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Holomorphic line bundles and Cartier divisors on domains in a Stein orbifold with discrete singularities

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Abstract. Let X be a Stein orbifold of pure dimension n such that $\mathrm{Sing}(X)$ is discrete. Let D be an open set of X such that $H^k(D, \mathscr{O}) = 0$ for $2 \leq k \leq n-1$ and every topologically trivial holomorphic line bundle on D is associated to some Cartier divisor on D. Then D is Stein.

1. Introduction

Abe [1] proved that an open set D of a Stein manifold X of dimension 2 is Stein if every holomorphic line bundle L on D is associated to some Cartier divisor $\mathfrak{d}$ on D. Ballico [7] proved that an open set D of a Stein manifold X of dimension more than 2 of the form $D = \{\varphi < c\}$, where $\varphi : X \to \mathbb{R}$ is a $\mathscr{C}^2$ weakly 2-convex function in the sense of Andreotti-Grauert [5], is Stein if every holomorphic line bundle L on D is associated to some Cartier divisor $\mathfrak{d}$ on D.

A complex space is said to be an *orbifold* (or a *V-manifold*) if every $x \in \mathrm{Sing}(X)$ is a quotient singular point. In this paper, we consider a Stein orbifold X of pure dimension n such that $\mathrm{Sing}(X)$ is discrete. Then, we prove that an open set D of X is Stein if $H^k(D, \mathscr{O}) = 0$ for $2 \leq k \leq n-1$ and for every topologically trivial holomorphic line bundle L on D is associated

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to some Cartier divisor $\mathfrak{d}$ on D (see Theorem 4.2), which generalizes two results above and improves Abe [2].

2. Preliminaries

We denote by $\mathscr{O}$ without subscript the reduced complex structure sheaf of a (not necessarily reduced) complex space. In other words, we always set $\mathscr{O} := \mathscr{O}_X/\mathscr{N}_X$ for a complex space X, where $\mathscr{O}_X$ is the structure sheaf of X and $\mathscr{N}_X$ is the nilradical of $\mathscr{O}_X$. For a reduced complex space X, we denote by $\mathscr{M}$ the sheaf of germs of meromorphic functions on X and by $\mathscr{O}^*$ (resp. $\mathscr{M}^*$) the multiplicative sheaf on X of germs of invertible holomorphic (resp. meromorphic) functions.

Let X be a reduced complex space. Let $\mathrm{Div}(X) := (\mathscr{M}^*/\mathscr{O}^*)(X)$. An element $\mathfrak{d} \in \mathrm{Div}(X)$ is said to be a *Cartier divisor* on X. If $\mathfrak{d} \in \mathrm{Div}(X)$ is defined by the meromorphic Cousin-II distribution $\{(U_i, m_i)\}_{i\in I}$ on X, then we denote by $[\mathfrak{d}]$ the holomorphic line bundle on X defined by the cocycle $\{m_i/m_j\} \in Z^1(\{U_i\}_{i\in I}, \mathscr{O}^*)$. We say that $[\mathfrak{d}]$ is the holomorphic line bundle associated to $\mathfrak{d}$. We say that $\mathfrak{d}$ is *positive* if $\mathfrak{d}$ can be defined by a holomorphic Cousin-II distribution.

By the extension theorem for analytic sets (see, for example, Grauert-Remmert [11, p. 181]), we have the following extension theorem for Cartier divisors on a complex manifold.

Lemma 2.1. *Let X be a complex manifold of pure dimension $n \geq 2$. Let T be an analytic set of X such that $\dim T \leq n-2$. Then for every $\mathfrak{d} \in \mathrm{Div}(X \setminus T)$ there exists $\mathfrak{c} \in \mathrm{Div}(X)$ such that $\mathfrak{c}|_{X\setminus T} = \mathfrak{d}$.*

Proof. Let $A = \sum_{\lambda\in\Lambda} \alpha_\lambda A_\lambda$ be the Weil divisor on $X \setminus T$ corresponding to $\mathfrak{d}$, where A_λ is an irreducible analytic set of $X \setminus T$ of dimension $n-1$ and $\alpha_\lambda \in \mathbb{Z}$ for every $\lambda \in \Lambda$. Then, the closure $\overline{|A|}$ of $|A| = \bigcup_{\lambda\in\Lambda} A_\lambda$ in X is an analytic set of X of pure dimension $n-1$ and the closure $\overline{A}_\lambda$ of A_λ in X is an irreducible analytic set of X of dimension $n-1$ for every $\lambda \in \Lambda$. We also have that $\overline{|A|} \cap (X \setminus T) = |A|$ and $\overline{A}_\lambda \cap (X \setminus T) = A_\lambda$ for every $\lambda \in \Lambda$. Since we can see that $\overline{|A|} = \bigcup_{\lambda\in\Lambda} \overline{A}_\lambda$, the system $\{\overline{A}_\lambda\}_{\lambda\in\Lambda}$ is locally finite in X. Let $\mathfrak{c}$ be the Cartier divisor on X corresponding to the Weil divisor $B := \sum_{\lambda\in\Lambda} \alpha_\lambda \overline{A}_\lambda$ on X. Since $B|_{X\setminus T} = A$, we have that $\mathfrak{c}|_{X\setminus T} = \mathfrak{d}$. $\square$

Let X be a reduced complex space. Let $e : \mathscr{O} \to \mathscr{O}^*$ be the homomorphism of sheaves defined by $e_x(h_x) := (\mathrm{e}^{2\pi\mathrm{i}\,h})_x$ for $h_x \in \mathscr{O}_x$ and $x \in X$. Then e induces the homomorphism $e^* : H^1(X, \mathscr{O}) \to H^1(X, \mathscr{O}^*)$. As usual, we identify the cohomology group $H^1(X, \mathscr{O}^*)$ with the set of holomorphic line bundles on X.

For a complex space X, we define the homomorphism

$$\Phi_X : H^1(X, \mathscr{O}_X) \to H^1(X, \mathscr{O}^*)$$

as follows: If $\alpha \in H^1(X, \mathscr{O}_X)$ is defined by the cocycle $\{h_{ij}\} \in Z^1(\{U_i\}, \mathscr{O}_X)$, where $\{U_i\}$ is an open covering of X, then let $\Phi_X(\alpha)$ be the cohomology class in $H^1(X, \mathscr{O}^*)$ defined by the cocycle $\{\mathrm{e}^{2\pi\mathrm{i}\,[h_{ij}]}\} \in Z^1(\{U_i\}, \mathscr{O}^*)$.[1] This definition does not depend on the choice of $\{U_i\}$ and $\{h_{ij}\}$. If X is reduced, then we have that $\Phi_X = e^*$.

Let $\Delta(r) := \{t \in \mathbb{C} \mid |t| < r\}$ for $r > 0$ and $\Delta := \Delta(1)$. Let

$$P = P(n, \varepsilon) := \Delta(1+\varepsilon)^n \quad \text{and}$$
$$H = H(n, \varepsilon) := \Delta^n \cup \left(\left(\Delta(1+\varepsilon) \setminus \overline{\Delta(1-\varepsilon)}\right) \times \Delta(1+\varepsilon)^{n-1}\right)$$

for $n \in \mathbb{N}$ and $0 < \varepsilon < 1$. The pair (P, H) is said to be a *Hartogs figure*.

An open set D of a complex space X is said to be *locally Stein* at a point $x \in \partial D$ if there exists a neighborhood U of x in X such that the open subspace $D \cap U$ is Stein. By Lemmas 6.1 and 6.2 of Abe [3], we have the following lemma.

Lemma 2.2. *Let X be a Stein space of pure dimension $n \geq 2$. Let D be an open set of X. Then the following two conditions are equivalent.*

(1) *D is not locally Stein at some point $p \in \partial D \setminus \mathrm{Sing}(X)$.*

(2) *There exist a holomorphic map $\theta : X \to \mathbb{C}^n$,[2] an open set $W \subset X \setminus \mathrm{Sing}(X)$, $\varepsilon \in (0,1)$, and $b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$ such that $\theta(W)$ is an open set of $\mathbb{C}^n$, the restriction $\theta|_W : W \to \theta(W)$ is biholomorphic, $P(n,\varepsilon) \Subset \theta(W)$, $(\theta|_W)^{-1}(H(n,\varepsilon)) \subset D$, $|b_1| \leq 1-\varepsilon$, $1 \leq |b_2| < 1+\varepsilon$, $|b_\nu| < 1$ for $3 \leq \nu \leq n$, and $(\theta|_W)^{-1}(b) \in \partial D$.*

[1] We denote by $[h]$ the valuation $x \mapsto h_x + \mathfrak{m}_x \in \mathscr{O}_{X,x}/\mathfrak{m}_x = \mathbb{C}$, $x \in U$, for $h \in \mathscr{O}_X(U)$, where U is an open set of X.

[2] As usual, we simply write $\theta : X \to \mathbb{C}^n$ instead of $(\theta, \tilde{\theta}) : (X, \mathscr{O}_X) \to (\mathbb{C}^n, \mathscr{O})$.

Lemma 2.3. *Let X be a reduced complex space. Then for every $x \in \mathrm{Sing}(X)$ has a sufficiently small neighborhood U of x such that U is Stein, U is contractible to x, and* $\mathrm{rank}\, H^1(U \setminus x, \mathbb{Z}) < +\infty$.

Proof. Take a neighborhood E of x in X such that E can be regarded as an analytic set of an open set B of some $\mathbb{C}^N$ in which x is the origin. Let $B_\varepsilon := \{z \in \mathbb{C}^N \mid \|z\| < \varepsilon\}$ for every $\varepsilon > 0$. Then, by the conic structure lemma of Burghelea-Verona [8, Lemma 3.2], there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0} \Subset B$ and the pair $(\overline{B}_\varepsilon, \overline{B}_\varepsilon \cap E)$ is homeomorphic to the cone over the pair $(\partial B_\varepsilon, \partial B_\varepsilon \cap E)$ for every $\varepsilon \in (0, \varepsilon_0]$. Take an $\varepsilon \in (0, \varepsilon_0]$ and let $U := B_\varepsilon \cap E$. Then U is Stein and is contractible to x. Since, by Łojasiewicz [17, Theorem 1], there exists a finite simplicial complex which decomposes $\partial B_\varepsilon \cap E$, we see that $U \setminus x$ has a finite covering by contractible open sets and therefore $\mathrm{rank}\, H^1(U \setminus x, \mathbb{Z}) < +\infty$. □

3. Domains in a Stein space

In this section, we revise the contents of Abe [2], for there are incorrect arguments in the proof of Abe [2, Lemma 3.3].

Lemma 3.1 (cf. Abe [2, Lemma 3.1]) *Let X be a Stein space of pure dimension 2 and D an open set of X. Let W be an open set of $X \setminus \mathrm{Sing}(X)$ and $\theta : X \to \mathbb{C}^2$ be a holomorphic map such that $\theta(W)$ is an open set of $\mathbb{C}^2$ and the restriction $\theta|_W : W \to \theta(W)$ is biholomorphic. Assume that there exist $\varepsilon \in (0,1)$ and $b = (b_1, b_2) \in \mathbb{C}^2$ such that $P(2,\varepsilon) \Subset \theta(W)$, $(\theta|_W)^{-1}(H(2,\varepsilon)) \subset D$, $|b_1| \le 1 - \varepsilon$, $1 \le |b_2| < 1 + \varepsilon$, and $(\theta|_W)^{-1}(b) \in \partial D$. Then there exists a cohomology class $\alpha \in H^1(D, \mathscr{O}_X|_D)$ such that the holomorphic line bundle $\Phi_{(D, \mathscr{O}_X|_D)}(\alpha)|_{D\cap R} \in H^1(D \cap R, \mathscr{O}^*)$ is not associated to any Cartier divisor on $D \cap R$, where $R := (\theta|_W)^{-1}(P(2,\varepsilon))$.*

Proof. The proof is essentially same as that of Abe [2, Lemma 3.1]. Let $\theta_\nu := \tilde{\theta} z_\nu$ for $\nu = 1, 2$, where z_1, z_2 are the coordinates of $\mathbb{C}^2$. Let $E_\nu := \{[\theta_\nu] \ne b_\nu\}$ for $\nu = 1, 2$. Since E_ν is Stein and $1/([\theta_\nu] - b_\nu) \in \mathscr{O}(E_\nu)$, there exists $u_\nu \in \mathscr{O}_X(E_\nu)$ such that $[u_\nu] = 1/([\theta_\nu] - b_\nu)$ on E_ν for $\nu = 1, 2$. Let $T := \{|[\theta_2]| < 1 + \varepsilon\}$ and $F := (E_1 \cap T) \cup (T \setminus \bar{R})$. Then T is Stein, $\{R, F\}$ is an open covering of T, and $R \cap F = E_1 \cap R$. Since $H^1(\{R, F\}, \mathscr{O}_X|_T) = 0$, there exist $v_0 \in \mathscr{O}_X(R)$ and $v_1 \in \mathscr{O}_X(F)$ such that $u_1 = v_1 - v_0$ on $R \cap F$. Let $D_1 := D \cap F$ and $D_2 := D \cap E_2$. Since $(\theta|_W)^{-1}(b) \notin D$, we see that $\{D_1, D_2\}$ is an open covering of D. Let $\alpha \in H^1(\{D_1, D_2\}, \mathscr{O}_X|_D)$ be the cohomology class defined by $\mathrm{e}^{v_1+u_2}|_{D_1 \cap D_2} \in \mathscr{O}_X(D_1 \cap D_2)$.[3] Assume that $\Phi_{(D, \mathscr{O}_X|_D)}(\alpha)|_{D\cap R}$ is associated to some Cartier divisor on $D \cap R$. Then there exist $g_\nu \in \mathscr{M}^*(D_\nu \cap R)$, $\nu = 1, 2$, such that $\exp(2\pi\mathrm{i}\,\mathrm{e}^{v_1+u_2}) = g_1/g_2$ on $D_1 \cap D_2 \cap R$. Let $P := P(2, \varepsilon)$ and $H := H(2, \varepsilon)$. Let $P_\nu := P \cap \{z_\nu \neq b_\nu\}$ and $H_\nu := H \cap \{z_\nu \neq b_\nu\}$ for $\nu = 1, 2$. Since $(\theta|_W)^{-1}(H_\nu) \subset D \cap E_\nu \cap R = D_\nu \cap R$, we have the function $f_\nu := g_\nu \circ (\theta|_W)^{-1} \in \mathscr{M}^*(H_\nu)$ for $\nu = 1, 2$. Since $(\theta|_W)^{-1}(P_1 \cap P_2) \subset E_1 \cap E_2 \cap R = F \cap E_2 \cap R$, we have the function $\xi := \exp(2\pi\mathrm{i}\,\mathrm{e}^{v_1+u_2}) \circ (\theta|_W)^{-1} \in \mathscr{O}(P_1 \cap P_2)$. Then we have that $\xi = f_1/f_2$ on $H_1 \cap H_2$. Since P is an envelope of holomorphy of H, the open set P_ν is an envelope of holomorphy of H_ν for $\nu = 1, 2$ by Grauert-Remmert [10, Satz 7] (see Jarnicki-Pflug [13, p. 182]). Therefore, by Kajiwara-Sakai [15, Proposition 3], there exists $\tilde{f}_\nu \in \mathscr{M}(P_\nu)$ such that $\tilde{f}_\nu = f_\nu$ on H_ν for $\nu = 1, 2$. Then, by the theorem of identity, we have that $\tilde{f}_\nu \in \mathscr{M}^*(P_\nu)$ for $\nu = 1, 2$ and $\xi = \tilde{f}_1/\tilde{f}_2$ on $P_1 \cap P_2$. Let $w_\nu := (z_\nu - b_\nu)/\delta$ for $\nu = 1, 2$, where $0 < \delta \leq 1 + \varepsilon - |b_2|$. Let $U_1 := \{0 < |w_1| < 1,\ |w_2| < 1\}$, $U_2 := \{|w_1| < 1,\ 0 < |w_2| < 1\}$, and $M := U_1 \cup U_2$. Since M is Cousin-II, we moreover have that $\xi|_{U_1 \cap U_2} \in B^1(\{U_1, U_2\}, \mathscr{O}^*)$. Let $\eta := v_0 \circ (\theta|_W)^{-1}$ on P. We have that

$$\xi = \exp\left(2\pi\mathrm{i}\,\mathrm{e}^{v_0+u_1+u_2}\right) \circ (\theta|_W)^{-1} = \exp\left(2\pi\mathrm{i}\,\mathrm{e}^{\eta}\mathrm{e}^{(1/\delta)(1/w_1+1/w_2)}\right)$$

on $U_1 \cap U_2$. By Abe [3, Lemma 3.3], we then have that

$$\mathrm{e}^{\eta}\mathrm{e}^{(1/\delta)(1/w_1+1/w_2)} \in B^1(\{U_1, U_2\}, \mathscr{O}),$$

which contradicts Kajiwara-Kazama [14, Lemma 9] (see Abe [3, Lemma 3.2]). It follows that the cohomology class $\Phi_{(D, \mathscr{O}_X|_D)}(\alpha)|_{D\cap R}$ is not associated to any Cartier divisor on $D \cap R$. □

Lemma 3.2 (cf. Abe [2, Lemma 3.3]) *Let X be a Cohen-Macaulay Stein space of pure dimension $n \geq 2$.*[4] *Let D be an open set of X such*

[3] See Kaup-Kaup [16, pp. 246] for the definition of $\mathrm{e}^h \in \mathscr{O}_X(U)$, where U is an open set of a complex space X and $h \in \mathscr{O}_X(U)$.

[4] A complex space X is said to be *Cohen-Macaulay* if the local $\mathbb{C}$-algebra $\mathscr{O}_{X,x}$ is Cohen-Macaulay for every $x \in X$.

that $H^k(D, \mathscr{O}_X|_D) = 0$ *for* $2 \le k \le n-1$. *Let* W *be an open set of* $X \setminus \mathrm{Sing}(X)$ *and* $\theta : X \to \mathbb{C}^n$ *be a holomorphic map such that* $\theta(W)$ *is an open set of* $\mathbb{C}^n$ *and the restriction* $\theta|_W : W \to \theta(W)$ *is biholomorphic. Assume that there exist* $\varepsilon \in (0,1)$ *and* $b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$ *such that* $P(n,\varepsilon) \Subset \theta(W)$, $(\theta|_W)^{-1}(H(n,\varepsilon)) \subset D$, $|b_1| \le 1-\varepsilon$, $1 \le |b_2| < 1+\varepsilon$, $|b_\nu| < 1$ *for* $3 \le \nu \le n$, *and* $(\theta|_W)^{-1}(b) \in \partial D$. *Then there exists a cohomology class* $\alpha \in H^1(D, \mathscr{O}_X|_D)$ *such that the holomorphic line bundle* $\Phi_{(D,\mathscr{O}_X|_D)}(\alpha)|_{D\cap R} \in H^1(D \cap R, \mathscr{O}^*)$ *is not associated to any Cartier divisor on* $D \cap R$, *where* $R := (\theta|_W)^{-1}(P(n,\varepsilon))$.

Proof. The proof proceeds by induction on n. By Lemma 3.1, the assertion is true if $n = 2$. We consider the case where $n \ge 3$. Let $\theta_\nu := \tilde{\theta} z_\nu$ for $\nu = 1, 2, \ldots, n$, where $z_1, z_2, \ldots, z_n$ are the coordinates of $\mathbb{C}^n$. We may assume that W is connected. Then, there exists $f \in \mathscr{O}_X(X)$ such that $[f] = [\theta_n] - b_n$ on the irreducible component X_0 of X containing W and $[f] \not\equiv 0$ on any irreducible component of X. Let $Y := \{[f] = 0\}$ and $\mathscr{O}_Y := (\mathscr{O}_X / f\mathscr{O}_X)|_Y$. By the argument in the proof of Abe [2, Lemma 3.3], the complex space Y is a Cohen-Macaulay Stein space of pure dimension $n-1$ and we have that $H^k(D \cap Y, \mathscr{O}_X|_{D\cap Y}) = 0$ for $2 \le k \le n-2$ and the restriction $\tilde{\iota}^* : H^1(D, \mathscr{O}_X|_D) \to H^1(D \cap Y, \mathscr{O}_Y|_{D\cap Y})$ is surjective. Let $\theta' : Y \to \mathbb{C}^{n-1}$ be the holomorphic map such that $\tilde{\theta}' z_\nu = (\tilde{\iota}\theta_\nu)|_Y$ for $\nu = 1, 2, \ldots, n-1$. Let $R' := R \cap Y$ and $W' := W \cap Y$. Then we have that $R' \Subset W' \subset Y \setminus \mathrm{Sing}(Y)$, $\theta(x) = (\theta'(x), b_n)$ for every $x \in W'$, the set $\theta'(W')$ is open in $\mathbb{C}^{n-1}$, the restriction $\theta'|_{W'} : W' \to \theta'(W')$ is biholomorphic,

$$\begin{aligned}
(\theta'|_{W'})^{-1}(P(n-1,\varepsilon)) &= R', \\
(\theta'|_{W'})^{-1}(H(n-1,\varepsilon)) &= (\theta|_W)^{-1}(H(n-1,\varepsilon) \times \{b_n\}) \\
&= (\theta|_W)^{-1}(H(n,\varepsilon)) \cap W' \subset D \cap Y,^5 \quad \text{and} \\
(\theta'|_{W'})^{-1}((b_1, \ldots, b_{n-1})) &= (\theta|_W)^{-1}(b) \notin D \cap Y.
\end{aligned}$$

Let $b' := (b_1, t_0 b_2, b_3, \ldots, b_{n-1})$, where

$$t_0 := \sup\{t \in [0,1] \mid (\theta'|_{W'})^{-1}((b_1, s b_2, b_3, \ldots, b_{n-1})) \in D \cap Y \text{ for every } s \in [0,t]\}.$$

[5] Because $|b_n| < 1$, we have that $H(n-1,\varepsilon) \times \{b_n\} = H(n,\varepsilon) \cap \{z_n = b_n\}$. The proof of Abe [2, Lemma 3.3] is not correct as the possibility of $|b_n| = 1$ is not avoided there.

Then we have that $1 \leq |t_0 b_2| < 1 + \varepsilon$ and the point $(\theta'|_{W'})^{-1}(b')$ belongs to the boundary of $D \cap Y$ in Y. By induction hypothesis, there exists $\alpha' \in H^1(D \cap Y, \mathscr{O}_Y|_{D\cap Y})$ such that $\Phi_{(D\cap Y,\, \mathscr{O}_Y|_{D\cap Y})}(\alpha')|_{D\cap R'}$ is not associated to any Cartier divisor on $D \cap R'$. Since $\tilde{\imath}^*$ is surjective, there exists $\alpha \in H^1(D, \mathscr{O}_X|_D)$ such that $\tilde{\imath}^*(\alpha)|_{D\cap Y} = \alpha'$. Then, by the argument in the proof of Abe [2, Lemma 3.3], the line bundle $\Phi_{(D,\, \mathscr{O}_X|_D)}(\alpha)|_{D\cap R}$ is not associated to any Cartier divisor on $D \cap R$. □

Theorem 3.3 (Abe [2, Theorem 4.1]) *Let X be a Stein space of pure dimension n. Assume further that X is Cohen-Macaulay if $n \geq 3$. Let D be an open set of X which satisfies the following two conditions:*

- $H^k(D, \mathscr{O}_X|_D) = 0$ *for* $2 \leq k \leq n-1$.[6]
- *For every* $\alpha \in H^1(D, \mathscr{O}_X|_D)$ *there exists* $\mathfrak{d} \in \mathrm{Div}(D)$ *such that* $\Phi_{(D,\mathscr{O}_X|_D)}(\alpha) = [\mathfrak{d}]$.

Then D is locally Stein at every point $x \in \partial D \setminus \mathrm{Sing}(X)$.

Proof. We may assume that $n \geq 2$. Assume that there exists $p \in \partial D \setminus \mathrm{Sing}(X)$ such that D is not locally Stein at p. Then, by Lemma 2.2, there exist a holomorphic map $\theta : X \to \mathbb{C}^n$, an open set $W \subset X \setminus \mathrm{Sing}(X)$, $\varepsilon \in (0,1)$, and $b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$ such that $\theta(W)$ is an open set of $\mathbb{C}^n$, the restriction $\theta|_W : W \to \theta(W)$ is biholomorphic, $P(n,\varepsilon) \Subset \theta(W)$, $(\theta|_W)^{-1}(H(n,\varepsilon)) \subset D$, $|b_1| \leq 1-\varepsilon$, $1 \leq |b_2| < 1+\varepsilon$, $|b_\nu| < 1$ for $3 \leq \nu \leq n$, and $(\theta|_W)^{-1}(b) \in \partial D$. By Lemmas 3.1 and 3.2, there exists $\alpha \in H^1(D, \mathscr{O}_X|_D)$ such that the line bundle $\Phi_{(D,\mathscr{O}_X|_D)}(\alpha)|_{D\cap R}$ is not associated to any Cartier divisor on $D \cap R$, where $R := (\theta|_W)^{-1}(P(n,\varepsilon))$. It is a contradiction. □

[6] This condition can be replaced by the weaker one that $\dim H^k(D, \mathscr{O}_X|_D) \leq \aleph_0$ for $2 \leq k \leq n-1$ (see Abe [2, Remark 4.2]).

4. Domains in a Stein orbifold

Lemma 4.1. *Let X be a Stein orbifold of pure dimension $n \geq 2$ such that $\operatorname{Sing}(X)$ is discrete. Let D be an open set of X which satisfies the following three conditions:*

- $H^k(D, \mathscr{O}) = 0$ *for* $2 \leq k \leq n-1$.
- *For every topologically trivial holomorphic line bundle L on D there exists $\mathfrak{d} \in \operatorname{Div}(D)$ such that $L = [\mathfrak{d}]$.*[7]
- $X \setminus \operatorname{Sing}(X) \subset D$.

Then we have that $D = X$.

Proof. Assume that $D \subsetneqq X$. Then $P := X \setminus D \neq \emptyset$. Since $\operatorname{Sing}(X)$ is discrete and $P \subset \operatorname{Sing}(X)$, the set P is also discrete. Take a system $\{U_x\}_{x \in P}$ of connected Stein open sets of X such that $U_x \cap \operatorname{Sing}(X) = \{x\}$ for every $x \in P$ and $U_x \cap U_y = \emptyset$ if $x \neq y$. By the Mayer-Vietoris exact sequence, we have the isomorphisms

$$H^k(D, \mathscr{O}) \xrightarrow{\sim} H^k(\bigcup_{x \in P} (U_x \setminus x), \mathscr{O}) \cong \prod_{x \in P} H^k(U_x \setminus x, \mathscr{O})$$

for every $k \geq 1$. Therefore $H^1(D, \mathscr{O}) \to H^1(U_x \setminus x, \mathscr{O})$ is surjective and $H^k(U_x \setminus x, \mathscr{O}) = 0$, $2 \leq k \leq n-1$, for every $x \in P$. Since X is normal,[8] the open set $U_x \setminus x$ is not Stein by the second Riemann extension theorem. It follows that $H^1(U_x \setminus x, \mathscr{O}) \neq 0$ for every $x \in P$ (see, for example, Coen [9]). We fix a point $p \in P$. By Prill [18] (see Abe [3, Lemma 2.4]), there exist a neighborhood U' of p in X, an open set W' of $\mathbb{C}^n$, and a finitely sheeted ramified covering $\pi' : W' \to U'$ such that $U' \cap \operatorname{Sing}(X) = \{p\}$ and π' is locally biholomorphic on $W' \setminus \pi'^{-1}(p)$. We may assume that $\pi'^{-1}(p) = \{0\}$ (see Grauert-Remmert [11, p. 48]). Take an open ball B centered at 0 such that $B \subset W'$. By Lemma 2.3, we may further assume that $U := U_p$ is contractible to p, $\operatorname{rank} H^1(U \setminus p, \mathbb{Z}) < +\infty$, $U \subset U'$, and $W := \pi'^{-1}(U) \subset B$.

[7] Note that the set $\operatorname{im} e^* = \ker \delta$ coincides with the set of topologically trivial holomorphic line bundles on D, where $H^1(D, \mathscr{O}) \xrightarrow{e^*} H^1(D, \mathscr{O}^*) \xrightarrow{\delta} H^2(D, \mathbb{Z})$.

[8] Every complex orbifold is Cohen-Macaulay and normal (see Abe [3, p. 706]).

Let $\pi := \pi'|_W : W \to U$. Let $b := \# \pi^{-1}(\xi)$, $\xi \in U \setminus p$, which is constant. The sequence $0 \to \mathbb{Z} \to \mathscr{O} \xrightarrow{e} \mathscr{O}^* \to 0$ is exact and we have the commutative diagram:

$$\begin{array}{ccccc} H^1(D,\mathbb{Z}) & \xrightarrow{\iota} & H^1(D,\mathscr{O}) & \xrightarrow{e^*} & H^1(D,\mathscr{O}^*) \\ \downarrow r & & \downarrow r & & \downarrow r \\ H^1(U\setminus p,\mathbb{Z}) & \xrightarrow{\iota} & H^1(U\setminus p,\mathscr{O}) & \xrightarrow{e^*} & H^1(U\setminus p,\mathscr{O}^*) \\ \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* \\ H^1(W\setminus 0,\mathbb{Z}) & \xrightarrow{\iota} & H^1(W\setminus 0,\mathscr{O}) & \xrightarrow{e^*} & H^1(W\setminus 0,\mathscr{O}^*), \end{array}$$

where the rows are exact. Take an arbitrary $\beta \in H^1(U \setminus p, \mathscr{O})$. Since the restriction $r : H^1(D, \mathscr{O}) \to H^1(U \setminus p, \mathscr{O})$ is surjective, there exists $\alpha \in H^1(D, \mathscr{O})$ such that $r(\alpha) = \beta/b$. By assumption, there exists $\mathfrak{d} \in \mathrm{Div}(D)$ such that $[\mathfrak{d}] = e^*(\alpha)$. By Lemma 2.1, there exists $\mathfrak{c} \in \mathrm{Div}(W')$ such that $\mathfrak{c}|_{W'\setminus 0} = \pi'_*(\mathfrak{d}|_{U'\setminus p})$. Since $H^1(B, \mathscr{O}^*) = 0$, we have that $[\mathfrak{c}]|_B = 0$. Then we have that

$$e^*(\pi_*(\beta/b)) = e^*(\pi_*(r(\alpha))) = \pi_*(r(e^*(\alpha))) = \pi_*([\mathfrak{d}]|_{U\setminus p}) = [\mathfrak{c}]|_{W\setminus 0} = 0.$$

Therefore there exists $\tilde{\nu} \in H^1(W \setminus 0, \mathbb{Z})$ such that $\iota(\tilde{\nu}) = \pi_*(\beta/b)$. Since $\pi : W \setminus 0 \to U \setminus p$ is a b-sheeted unramified covering over $U \setminus p$, there exists a simple open covering $\{V_i\}_{i\in I}$ of $U \setminus p$ such that $\pi^{-1}(V_i)$ consists of b connected components $\tilde{V}_{i1}, \tilde{V}_{i2}, \ldots, \tilde{V}_{ib}$ and $\pi|_{\tilde{V}_{i\lambda}} : \tilde{V}_{i\lambda} \to V_i$ is biholomorphic for every $i \in I$ and for every $\lambda = 1, 2, \ldots, b$. Then $\{\pi^{-1}(V_i)\}_{i\in I}$ is a Leray open covering of $W \setminus 0$ with respect to the constant sheaf $\mathbb{Z}$. Therefore there exists a cocycle $\{\tilde{\nu}_{ij}\} \in Z^1(\{\pi^{-1}(V_i)\}_{i\in I}, \mathbb{Z})$ such that $\tilde{\nu} = [\{\tilde{\nu}_{ij}\}] \in H^1(\{\pi^{-1}(V_i)\}_{i\in I}, \mathbb{Z})$. Since $\{V_i\}_{i\in I}$ can be chosen sufficiently fine, we may assume that there exists a cocycle $\{\beta_{ij}\} \in Z^1(\{V_i\}_{i\in I}, \mathscr{O})$ such that $\beta = [\{\beta_{ij}\}] \in H^1(\{V_i\}_{i\in I}, \mathscr{O})$. Since $\beta/b - \iota(\tilde{\nu}) = [\{(\beta_{ij} \circ \pi)/b - \tilde{\nu}_{ij}\}] = 0$ in $H^1(\{\pi^{-1}(V_i)\}_{i\in I}, \mathscr{O})$, there exists $\{\tilde{\gamma}_i\} \in C^0(\{\pi^{-1}(V_i)\}_{i\in I}, \mathscr{O})$ such that $(\beta_{ij} \circ \pi)/b - \tilde{\nu}_{ij} = \tilde{\gamma}_j - \tilde{\gamma}_i$ on $\pi^{-1}(V_i \cap V_j)$ for every $i, j \in I$. For an arbitrary $\xi \in \pi^{-1}(V_i \cap V_j)$ let $\{\eta_1, \eta_2, \ldots, \eta_b\} := \pi^{-1}(\xi)$. Since $\beta_{ij}(\xi)/b - \tilde{\nu}_{ij}(\eta_\lambda) = \tilde{\gamma}_j(\eta_\lambda) - \tilde{\gamma}_i(\eta_\lambda)$ for $\lambda = 1, 2, \ldots, b$, we obtain that

$$\beta_{ij}(\xi) - \sum_{\lambda=1}^{b} \tilde{\nu}_{ij}(\eta_\lambda) = \sum_{\lambda=1}^{b} \tilde{\gamma}_j(\eta_\lambda) - \sum_{\lambda=1}^{b} \tilde{\gamma}_i(\eta_\lambda).$$

Let $\gamma_i := \sum_{\lambda=1}^{b} \tilde{\gamma}_i \circ (\pi|_{\tilde{V}_{i\lambda}})^{-1}$ on V_i and let $\nu_{ij} := \sum_{\lambda=1}^{b} \tilde{\nu}_{ij} \circ ((\pi|_{\tilde{V}_{i\lambda}})^{-1}|_{V_i \cap V_j})$ on $V_i \cap V_j$. Then γ_i is a holomorphic function on V_i, ν_{ij} is a constant function on $V_i \cap V_j$ with values in $\mathbb{Z}$, and we have that $\beta_{ij} - \nu_{ij} = \gamma_j - \gamma_i$ on $V_i \cap V_j$ for every $i, j \in \mathbb{Z}$. Since $\delta\{\nu_{ij}\} = \delta\{\beta_{ij}\} - \delta\{\gamma_j - \gamma_i\} = 0$, we have that $\{\nu_{ij}\} \in Z^1(\{V_i\}_{i \in I}, \mathbb{Z})$. Then we have that $\beta = \iota(\nu) \in H^1(\{V_i\}_{i \in I}, \mathscr{O}) \subset H^1(U \setminus p, \mathscr{O})$, where $\nu := [\{\nu_{ij}\}] \in H^1(\{V_i\}_{i \in I}, \mathbb{Z})$. Thus we proved that $\iota : H^1(U \setminus p, \mathbb{Z}) \to H^1(V \setminus p, \mathscr{O})$ is surjective. Since $H^1(U \setminus p, \mathscr{O})$ is a non-trivial $\mathbb{C}$-vector space, we have that $\# H^1(U \setminus p, \mathscr{O}) \geq \aleph$. Since $\operatorname{rank} H^1(U \setminus p, \mathbb{Z}) < +\infty$, we also have that $\# H^1(U \setminus p, \mathbb{Z}) \leq \aleph_0$, which is a contradiction. It follows that $D = X$. □

Theorem 4.2. *Let X be a Stein orbifold of pure dimension n such that* $\operatorname{Sing}(X)$ *is discrete. Let D be an open set of X. Then the following two conditions are equivalent.*

(1) *D is Stein.*

(2) *D satisfies the following two conditions:*

- $H^k(D, \mathscr{O}) = 0$ *for* $2 \leq k \leq n-1$.[9]
- *For every topologically trivial holomorphic line bundle L on D there exists $\mathfrak{d} \in \operatorname{Div}(D)$ such that $L = [\mathfrak{d}]$.*

Proof. (1) → (2). Every holomorphic line bundle on a reduced Stein space is associated to some positive Cartier divisor (see Gunning [12, p. 124]).
(2) → (1). We may assume that $n \geq 2$. By Theorem 3.3, the open set D is locally Stein at every $x \in \partial D \setminus \operatorname{Sing}(X)$. Let D^* be the extension of D along $\operatorname{Sing}(X)$. We have that $D^* \setminus \operatorname{Sing}(X) = D \setminus \operatorname{Sing}(X)$. The open set D^* does not have boundary points removable along $\operatorname{Sing}(X)$ and is locally Stein at every $x \in \partial D^* \setminus \operatorname{Sing}(X)$. By Abe-Hamada [4, Lemma 2] (see Abe [3, Lemma 2.4]), the open set D^* is locally Stein at every $x \in \partial D^*$. Since $\operatorname{Sing}(X)$ is discrete, the open set D^* is Stein by Andreotti-Narasimhan [6]. Since $D^* \setminus \operatorname{Sing}(X) \subset D \subset D^*$, we have that $D = D^*$ by Lemma 4.1. Thus we proved that D is Stein. □

[9] This condition can be replaced by the weaker one that $\dim H^k(D, \mathscr{O}) \leq \aleph_0$ for $2 \leq k \leq n-1$ (see footnotes 6 and 8).

Corollary 4.3. *Let X be a Stein orbifold of pure dimension 2. Then for every open set D of X the following two conditions are equivalent.*

(1) *D is Stein.*

(2) *For every topologically trivial holomorphic line bundle L on D there exists $\mathfrak{d} \in \mathrm{Div}(D)$ on D such that $L = [\mathfrak{d}]$.*

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