Dynamic Characteristics of Pneumatic Transmission Lines and its Equivalent Systems

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Dynamic characteristics of pneumatic transmission line systems composed of terminal volumes and lines with multiple junctions are analyzed. Frequency characteristics are derived from the transfer functions and transient responses are calculated by the characteristic curve method, and they are compared with the experimental results.

Moreover, instead of the exact expression of complicated system with junction such as branching system, volume terminated single line with equivalent length and equivalent volume is proposed by using its similarity with the second order oscillatory system. Difference between the exact and its equivalent systems are compared by frequency and transient responses and good agreements are obtained.

1. Nomenclature

A : heat equivalence of mechanical energy [kcal/kgm]
a : radius of tube [m]
c : acoustic phase velocity [m/s]
c_p : specific heat capacity at constant pressure [kcal/kg°C]
c_v : specific heat capacity at constant volume [kcal/kg°C]
g : gravitational coefficient [m/s^2]
J_n : Bessel function of the 1st kind of order n
$j \equiv \sqrt{-1}$ [dl]
L : length of transmission line [m]
M : mass flow [(kg/s)/(m/s^2)]
n : polytropic exponent [dl]
P : pressure [kgf/m^2]
Pr : Prandtl number [dl]
Q : volume of terminal element [m^3]
r : radius position [m]
s : Laplace variable respect to time [1/s]
t : time[s]
v : velocity [m/s]
x : axial position [m]
Y : shunt admittance per unit length [ms/m]

Z : series impedance per unit length [1/m²s]
Z_c : characteristic impedance [1/ms]
Z_r : load impedance [1/ms]
Γ : propagation operator [1/m]
$\varkappa \equiv c_p/c_v$: ratio of specific heats for perfect gas [dl]
λ : thermal conductivity of perfect gas [kcal/ms°C]
μ : viscosity coefficient [kgf·s/m²]
ν : kinematic viscosity [m²/s]
ρ : gas density [(kg/m³)/(m/s²)]
θ : temperature of gas [°C]
※ subscript $_0$ means the steady state value

2. Dynamic Analysis

The following basic assumptions are made in deriving the fundamental equations of a compressible fluid:

(a) The flow and temperature distribution across the pipe sectional area are axi-symmetric.
(b) The radial velocity is very small compared to the axial velocity.
(c) There is no force from the outside.
(d) The pressure is uniform across the sectional area.
(e) There is no heat supply from the outside.

Therefore, from the equations of momentum, energy, continuity and state, the first order (small-amplitude) acoustic equations in the cylidrical line case are given below.

momentum;
$$\frac{\partial v}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x}-\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right)=0 \tag{1}$$

energy;
$$\frac{\partial \theta}{\partial t}-\frac{A}{\rho g C_P}\cdot\frac{\partial p}{\partial t}-\frac{\lambda}{\rho g C_P}\left(\frac{\partial^2 \theta}{\partial r^2}+\frac{1}{r}\frac{\partial \theta}{\partial r}\right)=0 \tag{2}$$

continuity;
$$\frac{\partial \rho}{\partial t}+\rho\frac{\partial v}{\partial x}=0 \tag{3}$$

state for gas;
$$\frac{dp}{p}=\frac{d\rho}{\rho}+\frac{d\theta}{\theta} \tag{4}$$

Taking the Laplace transforms of eqs. (1)~(4) with respect to t, solving them under the boundary condition local velocity must be zero at r=a and integrating the resulting equations over the cross sectional area, and combining them to obtain the relation between the mass flow and the pressure gradient along the longitudinal direction, yields

$$\frac{\partial P}{\partial x}=-Z\cdot M \tag{5}$$

where

$$Z=\frac{s}{\pi a^2}\cdot\frac{1}{1-\dfrac{2J_1(ja\sqrt{s/\nu_0})}{ja\sqrt{s/\nu_0}\,J_0(ja\sqrt{s/\nu_0})}}$$

Similarly solving the temperature distribution over the cross section under the boundary condition the temperature must be equal to zero at $r=a$, and combining eqs. (3) and (4) to obtain the relation between the pressure and the mass flow gradient along the longitudinal direction, yields

$$\frac{\partial M}{\partial x}=-Y\cdot P \tag{6}$$

where

$$Y=\frac{\pi a^2\rho_0 s}{\kappa P_0}\left\{1+\frac{2(\kappa-1)J_1(ja\sqrt{Pr\cdot s/\nu_0})}{ja\sqrt{Pr\cdot s/\nu_0}\,J_0(ja\sqrt{Pr\cdot s/\nu_0})}\right\}$$

Therefore, rearranging eqs. (5) and (6) into the vector partial differential equation form, the basic equation for the further analysis of acoustic behavior of a gas in a tube can be written in the form

$$\frac{\partial}{\partial x}\begin{bmatrix}P\\M\end{bmatrix}=\begin{bmatrix}0&-Z\\-Y&0\end{bmatrix}\begin{bmatrix}P\\M\end{bmatrix} \tag{7}$$

From eq. (7), another useful representation is given,

$$\begin{bmatrix}P(0,s)\\M(0,s)\end{bmatrix}=\begin{bmatrix}\cosh\Gamma x & Zc\sinh\Gamma x\\ 1/Zc\sinh\Gamma x & \cosh\Gamma x\end{bmatrix}\begin{bmatrix}P(x,s)\\M(x,s)\end{bmatrix} \tag{8}$$

The transfer function relating terminal points is, then

$$\frac{P(L,s)}{P(0,s)}=\frac{1}{\cosh\Gamma L+Zc/Zr\cdot\sinh\Gamma L} \tag{9}$$

whereas the transfer function can be expanded in the form

$$\frac{P(L,s)}{P(0,s)}=\frac{2}{1+Zc/Zr}e^{-\Gamma L}-\frac{2}{1+Zc/Zr}\left(\frac{1-Zc/Zr}{1+Zc/Zr}\right)e^{-3\Gamma L}+\frac{2}{1+Zc/Zr}\left(\frac{1-Zc/Zr}{1+Zc/Zr}\right)^2 e^{-5\Gamma L}-\cdots\cdots \tag{10}$$

where

$$\frac{Zc}{Zr}=\frac{Q}{\pi a^2}\sqrt{ZY} \tag{11}$$

and

$$Zr=\frac{P(L,s)}{M(L,s)}\propto\frac{1}{Q} \tag{12}$$

Therefore, in case of blocked line without terminal volume, the transfer function and its expanded fom will become

$$\frac{1}{\cosh\Gamma L}=2e^{-\Gamma L}-2e^{-3\Gamma L}+2e^{-5\Gamma L}-\cdots\cdots \tag{13}$$

while the propagation operator Γ is expressed as follows using the asymptotic expansion.

$$\Gamma=\sqrt{ZY}=\frac{s}{c}\left[1+\left(1-\frac{\kappa-1}{\sqrt{\sigma}}\right)\left(\frac{\nu}{a^2 s}\right)^{1/2}+\left\{1+\frac{\kappa-1}{\sqrt{\sigma}}-\frac{\kappa(\kappa-1)}{2\sigma}\right\}\left(\frac{\nu}{a^2 s}\right)\cdots\cdots\right] \tag{14}$$

where

$$\sigma = \frac{\nu_0 \cdot L/C}{a^2}$$

From eqs. (11), (14) Zc／Zr will be proportional to s^1 at high frequencies for volume terminated line and consequently at eq. (10)

$$\left| \frac{2}{1+Zc/Zr} \right| \ll 2.0 \tag{15}$$

Now consider the volume terminated network containing 2 junctions (Fig. 1)

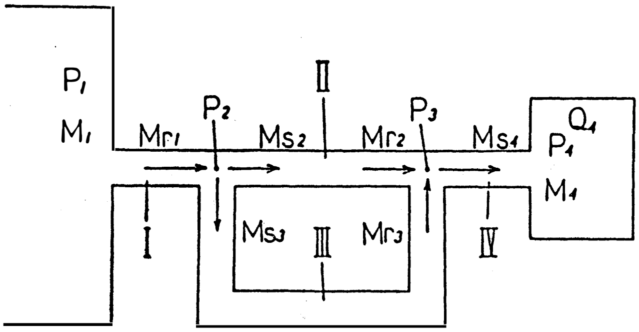

Fig. 1

Line matrix equations for I ～ IV are written as follows

$$\mathrm{I}: \begin{bmatrix} P_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} \cosh\Gamma_1 L_1 & Zc_1 \cdot \sinh\Gamma_1 L_1 \\ 1/Zc_1 \cdot \sinh\Gamma_1 L_1 & \cosh\Gamma_1 L_1 \end{bmatrix} \begin{bmatrix} P_2 \\ Mr_1 \end{bmatrix} \tag{16}$$

$$\mathrm{II}: \begin{bmatrix} P_2 \\ Ms_2 \end{bmatrix} = \begin{bmatrix} \cosh\Gamma_2 L_2 & Zc_2 \cdot \sinh\Gamma_2 L_2 \\ 1/Zc_2 \cdot \sinh\Gamma_2 L_2 & \cosh\Gamma_2 L_2 \end{bmatrix} \begin{bmatrix} P_3 \\ Mr_2 \end{bmatrix} \tag{17}$$

$$\mathrm{III}: \begin{bmatrix} P_2 \\ Ms_3 \end{bmatrix} = \begin{bmatrix} \cosh\Gamma_3 L_3 & Zc_3 \cdot \sinh\Gamma_3 L_3 \\ 1/Zc_3 \cdot \sinh\Gamma_3 L_3 & \cosh\Gamma_3 L_3 \end{bmatrix} \begin{bmatrix} P_3 \\ Mr_3 \end{bmatrix} \tag{18}$$

$$\mathrm{IV}: \begin{bmatrix} P_3 \\ Ms_4 \end{bmatrix} = \begin{bmatrix} \cosh\Gamma_4 L_4 & Zc_4 \cdot \sinh\Gamma_4 L_4 \\ 1/Zc_4 \cdot \sinh\Gamma_4 L_4 & \cosh\Gamma_4 L_4 \end{bmatrix} \begin{bmatrix} P_4 \\ M_4 \end{bmatrix} \tag{19}$$

from eqs. (16)～(19), transfer function relating P_1 and P_4 is

$$\frac{P_4}{P_1} = \frac{1}{\cosh\Gamma_1 L_1 + \frac{Zc_1}{Zr_1}\sinh\Gamma_1 L_1} \times \frac{1}{\cosh\Gamma_2 L_2 + \frac{Zc_2}{Zr_2}\sinh\Gamma_2 L_2} \times \frac{1}{\cosh\Gamma_4 L_4 + \frac{Zc_4}{Zr_4}\sinh\Gamma_4 L_4} \tag{20}$$

where, terminal impedance of line I is

$$Zr_1 = \frac{P_2}{Mr_1} \tag{21}$$

and mass balance at the junction of the line labeled I, II and III will be

$$Mr_1 = Ms_2 + Ms_3 \tag{22}$$

So Zr_1 is written as follows by applying eqs. (17), (18), (21) and (22).

$$Zr_1 = \frac{1}{\dfrac{1}{P_2/Ms_2} + \dfrac{1}{P_2/Ms_3}}$$

$$= \frac{1}{\dfrac{Zc_2\cdot\cosh\Gamma_2\ L_2 + Zr_2\cdot\sinh\Gamma_2\ L_2}{Zr_2\cdot\cosh\Gamma_2\ L_2 + Zc_2\cdot\sinh\Gamma_2\ L_2}\cdot\dfrac{1}{Zc_2} + \dfrac{Zc_3\cdot\cosh\Gamma_3\ L_3 + Zr_3\cdot\sinh\Gamma_3\ L_3}{Zr_3\cdot\cosh\Gamma_3\ L_3 + Zc_3\cdot\sinh\Gamma_3\ L_3}\cdot\dfrac{1}{Zc_3}} \tag{23}$$

More generally, consider the multiple junction where connects with k lines, mass balance will be

$$Mr_1 = Ms_2 + Ms_3 + \cdots\cdots + Ms_{k+1} \tag{24}$$

$$Mr_2 + Mr_3 + \cdots\cdots + Mr_{k+1} = Ms_{k+2} \tag{25}$$

Then, the terminal impedance of branch point: Zr_1

$$Zr_1 = \frac{1}{\sum_{i=2}^{K+1} \dfrac{Zci\cdot\cosh\Gamma i\ Li + Zri\cdot\sinh\Gamma i\ Li}{Zri\cdot\cosh\Gamma i\ Li + Zci\cdot\sinh\Gamma i\ Li}}$$

and the terminal impedance of confluence point: Zr_2

$$Zr_2 = \frac{\sum_{i=2}^{K+1} \dfrac{Zc_2\cdot \text{siuh}\,\Gamma_2\ L_2}{Zci\cdot\sinh\Gamma i\ Li}}{\dfrac{Zc_{k+2}\cdot\cosh\Gamma_{k+2}\ L_{k+2} + Zr_{k+2}\cdot\sinh\Gamma_{k+2}\ L_{k+2}}{Zr_{k+2}\cdot\cosh\Gamma_{k+2}\ L_{k+2} + Zc_{k+2}\cdot\sinh\Gamma_{k+2}\ L_{k+2}}\cdot\dfrac{1}{Zc_{k+2}} - \sum_{i=2}^{K+1}\dfrac{\cosh\Gamma_2\ L_2 - \cosh\Gamma i\ Li}{Zc_i\cdot\sinh\Gamma i\ Li}} \tag{27}$$

Terminal impedance of line Ⅳ will be written by applying the volume terminated line impedance

$$Zr_4 = \frac{nP_0}{Q_4\,\rho_0\,s} \tag{28}$$

In the case that the line specifications of Ⅱ, Ⅲ are exactly the same, the terminal impedances (26), (27) will be simplified as follows;

$$Zr_1 = \frac{1}{\dfrac{Zc_2\cdot\cosh\Gamma_2\ L_2 + Zr_2\cdot\sinh\Gamma_2 L_2}{Zr_2\cdot\cosh\Gamma_2\ L_2 + Zc_2\cdot\sinh\Gamma_2\ L_2}\cdot\dfrac{2}{Zc_2}} \tag{29}$$

$$Zr_2 = \frac{2}{\dfrac{Zc_4\cdot\cosh\Gamma_4\ L_4 + Zr_4\cdot\sinh\Gamma_4\ L_4}{Zr_4\cdot\cosh\Gamma_4\ L_4 + Zc_4\cdot\sinh\Gamma_4\ L_4}\cdot\dfrac{1}{Zc_4}} \tag{30}$$

By applying the impedances (29), (30), the transfer function P_4/P_1 (eq. (20)) will be rewritten in the form

$$\frac{P_4}{P_1} = \frac{1}{(\cosh\Gamma_1\ L_1\cdot\cosh\Gamma_2\ L_2 + 2\sinh\Gamma_1\ L_1\cdot\sinh\Gamma_2\ L_2)\cdot}$$

$$\overline{\qquad} \quad (31)$$

$$(\cosh \Gamma_4 L_4 + \frac{Zc_4}{Zr_4} \sinh \Gamma_4 L_4) +$$

$$\frac{1}{2}(\cosh \Gamma_1 L_1 \cdot \sinh \Gamma_2 L_2 + 2\sinh \Gamma_1 L_1 \cdot \cosh \Gamma_2 L_2) \cdot$$

$$(\sinh \Gamma_4 L_4 + \frac{Zc_4}{Zr_4} \cosh \Gamma_4 L_4)$$

Putting in order the denominator of eq. (31), it is easy to find by simple inspection, the transfer function is similar of volume terminated single line with equivalent length $\overline{L}$, (Fig. 2)

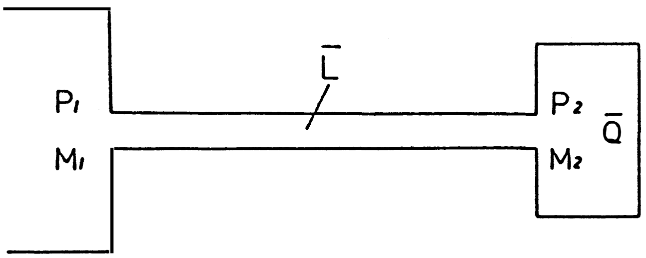

Fig. 2

$$\frac{P_{out}}{P_{in}} = \frac{1}{\cosh \Gamma(\overline{L_1} + \frac{1}{2}\overline{L_2} + \overline{L_4}) + \frac{Zc_4}{Zr_4} \sinh \Gamma(\overline{L_1} + \frac{1}{2}\overline{L_2} + \overline{L_4})} \tag{32}$$

and

$$\overline{L} = L_1 + \frac{1}{2}L_2 + L_4 \tag{33}$$

3. Experimental Results

Figure 6 is a schematic diagram of experimental set up. Tests were run using copper tubings with inner diameter 7.4mm. The sinusoidal pressure signals created by the combination of low-frequency signal generator and electro-pneumatic transducer and transmitted to the test line were detected at the sending and receiving end of the pneumatic system by use of semiconductor transducer and were recorded on the digital memoryscope simultaneously as the input and output signal. The amplitude of the pressure signals were kept small within 1~2mmAq. enough to satisfy the linearlity assumption. The step pressure signal was created by evacuating the line (to -25 mmAq) and then rupturing thin rubber membrane. The created pressure signals were kept small enough to satisfy the linearity assumption and sharp enough to be considered as stepwise signals with respect to the dynamic characteristics of the signal-subjected systems.

The pressure transducer at the input end was used for triggering the digital memoryscope to sweep the voltage change signal of the transducer at the output end, which was recorded by X−Y recorder.

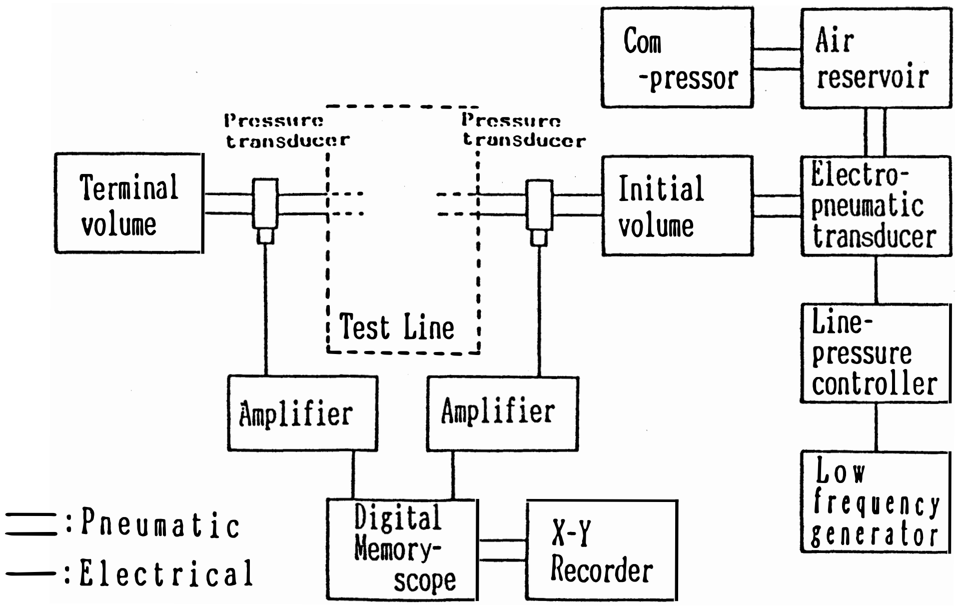

Fig. 3 Schematic diagram of experimental apparatus

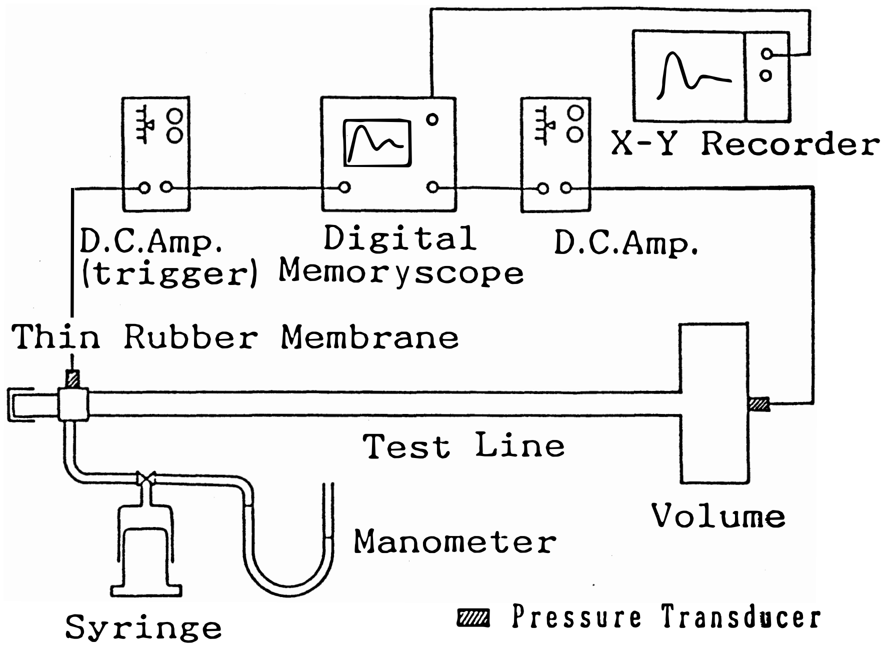

Fig. 4 Schematic diagram of test setup

Fig. 5～10 show the comparison between the theoretical results and the experimental results in the various kinds of combination of transmission lines for both frequency and transient characteristics.

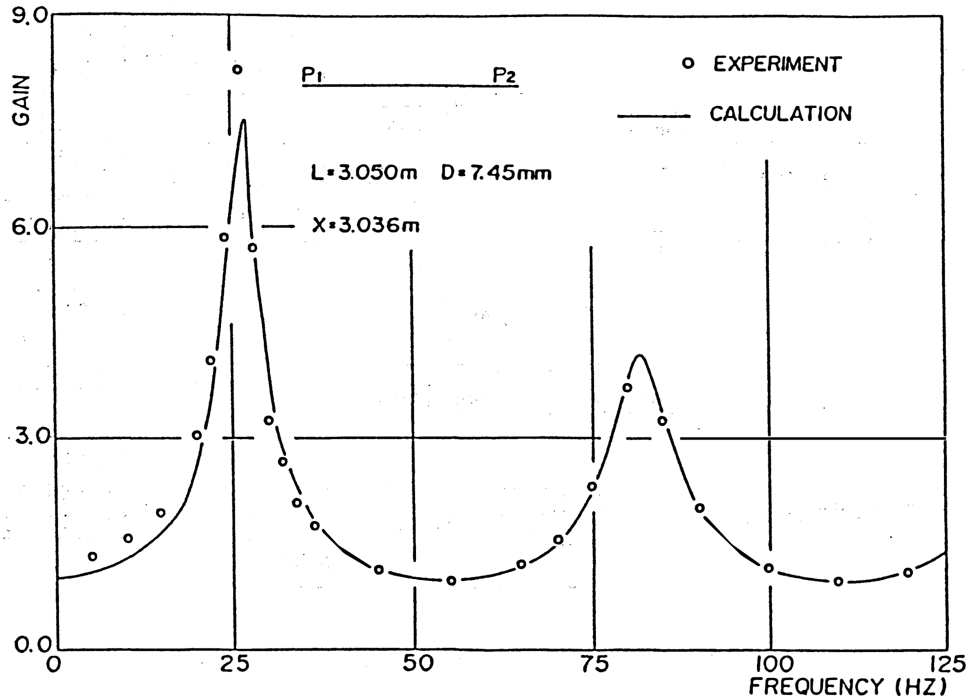
9.0
6.0
3.0
0.0
GAIN
P1
P2
o EXPERIMENT
CALCULATION
L=3.050m D=7.45mm
X=3.036m
0
25
50
75
100
125
FREQUENCY (HZ)

Fig. 5

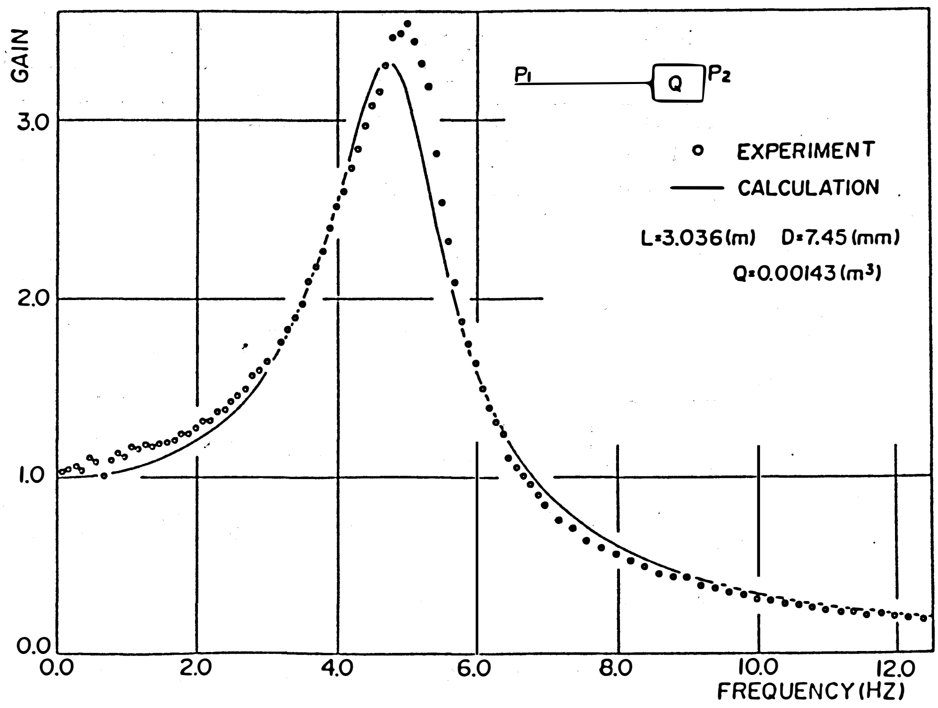
GAIN
3.0
2.0
1.0
0.0
P1
Q
P2
o EXPERIMENT
CALCULATION
L=3.036(m) D=7.45(mm)
Q=0.00143(m³)
0.0
2.0
4.0
6.0
8.0
10.0
12.0
FREQUENCY(HZ)

Fig. 6

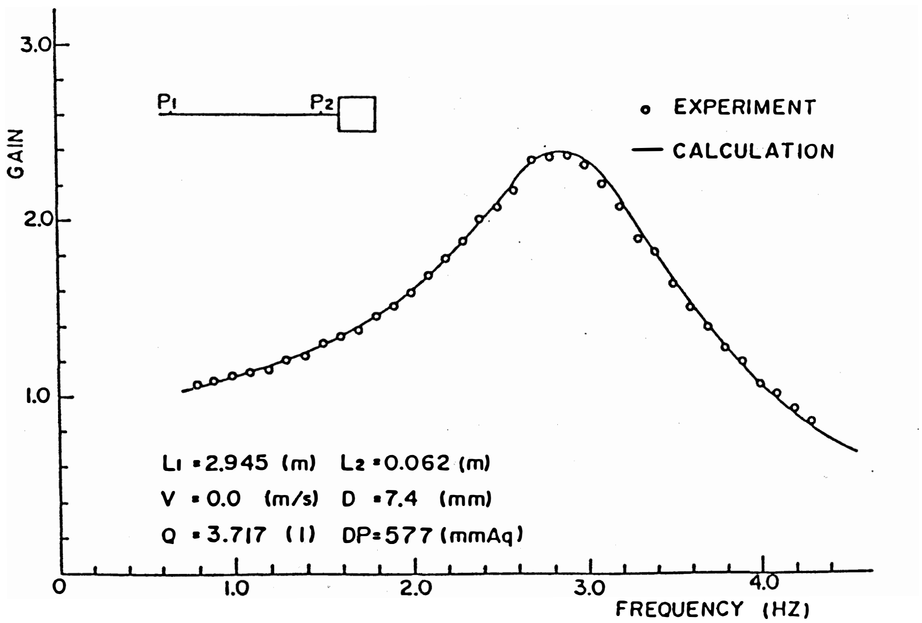
3.0
2.0
1.0
GAIN
P1
P2
o EXPERIMENT
— CALCULATION
L1 = 2.945 (m) L2 = 0.062 (m)
V = 0.0 (m/s) D = 7.4 (mm)
Q = 3.717 (l) DP = 577 (mmAq)
0
1.0
2.0
3.0
4.0
FREQUENCY (HZ)

Fig. 7

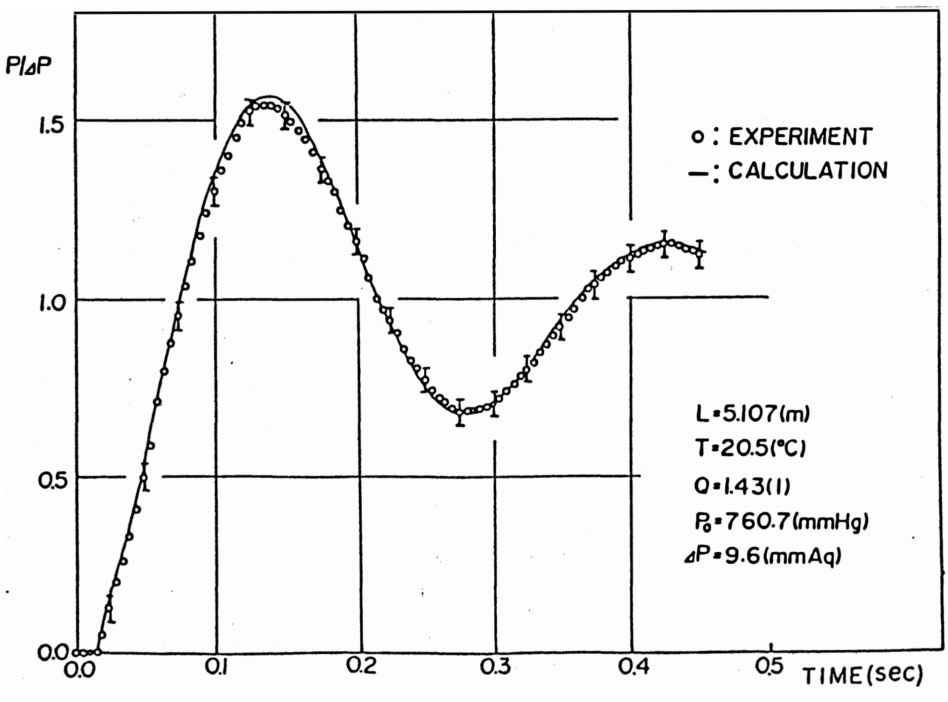
P/ΔP
1.5
1.0
0.5
0.0
o : EXPERIMENT
—: CALCULATION
L = 5.107 (m)
T = 20.5 (°C)
Q = 1.43 (l)
P0 = 760.7 (mmHg)
ΔP = 9.6 (mmAq)
0.0
0.1
0.2
0.3
0.4
0.5
TIME (sec)

Fig. 8

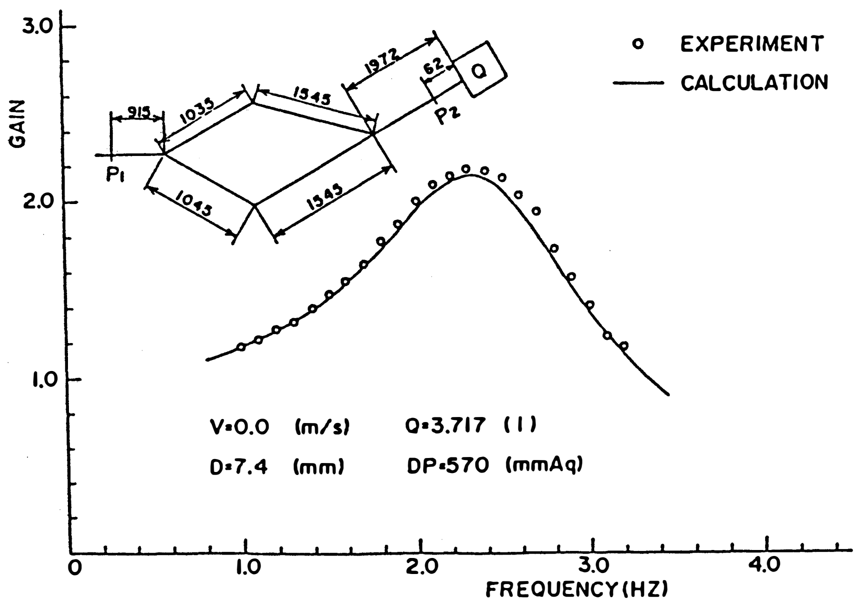
3.0
2.0
1.0
GAIN
915
1035
1545
1972
62
Q
P1
P2
1045
1545
EXPERIMENT
CALCULATION
V=0.0 (m/s)
Q=3.717 (l)
D=7.4 (mm)
DP=570 (mmAq)
0
1.0
2.0
3.0
4.0
FREQUENCY(HZ)

Fig. 9

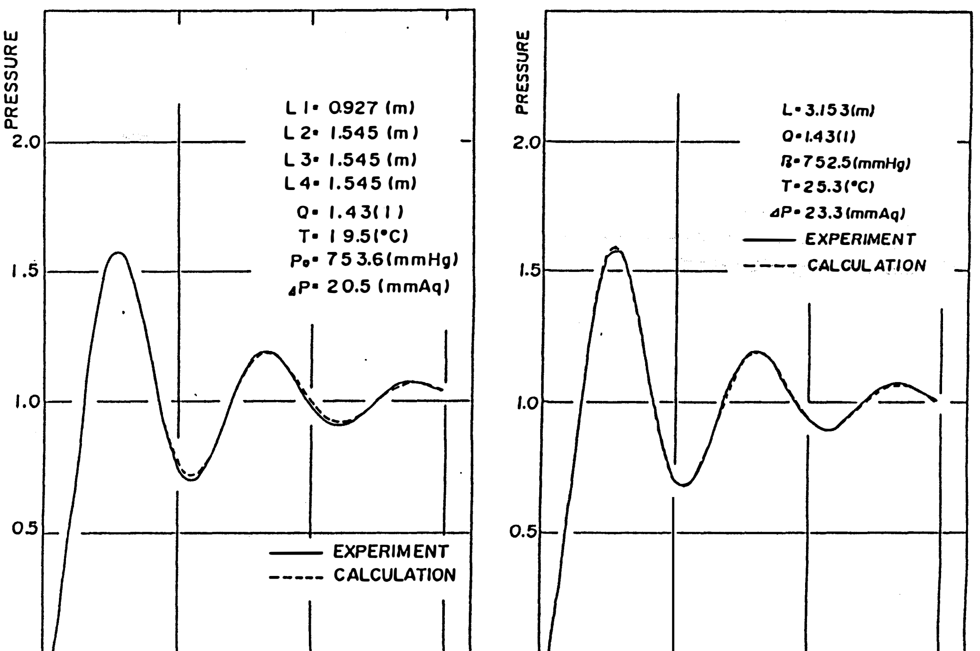
PRESSURE
2.0
1.5
1.0
0.5
0.0
L1= 0.927 (m)
L2= 1.545 (m)
L3= 1.545 (m)
L4= 1.545 (m)
Q= 1.43(l)
T= 19.5(°C)
P0= 753.6(mmHg)
ΔP= 20.5 (mmAq)
EXPERIMENT
CALCULATION
0.0
0.2
0.4
0.6
TIME(sec)
L=3.153(m)
Q=1.43(l)
P0=752.5(mmHg)
T=25.3(°C)
ΔP=23.3(mmAq)
EXPERIMENT
CALCULATION

Fig. 10

4. Conclusion

Instead of dealing with the exact complicated pneumatic systems, the equivalent single transmission line is proposed to make it easy to estimate the dynamic characteristics such as over shoot and delaytime in transient response.

The above-mentioned equivalent systems, however, must be applied to more complex network containing unsymmetricity of lines and terminal elements.

References

1) F.T.Brown: The Transient Response of Fluid Lines, Journal of Basic Engineering, Trans. ASME, Series D, Vol. 84, No. 4, December, 547/553 (1962).
2) R. Oldenburger and R.E.Goodson: Simplification of Hydraulic line Dynamics by Use of Infinite Products, Journal of Basic Engineering, Trans. ASME, Series D, Vol. 86, No. 1, March, 1/10 (1964).

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