\hat{\Sigma}_T$), and *U* tests one thru five, based on Roy's union intersection principle. Cornell et al. (1992) ran simulations, compared these tests, and found that, in most cases, the *V* test is most powerful in detecting sphericity. However, other authors suggest not using any of these tests because they do not provide enough information and often are faulty (O'Brien & Kaiser, 1985).

Because most authors agree that nearly all repeated measures experiments fail to meet the sphericity assumption, we assume an initial test for sphericity is not useful (O'Brien & Kaiser, 1985). When the experimental data fails the sphericity test and thus violates the assumptions of regression or RM ANOVA, a Box's epsilon (ϵ) correction on the degrees of freedom is commonly used (Box, 1954). Box described the correction on the degrees of freedom but never derived a formula for ϵ, so others have provided a variety of implementations for estimates of this correction factor. These correction factors, called Box's epsilon estimates, reduce the degrees of freedom of the RM ANOVA *F* test (Quintana & Maxwell, 1994).

We can perform the RM ANOVA and then calculate a Box's epsilon value to correct the biased F test, making the test for sphericity no longer necessary. The Box's epsilon estimate is multiplied by the degrees of freedom yielding new smaller degrees of freedom values and adjusting the F test (Crowder & Hand, 1990). In Model 1, the usual degrees of freedom associated with ANOVA would be $J - 1$ and $(J - 1)(I - 1)$. Our new degrees of freedom then are: $\nu_1 = \epsilon\,(J - 1)$ and $\nu_2 = \epsilon\,(J - 1)(I - 1)$. We would use the *F* value found for the ANOVA but then compare this to an *F*-critical value using the corrected degrees of freedom (Misangyi et al., 2006). In all cases, ϵ should never be greater than one, and if the sphericity assumption is met, then $\epsilon = 1$. Most estimators of ϵ are biased, because they can produce values greater than one and must be restricted to a domain of [0, 1] (Huynh & Feldt, 1976).

Greenhouse and Geisser (1958) suggest a set of conditions for the viability of the *F* test instead of checking the sphericity assumption or performing a Box's epsilon correction (Crowder & Hand, 1990). They argue that the *p*-value for the ANOVA being tested will only get larger with all of these adjustments. If the *p*-value is already large, we can retain the null hypothesis with confidence, even though the assumptions of sphericity may be violated. However, if the *p*-value is small, we can check the limit of the critical value $F(1, I * J - 1)$. If the *p*-value is still small, we can reject the null hypothesis with confidence. Now, the only cases remaining are when:

$$F_{\alpha,\,1,\,I*J-1} < F_{\text{observed}} < F_{\alpha,\,J-1,\,(J-1)(I-1)}. \tag{8}$$

In this case, we must test for sphericity and use a Box's epsilon correction or abandon the ANOVA test altogether (Huynh & Feldt, 1976).

Returning to the problem with sphericity tests, a Box's epsilon adjustment can be used to assess the deviation from the sphericity assumption. If a Box's epsilon estimate is approximately equal to one, we can conclude that the sphericity assumption is not violated (Baguley, 2004). Alternately, if a Box's epsilon estimate is less than one, sphericity is violated. However, we will only adjust the degrees of freedom with a Box's epsilon estimate if sphericity is seriously violated; in this case a Box's epsilon estimate is less than 0.75.

Several people have proposed different Box's epsilon estimates. Importantly, of the models examined in this thesis the assumption about sphericity and these estimated corrections only apply to Model 1 because the design is balanced and the contrasts are orthonormal (Crowder & Hand, 1990). One of the most commonly used Box's epsilon estimators was constructed by Greenhouse and Geisser:

$$\hat{\varepsilon} = \frac{\left(tr\left(\hat{\boldsymbol{\Sigma}}_{\mathrm{T}}\right)\right)^2}{(J-1)\; tr\left(\hat{\boldsymbol{\Sigma}}_{\mathrm{T}}^2\right)} = \frac{\left(\sum_{t}^{J-1} \lambda_t\right)^2}{(J-1)\sum_{t}^{J-1} \lambda_t^2} \tag{9}$$

where $\hat{\boldsymbol{\Sigma}}_{\mathrm{T}}$ is the estimated covariance matrix of the J - 1 orthonormal contrasts, and λ_t are the J - 1 eigenvalues of $\hat{\boldsymbol{\Sigma}}_{\mathrm{T}}$ (Greenhouse & Geisser, 1958). Greenhouse and Geisser's epsilon tends to be too conservative but adjusts well for the Type I error rate

(Quintana & Maxwell, 1995). Huynh and Feldt proposed a correction, denoted (HF or $\tilde{\varepsilon}$), to the Greenhouse and Geisser Box's epsilon as follows (Huynh & Feldt, 1976):

$$\tilde{\varepsilon} = \min(1\,,\, I(J-1)\hat{\varepsilon} - 2/(J-1)[I-1-(J-1)\hat{\varepsilon}]. \tag{10}$$

Some statistical software has included these two Box's epsilon estimators because they are so commonly used (O'Brien & Kaiser, 1985).

Quintana and Maxwell (1995) performed simulations and comparisons of eight possible Box's epsilon corrections. To the two Box's epsilon estimators already mentioned, they also advocated a correction by Lecoutre, denoted $\tilde{\varepsilon}^*$, when two or more groups are part of the experiment (Quintana, 1995). After much study, Quintana and Maxwell (1995) concluded that the most precise correction factor, however, uses either $\tilde{\varepsilon}$ or $\tilde{\varepsilon}^*$ depending on the value of $\tilde{\varepsilon}^*$; and no correction factor works well on data sets with small sample sizes.

Much of the time with RM ANOVA a correction on the degrees of freedom is necessary. Most people find the more common $\hat{\varepsilon}$ or $\tilde{\varepsilon}$ correction and use them to adjust the degrees of freedom. A less powerful MANOVA analysis might be used, which would not need the sphericity assumption because the only assumption made is that the data arises from a multivariate Normal distribution where the variances and covariances are unstructured (Misangyi et al., 2006). However, the multivariate approach will not work when more treatment levels than subjects exist in the experiment because the covariance matrix will be singular (Crowder & Hand, 1990). Looney and Stanley suggest running two analyses, one univariate and one multivariate, by halving the tolerance level on each test (1989).

In our simulations, we use $\hat{\varepsilon}$ as our correction factor because it is most commonly employed and fairly accurate. We also track the number of times the Box's epsilon correction is below 0.75 because researchers suggest that this is the threshold for determining whether sphericity was violated or not (Misangyi et al., 2006).

Regression Methods

Quite a few methods are referred to as repeated measures regression and no protocol exists to address the analysis of repeated measures experiments. Generalized least squares regression is a satisfactory method to analyze correlated data structures such as these.

Repeated Measures Regression

In the paper by Misangyi et al., (2006), repeated measures regression (RMR) is defined differently than in other texts. Both the observational units and treatment levels can be coded as dummy variables and are orthogonal in this special case, meaning **X'X = I** (Messer, 1993). In this particular case, Misangyi et al. (2006) perform a regression of the dependent variable using the treatment level as a dummy variable; we will denote this model with the subscript *TMT* since it is run on the treatment levels. Then they perform a regression of the dependent variable using the subjects as dummy variables; we will denote this model with a subscript *SUB*. Once these two regressions are performed the total sum of squares (*SST*), sum of squares of error (*SSE*), and the sum of squares of the regression (*SSR*) can be found. An R^2 value can be computed for both regression models as follows: $R^2 = (SST - SSE)/SST$ (Kleinbaum, Kupper, Muller, &

Nizam, 1998). They then have R^2_{TMT} and R^2_{SUB}, which they use to construct an *F* test (Misangyi et al., 2006):

$$F = \frac{(I-1)(J-1)R^2_{TMT}}{(J-1)(1-R^2_{SUB}-R^2_{TMT})}. \tag{11}$$

This *F* test appeared in Misangyi et al. (2006) with no further explanation and seemed quite unlike any other *F* test used to analyze repeated measures data.

Finally, it was found that this *F* test is a type of partial *F* test that only works for this specific model. We will now show how to obtain a more generalized form of this *F* test that will work for almost any model. First, we need to create a full regression model, which we give the subscript F, using the standard independent variables for the treatment as well as adding a dummy variable for the observational units (Misangyi et al., 2006). We then obtain a reduced model, which we will give the subscript R, with only the observational units as dummy coded variables. Note that the *SUB* model described previously is the same as our reduced model, so $R^2_R = R^2_{SUB}$.

For any models using the same data, the *SST*'s will be equal, so $SST_F = SST_{SUB} = SST_{TMT}$. In Model 1, treatment and subjects are independent (orthogonal) so that $SSR_F = SSR_{SUB} + SSR_{TMT}$ (Cohen & Cohen, 1975). In this first model, where every experimental unit is tested at every treatment level, we can show that:

$$R^2_F = \frac{SSR_F}{SST_F} \tag{12}$$

$$= \frac{SSR_{SUB} + SSR_{TMT}}{SST_F} \tag{13}$$

$$= \frac{SSR_{SUB}}{SST_{SUB}} + \frac{SSR_{TMT}}{SST_{TMT}} \tag{14}$$

$$= R^2_{SUB} + R^2_{TMT}\,. \tag{15}$$

The F test proposed by Misangyi et al. (2006) can now be compared to the more generalized form of the partial F test:

$$F = \frac{(I-1)(J-1)R^2_{TMT}}{(J-1)(1-R^2_{SUB}-R^2_{TMT})} \tag{16}$$

$$= \frac{(I-1)(J-1)\left(R^2_{SUB}+R^2_{TMT}-R^2_{SUB}\right)}{(J-1)(1-R^2_{SUB}-R^2_{TMT})} \tag{17}$$

$$= \frac{(I-1)(J-1)\left(R^2_F-R^2_R\right)}{(J-1)(1-R^2_F)} \tag{18}$$

$$= \frac{\left(R^2_F-R^2_R\right)/(J-1)}{(1-R^2_F)/[(I-1)(J-1)]} \tag{19}$$

We now have the more general partial F test, which can be used in other settings, where J - 1 is the reduction in parameters between the full and reduced model, and (I - 1)(J - 1) is the degrees of freedom associated with the SSE of the full model (Kleinbaum et al., 1998). For the full model we have the form: $y_{ij} = \alpha + \beta_i + \eta_j + \varepsilon_{ij}$, so this model accounts for an overall mean (α) and then a subject effect (β_i) and a treatment effect (η_j). And then we compare it to the model where we have $y_{ij} = \alpha^* + \beta_i^* + \varepsilon_{ij}^*$ which only accounts for the subject effect, β_i. The partial F test isolates the contribution of the treatment, and therefore tests if the treatment is significantly impacting the regression beyond what can be attributed to subject variability.

Misangyi et al. (2006) gave a modified partial *F* test that fits Model 1 only. In models two and three, we cannot use the *F* test proposed by Misangyi et al., (2006) because the independent variables may not be orthogonal. For this reason, in later models we must revert to the standard partial *F* test.

It seems that Misangyi et al. (2006) may have used this form of the partial F test so they could compare what they call RMR to RM ANOVA to show that these two types of analysis in this one situation are identical.

However, RMR poses a difficulty because it does not test for or address violations of sphericity. Unlike the RM ANOVA, this analysis does not have corrections for a violation of sphericity, even though it yields the same results and may violate the sphericity condition (Misangyi et al., 2006). RM ANOVA is a better choice because it attends to any violations to the assumption of sphericity. Thus, RMR should only be used when sphericity is not violated (Misangyi et al., 2006). We will add a partial *F* test (in this Model analogous to RMR) to our simulations, not because it is necessarily a valid method of analysis but for comparison purposes.

Ordinary Least Squares Regression

No studies of repeated measures advocate this type of analysis, however, some people seem to use it. In Ordinary Least Squares (OLS) regression we assume constant variance and independence of the errors. In this case, the covariance matrix of the dependent variable given the independent variables would be $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} = \sigma^2\mathbf{I}$. This structure, shown in matrix form, can be compared later to other such $\boldsymbol{\Sigma}$ matrices:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \begin{matrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{matrix} & & & \mathbf{0} \\ & \begin{matrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{matrix} & & \\ & & \ddots & \\ \mathbf{0} & & & \begin{matrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{matrix} \end{bmatrix} \tag{20}$$

OLS is useful when the assumptions about Normality, constant variance, independence and identical distribution of the errors hold. Therefore: $\boldsymbol{\varepsilon} \sim N(0, \sigma^2\mathbf{I})$ and $\mathbf{Y} \sim N(\mathbf{X\beta}, \mathbf{I}\sigma^2_{y|x})$, where $\sigma^2 = \sigma^2_{y|x}$. We will use the equation $\mathbf{Y} = \mathbf{X\beta} + \boldsymbol{\varepsilon}$ to perform the OLS analysis, which lead to: $\hat{\boldsymbol{\beta}} = (\mathbf{X'X})^{-1}\mathbf{X'Y}$ and $V(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X'X})^{-1}$ where $\hat{\boldsymbol{\beta}}$ is an unbiased estimator of $\boldsymbol{\beta}$, if the assumptions are met.

In OLS, we perform a statistical test for significance of the regression. For this, the null hypothesis is that the regression is not significant, or equivalently for our simulated models, $H_0 : \hat{\beta}_1 = 0$. We construct an *F* test to assess this null hypothesis. This *F* test for the regression will be composed of *MSR* / *MSE*. *MSE* is found by dividing *SSE* by its degrees of freedom and the *MSR* is found by dividing *SSR* by its degrees of freedom. Appendix B contains full proofs that the $SSE \sim \chi^2(IJ - k - 1)$, $SSR \sim \chi^2(k)$, and $SST \sim \chi^2(IJ - 1)$, so the *F* test will follow a Fisher's *F* distribution and *k* is the number of predictor variables. Table 2.2 displays the degrees of freedoms and also the calculation for the F test.

Table 2.2

ANOVA for Regular Regression

Source	*df*	*SS*	*MS*	*F*
Regression	$(k + 1) - 1$	*SSR*	*MSR*	*MSR / MSE*
Error	$IJ - k - 1$	*SSE*	*MSE*	
Total	$IJ - 1$	*SST*		

If our observations are correlated or the dependent variable has non-constant variance, OLS is not the proper tool for analysis. In a repeated measures experiment, it is regularly the case that the assumptions are invalid, so OLS is not the appropriate way to analyze the data presented here. In simulations, we include OLS, not because it is a valid analysis type, but because it is easily misused. The method is widely employed without evaluating the underlying assumptions.

Means Regression

Means regression is a modification of the first model's configuration, basically using OLS on summary statistics. We average the responses taken at each treatment level to get the mean for the treatment level. Then we can perform regular regression analysis on this new summary statistic dependent variable. Means regression is included because it deals with the data in a repeated measures experiment quickly and is perhaps one of the common types of analysis used, and it provides unbiased estimators (Lorch & Meyers, 1990). However, this eliminates subject variability from the analysis and thus more likely to reject the null hypothesis thereby inflating the Type I error rate (Lorch & Meyers, 1990).

In our simulations, we first perform standard means regression. Then we will add weights for the treatment to attempt to account for some of the variation. We will discuss these types of weights in the next section.

Generalized Least Squares

We can use a more generalized form of OLS regression, called Generalized Least Squares (GLS) regression that will assume the errors follow a Multivariate Normal distribution such that $\boldsymbol{\varepsilon} \sim MVN(\mathbf{0}, \boldsymbol{\Sigma})$ and $\mathbf{Y} \mid \mathbf{X} \sim MVN(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, ie. we no longer assume constant variance and zero correlations. GLS allows for the observations of the dependent variable to have any variance-covariance structure and does not assume the observations are independent. The diagonal of the $\boldsymbol{\Sigma}$ matrix will contain the variances and the off-diagonal elements will be the covariances of the error terms, as shown in equation (5), which can be rewritten as (Montgomery, Peck, & Vining, 2006):

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \rho_{n2}\sigma_n\sigma_2 & \cdots & \sigma_n^2 \end{bmatrix}. \tag{21}$$

In this set up, $\sigma_i^2 = V(\varepsilon_i) = \text{cov}(\varepsilon_i, \varepsilon_i)$ and $\text{cov}(\varepsilon_i, \varepsilon_j) = \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ where $\sigma_i = \sqrt{\sigma_i^2}$ and ρ_{ij} is the correlation between ε_i and ε_j.

The $\boldsymbol{\Sigma}$ matrix can be adjusted to account for any assumptions that are made. These assumptions may include constant variance or constant covariance of observational units being tested. It is also possible to have a completely unstructured $\boldsymbol{\Sigma}$ matrix. GLS analysis allows for more flexibility in the configuration of the experiment and is conducive to repeated measures designs because it can account for covariance between measurements. We will be testing the null hypothesis $H_0 : \hat{\beta}_1 = 0$ in all GLS analyses.

Weighted Least Squares

We will simulate several variations of GLS, one type being weighted regression (Montgomery et al., 2006). The assumption of constant variance is replaced by constant variance within each observational unit only:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \begin{matrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_1^2 \end{matrix} & & & \mathbf{0} \\ & \begin{matrix} \sigma_2^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_2^2 \end{matrix} & & \\ & & \ddots & \\ \mathbf{0} & & & \begin{matrix} \sigma_N^2 & 0 & \cdots & 0 \\ 0 & \sigma_N^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 \end{matrix} \end{bmatrix} \tag{22}$$

or, in this first model, each level can have its own constant variance:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \begin{matrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 \end{matrix} & & & \mathbf{0} \\ & \begin{matrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 \end{matrix} & & \\ & & \ddots & \\ \mathbf{0} & & & \begin{matrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 \end{matrix} \end{bmatrix} \tag{23}$$

GLS takes these variance inequalities into account by using weights. We will add simple estimated weights to see if accounting for individual or treatment variability is a good alternative analysis. We will perform two types of analysis: one with subject weights (Sub weights) as in equation (22) and one with treatment weights (Tr weights) as in equation (23). The first type of weights will be obtained by finding the sample variance of the subjects. The second type of weights will be found in a similar fashion but will be the sample variances of the treatment levels. The dependent variable will be weighted with the inverse of the sample variance; this means the objects with larger variance will have a smaller weight in the regression.

In this analysis the covariance of the measurements is not being accounted for so there is reason to doubt that the underlying assumptions will be valid in this regression method. We will simulate both types of weights in our analysis and compare this analysis with other types of analyses.

GLS Method I

Montgomery et al. (2006) presents the first type of GLS analysis that we will explore. $\mathbf{\Sigma}$ is a non-singular and semi-positive definite matrix, but we can assume for all models that it is a positive definite matrix (Kuan, 2001). By the definition of a covariance matrix and because we are assuming it to be positive definite then $\mathbf{\Sigma}$ is symmetric, $\mathbf{\Sigma}' = \mathbf{\Sigma}$, and invertible (see Appendix A). Therefore, $(\mathbf{\Sigma}')^{-1} = \mathbf{\Sigma}^{-1}$ or $(\mathbf{\Sigma}^{-1})' = \mathbf{\Sigma}^{-1}$. To find an estimator for the coefficients of the regression we begin with the equation $\mathbf{y} = \mathbf{X\beta}$ but must account for the non-zero covariance and possibly non-constant variance of the errors by multiplying both sides of the equation by $\mathbf{\Sigma}^{-1}$, the variance-covariance matrix. This modification results in the normal equation: $\mathbf{\Sigma}^{-1}\mathbf{X\beta} = \mathbf{\Sigma}^{-1}\mathbf{y}$ (Kuan, 2001). Manipulation of the equation yields an unbiased estimator for $\mathbf{\beta}$ (see Appendix C for proof of non-bias):

$$\hat{\mathbf{\beta}} = \left(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{y} \text{ (Montgomery et al., 2006).} \tag{24}$$

Before estimating $\mathbf{\beta}$, we must know the $\mathbf{\Sigma}$ matrix, but much of the time this is not possible. In simulations, however, we can perform one analysis where $\mathbf{\Sigma}$ is known. This is feasible in a simulation because we create the $\mathbf{\Sigma}$ matrix in advance to be able to generate our data. When constructing the $\mathbf{\Sigma}$ matrix, we assume that all experimental units are independent of one another and have the same variance and correlation and thus we specify two parameters, correlation ρ and variance σ^2 so that $\mathbf{\Sigma}$ is as shown in equation (25) (Crowder & Hand, 1990). This adheres to Model 1 where we are assuming that measurements are part of a longitudinal study where time is not necessarily a factor.

$$\Sigma = \begin{bmatrix} \begin{matrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & & \rho\sigma^2 \\ \vdots & & \ddots & \vdots \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{matrix} & & & \mathbf{0} \\ & \begin{matrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & & \rho\sigma^2 \\ \vdots & & \ddots & \vdots \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{matrix} & & \\ & & \ddots & \\ \mathbf{0} & & & \begin{matrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & & \rho\sigma^2 \\ \vdots & & \ddots & \vdots \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{matrix} \end{bmatrix} \quad (25)$$

We also perform a similar analysis where we estimate the **Σ** matrix. Many techniques exist to estimate **Σ**. In this case we assume that we know the structure of **Σ**. If the **Σ** matrix is entirely unstructured this leaves $[(I * J)^2 + I * J] / 2$ parameters to estimate and even with the block structure there are still $I * J * J$ parameters to estimate, which is more than the amount of data, $I * J$, being collected. So, it becomes necessary for **Σ** to have a structure and we must also infer what structure it has before attempting estimation techniques (Crowder & Hand, 1990).

In 2001, Koreisha & Fang conducted a study where they knew the true **Σ** matrix and estimated **Σ** assuming the correct structure as well as various other incorrect structures. The researchers found that incorrect parameterization provided estimates which were not as accurate as the correct one (Koreisha & Fang, 2001). For our simulations, we will assume the correct structure is known.

We must estimate **β**, ρ, and σ^2 by utilizing an iterative process to find the maximum likelihood estimates for these parameters. First, we supply general values for **β**, ρ, and σ^2, and we run the algorithm until we maximize the likelihood function for the errors (Crowder & Hand, 1990). The observations, given the independent variables, have a likelihood function of:

$$L = (2\pi)^{-IJ/2} |\boldsymbol{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right) \tag{26}$$

After several iterations, this function will be numerically maximized and we will have estimates for $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$, which is a function of the estimates of ρ and σ^2.

With the $\boldsymbol{\Sigma}$ matrix estimated, we can test the null hypothesis about this regression. In Montgomery et al., (2006) the researchers present an alternative calculation method for *SSE, SST*, and *SSR* to the one already presented in the OLS section. They prefer a method where $\bar{\mathbf{y}}$ is not subtracted from *SST* and *SSR;* the degrees of freedom are later adjusted for this fact. The Montgomery et al. $SSE \sim \chi^2(IJ-k-1)$, $SST \sim \chi^2(IJ)$, and $SSR \sim \chi^2(k+1)$ along with full proofs are given in Appendix C. Table 2.3 contains a summary of the calculations for the *F* test.

Table 2.3

ANOVA for GLS Method I

Source	*df*	*SS*	*MS*	*F*
Regression	$k+1$	$\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\mathbf{y}$	*MSR*	*MSR / MSE*
Error	$IJ-k-1$	$\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}-\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\mathbf{y}$	*MSE*	
Total	IJ	$\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}$		

Note. $\mathbf{A} = \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}$ (see Appendix C for more details) (Montgomery et al., 2006)

GLS Method II

The following GLS analysis was prompted by Montgomery et al., (2006) and Kuan (2002). $\mathbf{\Sigma}$, the covariance matrix of the errors, has the properties of symmetry, non-singularity, and positive definiteness because it is a covariance matrix (see Appendix A for full definitions) (Montgomery et al., 2006). Therefore, it can be orthogonally diagonalized (Kuan, 2002). One can decompose $\mathbf{\Sigma}$ into **KK**, where **K** is sometimes called the square root of $\mathbf{\Sigma}$. To find this **K** matrix, we first find the eigenvalues and eigenvectors of $\mathbf{\Sigma}$. The square root of the eigenvalues become the diagonal elements of a new matrix $\mathbf{D}_{\mathbf{nxn}}$, whose non-diagonal elements are zero, and the eigenvectors become the columns of the matrix **P**, such that $\mathbf{\Sigma} = \mathbf{PDP}^{-1}$ (Kuan, 2002). It can be shown that $\mathbf{\Sigma}$ is decomposable into two equal components (Kuan, 2002):

$$\mathbf{\Sigma} = \mathbf{PDP}^{-1} \tag{27}$$

$$= \mathbf{PD}^{1/2}\mathbf{D}^{1/2}\mathbf{P}^{-1} \tag{28}$$

$$= \mathbf{PD}^{1/2}\mathbf{P}^{-1}\mathbf{PD}^{1/2}\mathbf{P}^{-1} \tag{29}$$

$$= \mathbf{KK} \tag{30}$$

where $\mathbf{K} = \mathbf{PD}^{1/2}\mathbf{P}^{-1}$ and $\mathbf{D}^{1/2}$ is a diagonal matrix whose elements are the square roots of the corresponding elements of **D** and $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, since the columns of **P** are orthonormal (see Appendix A). Furthermore, $\mathbf{\Sigma} = \mathbf{KK} = \mathbf{K'K}$ and $\mathbf{\Sigma}^{-1} = \mathbf{K}^{-1}\mathbf{K}^{-1} = (\mathbf{KK})^{-1}$ (Kuan, 2001). We proceed by transforming the variables, we let $\mathbf{y}_{\mathbf{K}} = \mathbf{K}^{-1}\mathbf{y}$, $\mathbf{X}_{\mathbf{K}} = \mathbf{K}^{-1}\mathbf{X}$ and $\boldsymbol{\varepsilon}_{\mathbf{K}} = \mathbf{K}^{-1}\boldsymbol{\varepsilon}$. Where we once had the equation $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, we now have $\mathbf{y}_{\mathbf{K}} = \mathbf{X}_{\mathbf{K}}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_{\mathbf{K}}$, as $\boldsymbol{\varepsilon}_{\mathbf{K}}$ now satisfies the conditions of having independent elements with constant variance

(Montgomery et al., 2006). It is possible to run OLS regression on these new variables, $\mathbf{y}_\mathbf{K}$ and $\mathbf{X}_\mathbf{K}$ (Montgomery et al., 2006).

We must have a $\boldsymbol{\Sigma}$ matrix for this method to work, which, in most cases, is unknown. As in GLS Method I, we assume that we know $\boldsymbol{\Sigma}$ and perform an analysis in this manner. We also estimate $\boldsymbol{\Sigma}$, utilizing the same structure as before, and perform this analysis.

GLS Method III

The generalized least squares method Crowder and Hand use in their text is configured in a dissimilar manner (1990). Each observational unit has its own set of matrices: $\boldsymbol{\theta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\xi}$ matrix. The simplified matrices partitioned in (31-34) reveal that these two styles of GLS produce the same results. Using GLS Methods I and II with a model of $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, we can reconfigure matrices $\mathbf{X}$, $\mathbf{y}$, and $\boldsymbol{\Sigma}$ to have a matrix set for each observational unit: $\boldsymbol{\theta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\xi}$ with a model $\boldsymbol{\gamma} = \boldsymbol{\theta\pi} + \boldsymbol{\delta}$.

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} \\ 1 & x_{12} \\ \vdots & \vdots \\ 1 & x_{1J} \\ \hline 1 & x_{21} \\ \vdots & \vdots \\ \hline \vdots & \vdots \\ \hline 1 & x_{IJ} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta}_1 \\ \hline \boldsymbol{\theta}_2 \\ \hline \vdots \\ \hline \boldsymbol{\theta}_\mathbf{I} \end{bmatrix}, \qquad \mathbf{X}^\mathrm{T} = \left[\boldsymbol{\theta}_1^\mathrm{T} \mid \boldsymbol{\theta}_2^\mathrm{T} \mid \ldots \mid \boldsymbol{\theta}_\mathbf{I}^\mathrm{T}\right], \qquad \mathbf{Y} = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1J} \\ \hline y_{21} \\ \vdots \\ \hline \vdots \\ \hline y_{IJ} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \hline \boldsymbol{\gamma}_2 \\ \hline \vdots \\ \hline \boldsymbol{\gamma}_\mathbf{I} \end{bmatrix} \qquad (31\text{-}33)$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \begin{matrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1J} \\ \sigma_{21} & \sigma_2^2 & & \sigma_{2J} \\ \vdots & & \ddots & \vdots \\ \sigma_{J1} & \sigma_{J2} & \cdots & \sigma_J^2 \end{matrix} & & & \mathbf{0} \\ & \begin{matrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1J} \\ \sigma_{21} & \sigma_2^2 & & \sigma_{2J} \\ \vdots & & \ddots & \vdots \\ \sigma_{J1} & \sigma_{J2} & \cdots & \sigma_J^2 \end{matrix} & & \\ & & \ddots & \\ \mathbf{0} & & & \begin{matrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1J} \\ \sigma_{21} & \sigma_2^2 & & \sigma_{2J} \\ \vdots & & \ddots & \vdots \\ \sigma_{J1} & \sigma_{J2} & \cdots & \sigma_J^2 \end{matrix} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \boldsymbol{\xi}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\xi}_2 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\xi}_I \end{array}\right] \quad (34)$$

The inverse of a partitioned block diagonal matrix $\boldsymbol{\Sigma}$ is invertible so each block $\boldsymbol{\xi}_i$ of the partitioned matrix is also invertible, which leads to the following (Lay, 1996):

$$\boldsymbol{\Sigma}^{-1} = \left[\begin{array}{c|c|c|c} \boldsymbol{\xi}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\xi}_2 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\xi}_I \end{array}\right]^{-1} = \left[\begin{array}{c|c|c|c} \boldsymbol{\xi}_1^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\xi}_2^{-1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\xi}_I^{-1} \end{array}\right]. \quad (35)$$

The configuration presented here by Crowder and Hand (1990) at first appears quite different from GLS Methods I and II presented by Montgomery et al. (2006) and Kuan (2002); GLS Methods I and II utilize a single set of matrices for all observational units, whereas GLS Methods III assigns a distinct set of matrices for each observational unit. However, the use of either general design type produces identical algebraic functions and generates the same parameter estimates $\hat{\boldsymbol{\beta}}$, assuming $\xi_i = \xi_j$. Each experimental unit is provided with its own matrix set of $\boldsymbol{\theta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\xi}$ and are later summated to yield identical parameter estimates as in Method I and II. The following will show the progression from Method I and II to Method III:

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1}\left(\mathbf{X}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right) \tag{36}$$

$$= \left\{\begin{bmatrix}\boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \vdots \\ \boldsymbol{\theta}_I\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\xi_1^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \xi_2^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi_I^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \vdots \\ \boldsymbol{\theta}_I\end{bmatrix}\right\}^{-1}\left\{\begin{bmatrix}\boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \vdots \\ \boldsymbol{\theta}_I\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\xi_1^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \xi_2^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \xi_I^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \\ \vdots \\ \boldsymbol{\gamma}_I\end{bmatrix}\right\} \tag{37}$$

$$= \left\{\left[\boldsymbol{\theta}_1^{\mathrm{T}}\xi_1^{-1} \mid \boldsymbol{\theta}_2^{\mathrm{T}}\xi_2^{-1} \mid \ldots \mid \boldsymbol{\theta}_I^{\mathrm{T}}\xi_I^{-1}\right]\begin{bmatrix}\boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \vdots \\ \boldsymbol{\theta}_I\end{bmatrix}\right\}^{-1}\left\{\left[\boldsymbol{\theta}_1^{\mathrm{T}}\xi_1^{-1} \mid \boldsymbol{\theta}_2^{\mathrm{T}}\xi_2^{-1} \mid \ldots \mid \boldsymbol{\theta}_I^{\mathrm{T}}\xi_I^{-1}\right]\begin{bmatrix}\boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \\ \vdots \\ \boldsymbol{\gamma}_I\end{bmatrix}\right\} \tag{38}$$

$$= \left\{\boldsymbol{\theta}_1^{\mathrm{T}}\xi_1^{-1}\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2^{\mathrm{T}}\xi_2^{-1}\boldsymbol{\theta}_2 + \ldots + \boldsymbol{\theta}_I^{\mathrm{T}}\xi_I^{-1}\boldsymbol{\theta}_I\right\}^{-1}\left\{\boldsymbol{\theta}_1^{\mathrm{T}}\xi_1^{-1}\boldsymbol{\gamma}_1 + \boldsymbol{\theta}_2^{\mathrm{T}}\xi_2^{-1}\boldsymbol{\gamma}_2 + \ldots + \boldsymbol{\theta}_I^{\mathrm{T}}\xi_I^{-1}\boldsymbol{\gamma}_I\right\} \tag{39}$$

$$= \left\{\sum_{i=1}^{I}\boldsymbol{\theta}_i^{\mathrm{T}}\xi_i^{-1}\boldsymbol{\theta}_i\right\}^{-1}\left\{\sum_{i=1}^{I}\boldsymbol{\theta}_i^{\mathrm{T}}\xi_i^{-1}\boldsymbol{\gamma}_i\right\}. \tag{40}$$

As shown, the estimators produced in GLS method III are identical to the one generated from GLS Methods I and II.

In most cases, the researchers assume that the $\boldsymbol{\theta}$ matrix is equal for all observational units as is the ξ matrix. The $\boldsymbol{\gamma}_i$ vector is derived from a J-dimensional Normal distribution, denoted: $\boldsymbol{\gamma}_i \sim N_J(\boldsymbol{\mu},\xi)$. The vector $\bar{\boldsymbol{\gamma}}$ can be calculated as: $\bar{\boldsymbol{\gamma}} = (\boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2 + \ldots + \boldsymbol{\gamma}_i)/I$, and it too follows a Multivariate Normal distribution, $\bar{\boldsymbol{\gamma}} \sim N_J(\boldsymbol{\mu},\xi/I)$ (Crowder & Hand, 1990). A theoretical or known ξ follows a $\chi^2(J-2)$ distribution (Rao, 1959). The ξ matrix is assumed to be unstructured in these cases, however we assume each observational unit has the same ξ matrix. When all observational units share the ξ matrix, only $J * J$ entries must be estimated. If all observational units had their own ξ matrix, $J * J * I$ parameters would have to be estimated; this is more than the number of observations, however, and an inference of these parameters cannot be made (Crowder & Hand, 1990). A structure can be imposed upon ξ, so even fewer parameters estimates are needed. Rao (1959) suggests that ξ first be roughly estimated by the formula $\mathbf{S} = \frac{1}{I-1}\sum_{i=1}^{I}(\boldsymbol{\gamma}_i - \bar{\boldsymbol{\gamma}})(\boldsymbol{\gamma}_i - \bar{\boldsymbol{\gamma}})'$. Thus, **S** follows a Wishart distribution with I - 1 degrees of freedom, denoted $\mathbf{S} \sim W_J(I-1, \xi)$. If an estimate of **S** is desired, solve for $\hat{\boldsymbol{\pi}}$ and **S** iteratively until they converge using the following formulae (Crowder & Hand, 1990):

$$\mathbf{S} = \frac{1}{n}\sum_{i=1}^{I}(\hat{\gamma}_i - \boldsymbol{\theta}\hat{\boldsymbol{\pi}})(\hat{\gamma}_i - \boldsymbol{\theta}\hat{\boldsymbol{\pi}})' \tag{41}$$

$$\hat{\boldsymbol{\pi}} = (\boldsymbol{\theta}\mathbf{S}^{-1}\boldsymbol{\theta})^{-1}\boldsymbol{\theta}'\mathbf{S}^{-1}\bar{\boldsymbol{\gamma}}. \tag{42}$$

As before, we perform the GLS analysis assuming knowledge of ξ first, and then execute GLS analysis using an estimated ξ, denoted **S**. We impose a new structure, which forces all ξ to be identical. We find a structured **S** by using the iterative method given in equation (41-42) until $\hat{\boldsymbol{\pi}}$ converges.

Also, this GLS method presented in Crowder and Hand (1990) requires J + 1 to be less than I because, if this is not the case, the design matrix (**X**) will be I by J + 1 and **X'X** will be singular (see full proof in Appendix F). This would make several calculations that must be performed impossible, so we will restrict $J + 1 < I$ in all of our simulations.

Although the parameter estimates are identical in both methods, we now use a different test statistic to determine if the regression is significant. In most cases, an F test is performed to evaluate the null hypothesis. Instead, Crowder & Hand (1990) use an approximate χ_1^2 test statistic, denoted T. Due to the nature of Model 1 and the selected null hypothesis, the test statistic used in our simulations will be (Crowder & Hand, 1990):

$$T = \hat{\pi}_1 \left[Var(\hat{\pi}_1) \right]^{-1} \hat{\pi}_1 . \tag{43}$$

Method III produces different results than Method I and II for two reasons. First, it uses a χ^2 test instead of an F test. Secondly, when estimating the covariance-variance matrix a different structure is imposed. However, if the same structure was imposed then we know these methods would produce identical parameter estimates.

Calculating Type I Errors in Simulations

To estimate Type I error probabilities, a count is kept of how many times the hypothesis was rejected in our simulations with α tolerance level set at 0.05. Type I errors are only calculated in runs where $\beta_1 = 0$, meaning the null hypothesis was actually true (Wackerly, Mendenhall, & Scheaffer, 2002).

Calculating Type II Errors in Simulations

For estimates of Type II error probabilities, a count is kept of how many times the hypothesis was retained in our simulations with α set at 0.05. Type II errors are only calculated in runs where $\beta_1 \neq 0$, meaning the null hypothesis was false (Wackerly et al., 2002).

Verifying the Distribution and Estimating Degrees of Freedom

The $SSE \sim \chi^2_\upsilon$ and $SSR \sim \chi^2_\eta$ where υ and η are usually known, except in the cases where the variance matrix is estimated. We use the Kolmogorov-Smirnov test to verify that the simulated *SSE* and *SSR* follow a χ^2 distribution with the specified degrees of freedom. For this test, in the cases where we estimate the variance matrix, the correct degrees of freedom are assumed to be identical to the cases where the variance matrix is known.

Since the degrees of freedom are not always known, we find an estimate for the degrees of freedom using the generated data for the *SSE* and *SSR* (assuming they are χ^2). This is accomplished by finding the degree of freedom, which maximizes the χ^2 likelihood function.

In analyses where an *F* test is performed, we check the distribution using the Kolmogorov-Smirnov test with the theoretical degrees of freedom. For this test, when we estimate the variance matrix, the degrees of freedom are assumed to be identical to the cases where the variance matrix is known.

We can also find an estimate for the degrees of freedom using the maximum likelihood estimation for the *F* distribution. We use the simulated F values to find the degrees of freedom that maximize the likelihood equation. GLS Method III is a special case where an *F* test is not performed, but rather a *T* test following a χ^2 distribution, so we use the maximum likelihood method. Note that there is only one parameter or degree of freedom for the χ^2 distribution.

We can also find an estimate for the degrees of freedom of the *F* test by performing a method of moments calculation for the degrees of freedom, υ_1 and υ_2. The first moment and second central moment of an *F* distribution are (Wackerly et al., 2002):

$$E(F) = \frac{\upsilon_2}{\upsilon_2 - 2} \tag{44}$$

$$V(F) = \frac{2\upsilon_2^2(\upsilon_1 + \upsilon_2 - 2)}{\upsilon_1(\upsilon_2 - 2)^2(\upsilon_2 - 4)}. \tag{45}$$

Because the data is simulated, we can obtain a sample mean, $\overline{F}$, and a sample variance, s_F^2, from our simulated data and estimate the parameters as follows:

$$\overline{F} = \frac{\hat{\upsilon}_2}{\hat{\upsilon}_2 - 2} \tag{46}$$

$$s_F^2 = \frac{2\hat{\upsilon}_2^2(\hat{\upsilon}_1 + \hat{\upsilon}_2 - 2)}{\hat{\upsilon}_1(\hat{\upsilon}_2 - 2)^2(\hat{\upsilon}_2 - 4)}. \tag{47}$$

We now have two equations and two unknown parameters, so it is possible to solve for $\hat{\upsilon}_2$ and $\hat{\upsilon}_1$ (see Appendix D for full derivation):

$$\hat{\upsilon}_2 = \frac{2\overline{F}}{\overline{F} - 1} \tag{48}$$

$$\hat{\upsilon}_1 = \frac{2\overline{F}^2}{-\overline{F}^3 + \overline{F}^2 - \overline{F}s_F + 2s_F} \tag{49}$$

The resulting values are estimates for the degrees of freedom for our *F* test.

For GLS Method III, an F test is not performed, so we must set the first moment of a χ^2 distribution equal to the sample mean $\overline{\chi^2}$ rendering:

$$\hat{E}(\chi^2) = \hat{\upsilon} = \overline{\chi^2} \tag{50}$$

and thus $\hat{\upsilon} = \overline{\chi^2}$ (Wackerly et al., 2002).

Results

All simulations contain four subjects ($I = 4$), and each observational unit includes three repeated measures ($J = 3$) for this model. We hold the variance constant at two for all repeated measurements. The correlation ρ of the repeated measures for a person will vary and be either: 0, 0.2, 0.8, or 0.99, where 0.99 indicates that the data is highly correlated. When the correlation is set to zero all analysis is valid and no assumptions are violated. We can thus use these runs as our baseline or control. In the experiments, β_1, the true slope of the regression line will also be varied. When β_1 is set at zero we know the null hypothesis is true. A discussion section will follow after all of the simulation results are recorded.

In the following tables, we will simulate 10,000 different sets of repeated measures data for each combination of β_1 and ρ, then summarize results from fourteen types of analysis. Quite a few types of analysis are just variations of one another. We have the RM ANOVA and also the RM ANOVA with the Box's epsilon correction on its degrees of freedom. RMR in this model is identical to RM ANOVA and generalized partial *F* test analysis. WLS uses weights to account for some of the variation in the model. Two types of weights are used, one for the subject's variation and one for the treatment's variation. Means regression pools all of the observations in one treatment; we will also perform means regression with weights. GLS I, II, and III are executed as discussed in the GLS sections using the known variance-covariance matrix; we know the exact values of this matrix because we use it to generate our data. The analysis denoted: GLS I est, GLS II est, and GLS III est, use an estimate of the variance-covariance matrix, as is more likely to be the case in real data. The method of finding this estimated variance-covariance matrix is discussed in each GLS section.

We wanted to verify our analysis methods via other methods. First, we wanted to compare histograms of the F values for each of the fourteen methods of analysis to their theoretical F distribution. In some cases, there was not an F test but rather a χ^2 test. So in those cases we will compare the simulated χ^2 data to the theoretical values. Only a sample of such comparisons is added here (Figures 2.2 and 2.3). This analysis was only performed in cases where the true β_1 is set at zero, such that the null hypothesis is true, so that the true distributions should be central F or χ^2 distributions. There are full proofs in the Appendix B and C of the theory for the experimental results following χ^2 or F distributions.

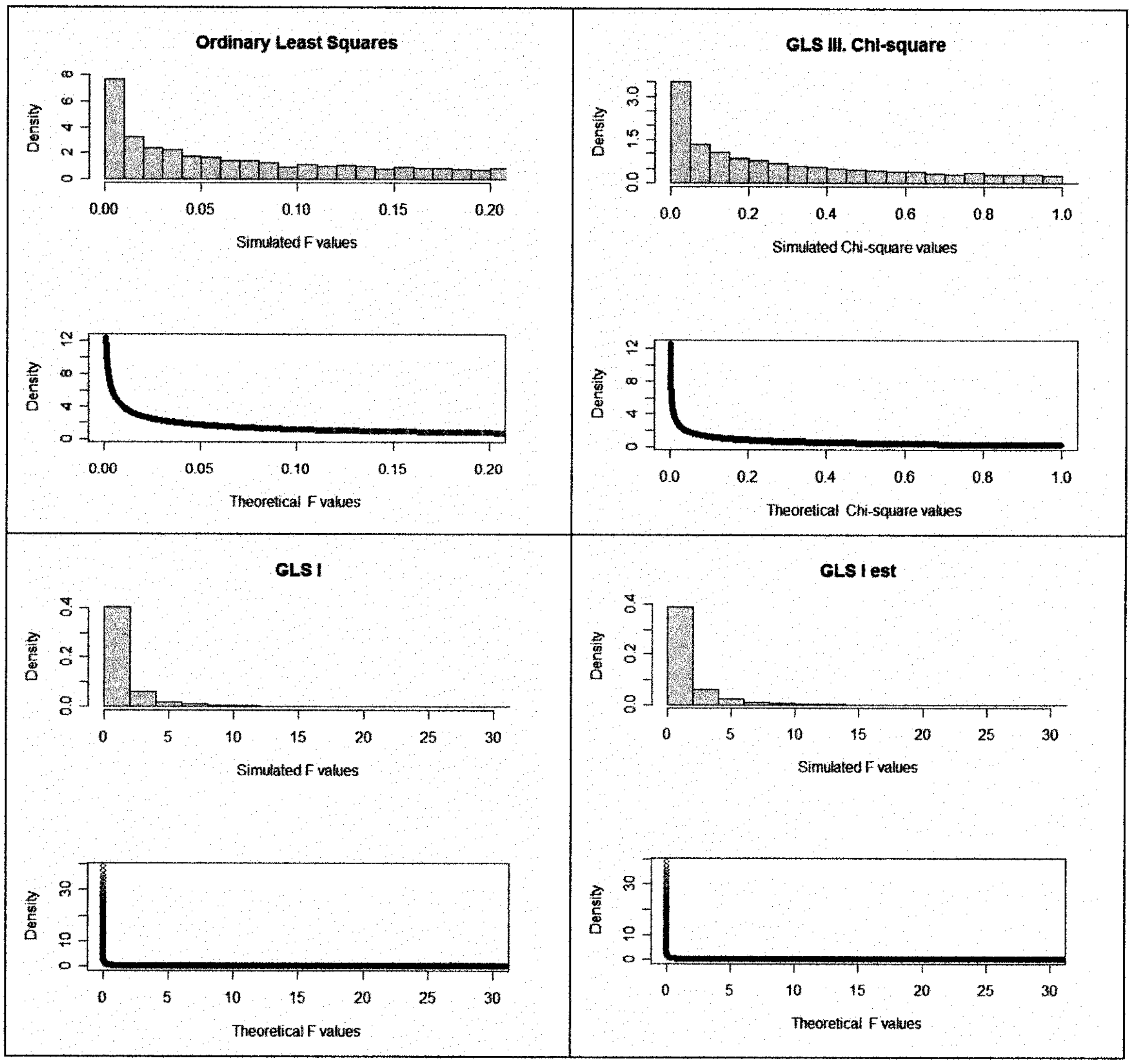

Figure 2.2. Simulated F or χ^2 values when $\beta_1=0$, $\rho = 0$.

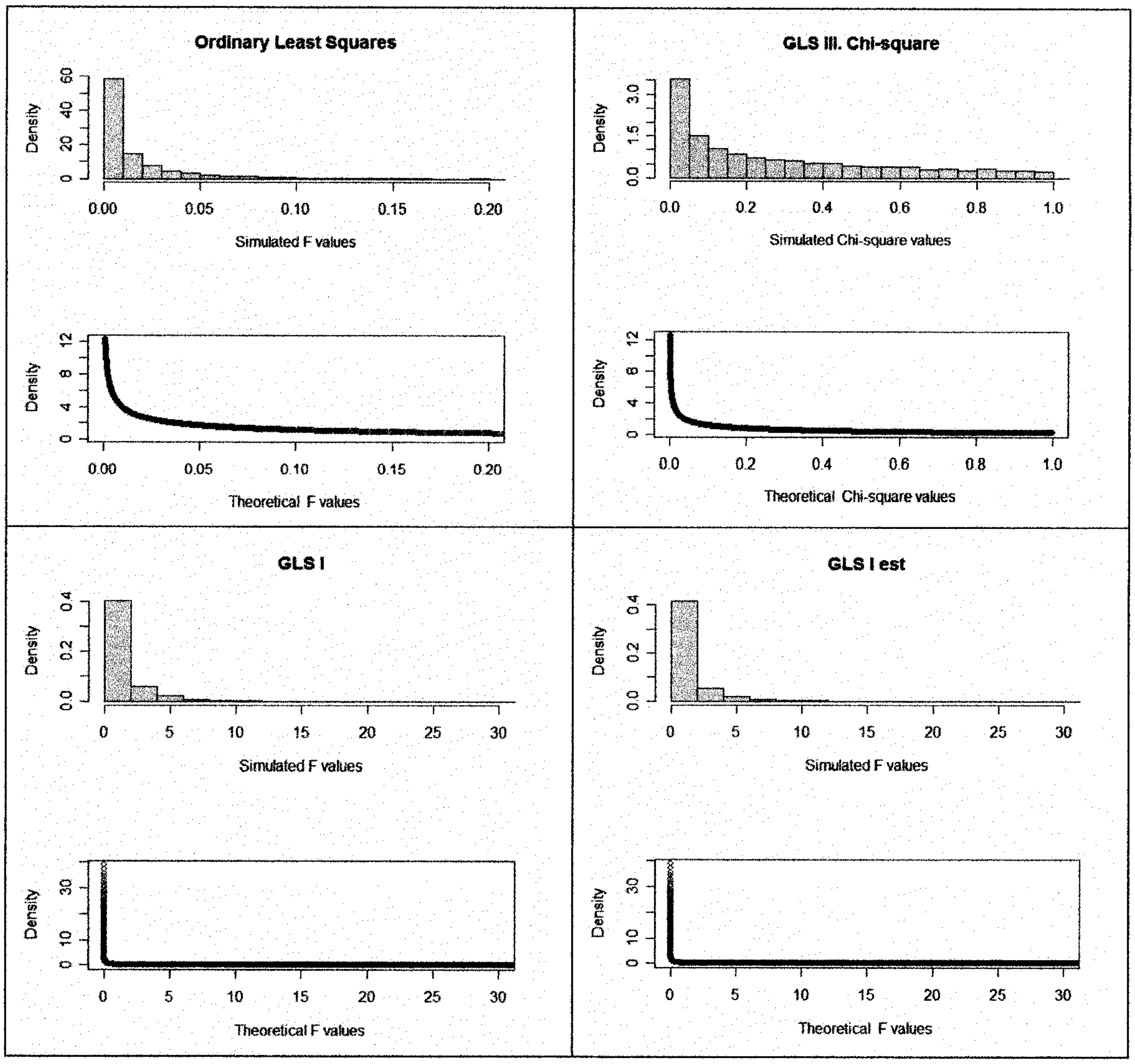

Figure 2.3. Simulated F or χ^2 values when β_1=0, ρ = 0.99.

The Kolmogorov-Smirnov (KS) test is also used to verify that the data follows a particular distribution. When using the KS test we provide a vector of data containing all of our *F* values or χ^2 values (we call this the overall distribution in our tables) from one type of analysis and also the distribution the data is coming from along with the distributions parameters. The simulated data is compared to the given distribution with stated parameters and a *p*-value is given in the table. A *p*-value over 0.05 means we believe the data does follow the given distribution with stated parameters. We not only tested our *F* values (KS overall) this way but also the components of the F values, which were χ^2 and in most cases were the *SSR* (KS num) and *SSE* (KS den), see Tables 2.4-2.7.

Table 2.4

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0$

Regression Type	KS overall	KS num	KS den
RM ANOVA	0.24	0	0
RM ANOVA-Box's ε	0.24	0	0
RMR	0.24	0	0
OLS	0.1	0	0
WLS- Tr weights	0	0	0
WLS-Sub weights	0	0	0
Mean OLS	0.29	0	0
Mean OLS-Tr weights	0	0	0
GLS I	0.1	0.16	0.86
GLS I est	0	0	0
GLS II	0.21	0.13	0.86
GLS II est	0.13	0	0
GLS III	0.16	NA	NA
GLS III est	0	NA	NA

Note: RM ANOVA, RMR, OLS, Means OLS, WLS, GLS are the same abbreviations as used in the analysis section. RM ANOVA-Box's ε is RM ANOVA with a Box's epsilon correction. Tr weights mean a weight is for each treatment level and Sub weights is means a weight is used for each subject. GLS I, II, and III est means we are using an estimated variance-covariance matrix for the errors.

Table 2.5

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.2$

Regression Type	KS overall	KS num	KS den
RM ANOVA	0.58	0	0
RM ANOVA-Box's ε	0.58	0	0
RMR	0.58	0	0
OLS	0	0	0
WLS- Tr weights	0	0	0
WLS-Sub weights	0	0	0
Mean OLS	0.1	0	0
Mean OLS-Tr weights	0.22	0	0
GLS I	0.2	0.39	0.56
GLS I est	0	0	0
GLS II	0.46	0.88	0.56
GLS II est	0	0	0
GLS III	0.39	NA	NA
GLS III est	0	NA	NA

Table 2.6

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.8$

Regression Type	KS overall	KS num	KS den
RM ANOVA	0.23	0	0
RM ANOVA-Box's ε	0.23	0	0
RMR	0.23	0	0
OLS	0	0	0
WLS- Tr weights	0	0	0
WLS-Sub weights	0	0	0
Mean OLS	0.06	0	0
Mean OLS-Tr weights	0	0	0
GLS I	0.03	0.1	0.06
GLS I est	0.07	0	0
GLS II	0.2	0.78	0.06
GLS II est	0	0	0
GLS III	0.1	NA	NA
GLS III est	0	NA	NA

Table 2.7

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.99$

Regression Type	KS overall	KS num	KS den
RM ANOVA	0.84	0	0
RM ANOVA-Box's ε	0.84	0	0
RMR	0.84	0	0
OLS	0	0	0
WLS- Tr weights	0	0	0
WLS-Sub weights	0	0	0
Mean OLS	0.92	0	0
Mean OLS-Tr weights	0	0	0
GLS I	0.42	0.62	0.68
GLS I est	0	0	0
GLS II	0.51	0.94	0.68
GLS II est	0	0	0
GLS III	0.62	NA	NA
GLS III est	0	NA	NA

After we check to see if the data follows the theoretical distribution we wanted to check the parameters (or degrees of freedom). We wanted to examine if the approximated *df* were the same as the theoretical *df*. In most cases, we have theoretical *df* but when the variance-covariance matrix is estimated the *df* were no longer known. In these cases we use the *df* from the theoretical case where the variance-covariance matrix is known. So, we wanted to estimate the *df* to see if they were close to the theoretical values. Also, there was some interest in the *df* of the RM ANOVA because the Box's epsilon correction is used on the *df* to account for the correlation in the data structure.

In the Tables 2.8-2.11 the theoretical degrees of freedom of the F or χ^2 distribution (Th *df1* and Th *df2*) are given in the first two columns. They are followed by *df* found by the χ^2 likelihood method (CSLM). CSLM *df1* is used to estimate the *df* coming from the *SSR* of the analysis and CSLM *df2* estimates the *df* from the *SSE*. This is followed by MOMF *df1* and MOMF *df2* which are the method of moments *df* for the F test as described in the methods section. Finally, we estimate the *df* a third way by the likelihood method again; this time using the F distribution likelihood method (we denote this FLM) and maximizing both parameters simultaneously to produce FLM *df1* and FLM *df2*. Occasionally, an NA was added where the test is not performed due to the set up of the analysis. This was the case in GLS III where we only performed a χ^2 test and not an F test, so there was no need for *df2*.

Table 2.8

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RM ANOVA	2	6	3.22	11	2.78	5.83	2.06	5.75
RM ANOVA-Box ε	1.21	3.62	3.22	11	2.78	5.83	2.06	5.75
RMR	2	6	3.22	11	2.78	5.83	2.06	5.75
OLS	1	10	1.38	19	1.13	9.35	1	9.69
WLS- Tr weights	1	10	0.99	10.42	0.92	14.59	0.97	13.38
WLS-Sub weights	1	10	*	*	-7.8	4.91	1.28	6.82
Mean OLS	1	1	0.78	0.78	0	2	0.99	0.99
Mean OLS- weights	1	1	0.62	0.6	0	2	1.02	0.98
GLS I	1	10	1.01	9.99	1.13	9.35	1	9.69
GLS I est	1	10	1.21	12.99	1.84	5.91	1.02	6.12
GLS II	2	10	2.02	9.99	2.16	9.4	2.02	9.58
GLS II est	2	10	2.43	12.99	2.17	8.54	2	8.63
GLS III	1	NA	1.01	NA	1.02	NA	NA	NA
GLS III est	1	NA	*	NA	79.15	NA	NA	NA

Note. * The function to find the degrees of freedom did not converge.

Table 2.9

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.2$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RM ANOVA	2	6	2.73	8.97	2.28	5.68	2.02	5.69
RM ANOVA-Box ε	1.07	3.22	2.73	8.97	2.28	5.68	2.02	5.69
RMR	2	6	2.73	8.97	2.28	5.68	2.02	5.69
OLS	1	10	1.21	18.52	0.67	58.19	0.94	20.67
WLS- Tr weights	1	10	0.92	10.67	0.61	-52.99	0.92	32.03
WLS-Sub weights	1	10	*	*	6.13	6.38	1.15	11.13
Mean OLS	1	1	0.71	0.73	0	2	0.98	1
Mean OLS-weights	1	1	0.59	0.58	0	2	1.02	0.99
GLS I	1	10	0.99	9.98	0.97	9.83	0.98	9.92
GLS I est	1	10	1.15	12.98	1.17	6.59	0.99	6.73
GLS II	2	10	2	9.98	1.99	10.08	2.01	10.13
GLS II est	2	10	2.57	12.98	3.31	6.68	2.07	7
GLS III	1	NA	0.99	NA	1	NA	NA	NA
GLS III est	1	NA	*	NA	32.52	NA	NA	NA

Note. * The function to find the degrees of freedom did not converge.

Table 2.10

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.8$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RM ANOVA	2	6	1.23	2.91	2.31	5.81	2.02	5.84
RM ANOVA-Box ε	0.47	1.42	1.23	2.91	2.31	5.81	2.02	5.84
RMR	2	6	1.23	2.91	2.31	5.81	2.02	5.84
OLS	1	10	0.73	15.48	0.24	-1.12	*	*
WLS- Tr weights	1	10	0.66	13.21	0.24	-1.02	*	*
WLS-Sub weights	1	10	*	*	0.71	-7.78	*	*
Mean OLS	1	1	0.5	0.49	0	2	1.03	1.01
Mean OLS-weights	1	1	0.47	0.44	0	2	1.09	0.96
GLS I	1	10	1.02	9.93	1.01	9.75	1.03	10.05
GLS I est	1	10	1.16	12.91	0.93	7.51	1.02	7.5
GLS II	2	10	2.01	9.93	2.09	9.99	2.04	10.24
GLS II est	2	10	2.85	12.91	-3.68	4.23	2.19	4.21
GLS III	1	NA	1.02	NA	1.01	NA	NA	NA
GLS III est	1	NA	*	NA	27.86	NA	NA	NA

Note. * The function to find the degrees of freedom did not converge.

Table 2.11

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.99$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RM ANOVA	2	6	0.47	0.7	2.46	6.04	1.95	6.07
RM ANOVA-Box ε	0.16	0.49	0.47	0.7	2.46	6.04	1.95	6.07
RMR	2	6	0.47	0.7	2.46	6.04	1.95	6.07
OLS	1	10	0.36	13.55	0.09	-0.06	*	*
WLS- Tr weights	1	10	0.36	*	0.08	-0.05	*	*
WLS-Sub weights	1	10	*	*	0.14	-0.52	*	*
Mean OLS	1	1	0.29	0.29	0	2	0.98	1.01
Mean OLS-weights	1	1	0.29	0.28	0	2	1.09	0.96
GLS 1	1	10	0.99	10.04	0.96	9.56	0.97	9.74
GLS 1 est	1	10	1.05	12.94	0.73	18	0.94	14.23
GLS 2	2	10	2	10.04	1.94	9.75	1.97	9.76
GLS 2 est	2	10	2.84	12.94	-1.97	3.99	2.16	4
GLS 3	1	NA	0.99	NA	1.01	NA	NA	NA
GLS 3 est	1	NA	*	NA	51.68	NA	NA	NA

Note. * The function to find the degrees of freedom did not converge.

Table 2.12 contains the percentage of the time an HF Box's epsilon estimation is suggested to correct for a violation of sphericity, rounded to three decimal places. It was recommended by Quintana and Maxwell (1994) that any HF Box's epsilon estimator less than 0.75 be used to correct for a serious violation of sphericity. Our data suggests that 80-100% of simulated data sets needed a Box's epsilon correction for deviation from the sphericity assumption. Actually, none of our runs violate the sphericity assumption since the variance-covariance matrix used to create the data has the property of compound symmetry.

Table 2.12

Proportion of data sets where a Box's Epsilon correction factor is suggested

True β_1	0	0.1	1	2	5
$\rho = 0$	0.817	0.811	0.815	0.809	0.816
$\rho = 0.2$	0.875	0.875	0.876	0.873	0.874
$\rho = 0.8$	0.989	0.989	0.986	0.988	0.988
$\rho = 0.99$	1.000	1.000	1.000	1.000	1.000

We want to test the different types of analysis presented in the paper for their ability to handle Type I and Type II errors. The Type I errors should be near 5% if the analysis is working properly. We want the Type II error rate to be as small as possible. Usually as the true value of β_1 gets larger then the Type II errors will become smaller. We look for methods of analysis where the Type II error is small compared to other methods for all values of β_1. Tables 2.13-2.16 contain Type I and Type II error rates for the experiments for Model 1 where each is for a different level of correlation.

Table 2.13

Proportion of Observed Type I or Type II errors when $\rho = 0$

	Type I error	Type II error rates					
The true β_1 value	0	0.001	0.01	0.1	1	2	5
Regression Type							
RM ANOVA	0.048	0.951	0.95	0.73	0	0	0
RM ANOVA-Box's ε	0	1	1	1	0.978	0.926	0.175
RMR	0.048	0.951	0.95	0.73	0	0	0
OLS	0	1	1	0.999	0.44	0.015	0
WLS- Tr weights	0	1	1	0.999	0.539	0.078	0
WLS-Sub weights	0	1	1	0.999	0.446	0.015	0
Mean OLS	0.048	0.953	0.948	0.876	0.121	0.001	0
Mean OLS-Tr weights	0.054	0.948	0.943	0.865	0.154	0.012	0
GLS I	0.05	0.951	0.946	0.562	0	0	0
GLS I est	0.038	0.961	0.957	0.61	0	0	0
GLS II	0.049	0.952	0.949	0.676	0	0	0
GLS II est	0.102	0.895	0.891	0.652	0	0	0
GLS III	0.05	0.952	0.948	0.487	0	0	0
GLS III est	0.05	0.952	0.948	0.487	0	0	0

Table 2.14

Proportion of Observed Type I or Type II errors when ρ = 0.2

	Type I error	Type II error rates					
The true β_1 value	0	0.01	0.1	1	2	5	10
Regression Type							
RM ANOVA	0.052	0.948	0.948	0.666	0.127	0	0
RM ANOVA-Box's ε	0.008	0.992	0.993	0.897	0.437	0.006	0
RMR	0.052	0.948	0.948	0.666	0.127	0	0
OLS	0.035	0.964	0.961	0.553	0.036	0	0
WLS- Tr weights	0.029	0.97	0.965	0.637	0.112	0	0
WLS-Sub weights	0.028	0.971	0.972	0.749	0.207	0	0
Mean OLS	0.051	0.951	0.947	0.863	0.719	0.372	0.079
Mean OLS-Tr weights	0.053	0.949	0.946	0.869	0.755	0.476	0.168
GLS I	0.05	0.949	0.943	0.47	0.021	0	0
GLS I est	0.064	0.937	0.93	0.452	0.02	0	0
GLS II	0.049	0.95	0.935	0.041	0	0	0
GLS II est	0.07	0.933	0.916	0.04	0	0	0
GLS III	0.051	0.949	0.941	0.388	0.005	0	0
GLS III est	0.051	0.949	0.941	0.388	0.005	0	0

Table 2.15

Proportion of Observed Type I or Type II errors when ρ = 0.8

	Type I error	Type II error rates					
The true β_1 value	0	0.001	0.01	0.1	1	2	5
Regression Type							
RM ANOVA	0.05	0.95	0.946	0.94	0.126	0	0
RM ANOVA-Box's ε	0.001	0.999	1	1	0.911	0.632	0.216
RMR	0.05	0.95	0.946	0.94	0.126	0	0
OLS	0.004	0.996	0.997	0.994	0.484	0.016	0
WLS- Tr weights	0.004	0.996	0.997	0.994	0.582	0.072	0
WLS-Sub weights	0.004	0.997	0.997	0.996	0.584	0.03	0
Mean OLS	0.047	0.95	0.95	0.946	0.731	0.479	0.074
Mean OLS-Tr weights	0.052	0.946	0.947	0.942	0.732	0.543	0.161
GLS I	0.05	0.949	0.95	0.929	0.02	0	0
GLS I est	0.058	0.941	0.943	0.921	0.026	0	0
GLS II	0.048	0.948	0.951	0.934	0.011	0	0
GLS II est	0.105	0.891	0.892	0.874	0.008	0	0
GLS III	0.05	0.949	0.948	0.924	0.006	0	0
GLS III est	0.05	0.949	0.948	0.924	0.006	0	0

Table 2.16

Proportion of Observed Type I or Type II errors when ρ = 0.99

	Type I error	Type II error rates					
The true β_1 value	0	0.001	0.01	0.1	1	2	5
Regression Type							
RM ANOVA	0.048	0.951	0.95	0.73	0	0	0
RM ANOVA-Box's ε	0	1	1	1	0.978	0.926	0.175
RMR	0.048	0.951	0.95	0.73	0	0	0
OLS	0	1	1	0.999	0.44	0.015	0
WLS- Tr weights	0	1	1	0.999	0.539	0.078	0
WLS-Sub weights	0	1	1	0.999	0.446	0.015	0
Mean OLS	0.048	0.953	0.948	0.876	0.121	0.001	0
Mean OLS-Tr weights	0.054	0.948	0.943	0.865	0.154	0.012	0
GLS I	0.05	0.951	0.946	0.562	0	0	0
GLS I est	0.038	0.961	0.957	0.61	0	0	0
GLS II	0.049	0.952	0.949	0.676	0	0	0
GLS II est	0.102	0.895	0.891	0.652	0	0	0
GLS III	0.05	0.952	0.948	0.487	0	0	0
GLS III est	0.05	0.952	0.948	0.487	0	0	0

Discussion

In the results, a Box's epsilon correction was suggested 80-100% of the time. We know that sphericity is not being violated when the correlation is zero. But the test was still recommending a significant Box's epsilon correction (a correction factor smaller than 0.75) about 80% of the time. When correlation between measures existed sphericity still was not violated but the test was recommending a Box's epsilon correction almost all of the time. So there is some question about whether using the RM ANOVA with a correction factor is a reasonable method. The RM ANOVA with a correction factor performed the worst of all the analysis on Type I and Type II errors. Quintana and Maxwell (1994) hold that the corrections for the sphericity assumption do not work when the sample size is small, or if there are nearly the same number of factor levels as repeated observations, which was the case in our experiment.

However, RM ANOVA and RMR (which happen to be the same thing in this model) without any correction have slightly higher Type II error rate than GLS but overall seem to perform well. Overall, the estimates of the RM ANOVA and RMR seem to follow an F distribution where the estimated df are close to the theoretical values. There is no violation of sphericity in this model so this method should work well. The main problem with using this method is that in real data it is hard to know whether sphericity is violated. In this case, many statistical packages have the correction factors built in and we would be using a Box's epsilon correction factor 80-100% of the time, which in this case is producing terrible results. Unless there is a good way to test for sphericity, it is hard to advocate the use of RM ANOVA or RMR.

OLS, both types of WLS, means OLS, and means OLS with weights regression all have the assumption that the correlation should be zero. These methods had the most Type II errors of all but they were consistent from the case with no correlation to the cases with high correlation, and were outperformed by GLS I, GLS II, GLS III, RM ANOVA and RMR. GLS II produces too many Type I errors to be a good method. GLS I and III do not have this same problem and should be used over GLS II for this model.

When checking the types of analysis to see if their estimated test statistics were following the theoretical distribution we found that OLS did so when the correlation was zero but not in other cases, as might be expected. All of the GLS methods produced F tests that followed an F distribution with their theoretical *df*. But the GLS methods where the variance covariance matrix was estimated did not follow the F distribution or χ^2 distribution. This was no surprise since we had used the theoretical *df* for the case where the variance-covariance matrix was known. It probably is not true that the *df* for the analysis where the variance-covariance matrix was estimated and the analysis where variance-covariance matrix is known should be the same. This was also true with the WLS. We used the theoretical *df* from OLS but the weights had to be estimated so we would expect the *df* to change. This is why we went on to estimate the *df*. We would have to run many more simulations for different settings of I and J to be able to approximate the *df* well for the cases where the variance-covariance matrix is estimated. Finding approximate *df* for the cases where the variance-covariance matrix is estimated would also change our Type I and Type II error rates.

If the variance-covariance matrix is known, all of the GLS *F* statistics follow a theoretical *F* distribution with the proper *df*. The estimated *df* for the GLS with estimated variance-covariance matrix seem to be higher than the *df* for the case where the variance-covariance matrix is known; this is a good explanation for why they were not doing well with the KS test.

CHAPTER III - MODEL 2: EVERY SUBJECT AT THEIR OWN LEVEL

We examine a repeated measurement study with multiple observations per person at a set factor level. For example, researchers test how a person's weight affects kidney function, the person's weight is recorded and several urine samples are collected. Generally, the subject's weight remains constant over the short time the study is conducted, but researchers collect and analyze several urine samples to ensure that any one meal does not affect the analysis unduly (Liu & Liang, 1992). Other medical studies employ this configuration type to analyze multiple readings from the same test subject over several days to guarantee that diet or other factors do not affect measurements during the administration of a drug. Other studies include manufactured items tested several times to guarantee accurate readings and preclude additional factors from affecting the measurements.

Model 2 consists of I people with indices $i = 1,2,\ldots,I$ and each has J repeated measures with indices $j = 1,2,\ldots,J$, where $N = I * J$ denotes the total number of observations. y_{ij} is jth measurement from the ith observational unit. $\mathbf{x}_i$ is the ith column of the design matrix, **X**, in this model and could signify the dosage of drug administered or the subject's weight, but it will remain constant for each observational unit throughout the experiment. Basically, each observational unit will have a treatment level associated with it that remains constant, so that the treatment level given will be perfectly correlated with the observational unit. ε denotes the vector of errors. The $\boldsymbol{\beta}$ matrix contains the coefficients of the analysis and the $\boldsymbol{\Sigma}$ matrix is comprised of the variance and covariance of the errors. Our model follows the equation: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. In matrix notation:

$$\mathbf{y}=\begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1J} \\ y_{21} \\ \vdots \\ y_{IJ} \end{bmatrix} \qquad \mathbf{X}=\begin{bmatrix} 1 & x_1 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_I \end{bmatrix} \qquad \boldsymbol{\beta}=\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \qquad \boldsymbol{\varepsilon}=\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \vdots \\ \varepsilon_{1J} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{IJ} \end{bmatrix} \tag{51-54}$$

$$\boldsymbol{\Sigma}=\begin{bmatrix} \text{cov}(\varepsilon_{11},\varepsilon_{11}) & \text{cov}(\varepsilon_{12},\varepsilon_{11}) & \dots & \text{cov}(\varepsilon_{1J},\varepsilon_{11}) & \text{cov}(\varepsilon_{21},\varepsilon_{11}) & \dots & \text{cov}(\varepsilon_{IJ},\varepsilon_{11}) \\ \text{cov}(\varepsilon_{11},\varepsilon_{12}) & \text{cov}(\varepsilon_{12},\varepsilon_{12}) & \dots & \text{cov}(\varepsilon_{1J},\varepsilon_{12}) & \text{cov}(\varepsilon_{21},\varepsilon_{12}) & \dots & \text{cov}(\varepsilon_{IJ},\varepsilon_{12}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \text{cov}(\varepsilon_{11},\varepsilon_{1J}) & \text{cov}(\varepsilon_{12},\varepsilon_{1J}) & \dots & \text{cov}(\varepsilon_{1J},\varepsilon_{1J}) & \text{cov}(\varepsilon_{21},\varepsilon_{1J}) & \dots & \text{cov}(\varepsilon_{IJ},\varepsilon_{1J}) \\ \text{cov}(\varepsilon_{11},\varepsilon_{21}) & \text{cov}(\varepsilon_{12},\varepsilon_{21}) & \dots & \text{cov}(\varepsilon_{1J},\varepsilon_{21}) & \text{cov}(\varepsilon_{21},\varepsilon_{21}) & \dots & \text{cov}(\varepsilon_{IJ},\varepsilon_{21}) \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ \text{cov}(\varepsilon_{11},\varepsilon_{IJ}) & \text{cov}(\varepsilon_{12},\varepsilon_{IJ}) & \dots & \text{cov}(\varepsilon_{1J},\varepsilon_{IJ}) & \text{cov}(\varepsilon_{21},\varepsilon_{IJ}) & \dots & \text{cov}(\varepsilon_{IJ},\varepsilon_{IJ}) \end{bmatrix} \tag{55}$$

It is also noted that these matrices were created for a regression analysis. If ANOVA is utilized, which requires categorical variables, the **X** matrix must be adjusted. In this model, the **X** matrix used for ANOVA would have one categorical variable, which would be dummy coded with J - 1 levels and result in rank(**X**) = $(J-1) + 1 = J$.

A graphical view of Model 2's configuration aids in clarity. In this model, we collect multiple readings on each observational unit. Each observational unit is given their own symbol in Figure 3.1.

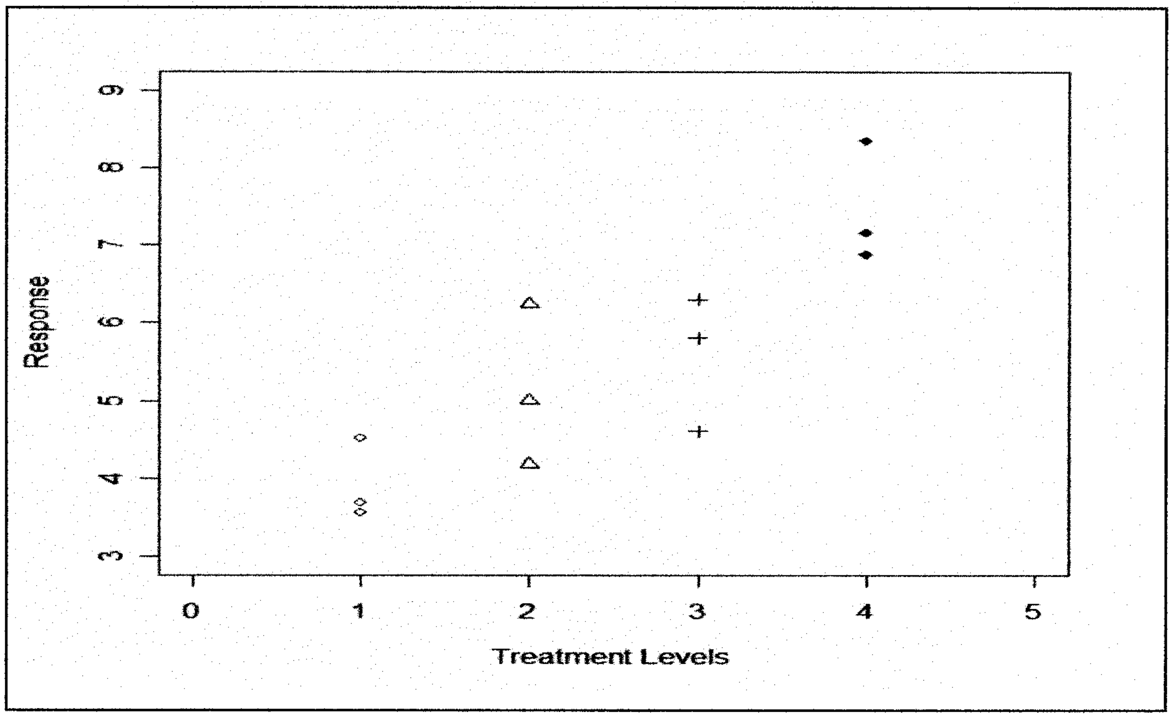

Figure 3.1. *Graphical Representation of Model 2.*

Methods

We employ fewer analysis methods for Model 2 than Model 1 because some are not possible in this setting. All mathematical proofs contained in the methods section of Chapter II are assumed in Chapter III.

ANOVA Analysis

In Model 2, performing RM ANOVA is impossible because there is only one independent variable, but standard ANOVA is a viable method of analysis. Because the treatment remains constant for each observational unit, we can assume it is a categorical variable and can utilize ANOVA. However, this configuration is rather limited; only one variable of interest exists. In Model 2, the independent variables, subject and treatment, are synonymous so we must execute ANOVA using only one of these variables. Model 1 contained two independent variables, subject and treatment, so we performed RM ANOVA.

ANOVA requires the assumptions of Normally, independent and identically distributed errors with constant variance. This assumption set diverges from RM ANOVA in Model 1, which assumed sphericity. Model 2 does not depend on the sphericity assumption because it does not contain two independent variables and RM ANOVA cannot be performed. By extension, we do not require Box's epsilon corrections either. We include ANOVA analysis in our simulations of this model, even though the assumptions of independence and constant variance may not be valid, because it is functional as a means of comparison to other analyses and is often used by analysts who do not take into account the lack of independence. The F test for ANOVA is shown in Table 3.1.

Table 3.1

Univariate ANOVA

Source	*df*	*SS*	*MS*	*F*
Treatments	$I - 1$	*SSR*	*MSR*	*MSR/MSE*
Error	$I(J-1)$	*SSE*	*MSE*	
Total	$IJ - 1$	*SST*		

*N*ote: Proofs in Appendix E.

Regression Methods

Numerous methods referred to as repeated measures regression exist, and we use several of them to analyze simulated data for Model 2.

Repeated Measures Regression

We cannot utilize the analysis method proposed by Misangyi et al. (2006) in Model 2. The Misangyi et al. (2006) method requires us to examine a full model and a reduced model, but Model 2 only contains one independent variable because the treatment variable and subject variable are identical when dummy coded. Although Misangyi et al. (2006) refers to his method as repeated measures regression (RMR), it cannot address all repeated measures experiments. There was difficulty finding valid methods of analysis for repeated measures since some of the methods and papers had titles such as RMR but only dealt with one specific repeated measures design. Many other papers on repeated measures required time to be a factor to use the methods. Because Misangyi et al. (2006) analysis is incompatible with Model 2, it is omitted from the simulations.

Ordinary Least Squares Regression

Repeated measures analysis can violate the assumptions of independence, constant variance, and normality when using OLS regression. However, we include this analysis in our simulations for comparison purposes because it is the most common statistical regression analysis. We address all calculations for the coefficients and F tests in the OLS section of Model 1. In Model 2, we also perform OLS analysis using simple weights, just as in Model 1, to account for deviation from the variance assumption. The inverse of each subject's sample variance is applied as a weight on each of its observations. Although we used two types of weighted regression in our first model, in Model 2, we only utilize one because only one independent variable exists. In Model 1 we were able to add a second variable due to the subjects but in this model the variable for subjects is identical and therefore perfectly correlated to the variable for treatment so only one can be added.

Means Regression

Our simulations include means regression for Model 2. To do this, we calculate the mean of each treatment level of the dependent variable and then run a regression analysis on these means. We also administer some simple weights to this analysis obtained from each treatment.

Generalized Least Squares

GLS methods for Model 2 are identical to the ones employed in Model 1.

Further Analysis

Techniques for confirming our simulations using Type I and Type II errors, estimating the degrees of freedom, and verifying the distributions are identical to the methods presented in Model 1.

Results

For all of the experiments in Model II, there are four subjects and three repeated measures from each. These measures are created at the same factor level for a subject. The variation for each subject is set to two. The correlation will vary and be equal to 0, 0.2, 0.8, or 0.99. When the correlation is equal to zero all of the assumptions should hold for the analyses and we can use this case as our baseline for comparison. Each experiment will have 10,000 simulated data sets and we will perform eleven types of analysis. Unlike Model 1, here we have used regular ANOVA instead of RM ANOVA or RM ANOVA with a Box's epsilon correction because treatment and subject were identical categorical variables. For this same reason only one type of weights was used. The RMR could not be performed since we only have one independent variable.

We wanted to verify our analysis methods via other methods. First, we wanted to compare histograms of the *F* values for each of the eleven methods of analysis to their theoretical *F* distribution. In some cases, there was not an *F* test but rather a χ^2 distribution. So, in those cases we will compare the simulated χ^2 data to the theoretical values. A few of these comparisons will be added but space keeps us from adding all of them (see Figures 3.2 and 3.3). This analysis was only performed in cases where the true β_1 is set at zero, such that the null hypothesis is true.

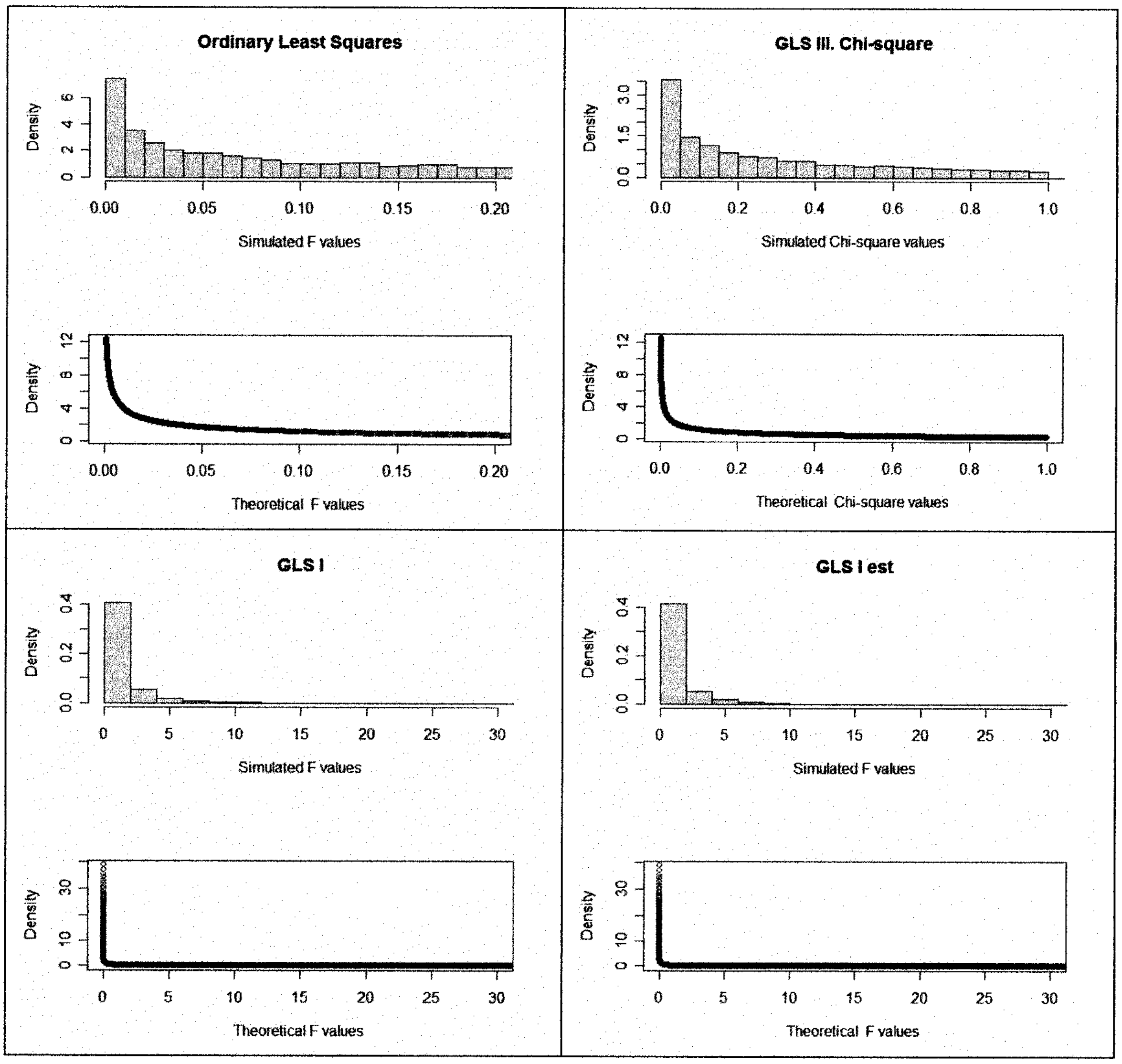

Figure 3.2. Simulated F or χ^2 values when $\beta_1=0$, $\rho = 0$.

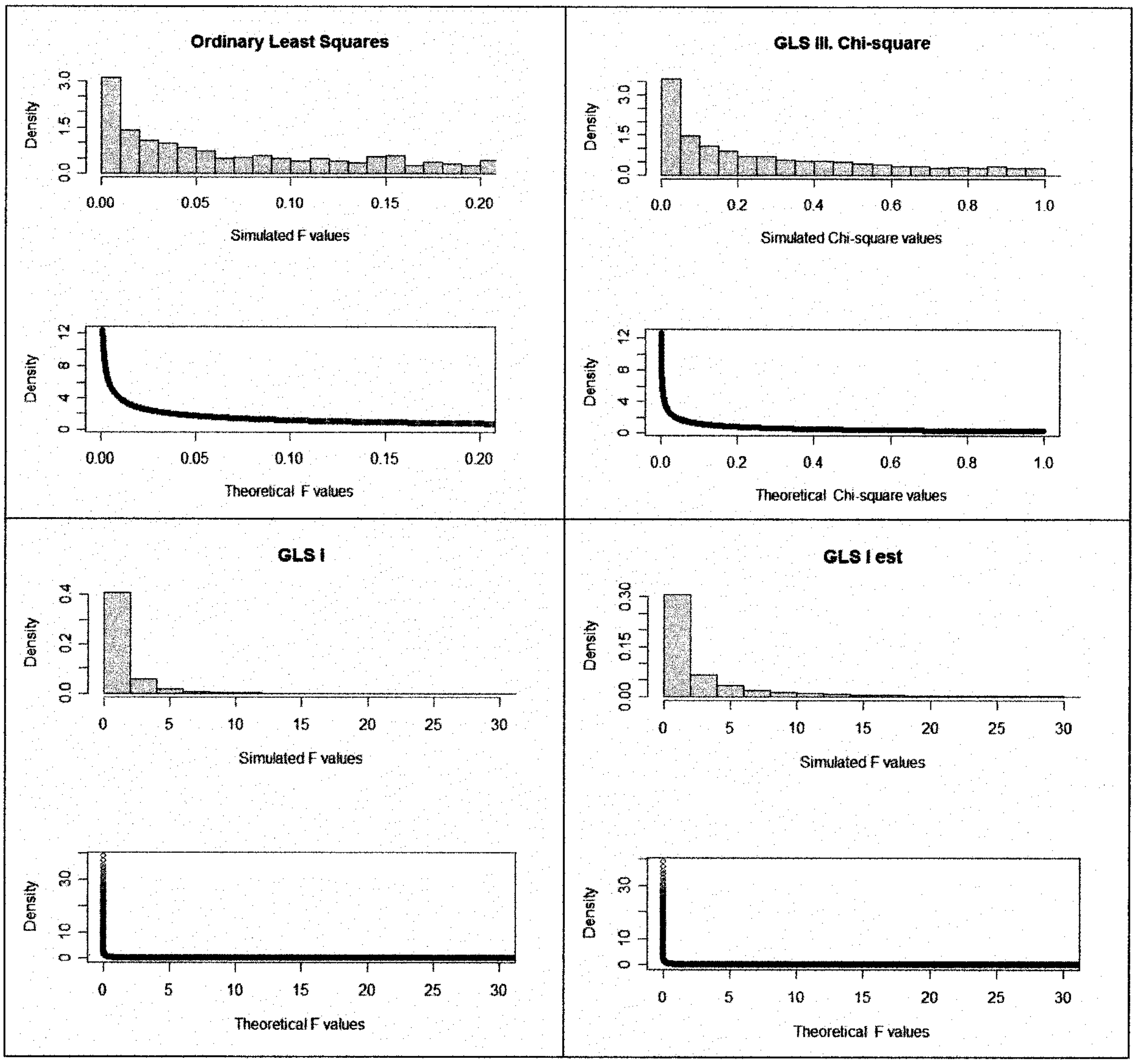

Figure 3.3. Simulated F or χ^2 values when $\beta_1=0$, $\rho = 0.99$.

The Kolmogorov-Smirnov (KS) test is also used to verify that the data follows a particular distribution. When using the KS test we provide a vector of data containing all of our simulated F values or χ^2 values (in our table we denote this as the overall distribution) from one type of analysis. The simulated data is compared to the theoretical distribution with stated parameters and a *p*-value is given in the table. A *p*-value over 0.05 means we believe the data does follow the given distribution with stated parameters. We not only tested our *F* values (KS overall) this way but also the components of the F values, which were χ^2 and in most cases were the *SSR* (KS num) and *SSE* (KS den), see Tables 3.2-3.5.

Table 3.2

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0$

Regression Type	KS overall	KS num	KS den
ANOVA	0	0	0
OLS	0.21	0	0
WLS	0	0	0
Mean OLS	0	0	0
Mean OLS- weights	0	0	0
GLS I	0.21	0.13	0.6
GLS I est	0	0	0
GLS II	0.8	0.87	0.6
GLS II est	0	0	0
GLS III	0.13	NA	NA
GLS III est	0	NA	NA

Table 3.3

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.2$

Regression Type	KS overall	KS num	KS den
ANOVA	0	0	0
OLS	0	0	0
WLS	0	0	0
Mean OLS	0	0	0
Mean OLS- weights	0	0	0
GLS I	0.93	0.9	0.28
GLS I est	0	0	0
GLS II	0.26	0.27	0.28
GLS II est	0	0	0
GLS III	0.9	NA	NA
GLS III est	0	NA	NA

Table 3.4

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.8$

Regression Type	KS overall	KS num	KS den
ANOVA	0	0	0
OLS	0	0	0
WLS	0	0	0
Mean OLS	0	0	0
Mean OLS- weights	0	0	0
GLS I	0.11	0.33	0.68
GLS I est	0	0	0
GLS II	0.72	0.86	0.68
GLS II est	0	0	0
GLS III	0.33	NA	NA
GLS III est	0	NA	NA

Table 3.5

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.99$

Regression Type	KS overall	KS num	KS den
ANOVA	0	0	0
OLS	0	0	0
WLS	0	0	0
Mean OLS	0	0	0
Mean OLS- weights	0	0	0
GLS I	0.17	0.26	0.2
GLS I est	0	0	0
GLS II	0.04	0.19	0.2
GLS II est	0	0	0
GLS III	0.26	NA	NA
GLS III est	0	NA	NA

After we check to see if the statistics from the simulated data follows the theoretical distribution, we wanted to check the parameters (or *df*). In most cases, we have theoretical *df* but when the variance-covariance matrix is estimated then the *df* were no longer known. We continued to use the *df* from the theoretical case where the variance-covariance matrix is known in the cases where the variance-covariance matrix is estimated. As previously, we estimated the *df* to see if they were close to the theoretical values.

In the Tables 3.6 - 3.9, the theoretical degrees of freedom (Th *df1* and Th *df2*) are given in the first two columns. They are followed by *df* found by the χ^2 likelihood method (CSLM). CSLM *df1* is used to estimate the *df* coming from the *SSR* of the analysis and CSLM *df2* estimates the *df* from the *SSE*. This is followed by MOMF *df1* and MOMF *df2* which are the method of moments *df* for the *F* test as described in the methods section. Finally, we estimate the df a third way by the likelihood method again; this time using the *F* distribution (we denote it FLM) and maximizing both parameters simultaneously to produce FLM *df1* and FLM *df2*. Occasionally, an NA was added where the test is not performed due to the set up of the analysis. This was most often the case in GLS III where we only performed a χ^2 test and not an *F* test.

Table 3.6

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
ANOVA	2	6	1.35	19	0.9	11.49	1	10.96
OLS	1	10	1.35	19	0.9	11.49	1	10.96
WLS	1	10	*	*	-0.36	3.45	0.85	3.27
Mean OLS	1	1	0.85	1.58	0	2.26	1	2.07
Mean OLS- weights	1	1	*	1.61	0	2.02	1.25	1.14
GLS I	1	10	0.99	9.99	0.9	11.49	1	10.96
GLS I est	1	10	1.08	12.99	0.75	17.56	0.98	13.73
GLS II	2	10	1.99	9.99	1.97	10.62	1.99	10.72
GLS II est	2	10	2.26	12.99	1.55	15.51	1.91	13.46
GLS III	1	NA	0.99	NA	0.97	NA	NA	NA
GLS III est	1	NA	*	NA	457.66	NA	NA	NA

Note. * The function to find the degrees of freedom did not converge.

Table 3.7

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.2$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
ANOVA	2	6	1.63	17.64	-3.68	4.26	1.1	4.25
OLS	1	10	1.63	17.64	-3.68	4.26	1.1	4.25
WLS	1	10	*	*	-0.08	2.75	0.92	2.19
Mean OLS	1	1	0.97	1.94	0	2.33	1	2.06
Mean OLS- weights	1	1	*	*	0	2.02	1.26	1.14
GLS I	1	10	1	10.08	0.99	10.85	1	10.83
GLS I est	1	10	1.27	12.99	5.43	4.97	1.05	5.09
GLS II	2	10	2.04	10.08	2.17	10.2	2.07	10.28
GLS II est	2	10	2.92	12.99	-11.69	4.88	2.32	4.98
GLS III	1	NA	1	NA	0.99	NA	NA	NA
GLS III est	1	NA	*	NA	987.32	NA	NA	NA

Note. * The function to find the degrees of freedom did not converge.

Table 3.8

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.8$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
ANOVA	2	6	2.37	11.33	-0.08	2.35	1.35	1.39
OLS	1	10	2.37	11.33	-0.08	2.35	1.35	1.39
WLS	1	10	*	*	-0.01	2.15	1.22	1.04
Mean OLS	1	1	1.28	2.84	0	2.16	0.99	1.99
Mean OLS-weights	1	1	*	*	0	2.03	1.22	1.12
GLS I	1	10	1	9.96	1.15	9.03	0.99	9.55
GLS I est	1	10	1.75	12.98	-0.1	2.57	1.12	1.87
GLS II	2	10	1.99	9.96	2.17	9.84	1.97	10.11
GLS II est	2	10	4.51	12.98	-0.15	2.58	2.61	1.88
GLS III	1	NA	1	NA	1.03	NA	NA	NA
GLS III est	1	NA	*	NA	398.53	NA	NA	NA

Note. * The function to find the degrees of freedom did not converge.

Table 3.9

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.99$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
ANOVA	2	6	2.55	8.21	-0.01	2.11	1.48	1.01
OLS	1	10	2.55	8.21	-0.01	2.11	1.48	1.01
WLS	1	10	*	*	0	2.04	1.65	0.74
Mean OLS	1	1	1.35	3.19	0	2.05	0.99	2.03
Mean OLS- weights	1	1	*	*	0	2.02	1.25	1.15
GLS I	1	10	0.99	10.08	0.91	11.15	1	10.51
GLS I est	1	10	1.82	12.77	-0.06	2.44	1.16	1.72
GLS II	2	10	1.98	10.08	1.8	11.2	2	10.57
GLS II est	2	10	4.8	12.77	-0.09	2.45	2.94	1.72
GLS III	1	NA	0.99	NA	0.98	NA	NA	NA
GLS III est	1	NA	*	NA	566.27	NA	NA	NA

Note. * The function to find the degrees of freedom did not converge.

As before we want to compare these forms of analysis by calculating their Type I and Type II errors; this will be summarized in Tables 3.10 – 3.13. The Type I errors should be near 0.05 and the Type II errors should be as small as possible.

Table 3.10

Proportion of Observed Type I or Type II errors when $\rho = 0$

	Type I error	Type II error rates			
The true β_1 value	0	0.01	0.1	1	5
Regression Type					
ANOVA	0.047	0.952	0.944	0.302	0
OLS	0.047	0.952	0.944	0.302	0
WLS	0.112	0.888	0.882	0.459	0.053
Mean OLS	0.046	0.949	0.95	0.655	0
Mean OLS- weights	0.134	0.864	0.866	0.51	0.007
GLS I	0.047	0.952	0.944	0.302	0
GLS I est	0.041	0.958	0.95	0.335	0
GLS II	0.048	0.949	0.927	0	0
GLS II est	0.041	0.957	0.937	0.001	0
GLS III	0.046	0.95	0.943	0.215	0
GLS III est	0.046	0.95	0.943	0.215	0

Table 3.11

Proportion of Observed Type I or Type II errors when ρ = 0.2

	Type I error	Type II error rates			
The true β_1 value	0	0.01	0.1	1	5
Regression Type					
ANOVA	0.099	0.895	0.887	0.302	0
OLS	0.099	0.895	0.887	0.302	0
WLS	0.179	0.813	0.813	0.44	0.044
Mean OLS	0.048	0.949	0.948	0.733	0.002
Mean OLS- weights	0.136	0.863	0.859	0.587	0.018
GLS I	0.048	0.951	0.943	0.443	0
GLS I est	0.085	0.912	0.905	0.358	0
GLS II	0.05	0.951	0.93	0.005	0
GLS II est	0.094	0.901	0.869	0.007	0
GLS III	0.051	0.949	0.941	0.362	0
GLS III est	0.051	0.949	0.941	0.362	0

Table 3.12

Proportion of Observed Type I or Type II errors when ρ = 0.8

	Type I error	Type II error rates			
The true β_1 value	0	0.01	0.1	1	5
Regression Type					
ANOVA	0.329	0.678	0.664	0.276	0
OLS	0.329	0.678	0.664	0.276	0
WLS	0.416	0.584	0.58	0.34	0.023
Mean OLS	0.049	0.952	0.947	0.828	0.027
Mean OLS- weights	0.141	0.86	0.861	0.694	0.06
GLS I	0.052	0.951	0.942	0.664	0
GLS I est	0.213	0.796	0.779	0.44	0
GLS II	0.049	0.953	0.939	0.103	0
GLS II est	0.275	0.728	0.704	0.06	0
GLS III	0.052	0.949	0.944	0.605	0
GLS III est	0.052	0.949	0.944	0.605	0

Table 3.13

Proportion of Observed Type I or Type II errors when ρ = 0.99

	Type I error	Type II error rates			
The true β_1 value	0	0.01	0.1	1	5
Regression Type					
ANOVA	0.407	0.582	0.578	0.248	0
OLS	0.407	0.582	0.578	0.248	0
WLS	0.561	0.439	0.44	0.247	0.006
Mean OLS	0.047	0.952	0.954	0.834	0.043
Mean OLS- weights	0.133	0.864	0.864	0.708	0.069
GLS I	0.048	0.951	0.949	0.694	0
GLS I est	0.218	0.776	0.772	0.441	0
GLS II	0.046	0.947	0.943	0.142	0
GLS II est	0.283	0.706	0.699	0.049	0
GLS III	0.05	0.95	0.948	0.643	0
GLS III est	0.05	0.95	0.948	0.643	0

Discussion

Most of the methods of analysis in this model had a hard time controlling the Type I errors. When the correlation was higher, the more the Type I errors for some of the models increased. Overall, only five analyses kept the Type I errors near 0.05: mean OLS, GLS III where the variance-covariance matrix is estimated, and all types of GLS when the variance-covariance matrix is known. The other methods will be considered inferior since this is an important criterion for using a method.

Mean OLS however, does a poor job at controlling the Type II errors. GLS II controls the Type II errors the best. But when the variance-covariance matrix is estimated, the Type I errors are high. It seems that GLS III controls both Type I and Type II errors when variance-covariance matrix is estimated, which is usually the case. It may be best in this model to use GLS III.

The GLS methods where the variance-covariance matrix is known do follow an F distribution with the theoretical *df.* As noted in the Model 1, it is not expected that some of the methods of analysis do well with the KS test since we do not know the theoretical *df* when a variance-covariance matrix has to be estimated. In the GLS analysis we estimate the *df* to be higher when the variance-covariance matrix is estimated. GLS III produces an unusually large estimated *df* when the variance-covariance matrix is estimated. But as far as controlling the Type I and Type II errors, GLS III still performed the best.

CHAPTER IV- MODEL 3: UNSTRUCTURED REPEATED MEASURES

In Model 3, we will examine the case where there are repeated measures on an observational unit but there is no structure to the independent variables values. This situation arises, for example, in clinical studies where patients are asked to come in at various times for check ups and the check up time is the independent variable. So, one patient may come in after two days for a follow up visit and then not return for eight more days, while another patient may come in three days after then every five days thereafter. We will assume no structure of the independent variables or even the number of repeated measures per observational unit. Another possible situation is where the weight, height, and age of a person is collected and they are put on a drug, then the dosage is adjusted and repeated measures are taken at different dosage levels. It is not assumed that each person would be taking every dose of the drug or taking a constant dose of the drug like the previous two models would have assumed.

This is a regression situation where we have I observational units with indices $i = 1,2,\ldots,I$ and they have some amount of repeated measures J_i with indices $j = 1,2,\ldots,J_i$. J_i denotes the ith observational unit's number of repeated measurements. This is an unbalanced design, such that each observational unit need not have the same number of measures as any of the other observational units. For ease, we will let n denote the total number of observations in the entire experiment. Our model will be of the form $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where y_{ij} is the jth measurement of the ith observational unit. x_{ijk} is the jth measurement on the ith person for the kth independent variable. $\boldsymbol{\beta}$ is a vector of coefficients and $\boldsymbol{\varepsilon}$ is the vector of errors. And we will let $\boldsymbol{\Sigma}$ be the covariance matrix of the errors. In matrix notation we have:

$$\mathbf{y}=\begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1J_1} \\ y_{21} \\ \vdots \\ y_{IJ_I} \end{bmatrix}, \qquad \mathbf{X}=\begin{bmatrix} 1 & x_{111} & \cdots & x_{11k} \\ 1 & x_{121} & \cdots & x_{12k} \\ \vdots & \vdots & & \vdots \\ 1 & x_{1J_11} & \cdots & x_{1J_1k} \\ 1 & x_{211} & \cdots & x_{21k} \\ \vdots & \vdots & & \vdots \\ 1 & x_{IJ_I1} & \cdots & x_{IJ_Ik} \end{bmatrix}, \qquad \boldsymbol{\beta}=\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}, \qquad \boldsymbol{\varepsilon}=\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \vdots \\ \varepsilon_{1J_1} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{IJ_I} \end{bmatrix} \qquad (56\text{-}59)$$

$$\boldsymbol{\Sigma}=\begin{bmatrix} \operatorname{cov}(\varepsilon_{11},\varepsilon_{11}) & \operatorname{cov}(\varepsilon_{12},\varepsilon_{11}) & \cdots & \operatorname{cov}(\varepsilon_{1J_1},\varepsilon_{11}) & \operatorname{cov}(\varepsilon_{21},\varepsilon_{11}) & \cdots & \operatorname{cov}(\varepsilon_{IJ_I},\varepsilon_{11}) \\ \operatorname{cov}(\varepsilon_{11},\varepsilon_{12}) & \operatorname{cov}(\varepsilon_{12},\varepsilon_{12}) & \cdots & \operatorname{cov}(\varepsilon_{1J_1},\varepsilon_{12}) & \operatorname{cov}(\varepsilon_{21},\varepsilon_{12}) & \cdots & \operatorname{cov}(\varepsilon_{IJ_I},\varepsilon_{12}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \operatorname{cov}(\varepsilon_{11},\varepsilon_{1J_1}) & \operatorname{cov}(\varepsilon_{12},\varepsilon_{1J_1}) & \cdots & \operatorname{cov}(\varepsilon_{1J_1},\varepsilon_{1J_1}) & \operatorname{cov}(\varepsilon_{21},\varepsilon_{1J_1}) & \cdots & \operatorname{cov}(\varepsilon_{IJ_I},\varepsilon_{1J_1}) \\ \operatorname{cov}(\varepsilon_{11},\varepsilon_{21}) & \operatorname{cov}(\varepsilon_{12},\varepsilon_{21}) & \cdots & \operatorname{cov}(\varepsilon_{1J_1},\varepsilon_{21}) & \operatorname{cov}(\varepsilon_{21},\varepsilon_{21}) & \cdots & \operatorname{cov}(\varepsilon_{IJ_I},\varepsilon_{21}) \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ \operatorname{cov}(\varepsilon_{11},\varepsilon_{IJ_I}) & \operatorname{cov}(\varepsilon_{12},\varepsilon_{IJ_I}) & \cdots & \operatorname{cov}(\varepsilon_{1J_1},\varepsilon_{IJ_I}) & \operatorname{cov}(\varepsilon_{21},\varepsilon_{IJ_I}) & \cdots & \operatorname{cov}(\varepsilon_{IJ_I},\varepsilon_{IJ_I}) \end{bmatrix} \qquad (60)$$

Figure 4.1 is a graphical representation of possible data that would fit this model. Each observational unit has its own symbol and individual number of repeated measures.

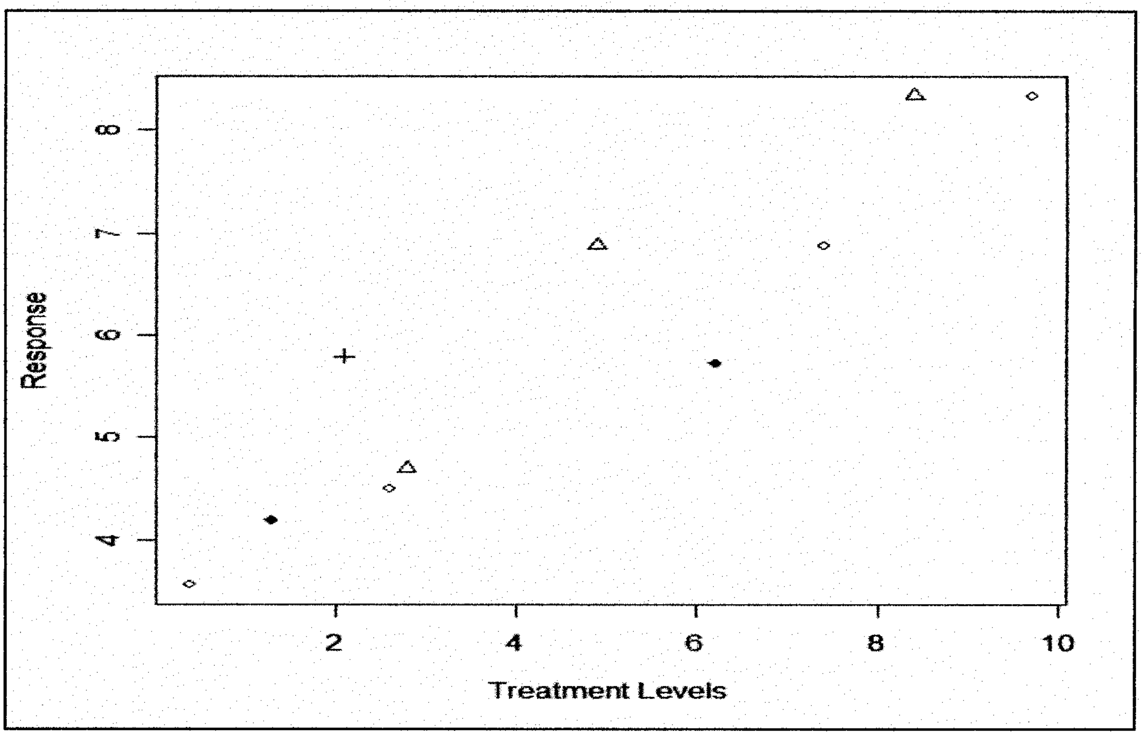

Figure 4.1. Graphical Representation of Model 3.

Methods

As in model one there are many inefficient and invalid ways of analyzing this type of data. There are concerns with performing a regression for each of I observational units and then trying to find a way to compare I regressions. The best way to analyze data is to have one all-inclusive ANOVA or regression model, so as not to have multiple statistical tests (Crowder & Hand, 1990).

ANOVA and RM ANOVA

It is impossible to perform any type of ANOVA test on an unstructured set up such as this. ANOVA requires that the independent variables be able to be classified as categorical variables and in Model 3 no such structure is imposed upon the data. So, there will be no ANOVA performed and also no discussion of the associated sphericity or Box's epsilon adjustments.

Regression Methods

Several regression methods will work with this model. Regression does not require any particular structure of the independent variables, as does ANOVA. As usual, quite a few methods are referred to as repeated measures regression in the literature but not all apply to this model.

Repeated Measures Regression

The type of regression presented by Misangyi et al. (2006) cannot be performed as they suggest on this model. However, a few changes can be made to their method to accommodate our particular set up. This type of RMR suggested by Misangyi et al. (2006) requires two categorical variables, one for the treatments and one for the subjects. In our case, however, we do not want to require the treatments to be categorical and we have not yet included subjects in our design matrix. In Chapter II it was shown this method is analogous to a more flexible type of regression using a full model and reduced model. In this analysis, we will instead discuss the partial F test with the reduced and full models for regression instead of mentioning the more restrictive subject (*SUB*) and treatment (*TMT*) F test used in Model 1.

We will be testing whether there is a significant effect by the independent variables in the model beyond that due to the observational units. The full model will include all independent variables plus an added categorical variable for the observational units. Then in the reduced model we will only regress on the categorical variable for the observational units. These two models will be compared by a partial F test, where (Kleinbaum et al., 1998):

$$F = \frac{\left(R_F^2 - R_R^2\right)/(k)}{\left(1 - R_F^2\right)/\left(IJ - I - k\right)} \tag{61}$$

where k is the number of independent variables, in our case we will use the case where k equals one.

Ordinary Least Squares Regression

OLS will be performed by the same method as in Chapter II. It is also noted here that there is no reason to believe our model will hold to the assumptions of OLS but this analysis will be performed for the purpose of comparison; since this method is so widely known and possibly misused.

Means Regression

Means regression is not feasible in this model because of the unstructured data. Possibly, there could be some structure within a particular data set that may allow for some sort of averaging but we will not assume this is true for all data sets. Previously, we had performed simulations using weights however this analysis will not be performed on this type of model.

Generalized Least Squares

GLS for Model 3 will be performed in the same manner as discussed in Chapter II. However, there will be one change to GLS III since we cannot assume the covariance matrix for each observational unit is identical since Model 3 is unbalanced (every observational unit has a different number of repeated measures). So we will have to impose a different structure than the one used in Models 1 and 2. It is impossible to estimate all of the parameters of the covariance matrix of an observational unit since this would require more estimates than we have data. So, we still need a structure to our covariance matrix, so instead we will assume the same structure as we use in GLS I and II, the correlation and variance are held constant. GLS III will still yield different results than GLS I and GLS II because we are performing the χ^2 test suggested by Crowder and Hand (1990) instead of a typical F test.

Further Analysis

Verifying the simulations with Type I and Type II error rates will be identical to the methods presented for Model 1 in Chapter II; as will estimate the degrees of freedom and verify the distributions. In this case, we do need one constraint on our simulations to be able to estimating the degrees of freedom and verifying the distributions; the total number of observations in each run needs to be the same. But each subject can still have a different number of observations.

Results

Model 3 was the most flexible and more of a general form for repeated measures and therefore we will spend more time and run two sets of experiments. The first set will have the same sample size as in earlier simulations, while the second will have more observational units and total observations. The first sets of data will be created for four observational units with a total of 12 observations. Each observational unit in this model can have a different number of repeated measures. The variance for all error terms was set to be two. The correlation was set at one of four levels for each experiment: 0, 0.2, 0.8, or 0.99. As noted before, when the correlation is set to zero then the assumptions of all of the types of analysis hold and we can use this case as our baseline or control. Each experiment will have 10,000 simulated data sets and we will perform eleven types of analysis.

Unlike Models 1 and 2, the independent variable was not set to any particular levels of a factor but rather was allowed to be continuous and randomly generated between zero and ten. Therefore, no ANOVA can be performed in this case. RMR will be performed but not as described by Misangyi et al. (2006) but rather using the more general partial *F* test. Finally, all six types of GLS analysis will be performed.

We will verify our analysis methods via other methods. First, we will compare histograms of the *F* values for each of the eleven methods of analysis to their theoretical *F* distribution. In some cases, there was not an *F* test but rather a χ^2 distribution. So, in those cases we will compare the simulated χ^2 data to the theoretical values. A few of these comparisons will be added but space keeps us from adding all of them (see Figures 4.2 and 4.3). This analysis was only performed in cases where the true β_1 is set at zero, such that the null hypothesis is true.

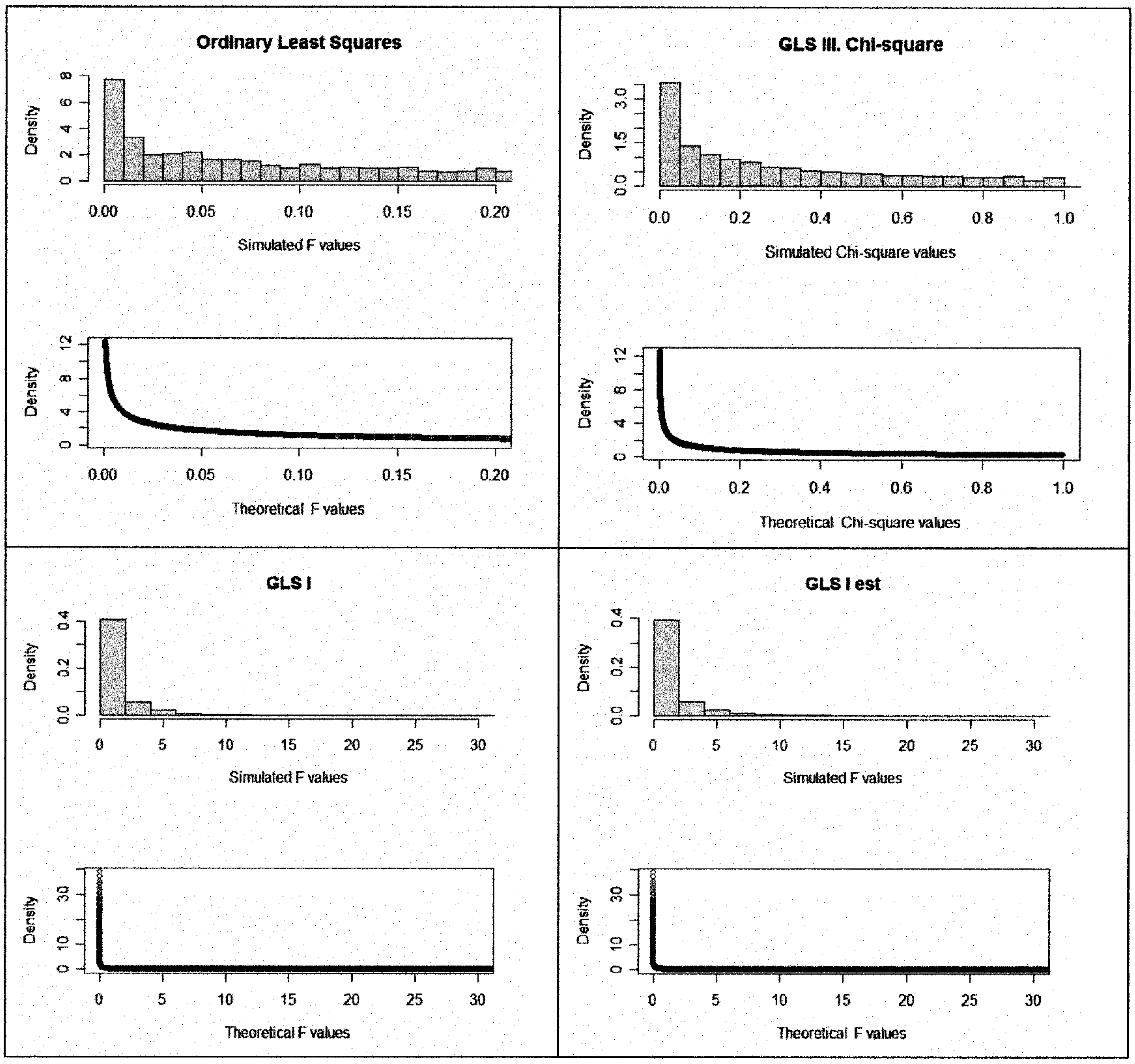

Figure 4.2. Simulated F or χ^2 values when β_1=0, ρ = 0.

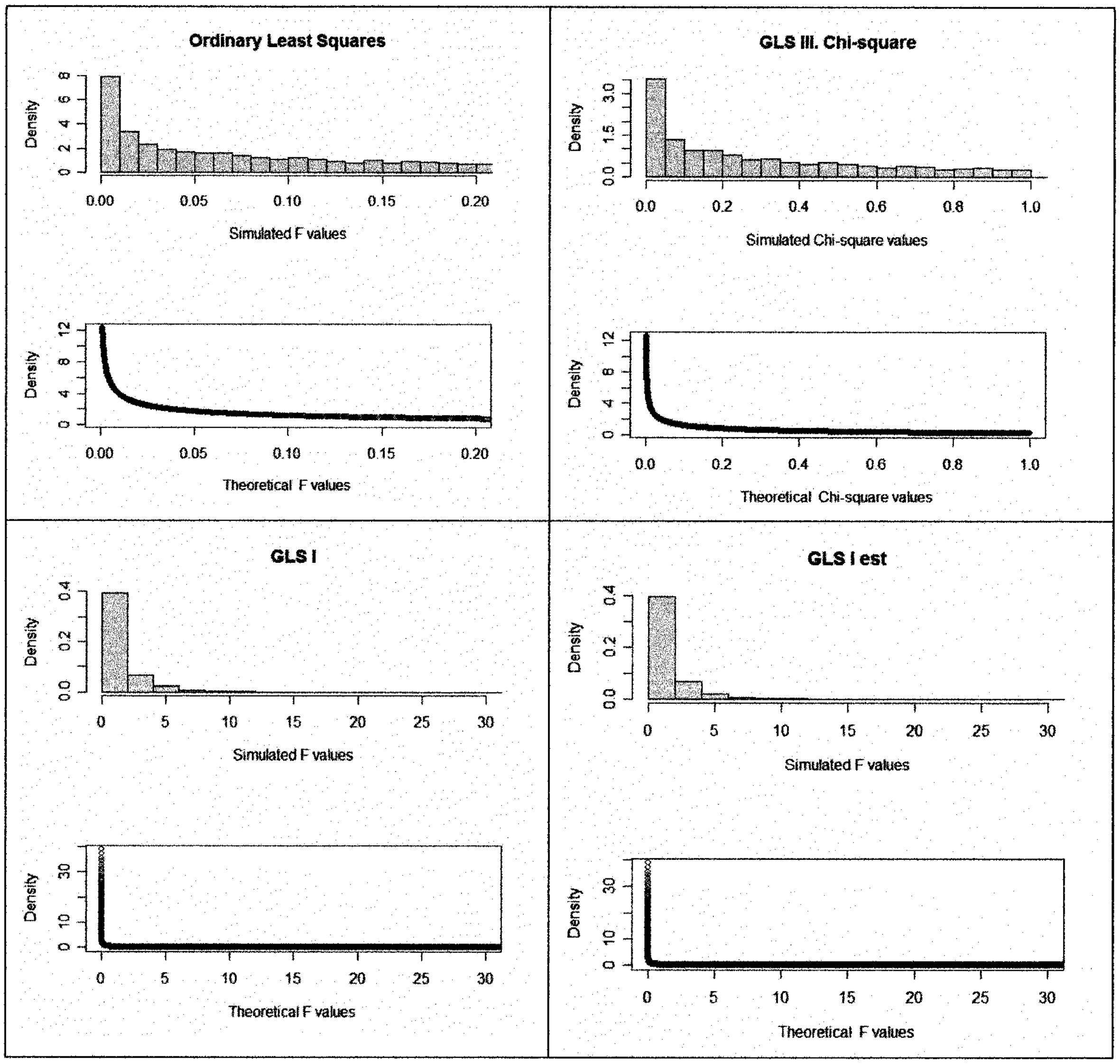

Figure 4.3. Simulated F or χ^2 values when β_1=0, ρ = 0.99.

The Kolmogorov-Smirnov (KS) test is also used to verify that the data follows a particular distribution. When using the KS test we provide a vector of data containing all of our simulated F values or χ^2 value (in our table we denote this as the overall distribution) from one type of analysis. The simulated data is compared to the theoretical distribution with stated parameters, and a p-value is given in the table. A p-value over

0.05 means we believe the data does follow the given distribution with stated parameters. We not only tested our *F* values (KS overall) this way but also the components of the F values, which were χ^2 and in most cases were the *SSR* (KS num) and *SSE* (KS den), see Tables 4.1-4.4.

Table 4.1

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0$

Regression Type	KS overall	KS num	KS den
RMR	0.53	0	0
OLS	0.38	0	0
GLS I	0.38	0.27	0.77
GLS I est	0	0	0
GLS II	0.19	0.13	0.77
GLS II est	0.32	0	0
GLS III	0.27	NA	NA
GLS III est	0	NA	NA

Table 4.2

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.2$

Regression Type	KS overall	KS num	KS den
RMR	0.32	0	0
OLS	0.79	0	0
GLS I	0	0	0.01
GLS I est	0	0	0
GLS II	0.63	0.98	0.07
GLS II est	0	0	0
GLS III	0.79	NA	NA
GLS III est	0	NA	NA

Table 4.3

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.8$

Regression Type	KS overall	KS num	KS den
RMR	0.14	0	0
OLS	0.65	0	0
GLS I	0	0	0.03
GLS I est	0	0	0
GLS II	0.58	0.56	0.5
GLS II est	0	0	0
GLS III	0.12	NA	NA
GLS III est	0	NA	NA

Table 4.4

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.99$

Regression Type	KS overall	KS num	KS den
RMR	0.48	0	0
OLS	0.46	0	0
GLS I	0	0	0.23
GLS I est	0	0	0
GLS II	0.46	0.77	0.6
GLS II est	0	0	0
GLS III	0.26	NA	NA
GLS III est	0	NA	NA

After we check to see if the data follows the theoretical distribution we wanted to check the parameters (or *df*). In most cases, we have theoretical *df* but when the variance-covariance matrix is estimated then the *df* were no longer known. We continued to use the *df* from the theoretical case where the variance-covariance matrix is known in the cases where the variance-covariance matrix is estimated. As previously, we estimated the *df* to see if they were close to the theoretical values.

In the Tables 4.5 – 4.8, the theoretical degrees of freedom (Th *df1* and Th *df2*) are given in the first two columns. They are followed by *df* found by the χ^2 likelihood method (CSLM). CSLM *df1* is used to estimate the *df* coming from the *SSR* of the analysis and CSLM *df2* estimates the *df* from the *SSE*. This is followed by MOMF *df1* and MOMF *df2* which are the method of moments *df* for the *F* test as described in the methods section. Finally, we estimate the df a third way by the likelihood method again; this time using the *F* distribution (we denote it FLM) and maximizing the likelihood over both parameters simultaneously to produce FLM *df1* and FLM *df2*. Occasionally, an NA was added where the test could not be performed due to the set up of the analysis. This was most often the case in GLS III where we only performed a χ^2 test and not an *F* test.

Table 4.5

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RMR	1	7	1.37	12.97	0.98	6.54	1	6.63
OLS	1	10	1.35	19.06	0.92	10.31	0.99	10.11
GLS I	1	10	0.99	10.01	0.92	10.31	0.99	10.11
GLS I est	1	10	1.22	13	1.17	6.27	1.08	6.27
GLS II	2	10	1.98	10.01	1.84	10.86	2	10.56
GLS II est	2	10	2.38	12.99	1.62	9.57	1.99	9.29
GLS III	1	NA	0.99	NA	0.99	NA	NA	NA
GLS III est	1	NA	1.18	NA	1.75	NA	NA	NA

Table 4.6

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.2$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RMR	1	7	1.22	10.55	1.17	6.67	0.99	6.78
OLS	1	10	1.34	17.96	1.1	9.72	1.01	9.79
GLS I	1	10	1.11	9.93	1.17	9	1.24	7.95
GLS I est	1	10	1.3	12.98	1.88	5.8	1.18	5.77
GLS II	2	10	2	9.95	2.1	9.7	1.99	9.81
GLS II est	2	10	2.73	12.99	8.5	5.52	2.17	5.76
GLS III	1	NA	1	NA	1	NA	NA	NA
GLS III est	1	NA	1.22	NA	1.81	NA	NA	NA

Table 4.7

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.8$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RMR	1	7	0.73	3.35	1.18	6.88	1.01	6.99
OLS	1	10	1.15	13.52	0.93	9.76	1.01	9.46
GLS I	1	10	1.29	9.86	1.78	7.15	1.69	6.49
GLS I est	1	10	1.53	12.85	3.09	5.33	1.66	5.01
GLS II	2	10	2	9.96	2.2	9.73	1.99	9.95
GLS II est	2	10	3.03	12.98	-0.97	3.64	2.24	3.49
GLS III	1	NA	1.01	NA	1.01	NA	NA	NA
GLS III est	1	NA	1.21	NA	1.79	NA	NA	NA

Table 4.8

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.99$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RMR	1	7	0.36	0.74	1.08	6.72	1	6.79
OLS	1	10	1.05	11.19	0.94	8.84	0.99	9.01
GLS I	1	10	1.32	9.9	1.91	6.65	1.8	5.97
GLS I est	1	10	1.58	12.7	1.58	6.55	2.01	5.96
GLS II	2	10	2	10.01	1.79	9.72	2	9.6
GLS II est	2	10	2.83	12.92	-1.86	3.97	2.15	3.89
GLS III	1	NA	1.01	NA	1.02	NA	NA	NA
GLS III est	1	NA	1.07	NA	1.37	NA	NA	NA

As before we want to compare these forms of analysis by calculating their Type I and Type II errors; this will be summarized in Tables 4.9 - 4.12. The Type I errors should be near 0.05 and the Type II errors should be as small as possible.

Table 4.9

Proportion of Observed Type I or Type II errors when $\rho = 0$

	Type I error	Type II error rates				
The true β_1 value	0	0.01	0.1	0.5	1	5
Regression Type						
RMR	0.051	0.948	0.917	0.023	0	0
OLS	0.049	0.952	0.902	0.001	0	0
GLS I	0.049	0.952	0.902	0.001	0	0
GLS I est	0.065	0.937	0.885	0.002	0.007	0.049
GLS II	0.045	0.949	0.815	0	0	0
GLS II est	0.051	0.948	0.826	0	0	0
GLS III	0.051	0.949	0.896	0	0	0
GLS III est	0.129	0.876	0.803	0	0	0

Table 4.10

Proportion of Observed Type I or Type II errors when ρ = 0.2

	Type I error	Type II error rates				
The true β_1 value	0	0.01	0.1	0.5	1	5
Regression Type						
RMR	0.053	0.947	0.91	0.011	0	0
OLS	0.051	0.949	0.9	0.002	0	0
GLS I	0.051	0.948	0.895	0.002	0.012	0.138
GLS I est	0.068	0.928	0.872	0.002	0.014	0.087
GLS II	0.052	0.948	0.847	0	0	0
GLS II est	0.083	0.918	0.793	0	0	0
GLS III	0.048	0.949	0.886	0	0	0
GLS III est	0.131	0.869	0.79	0	0	0

Table 4.11

Proportion of Observed Type I or Type II errors when ρ = 0.8

	Type I error	Type II error rates				
The true β_1 value	0	0.01	0.1	0.5	1	5
Regression Type						
RMR	0.051	0.947	0.797	0	0	0
OLS	0.05	0.951	0.88	0.003	0	0
GLS I	0.055	0.946	0.761	0.008	0.074	0.253
GLS I est	0.071	0.929	0.744	0.009	0.075	0.259
GLS II	0.051	0.951	0.797	0	0	0
GLS II est	0.128	0.87	0.697	0	0	0
GLS III	0.051	0.948	0.73	0	0	0
GLS III est	0.124	0.871	0.629	0	0	0

Table 4.12

Proportion of Observed Type I or Type II errors when ρ = 0.99

	Type I error	Type II error rates				
The true β_1 value	0	0.01	0.1	0.5	1	5
Regression Type						
RMR	0.052	0.922	0.018	0	0	0
OLS	0.052	0.95	0.829	0.002	0	0
GLS I	0.057	0.914	0.01	0.011	0.076	0.25
GLS I est	0.054	0.923	0.014	0.044	0.163	0.338
GLS II	0.049	0.932	0.028	0	0	0
GLS II est	0.111	0.876	0.032	0	0	0
GLS III	0.053	0.915	0.005	0	0	0
GLS III est	0.088	0.866	0.007	0	0	0

For the second experiment, data will be created for ten observational units with an average of four repeated measures each. Each observational unit in this model can have a different number of repeated measures but the total number of observations here will add up to forty. The variance for all error terms was set to be two. The correlation was set at one of four levels for each experiment: 0, 0.2, 0.8, or 0.99. As noted before, when the correlation is set to zero then the assumptions of all of the types of analysis hold and we can use this case as our baseline or control. Each experiment will have 10,000 simulated data sets and we will perform eleven types of analysis.

We want to verify our analysis methods via other methods. First, we want to compare histograms of the F values for each of the eleven methods of analysis to their theoretical F distribution. In some cases, there was not an F test but rather a χ^2 test. So, in those cases we will compare the χ^2 results from the simulated data to the theoretical values. A few of these comparisons will be added but space keeps us from adding all of them (see Figures 4.4 and 4.5). This analysis was only performed in cases where the true β_1 is set at zero, such that the null hypothesis is true.

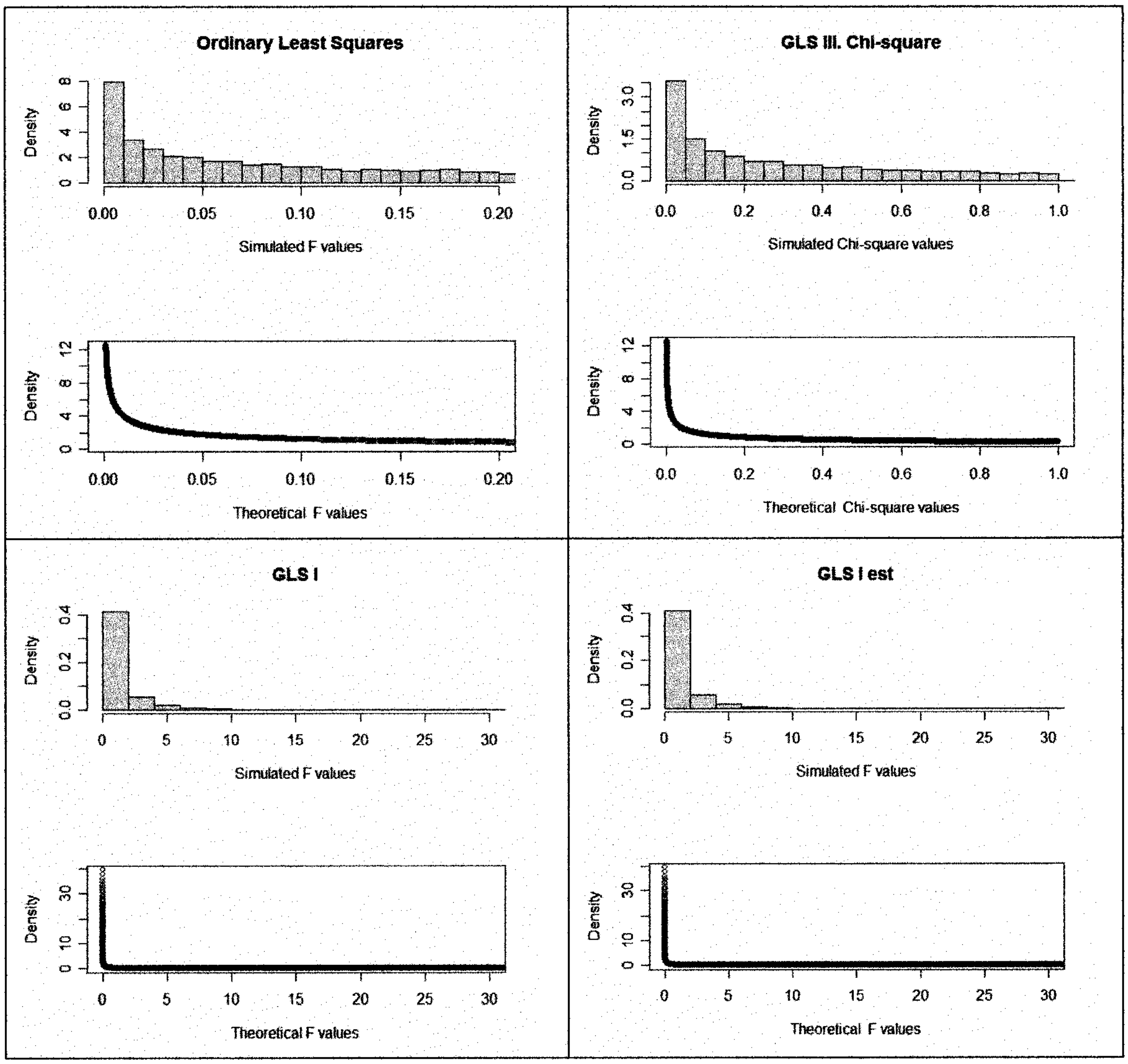

Figure 4.4. Simulated F or χ^2 values when $\beta_1=0$, $\rho = 0$.

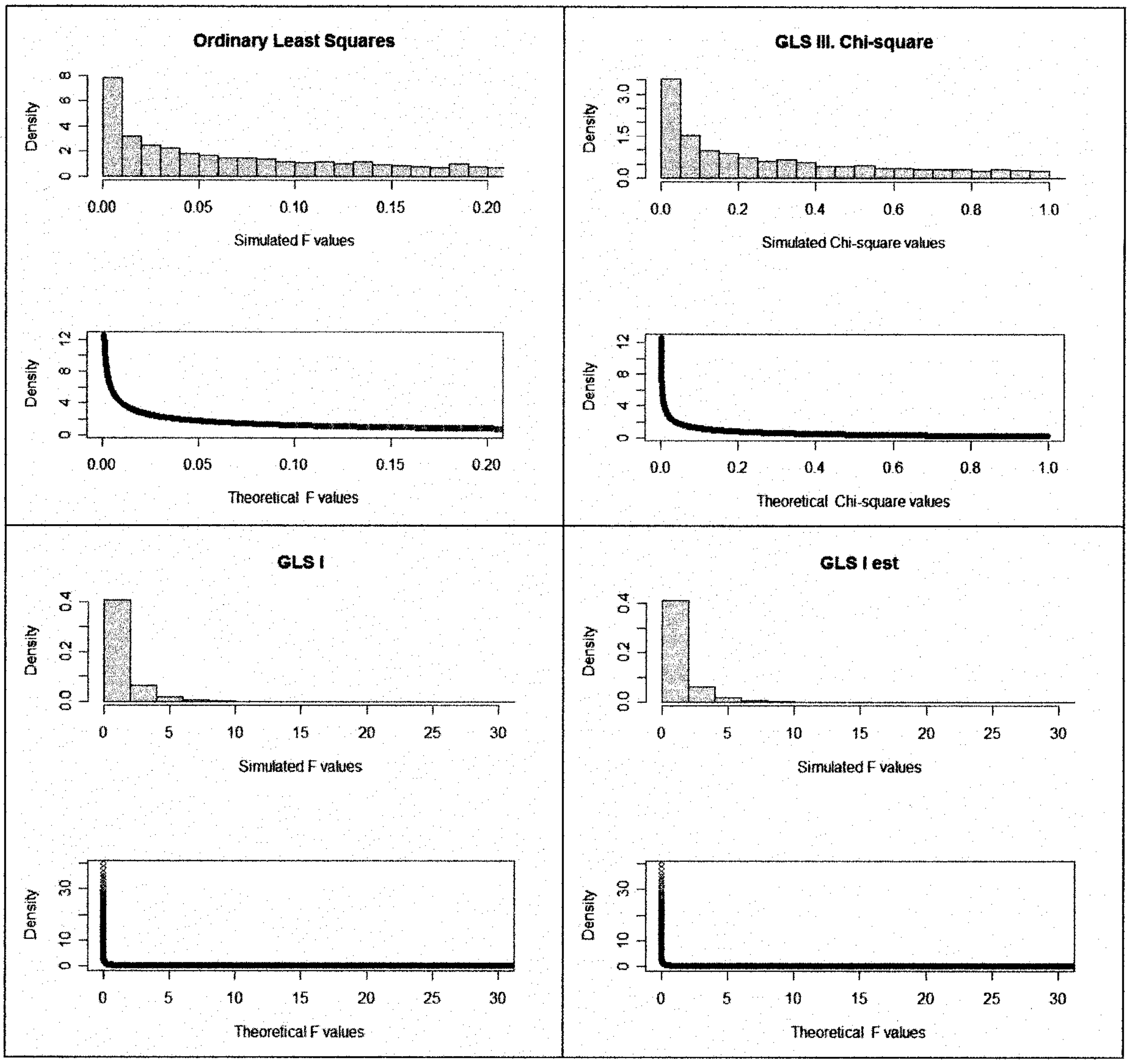

Figure 4.5. Simulated F or χ^2 values when β_1=0, $\rho = 0.99$.

The Kolmogorov-Smirnov (KS) test is also used to verify that the data follows a particular distribution. When using the KS test we provide a vector of data containing all of our simulated F values or χ^2 values (in our table we denote this as the overall distribution) from one type of analysis. The simulated data is compared to the theoretical distribution with stated parameters and a p-value is given in the table. A p-value over 0.05 means we believe the data does follow the given distribution with stated parameters.

We not only tested our *F* values (KS overall) this way but also the components of the F values, which were χ^2 and in most cases were the *SSR* (KS num) and *SSE* (KS den), see Tables 4.13-4.16.

Table 4.13

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0$

Regression Type	KS overall	KS num	KS den
RMR	0.88	0	0
OLS	0.55	0	0
GLS I	0.55	0.62	0.77
GLS I est	0.12	0	0
GLS II	0.54	0.55	0.77
GLS II est	0.04	0	0
GLS III	0.62	NA	NA
GLS III est	0	NA	NA

Table 4.14

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.2$

Regression Type	KS overall	KS num	KS den
RMR	0.94	0	0
OLS	0.65	0	0
GLS I	0	0	0.19
GLS I est	0	0	0
GLS II	0.65	0.76	0.42
GLS II est	0	0	0
GLS III	0.93	NA	NA
GLS III est	0	NA	NA

Table 4.15

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.8$

Regression Type	KS overall	KS num	KS den
RMR	0.33	0	0
OLS	0.11	0	0
GLS I	0	0	0.23
GLS I est	0	0	0
GLS II	0.06	0.1	0.34
GLS II est	0	0	0
GLS III	0.74	NA	NA
GLS III est	0	NA	NA

Table 4.16

Kolmogorov-Smirnov Tests when $\beta_1 = 0$ and $\rho = 0.99$

Regression Type	KS overall	KS num	KS den
RMR	0.74	0	0
OLS	0.19	0	0
GLS I	0	0	0.57
GLS I est	0	0	0
GLS II	0.54	0.77	0.16
GLS II est	0	0	0
GLS III	0.74	NA	NA
GLS III est	0.35	NA	NA

After we check to see if the data follows the theoretical distribution we want to check the parameters (or *df*). In most cases, we have theoretical *df* but when the variance-covariance matrix is estimated then the *df* were no longer known. We continue to use the *df* from the theoretical case where the variance-covariance matrix is known in the cases where the variance-covariance matrix is estimated. As previously, we estimated the *df* to see if they are close to the theoretical values.

In the Tables 4.17 – 4.20, the theoretical degrees of freedom (Th *df1* and Th *df2*) are given in the first two columns. They are followed by *df* found by the χ^2 likelihood method (CSLM). CSLM *df1* is used to estimate the *df* coming from the *SSR* of the analysis and CSLM *df2* estimates the *df* from the *SSE*. This is followed by MOMF *df1* and MOMF *df2* which are the method of moments *df* for the *F* test as described in the methods section. Finally, we estimate the df a third way by the likelihood method again; this time using the *F* distribution (we denote it FLM) and maximizing both parameters simultaneously to produce FLM *df1* and FLM *df2*. Occasionally, an NA was added where the test could not be performed due to the set up of the analysis. This was most often the case in GLS III where we only performed a χ^2 test and not an *F* test.

Table 4.17

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RMR	1	29	1.35	56.98	1.02	25.6	0.98	28.71
OLS	1	38	1.35	74.86	1.02	32.72	0.98	36.16
GLS I	1	38	0.99	37.93	1.02	32.72	0.98	36.16
GLS I est	1	38	1.07	41	1.08	21.42	1.05	21.45
GLS II	2	38	2	37.93	2.02	31.58	1.98	34.37
GLS II est	2	38	2.05	41	1.85	75.44	1.93	56.54
GLS III	1	NA	0.99	NA	1	NA	NA	NA
GLS III est	1	NA	1.03	NA	1.16	NA	NA	NA

Table 4.18

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.2$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RMR	1	29	1.23	45.82	1.02	29.01	1.01	29.53
OLS	1	38	1.35	73.12	0.97	55.32	1.01	44.58
GLS I	1	38	1.16	37.95	1.1	28.23	1.35	16.94
GLS I est	1	38	1.18	40.97	1.19	17.61	1.25	15.26
GLS II	2	38	2.02	37.98	2.02	37.26	2.04	37.13
GLS II est	2	38	2.27	41	2.55	10.9	2.11	12.51
GLS III	1	NA	1	NA	0.99	NA	NA	NA
GLS III est	1	NA	1.06	NA	1.16	NA	NA	NA

Table 4.19

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.8$

Regression Type	Th *df1*	Th *df2*	CSLM *df1*	CSLM *df2*	MOM *df1*	MOM *df2*	FLM *df1*	FLM *df2*
RMR	1	29	0.72	12.25	0.94	30.48	1.02	25.18
OLS	1	38	1.24	64.06	0.92	86.32	0.99	48.53
GLS I	1	38	1.34	37.96	1.46	13.14	1.85	10.02
GLS I est	1	38	1.4	40.86	1.52	12.12	1.9	9.47
GLS II	2	38	1.98	38.08	1.86	54.19	1.98	40.14
GLS II est	2	38	2.21	41	2.02	11.83	2.06	12.02
GLS III	1	NA	1	NA	1	NA	NA	NA
GLS III est	1	NA	1.04	NA	1.12	NA	NA	NA

Table 4.20

Degrees of Freedom when $\beta_1 = 0$ and $\rho = 0.99$

Regression Type	Th df1	Th df2	CSLM df1	CSLM df2	MOM df1	MOM df2	FLM df1	FLM df2
RMR	1	29	0.36	1.37	0.99	29.78	0.99	30.32
OLS	1	38	1.23	59.76	1.07	26.02	1	30.81
GLS I	1	38	1.38	37.94	1.74	11.46	1.96	9.42
GLS I est	1	38	1.46	40.73	1.84	12.03	2.12	9.85
GLS II	2	38	2	38.08	1.96	40.45	2	37.77
GLS II est	2	38	2.25	40.91	2.16	10.66	2.09	11.5
GLS III	1	NA	0.99	NA	1	NA	NA	NA
GLS III est	1	NA	1	NA	1.05	NA	NA	NA

As before we want to compare these forms of analysis by calculating their Type I and Type II errors; this will be summarized in Tables 4.21 - 4.24. The Type I errors should be near 0.05 and the Type II errors should be as small as possible.

Table 4.21

Proportion of Observed Type I or Type II errors when $\rho = 0$

	Type I error	Type II error rates				
The true β_1 value	0	0.01	0.1	0.5	1	5
Regression Type						
RMR	0.054	0.948	0.813	0.001	0	0
OLS	0.055	0.945	0.764	0	0	0
GLS I	0.055	0.945	0.764	0	0	0
GLS I est	0.059	0.94	0.757	0	0	0.005
GLS II	0.052	0.945	0.408	0	0	0
GLS II est	0.048	0.951	0.447	0	0	0
GLS III	0.053	0.947	0.755	0	0	0
GLS III est	0.07	0.927	0.723	0	0	0

Table 4.22

Proportion of Observed Type I or Type II errors when ρ = 0.2

	Type I error	Type II error rates				
The true β_1 value	0	0.01	0.1	0.5	1	5
Regression Type						
RMR	0.049	0.944	0.768	0	0	0
OLS	0.046	0.949	0.755	0	0	0
GLS I	0.046	0.944	0.733	0	0	0.036
GLS I est	0.053	0.939	0.725	0	0	0.031
GLS II	0.05	0.948	0.545	0	0	0
GLS II est	0.074	0.918	0.497	0	0	0
GLS III	0.048	0.945	0.731	0	0	0
GLS III est	0.068	0.924	0.692	0	0	0

Table 4.23

Proportion of Observed Type I or Type II errors when ρ = 0.8

	Type I error	Type II error rates				
The true β_1 value	0	0.01	0.1	0.5	1	5
Regression Type						
RMR	0.049	0.94	0.322	0	0	0
OLS	0.046	0.947	0.723	0	0	0
GLS I	0.054	0.937	0.289	0	0	0.097
GLS I est	0.055	0.932	0.289	0	0	0.114
GLS II	0.05	0.942	0.331	0	0	0
GLS II est	0.075	0.917	0.31	0	0	0
GLS III	0.049	0.942	0.288	0	0	0
GLS III est	0.063	0.922	0.273	0	0	0

Table 4.24

Proportion of Observed Type I or Type II errors when ρ = 0.99

	Type I error	Type II error rates				
The true β_1 value	0	0.01	0.1	0.5	1	5
Regression Type	0	0.01	0.1	0.5	1	5
RMR	0.05	0.805	0	0	0	0
OLS	0.054	0.949	0.699	0.001	0	0
GLS I	0.054	0.793	0	0	0	0.098
GLS I est	0.051	0.803	0	0	0	0.149
GLS II	0.049	0.855	0	0	0	0
GLS II est	0.074	0.833	0	0	0	0
GLS III	0.05	0.795	0	0	0	0
GLS III est	0.056	0.789	0	0	0	0

Discussion

A few things stand out in the results section. As before, the GLS I, GLS II, and GLS III simulated *F* or χ^2 statistics followed their respective distributions well in all situations. GLS I est, GLS II est, and GLS III est simulated *F* or χ^2 statistic for the most part did not seem to follow the specified distribution with theoretical *df*. This is not surprising since the theoretical *df* used were obtained from the case when the variance-covariance matrix is known. There most likely needs to be some sort of correction on the df when estimating the variance-covariance matrix. The OLS and RMR simulated *F* or χ^2 values did follow the F distribution with the theoretical *df* but the SSE and SSR did not follow the χ^2 with the theoretical *df*.

The *df* were estimated for all of the analysis using several methods. GLS I, GLS II, GLS III, OLS, and RMR had estimated *df* that were very similar to the theoretical ones. RMR and OLS had slightly unstable estimated *df2* when the correlation got larger. It is no surprise that the estimated *df* are different than the theoretical ones since the assumptions for this type of analysis are violated when correlation is present in the data. GLS I, II, and III est all had larger *df* than the theoretical ones, which was a bit of a surprise since we thought they might be smaller since we were estimating more parameters. There is good reason to believe that the KS test failed because the wrong *df* are used to compare the simulated values to the theoretical distributions.

When the correlation is non-zero, the assumptions for OLS do not hold. However, in our simulations, the Type II error rates do not get worse as the correlation increases when using OLS. OLS has pretty much constant results as the correlation changes, but all of the other tests have smaller Type II error rates as the correlation increases as would be expected. So, in the end OLS performs the worst of all the analysis in controlling Type II errors when the correlation is high. It seems that OLS is overall a more conservative test.

GLS II and III had rather high Type I errors when estimating the variance covariance matrix; this was the case when the sample size was small. In the second set of simulations when the sample size was larger there was not as noticeable a problem with the Type I error rate. One might opt to use GLS I methods because it had better Type I error rates when the sample size was small. However, GLS I causes a higher Type II error rate when β_1 is rather large. So, if the model is producing rather large estimates of β_1 it maybe best to opt for GLS II or GLS III. GLS II was overall the best test at controlling for Type II errors.

Finally, RMR or in this case the partial F test did work as well as the GLS methods at controlling Type II errors and does not require us to estimate a variance-covariance matrix. As we saw in Model 1, RMR is susceptible to problems when the sphericity assumption fails. Our simulations were all run where the sphericity assumption was satisfied; because the variance of our errors had compound symmetry.

CHAPTER V - CONCLUSION

The types repeated measures experiments explored in this paper are classified as longitudinal studies and have been employed across many disciplines and have a plethora of practical applications. We found examples in the disciplines of psychology and cognitive development, such as the aforementioned experiment with subjects reading sentences (Lorch & Myers, 1990). Also, repeated measures designs have been used in survey analysis as in the government survey example (Kotz & Johnson, 1988). In the life sciences, people may be measured sporadically for kidney disease; a single reading could be affected by one particular meal so multiple readings are taken instead (Liu & Liang, 1992). In many cases where single measurements are taken, the experiment could be redesigned to collect multiple readings on a subject. This can reduce the number of observational units needed when conducting a study and already has been used in many fields such as psychology, education, and medicine. Repeated measurement experiments are common in most situations and fields of study. However, standard analysis will not suffice because the measurements are correlated. The standard errors, t statistics, and p values in most statistical tests are invalid when the measurements are not independent (Misangyi, LePine, Algina, & Goeddeke, 2006).

We found that many methods that claim to work on repeated measures studies only can be applied in some cases. There are many classifications for repeated measures designs and it is important to use the correct analysis for a particular type of designed experiment. We found that generalized least squares regression (GLS) was the best method for analyzing all type of repeated measures experiments because the variance-covariance matrix structure could be tailored to the type of experiment. We also note that some of the types of analysis, such as OLS, RMR, and ANOVA, have particular assumptions that must be met. If these assumptions are not met then the analysis is invalid.

Specifically, in the case of RMR, this test only worked when the sphericity assumption was not violated. It turned out that some authors were calling RMR some particular form of a partial F test that only applied in a few repeated measures situations. Once this was discovered we were able to use the more generalized partial F test as a means of analysis for all types of repeated measures experiments.

In our analysis it seems that the best method of analysis was either the methods we called GLS II or GLS III for these models. It will be noted that both of these types of analysis had larger Type I errors than they should when the sample size was small. In the case of the GLS analysis, more research could be pursued in finding the correct degrees of freedom for the F test when estimating the variance-covariance matrix.

Note that our simulations were using a variance-covariance matrix to create the errors that had compound symmetry. More experiments could be performed and this could be changed to a more unstructured case where the "true" variance-covariance matrix did not meet the sphericity assumption. Only two types of structures were used when estimating the variance-covariance matrix. These too could be expanded to include more structures: such as, diminishing correlation as the factor levels increase or structure on the eigenvalues of $\mathbf{\Sigma}$ instead of the actual values of $\mathbf{\Sigma}$.

Analysis was only performed on small sample sizes and in Model 1 and 2 only one sample size was explored. More trials could be done on different sample sizes in order to explore the effect on the analysis performance.

Any further study should most likely be spent on GLS and maybe partial *F* tests since the assumptions are not violated in these cases whereas they are in OLS and ANOVA.

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APPENDIXES

Appendix A - Matrix Definitions, Lemmas, and Theorems

A. Idempotent Matrix- for **M** an nxn matrix to be idempotent it must have the property: **M=MM** (Montgomery et al., 2006).

B. Orthogonal Matrix –a matrix **M** is said to be orthogonal if and only if **M'M=I** (Messer, 1993).

C. Symmetric Matrix- for **M** an nxn matrix to be symmetric it must have the property **M=M'** (Montgomery et al., 2006).

D. Orthogonally Diagonalizable Matrix - **M** an nxn matrix is orthogonally diagonalizable if and only if it is symmetric. Then there exists an orthogonal matrix **P** such that $\mathbf{P}^{-1} = \mathbf{P}'$ and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$ (Lay,1997).

E. Symmetric Matrix Properties – for **M** an nxn symmetric matrix the characteristic polynomial has n real roots and its eigenvectors are orthogonal, but **M** does not have to be positive definite (Messer, 1993).

F. Positive Definite Matrix- **M** an nxn matrix is positive definite if **M** has all positive eigenvalues (Lay,1997).

G. Properties of the Trace

   (i) Rank of an Idempotent Matrix – If **M** is an idempotent matrix then the rank(**M**) is its trace. ie. rank(**M**)=tr(**M**). (Montgomery et al., 2006).

   (ii) tr(**ABC**)=tr(**CAB**)=tr(**BAC**) if the matrices are appropriately conformable (Montgomery et al., 2006).

   (iii) tr(**A**+**B**) = tr**A** + tr**B**, if **A** and **B** are n x n (Messer, 1993).

(iv) tr(**A**) = tr($\mathbf{P}^{-1}\mathbf{AP}$), if **A** is n x n and **P** is an invertible n x n matrix

(Messer, 1993).

H. For **A**, a fixed matrix, and **Y**, a vector of random variables, Var(**AY**)=**A**Var(**Y**)**A**'

(Montgomery et al., 2006).

I. Perpendicular projection - **M** is a perpendicular projection operator on the column

space of **M**, denoted C(**M**), if and only if $\mathbf{M}^2 = \mathbf{M}$ and $\mathbf{M}' = \mathbf{M}$ (Christensen, 1987).

J. If **Y** is a random vector with $\mathbf{Y} \sim N(\boldsymbol{\mu}, \mathbf{I})$ and if **M** is a perpendicular projection

matrix then $\mathbf{Y}'\mathbf{M}\mathbf{Y} \sim \chi^2(tr(\mathbf{M}), \boldsymbol{\mu}'\mathbf{M}\boldsymbol{\mu}/2)$ (Christensen, 1987).

K. If $\mathbf{Y} \sim N(\boldsymbol{\mu}, \mathbf{M})$ where $\boldsymbol{\mu} \in C(\mathbf{M})$ and if **M** is a perpendicular projection matrix, then

$\mathbf{Y}'\mathbf{Y} \sim \chi^2(r(\mathbf{M}), \boldsymbol{\mu}'\boldsymbol{\mu}/2)$ (Christensen, 1987).

L. If $\mathbf{Y} \sim N(\boldsymbol{\mu}, \mathbf{V})$ and **A** is a matrix then $\mathbf{Y}'\mathbf{A}\mathbf{Y} \sim \chi^2(tr(\mathbf{AV}), \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}/2)$ if (1)

$\mathbf{VAVAV} = \mathbf{VAV}$, (2) $\boldsymbol{\mu}'\mathbf{AVA}\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$, and (3) $\mathbf{VAVA}\boldsymbol{\mu} = \mathbf{VA}\boldsymbol{\mu}$ (Christensen,

1987).

Appendix B - Ordinary Least Squares Proofs

Matrix Definitions for OLS

OLS is useful when the assumptions of Normality, constant variance, independence and identical distribution of the errors hold. Therefore: $\boldsymbol{\varepsilon} \sim N(0, \sigma^2 \mathbf{I})$ and $\mathbf{Y} \sim N(\mathbf{X\beta}, \mathbf{I}\sigma^2_{y|x})$, where $\sigma^2 = \sigma^2_{y|x}$. We will use the equation $\mathbf{Y} = \mathbf{X\beta} + \boldsymbol{\varepsilon}$ to perform the OLS analysis, which lead to: $\hat{\boldsymbol{\beta}} = (\mathbf{X'X})^{-1}\mathbf{X'Y}$ and $V(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X'X})^{-1}$ where $\hat{\boldsymbol{\beta}}$ is an unbiased estimator of $\boldsymbol{\beta}$, if the assumptions are met (Montgomery et al., 2006). Define a matrix **H** to be $\mathbf{H} = \mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'}$, which is typically called the Hat matrix because $\hat{\mathbf{y}} = \mathbf{Hy}$. We can see **H** has the properties:

$$\mathbf{H} = \mathbf{H'} \tag{62}$$

$$\mathbf{H} = \mathbf{HH} \tag{63}$$

$$(\mathbf{I} - \mathbf{H})^2 = \mathbf{II} - \mathbf{IH} - \mathbf{HI} + \mathbf{HH} = \mathbf{I} - \mathbf{H} - \mathbf{H} + \mathbf{H} = \mathbf{I} - \mathbf{H} \tag{64}$$

$$(\mathbf{I} - \mathbf{H})' = \mathbf{I'} - \mathbf{H'} = \mathbf{I} - \mathbf{H} \tag{65}$$

which means **H** and **I-H** are symmetric idempotent matrices (see Appendix A). Other useful properties we will use are:

$$\mathbf{X'H} = \mathbf{X'X}(\mathbf{X'X})^{-1}\mathbf{X'} = \mathbf{X'} \tag{66}$$

$$\mathbf{HX} = \mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'X} = \mathbf{X} \tag{67}$$

Other identities and definitions we will need in the proofs of the *SST* and *SSR* are as follows: $\bar{\mathbf{y}} = (\mathbf{1'1})^{-1}\mathbf{1'y}$ where **1** is an n x 1 vector of ones and $\mathbf{I}_n$ is an n x n identity matrix. We also need the facts **H1=1** and **1'H=1'** this follows from an identity involving the partitioning of the **X** matrix (Montgomery et al., 2006). And $(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')$ is idempotent and thus a perpendicular projection matrix because:

$$(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') = (\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') \quad (68)$$

$$(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')' = (\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') \quad (69)$$

$(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')$ is also idempotent and a projection matrix because:

1. $(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')' = (\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') \quad (70)$

2. $(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') = \mathbf{HH} - \mathbf{H1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}' - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{H} + \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}' \quad (71)$

$$= \mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}' - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}' + \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}' \quad (72)$$

$$= \mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}' \quad (73)$$

SSE for OLS

Using all of these facts we will now show that the *SSE* follows a Chi-square distribution with $IJ - k - 1$ degrees of freedom. To do that we must first manipulate the general form of the *SSE* so that we can use some of the theorems previously stated in Appendix A.

$$\text{SSE} = (\mathbf{y} - \hat{\mathbf{y}})' \frac{1}{\sigma^2} (\mathbf{y} - \hat{\mathbf{y}}) \quad (74)$$

$$= \frac{1}{\sigma^2} (\mathbf{y}' - \mathbf{y}'\mathbf{H}')(\mathbf{y} - \mathbf{H}\mathbf{y}) \quad (75)$$

$$= \frac{1}{\sigma^2} (\mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{H}\mathbf{y} - \mathbf{y}'\mathbf{H}'\mathbf{y} + \mathbf{y}'\mathbf{H}'\mathbf{H}\mathbf{y}) \quad (76)$$

$$= \frac{1}{\sigma^2} (\mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{H}\mathbf{y} - \mathbf{y}'\mathbf{H}\mathbf{y} + \mathbf{y}'\mathbf{H}\mathbf{y}) \quad (77)$$

$$= \frac{1}{\sigma^2} (\mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{H}\mathbf{y}) \quad (78)$$

$$= \frac{1}{\sigma^2}\mathbf{y}'(\mathbf{I}-\mathbf{H})\mathbf{y} \tag{79}$$

We can use Theorem K under three certain conditions to show that the *SSE* is a Chi-square with previously stated *df*:

$$\text{(i) } \sigma^2\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\sigma^2\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\sigma^2 = \sigma^2\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})(\mathbf{I}-\mathbf{H})\sigma^2 \tag{80}$$

$$= \sigma^2\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\sigma^2$$

$$\text{(ii) } (\mathbf{X\beta})'\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\sigma^2\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})(\mathbf{X\beta}) = (\mathbf{X\beta})'\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})(\mathbf{X\beta}) \tag{81}$$

$$\text{(iii) } \sigma^2\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\sigma^2\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\mathbf{X\beta} = \sigma^2\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\mathbf{X\beta} \tag{82}$$

Since these conditions hold, we can now apply by Theorem K:

$$\mathbf{y}'\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\mathbf{y} \sim \chi^2\left(tr\left(\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\sigma^2\right), (\mathbf{X\beta})'\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})(\mathbf{X\beta})/2\right) \tag{83}$$

Now it will be shown that the non-centrality parameter reduces to zero:

$$(\mathbf{X\beta})'\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})(\mathbf{X\beta})/2 = \left[(\mathbf{X\beta})'\frac{1}{\sigma^2}\mathbf{I}(\mathbf{X\beta}) - (\mathbf{X\beta})'\frac{1}{\sigma^2}\mathbf{H}(\mathbf{X\beta})\right]/2 \tag{84}$$

$$= \left[(\mathbf{X\beta})'\frac{1}{\sigma^2}(\mathbf{X\beta}) - \boldsymbol{\beta}'\mathbf{X}\frac{1}{\sigma^2}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X\beta})\right]/2 \tag{85}$$

$$= \left[(\mathbf{X\beta})'\frac{1}{\sigma^2}(\mathbf{X\beta}) - \boldsymbol{\beta}'\mathbf{X}'\frac{1}{\sigma^2}(\mathbf{X\beta})\right]/2 \tag{86}$$

$$= 0 \tag{87}$$

And then we will go on to manipulate the *df* given by the theorem into something a bit more useful:

$$tr\left(\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{H})\sigma^2\right)=tr(\mathbf{I}-\mathbf{H}) \tag{88}$$

$$=tr(\mathbf{I}_{\mathbf{IJ}})-tr(\mathbf{H}) \text{ by property G (iii)} \tag{89}$$

$$= IJ-tr(\mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'}) \text{ by property G (iv)} \tag{90}$$

$$=IJ-tr(\mathbf{X'X}(\mathbf{X'X})^{-1}) \text{ by property G (ii)} \tag{91}$$

$$= IJ-tr(\mathbf{I}_{\mathbf{k+1\times k+1}}) \tag{92}$$

$$=IJ-(k+1) \tag{93}$$

$$=IJ-k-1 \tag{94}$$

If we put all of this together we conclude that: $SSE\sim\chi^2(IJ-k-1)$.

SST for OLS

Next we need to show that the *SST* follows a Chi-square distribution with IJ - 1 *df*. We must first manipulate the form of the *SST* so that we can apply the Theorems from Appendix A:

$$SST = (\mathbf{y} - \mathbf{1}\bar{\mathbf{y}})'\frac{1}{\sigma^2}(\mathbf{y} - \mathbf{1}\bar{\mathbf{y}}) \tag{95}$$

$$= (\mathbf{y} - \mathbf{1}(\mathbf{1'1})^{-1}\mathbf{1'y})'\frac{1}{\sigma^2}(\mathbf{y} - \mathbf{1}(\mathbf{1'1})^{-1}\mathbf{1'y}) \tag{96}$$

$$= \mathbf{y}'\frac{1}{\sigma^2}(\mathbf{I_n} - \mathbf{1}(\mathbf{1'1})^{-1}\mathbf{1'})'(\mathbf{I_n} - \mathbf{1}(\mathbf{1'1})^{-1}\mathbf{1'})\mathbf{y} \tag{97}$$

$$= \frac{1}{\sigma^2}\mathbf{y}'(\mathbf{I_n} - \mathbf{1}(\mathbf{1'1})^{-1}\mathbf{1'})\mathbf{y}. \tag{98}$$

We can use Theorem K under three certain conditions to show that the *SST* is a Chi-square with previously stated *df*:

$$\text{(i) } \sigma^2 \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2 \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2 = \sigma^2 \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2 \quad (99)$$

$$\text{(ii) } (\mathbf{X\beta})' \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2 \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{X\beta}) \quad (100)$$

$$= (\mathbf{X\beta})' \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{X\beta})$$

$$\text{(iii) } \sigma^2 \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2 \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{X\beta}) \quad (101)$$

$$= \sigma^2 \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{X\beta})$$

Since these conditions hold, we can now apply Theorem K:

$$\mathbf{y}' \frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\,\mathbf{y} \sim \chi^2\Big(tr\,\big(\frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2\big),\ (\mathbf{X\beta})'(\mathbf{X\beta}) / 2\sigma^2\Big) \quad (102)$$

And the degrees of freedom can be reduced as follows:

$$tr\,\big(\frac{1}{\sigma^2}(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\,\sigma^2\big) = tr\,(\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') \quad (103)$$

$$= tr(\mathbf{I}_n) - \mathrm{tr}(\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') \text{ by property G (iii)} \quad (104)$$

$$= IJ - \mathrm{tr}((\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{1}) \quad (105)$$

$$= IJ - 1. \quad (106)$$

Also, the non-centrality parameter can be evaluated at $\boldsymbol{\beta} = \mathbf{0}$ since we are assuming the null hypothesis is true, so $(\mathbf{X\beta})'(\mathbf{X\beta}) / 2\sigma^2 = 0$. If we put all of this together we conclude that: $SST \sim \chi^2(IJ - 1)$ when H_0 is true.

SSR for OLS

Next we need to show that the *SSR* follows a Chi-square distribution with k degrees of freedom. We must first manipulate the form of the *SSR* so that we can apply the Theorems from Appendix A:

$$SSR = (\hat{\mathbf{y}} - \mathbf{1}\bar{\mathbf{y}})'\frac{1}{\sigma^2}(\hat{\mathbf{y}} - \mathbf{1}\bar{\mathbf{y}}) \quad (107)$$

$$= (\mathbf{Hy} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{y})'\frac{1}{\sigma^2}(\mathbf{Hy} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{y}) \quad (108)$$

$$= \mathbf{y}'\frac{1}{\sigma^2}(\mathbf{H}' - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')'(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\mathbf{y} \quad (109)$$

$$= \frac{1}{\sigma^2}\mathbf{y}'(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\mathbf{y}. \quad (110)$$

We can use Theorem K under three certain conditions to show that the *SSR* is a Chi-square with previously stated *df*:

$$\text{(i) } \sigma^2\frac{1}{\sigma^2}(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\,\sigma^2\frac{1}{\sigma^2}(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2 = \sigma^2\frac{1}{\sigma^2}(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2 \quad (111)$$

$$\text{(ii) } (\mathbf{X}\boldsymbol{\beta})'\frac{1}{\sigma^2}(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2\frac{1}{\sigma^2}(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{X}\boldsymbol{\beta}) \quad (112)$$

$$=(\mathbf{X}\boldsymbol{\beta})'\frac{1}{\sigma^2}(\mathbf{H}-\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{X}\boldsymbol{\beta})$$

$$\text{(iii) } \sigma^2\frac{1}{\sigma^2}(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2\frac{1}{\sigma^2}(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{X}\boldsymbol{\beta}) \quad (113)$$

$$= \sigma^2\frac{1}{\sigma^2}(\mathbf{H} - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')(\mathbf{X}\boldsymbol{\beta})$$

Since these conditions hold, we can now apply Theorem K:

$$\mathbf{y}'\frac{1}{\sigma^2}(\mathbf{H}-\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\mathbf{y} \sim \chi^2(\mathrm{tr}(\frac{1}{\sigma^2}(\mathbf{H}-\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2), (\mathbf{X\beta})'(\mathbf{X\beta})/2\sigma^2) \qquad (114)$$

And the degrees of freedom can be reduced as follows:

$$\mathrm{tr}(\frac{1}{\sigma^2}(\mathbf{H}-\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')\sigma^2) = \mathrm{tr}(\mathbf{H}-\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') \qquad (115)$$

$$= \mathrm{tr}(\mathbf{H}) - \mathrm{tr}(\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') \text{ by property G (iii)} \qquad (116)$$

$$= k + 1 - \mathrm{tr}((\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{1}) \qquad (117)$$

$$= k + 1 - 1 \qquad (118)$$

$$= k. \qquad (119)$$

Also, the non-centrality parameter can be evaluated at $\boldsymbol{\beta} = \mathbf{0}$ because we are assuming the null hypothesis is true so $(\mathbf{X\beta})'(\mathbf{X\beta})/2\sigma^2 = 0$ thus $SSR \sim \chi^2(k)$.

Appendix C - Generalized Least Squares Proof

In Generalized Least Squares (GLS) regression we will assume the errors follow a Multivariate Normal distribution such that $\boldsymbol{\varepsilon} \sim MVN(\mathbf{0}, \boldsymbol{\Sigma})$ and thus $\mathbf{Y} \mid \mathbf{X} \sim MVN(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, ie. we no longer assume constant variance and zero correlations.

Properties of $\hat{\boldsymbol{\beta}}$ for GLS I and II

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{y} \tag{120}$$

$$E(\mathbf{Y}) = E(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}) = E(\mathbf{X}\boldsymbol{\beta}) + E(\boldsymbol{\varepsilon}) = \mathbf{X}\boldsymbol{\beta} = \boldsymbol{\mu} \tag{121}$$

$$V(\mathbf{Y} \mid \mathbf{X}) = V(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}) = V(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma} \tag{122}$$

We will show that $\hat{\boldsymbol{\beta}}$ is an unbiased estimator for $\boldsymbol{\beta}$:

$$E(\hat{\boldsymbol{\beta}}) = \mathbf{E}\left[(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{y}\right] \tag{123}$$

$$= (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}E(\mathbf{y}) \tag{124}$$

$$= (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}E(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}) \tag{125}$$

$$= (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\mathbf{X}\boldsymbol{\beta} + \mathbf{0}) \tag{126}$$

$$= (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boldsymbol{\beta} \tag{127}$$

$$= \boldsymbol{\beta} \tag{128}$$

The variance-covariance matrix for $\hat{\boldsymbol{\beta}}$ is:

$$V(\hat{\boldsymbol{\beta}}) = V\left[(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{y}\right] \tag{129}$$

$$= (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{V}(\mathbf{y})\left[(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\right]' \tag{130}$$

$$= (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\left[(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\right] \tag{131}$$

$$= \left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'^{-1}\mathbf{X}\left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1} \quad (132)$$

$$= \left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1} \quad (133)$$

$$= \left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1} \quad (134)$$

Matrix Definitions for GLS

$$\mathbf{Y} \mid \mathbf{X} \sim MVN(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}) \quad (135)$$

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{y} \quad (136)$$

Define **A** to be:

$$\mathbf{A} = \mathbf{X}\left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}. \quad (137)$$

We will now examine some properties of these matrices that we will need for latter proofs:

$$\boldsymbol{\Sigma}^{-1}\mathbf{A} = \boldsymbol{\Sigma}^{-1}\mathbf{X}\left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1} = \mathbf{A}'\boldsymbol{\Sigma}^{-1} = (\boldsymbol{\Sigma}^{-1}\mathbf{A})' \quad (138)$$

$$\mathbf{AA} = \mathbf{X}\left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1} = \mathbf{A} \quad (139)$$

$$(\mathbf{I}-\mathbf{A})(\mathbf{I}-\mathbf{A}) = \mathbf{I}\,\mathbf{I} - \mathbf{I}\,\mathbf{A} - \mathbf{A}\,\mathbf{I} + \mathbf{A}\,\mathbf{A} = \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} = \mathbf{I} - \mathbf{A} \quad (140)$$

Because **A=AA,** then **A** is idempotent however since **A≠A'** then **A** is not symmetric.

SSE for GLS II

We show that the *SSE* follows a Chi-square distribution with $IJ - k - 1$ degrees of freedom. We must first manipulate the form of the *SSE* so that we can apply the theorems from Appendix A:

$$SSE = \left(\mathbf{y} - \hat{\mathbf{y}}\right)'\boldsymbol{\Sigma}^{-1}\left(\mathbf{y} - \hat{\mathbf{y}}\right) \quad (141)$$

$$= \left(\mathbf{y}'\boldsymbol{\Sigma}^{-1} - \mathbf{y}'\mathbf{A}'\boldsymbol{\Sigma}^{-1}\right)\left(\mathbf{y} - \mathbf{A}\mathbf{y}\right) \quad (142)$$

$$=\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y}-\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{y}-\mathbf{y}'\mathbf{A}'\mathbf{\Sigma}^{-1}\mathbf{y}+\mathbf{y}'\mathbf{A}'\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{y} \tag{143}$$

$$=\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y}-\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{y}-\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{y}+\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{y} \tag{144}$$

$$=\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y}-\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{y} \tag{145}$$

$$=\mathbf{y}'\mathbf{\Sigma}^{-1}(\mathbf{I}-\mathbf{A})\mathbf{y}\,. \tag{146}$$

We can use Theorem K under three certain conditions to show that the *SSE* follows a Chi-square distribution with previously stated *df*:

$$\text{(i)}\ \mathbf{\Sigma\Sigma^{-1}(I - A)\,\Sigma\Sigma^{-1}(I - A)\,\Sigma = \Sigma\Sigma^{-1}(I - A)\,(I - A)\,\Sigma = \Sigma\Sigma^{-1}(I - A)\,\Sigma} \tag{147}$$

$$\text{(ii)}\ \mathbf{(X\beta)'\,\Sigma^{-1}(I - A)\,\Sigma\,\Sigma^{-1}(I - A)\,(X\beta) = (X\beta)'\,\Sigma^{-1}(I - A)\,(X\beta)} \tag{148}$$

$$\text{(iii)}\ \mathbf{\Sigma\Sigma^{-1}(I - A)\,\Sigma\Sigma^{-1}(I - A)\,(X\beta) = \Sigma\Sigma^{-1}(I - A)\,(X\beta)}. \tag{149}$$

All of these conditions hold so we will now apply Theorem K:

$$\mathbf{y}'\mathbf{\Sigma}^{-1}(\mathbf{I}-\mathbf{A})\mathbf{y}\sim\chi^2\left(tr\left(\mathbf{\Sigma}^{-1}(\mathbf{I}-\mathbf{A})\mathbf{\Sigma}\right),(\mathbf{X}\boldsymbol{\beta})'\mathbf{\Sigma}^{-1}(\mathbf{I}-\mathbf{A})(\mathbf{X}\boldsymbol{\beta})/2\right). \tag{150}$$

And the degrees of freedom can be reduced as follows:

$$tr\left(\mathbf{\Sigma}^{-1}(\mathbf{I}-\mathbf{A})\mathbf{\Sigma}\right)=tr\left(\mathbf{\Sigma}^{-1}\mathbf{\Sigma}-\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{\Sigma}\right) \tag{151}$$

$$=tr(\mathbf{\Sigma}^{-1}\mathbf{\Sigma})-tr(\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{\Sigma})\ \text{by property G (iii)} \tag{152}$$

$$=tr(\mathbf{\Sigma}^{-1}\mathbf{I}_{\mathbf{IJ}}\mathbf{\Sigma})-tr(\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{\Sigma}) \tag{153}$$

$$=tr(\mathbf{I}_{\mathbf{IJ}})-tr(\mathbf{A})\ \text{by property G (iv)} \tag{154}$$

$$=I\,J-tr\left(\mathbf{X}\left(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\right) \tag{155}$$

$$=I\,J-tr\left(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\right)^{-1}\right)\ \text{by property G (i)} \tag{156}$$

$$=I\,J-tr(\mathbf{I}_{\mathbf{k+1,k+1}}) \tag{157}$$

$$=I\,J-(k+1) \tag{158}$$

$$= I\,J - k - 1. \tag{159}$$

And the non-centrality parameter can be reduced as follows:

$$(\mathbf{X\beta})'\mathbf{\Sigma}^{-1}(\mathbf{I}-\mathbf{A})(\mathbf{X\beta})/2 = \left[(\mathbf{X\beta})'\mathbf{\Sigma}^{-1}\mathbf{I}(\mathbf{X\beta}) - (\mathbf{X\beta})'\mathbf{\Sigma}^{-1}\mathbf{A}(\mathbf{X\beta})\right]/2 \tag{160}$$

$$= \left[(\mathbf{X\beta})'\mathbf{\Sigma}^{-1}(\mathbf{X\beta}) - \mathbf{\beta}'\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}(\mathbf{X\beta})\right]/2 \tag{161}$$

$$= \left[(\mathbf{X\beta})'\mathbf{\Sigma}^{-1}(\mathbf{X\beta}) - \mathbf{\beta}'\mathbf{X}'\mathbf{\Sigma}^{-1}(\mathbf{X\beta})\right]/2 \tag{162}$$

$$= 0. \tag{163}$$

Thus our $SSE \sim \chi^2(I\,J - k - 1)$.

SST for GLS II

We show that the *SST* follows a Chi-square distribution with $I\,J$ degrees of freedom. We can use Theorem K under three certain conditions to show that the $SST = \mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y}$ is a Chi-square with previously stated *df*:

$$\text{(i) } \mathbf{\Sigma\Sigma}^{-1}\mathbf{\Sigma\Sigma}^{-1}\mathbf{\Sigma} = \mathbf{\Sigma\Sigma}^{-1}\mathbf{\Sigma} \tag{164}$$

$$\text{(ii) } (\mathbf{X\beta})'\mathbf{\Sigma}^{-1}\mathbf{\Sigma\Sigma}^{-1}(\mathbf{X\beta}) = (\mathbf{X\beta})'\mathbf{\Sigma}^{-1}(\mathbf{X\beta}) \tag{165}$$

$$\text{(iii) } \mathbf{\Sigma\Sigma}^{-1}\mathbf{\Sigma\Sigma}^{-1}\mathbf{X\beta} = \mathbf{\Sigma\Sigma}^{-1}\mathbf{X\beta} \tag{166}$$

All of these conditions hold so: $\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y} \sim \chi^2(tr(\mathbf{\Sigma}^{-1}\mathbf{\Sigma}), (\mathbf{X\beta})'\mathbf{\Sigma}^{-1}(\mathbf{X\beta})/2)$. And the degrees of freedom can be reduced as follows:

$$tr(\mathbf{\Sigma}^{-1}\mathbf{\Sigma}) = tr(\mathbf{I}_{\mathbf{IJ}}) \tag{167}$$

$$= I\,J \tag{168}$$

Also, the non-centrality parameter can be evaluated at $\mathbf{\beta} = \mathbf{0}$ because we are assuming the null hypothesis is true so:

$$(\mathbf{X\beta})'\mathbf{\Sigma}^{-1}(\mathbf{X\beta})/2 = 0. \tag{169}$$

Then $\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y} \sim \chi^2(IJ)$

SSR for GLS II

We show that the *SSR* follows a Chi-square distribution with $k + 1$ *df*. We must first manipulate the form of the *SSR* into something that we can apply the Theorems from Appendix A:

$$SSR = SST - SSE \tag{170}$$

$$= \mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y} - \left(\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y} - \mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{y}\right) \tag{171}$$

$$= \mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{y} \tag{172}$$

We can use Theorem K under three certain conditions to show that the *SSE* is a Chi-square with previously stated *df*:

$$\text{(i) } \mathbf{\Sigma\Sigma}^{-1}\mathbf{A\Sigma\Sigma}^{-1}\mathbf{A\Sigma} = \mathbf{\Sigma\Sigma}^{-1}\mathbf{AA\Sigma} = \mathbf{\Sigma\Sigma}^{-1}\mathbf{A\Sigma} \tag{173}$$

$$\text{(ii) } (\mathbf{X\beta})'\mathbf{\Sigma}^{-1}\mathbf{A\Sigma\Sigma}^{-1}\mathbf{A}(\mathbf{X\beta}) = (\mathbf{X\beta})'\mathbf{\Sigma}^{-1}\mathbf{A}(\mathbf{X\beta}) \tag{174}$$

$$\text{(iii) } \mathbf{\Sigma\Sigma}^{-1}\mathbf{A\Sigma\Sigma}^{-1}\mathbf{AX\beta} = \mathbf{\Sigma\Sigma}^{-1}\mathbf{AX\beta} \tag{175}$$

All of these conditions hold so:

$$\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{y} \sim \chi^2\left(tr\left(\mathbf{\Sigma}^{-1}\mathbf{A\Sigma}\right), (\mathbf{X\beta})'\mathbf{\Sigma}^{-1}\mathbf{A}(\mathbf{X\beta})/2\right). \tag{176}$$

And the degrees of freedom can be reduced as follows:

$$tr\left(\mathbf{\Sigma}^{-1}\mathbf{A\Sigma}\right) = tr(\mathbf{A}) \text{ by property G (iv)} \tag{177}$$

$$= tr(\mathbf{X}\left(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}) \tag{178}$$

$$= tr(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\right)^{-1}) \text{ by property G (ii)} \tag{179}$$

$$= tr(\mathbf{I}_{\mathbf{k+1,k+1}}) \tag{180}$$

$$= \mathrm{k} + 1. \tag{181}$$

Also, the non-centrality parameter can be evaluated at $\boldsymbol{\beta} = \mathbf{0}$ because we are assuming the null hypothesis is true so: $(\mathbf{X}\boldsymbol{\beta})'\boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{X}\boldsymbol{\beta})/2 = 0$. Finally, we have that the $SSR \sim \chi^2(k+1)$.

Appendix D - Method of Moments for Fisher's F Distribution

We can find an estimate for the degrees of freedom of the *F* test by performing a method of moments calculation for the degrees of freedom, υ_1 and υ_2. The first moment and second central moment of an *F* distribution are (Wackerly et al., 2002):

$$E(F) = \frac{\upsilon_2}{\upsilon_2 - 2} \tag{182}$$

$$V(F) = \frac{2\upsilon_2^2(\upsilon_1 + \upsilon_2 - 2)}{\upsilon_1(\upsilon_2 - 2)^2(\upsilon_2 - 4)}. \tag{183}$$

Because the 10,000 sets of data are simulated, we can obtain a sample mean of the F statistics, $\overline{F}$, and a sample variance, s_F^2, from our simulated data and estimate the parameters as follows:

$$\overline{F} = \frac{\hat{\upsilon}_2}{\hat{\upsilon}_2 - 2} \tag{184}$$

$$s_F^2 = \frac{2\hat{\upsilon}_2^2(\hat{\upsilon}_1 + \hat{\upsilon}_2 - 2)}{\hat{\upsilon}_1(\hat{\upsilon}_2 - 2)^2(\hat{\upsilon}_2 - 4)}. \tag{185}$$

We obtain an estimate of the denominator degree of freedom using equation (185) :

$$\hat{\upsilon}_2 = \frac{2\overline{F}}{\overline{F} - 1}. \tag{186}$$

Next we set the second moment of the Fisher's F distribution equal to the sample variance. Now we can solve for $\hat{\upsilon}_1$ by using both equations (185) and (186).

$$s_F^2 = \frac{2\left(\frac{2\overline{F}}{\overline{F}-1}\right)^2\left(\hat{\upsilon}_1 + \frac{2\overline{F}}{\overline{F}-1} - 2\right)}{\hat{\upsilon}_1\left(\frac{2\overline{F}}{\overline{F}-1} - 2\right)^2\left(\frac{2\overline{F}}{\overline{F}-1} - 4\right)} \tag{187}$$

$$s_F^2 = \frac{2\left(\frac{2\overline{F}}{\overline{F}-1}\right)^2\left(\hat{\upsilon}_1 + \frac{2}{\overline{F}-1}\right)}{\hat{\upsilon}_1\left(\frac{2}{\overline{F}-1}\right)^2\left(\frac{2\overline{F}}{\overline{F}-1} - 4\right)} \tag{188}$$

$$s_F^2 = \frac{2\overline{F}^2\left(\hat{\upsilon}_1 + \frac{2}{\overline{F}-1}\right)}{\hat{\upsilon}_1\left(\frac{-2\overline{F}+4}{\overline{F}-1}\right)} \tag{189}$$

$$\hat{\upsilon}_1\left(\frac{-2\overline{F}+4}{\overline{F}-1}\right)s_F^2 = 2\overline{F}^2\hat{\upsilon}_1 + \frac{4\overline{F}^2}{\overline{F}-1} \tag{190}$$

$$\hat{\upsilon}_1\left(-2\overline{F}+4\right)s_F^2 - 2\overline{F}^2\hat{\upsilon}_1\left(\overline{F}-1\right) = 4\overline{F}^2 \tag{191}$$

$$\hat{\upsilon}_1 = \frac{4\overline{F}^2}{\left(-2\overline{F}+4\right)s_F^2 - 2\overline{F}^2\left(\overline{F}-1\right)} \tag{192}$$

$$\hat{\upsilon}_1 = \frac{4\overline{F}^2}{-2\overline{F}s_F + 4s_F^2 - 2\overline{F}^3 + 2\overline{F}^2} \tag{193}$$

$$\hat{\upsilon}_1 = \frac{2\overline{F}^2}{-\overline{F}^3 + \overline{F}^2 - \overline{F}s_F + 2s_F^2} \tag{194}$$

The resulting values are estimates for the degrees of freedom for our *F* test; our two estimated parameters are :

$$\hat{\upsilon}_2 = \frac{2\overline{F}}{\overline{F}-1} \tag{195}$$

$$\hat{\upsilon}_1 = \frac{2\overline{F}^2}{-\overline{F}^3 + \overline{F}^2 - \overline{F}s_F + 2s_F} \tag{196}$$

Appendix E - ANOVA Sum of Squares

Definitions of the sample means:

$$\bar{y}_{\bullet\bullet} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij}\ ,\quad \bar{y}_{i\bullet} = \frac{1}{J}\sum_{j=1}^{J} y_{ij}\ ,\ \text{and}\ \bar{y}_{\bullet\bullet} = \frac{1}{I}\sum_{i=1}^{I}\bar{y}_{i\bullet}\ . \qquad (197\text{-}199)$$

The distributions of the various components of the *Sum of Squares*:

$$y_{ij} \sim \mathrm{N}(\mu,\sigma^2),\ \bar{y}_{i\bullet} \sim \mathrm{N}(\mu,\sigma^2),\ \text{and}\ \bar{y}_{\bullet\bullet} \sim \mathrm{N}(\mu,\sigma^2). \qquad (200\text{-}202)$$

Description of Table 3.1 – Univariate ANOVA

$$SSE = \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{i\bullet})^2 \qquad (203)$$

$$SSE \sim \chi^2_{IJ-I} \qquad (204)$$

$$SSR = \sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet})^2 \qquad (205)$$

$$SSR \sim \chi^2_{I-1} \qquad (206)$$

$$SST = \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{\bullet\bullet})^2 \qquad (207)$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{i\bullet} + \bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet})^2 \qquad (208)$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{i\bullet})^2 + 2\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{i\bullet})(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet}) + \sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet})^2 \qquad (209)$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{i\bullet})^2 + 2\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}\bar{y}_{i\bullet} - y_{ij}\bar{y}_{\bullet\bullet} - \bar{y}_{i\bullet}\bar{y}_{i\bullet} + \bar{y}_{i\bullet}\bar{y}_{\bullet\bullet}) + \sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet})^2 \qquad (210)$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{i\bullet})^2 + 2\sum_{i=1}^{I}\bar{y}_{i\bullet}\sum_{j=1}^{J}y_{ij} - 2\bar{y}_{\bullet\bullet}\sum_{i=1}^{I}\sum_{j=1}^{J}y_{ij} - 2J\sum_{i=1}^{I}\bar{y}_{i\bullet}\bar{y}_{i\bullet} + 2J\bar{y}_{\bullet\bullet}\sum_{i=1}^{I}\bar{y}_{i\bullet} + \sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet})^2 \qquad (211)$$

$$=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{i\bullet})^2+2J\sum_{i=1}^{I}\bar{y}_{i\bullet}\bar{y}_{i\bullet}-2IJ\bar{y}_{\bullet\bullet}\bar{y}_{\bullet\bullet}-2J\sum_{i=1}^{I}\bar{y}_{i\bullet}\bar{y}_{i\bullet}+2J\bar{y}_{\bullet\bullet}\sum_{i=1}^{I}\bar{y}_{i\bullet}+\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\bullet}-\bar{y}_{\bullet\bullet})^2 \quad (212)$$

$$=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{i\bullet})^2-2IJ\,\bar{y}_{\bullet\bullet}\bar{y}_{\bullet\bullet}+2J\,\bar{y}_{\bullet\bullet}\sum_{i=1}^{I}\bar{y}_{i\bullet}+\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\bullet}-\bar{y}_{\bullet\bullet})^2 \quad (213)$$

$$=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{i\bullet})^2-2IJ\,\bar{y}_{\bullet\bullet}\bar{y}_{\bullet\bullet}+2IJ\,\bar{y}_{\bullet\bullet}\bar{y}_{\bullet\bullet}+\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\bullet}-\bar{y}_{\bullet\bullet})^2 \quad (214)$$

$$=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{i\bullet})^2+\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\bullet}-\bar{y}_{\bullet\bullet})^2 \quad (215)$$

$$=SSE+SSR \quad (216)$$

$$SST\sim\chi^2_{IJ-1} \quad (217)$$

Descriptions for Table 2.1- ANOVA with one Within-Subject Factor

$$SSBS=\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i\bullet}-\bar{y}_{\bullet\bullet})^2 \quad (218)$$

$$SSBS\sim\chi^2_{I-1} \quad (219)$$

$$SSWS=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{i\bullet})^2 \quad (220)$$

$$SSWS\sim\chi^2_{I(J-1)} \quad (221)$$

$$SSTR=\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{\bullet j}-\bar{y}_{\bullet\bullet})^2 \quad (222)$$

$$SSTR\sim\chi^2_{J-1} \quad (223)$$

$$SSE=\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{i\bullet})^2-\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{\bullet j}-\bar{y}_{\bullet\bullet})^2 \quad (224)$$

$$=SSWS-SSTR \quad (225)$$

$$SSE \sim \chi^2_{(I-1)(J-1)} \tag{226}$$

$$SST = \sum_{i=1}^{I} \sum_{j=1}^{J} \left(y_{ij} - \bar{y}_{\bullet\bullet}\right)^2 \tag{227}$$

$$SST \sim \chi^2_{IJ-1} \tag{228}$$

Appendix F - Possible Restrictions on the Design Matrix

The GLS III requires $J + 1$ to be less than I because if this is not the case, the design matrix (**X**) will be I by $J + 1$ and **X'X** will be singular. This would several calculations that must be performed impossible, so we will restrict $J + 1 < I$ in all of our simulations.

Here we will do a proof by counter example showing that J + 1 < I must be the case. For example, if we let $J + 1 > I$ then $rank(\mathbf{X}) \leq I$ and it can be shown that:

$$rank(\mathbf{X'X}) \leq \min(rank(\mathbf{X'}), rank(\mathbf{X})) \leq \min(I, J+1) = I. \tag{229}$$

X'X is a $(J + 1)$ x $(J + 1)$ matrix where $J + 1 > I$. Therefore, the rank (**X'X**) will always be less than its column space, which makes it singular and non-invertible. This would be a large problem in several calculations that must be performed so we will restrict $J + 1 < I$ in all of our simulations.