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Order relations among some interpolating families of means

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Abstract. Some interpolating families of means of two positive numbers with a parameter are studied. They are concerned with some of the most familiar means, i.e., the arithmetic, geometric, harmonic, logarithmic means and so on. Their monotonicity with respect to the parameter and the order relations among them are discussed.

1. Introduction

Together with the most familiar means, the arithmetic, geometric, harmonic means, the logarithmic mean $L(a,b)$ for two positive numbers a and b is well-known, and is defined by

$$L(a,b) = \frac{b-a}{\ln b - \ln a} \ (a \neq b), \quad L(a,a) = a.$$

It can be seen that

$$\frac{1}{L(a,b)} = \int_0^\infty \frac{dt}{(t+a)(t+b)}. \tag{1.1}$$

The arithmetic-geometric mean $AG(a,b)$ of Gauss is well-known and it is defined as follows: the two sequences $\{a_n\}$ and $\{b_n\}$ defined inductively as $a_0 = a, b_0 = b, a_{n+1} = \frac{a_n+b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$ have a common limit

$$AG(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{1.2}$$

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By an ingenious calculation due to Gauss, (1.2) can be seen that

$$\frac{1}{AG(a,b)} = \frac{2}{\pi}\int_0^\infty \frac{dt}{\sqrt{(t^2+a^2)(t^2+b^2)}}. \tag{1.3}$$

A family of means $M_p(a,b)$ with a parameter p was introduced by R. Bhatia and R.-C. Li [3] as follows, noticing the similarity between the above expressions (1.1) and (1.3):

$$\frac{1}{M_p(a,b)} := c_p \int_0^\infty \frac{dt}{\{(t^p+a^p)(t^p+b^p)\}^{1/p}}, \quad 0<p<\infty, \tag{1.4}$$

where the constant c_p, depending on p, will be chosen to have $M_p(a,a)=a$. Thus

$$\frac{1}{c_p} = a\int_0^\infty \frac{dt}{(t^p+a^p)^{2/p}} = \int_0^\infty \frac{ds}{(s^p+1)^{2/p}}.$$

Here *a (symmetric) mean* $m(a,b)$ *for* $a,b>0$ is defined as a function satisfying the following properties:

(i) $\min\{a,b\} \le m(a,b) \le \max\{a,b\}$ (*In particular*, $m(a,a)=a$);

(ii) $m(a,b)=m(b,a)$ (*Symmetry*);

(iii) $m(\alpha a, \alpha b) = \alpha m(a,b)$ *for all* $\alpha>0$ (*Homogeneity*);

(iv) $m(a,b)$ *is non-decreasing in* a *and* b.

It is easy to see that the familiar means stated before and $M_p(a,b)$ satisfy the properties (i)-(iv).

Now for other families of means, first, the power difference mean $K_p(a,b)$ is defined for any real number p and two positive numbers a and b as follows ([3],[9]):

$$K_p(a,b) := \frac{p-1}{p}\frac{a^p-b^p}{a^{p-1}-b^{p-1}} \quad (a\ne b), \quad p\ne 0,1.$$

As for particular cases, $K_p(a,a)=a$ (for any p) and for $p=0,1$

$$K_0(a,b) := \lim_{p\to 0} K_p(a,b) = ab\frac{\ln b - \ln a}{b-a}, \;\; K_1(a,b) := \lim_{p\to 1} K_p(a,b) = L(a,b).$$

This mean admits the following integral expression ([3]):

$$\frac{1}{K_p(a,b)} = \int_0^1 \frac{dt}{\{(1-t)a^p + tb^p\}^{1/p}} \quad (p \neq 0).$$

Besides, $K_{p+1}(a,b)$ can be defined directly as an integral form as follows ([9], [5]):

$$K_{p+1}(a,b) := \int_0^1 \{(1-t)a^p + tb^p\}^{1/p} dt = \frac{p}{p+1} \frac{a^{p+1} - b^{p+1}}{a^p - b^p}.$$

The power logarithmic mean $D_p(a,b)$ and the power binomial mean $N_p(a,b)$ for $-\infty < p < \infty$ and two positive numbers a and b, are defined as follows, respectively ([9]),

$$D_p(a,b) := \left(\int_0^1 \{(1-t)a + tb\}^p dt\right)^{1/p} = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p} \quad (a \neq b, p \neq -1, 0),$$

$$D_p(a,a) = a, \quad D_{-1} = L(a,b), \quad D_0(a,b) = \frac{1}{e}(a^a b^{-b})^{1/(a-b)} \quad (a \neq b),$$

and

$$N_p(a,b) := \left(\int_0^1 \{(1-t)a^p + tb^p\} dt\right)^{1/p} = \left(\frac{a^p + b^p}{2}\right)^{1/p} \quad (p \neq 0),$$

$$N_0(a,b) = \sqrt{ab}.$$

It is obvious from each definition that the means $K_p(a,b), D_p(a,b)$ and $N_p(a,b)$ satisfy the properties (i)-(iv).

For the mean $M_p(a,b)$, considering (1.4), R. Bhatia and R.-C. Li ([3], Theorems 2.4 and 2.5) showed that

$$M_0(a,b) \left(:= \lim_{p \to 0} M_p(a,b)\right) = \sqrt{ab},$$

$$M_\infty(a,b) \left(:= \lim_{p \to \infty} M_p(a,b)\right) = \frac{2\max\{a,b\}}{2 + \ln \frac{\max\{a,b\}}{\min\{a,b\}}},$$

and that

$$M_0(a,b) \leq M_p(a,b) \leq M_\infty(a,b). \tag{1.5}$$

The above fact (1.5) and inequalities already known then lead to the chain

$$M_0(a,b) \le M_1(a,b) \le M_2(a,b) \le M_\infty(a,b). \tag{1.6}$$

Here the second inequality in (1.6) has been given, say, in [10]. R. Bhatia and R.-C. Li ([3]) have conjectured that $M_p(a,b)$ for fixed a and b is an increasing function of p.

In this paper, we show monotonicity of $K_p(a,b), D_p(a,b)$ and $N_p(a,b)$ with respect to the parameter p. And we also investigate relations among the above four interpolating families of means. Some facts are directly obtained by calculating Taylor's expansion of each mean.

2. Monotonicity of interpolating families of means

First, as for $N_p(a,b)$:

Lemma 2.1 (cf. [7], Theorem 2.1). *$N_p(a,b)$ is an increasing function of p $(-\infty < p < \infty)$ with $a,b > 0$. I.e., if $p \le q$, then $N_p(a,b) \le N_q(a,b)$.*

Let $\Phi(p) := \Phi_p(a,b;t) = ((1-t)a^p + tb^p)^{1/p}$ for $t \in [0,1]$. Then as a remark, we can, similarly as $N_p(a,b)$, show that $\Phi(p)$ is an increasing function of p.

For $K_p(a,b)$ we have:

Proposition 2.2. *$K_p(a,b)$ is an increasing function of p $(-\infty < p < \infty)$ with $a,b > 0$.*

Proof. If $p \le q, pq \ne 0$, then $\{(1-t)a^p + tb^p\}^{1/p} \le \{(1-t)a^q + tb^q\}^{1/q}$ by the remark after Lemma 2.1. Hence from

$$\int_0^1 \frac{dt}{\{(1-t)a^p + tb^p\}^{1/p}} \ge \int_0^1 \frac{dt}{\{(1-t)a^q + tb^q\}^{1/q}},$$

the inequality $\frac{1}{K_p(a,b)} \ge \frac{1}{K_q(a,b)}$ holds, therefore $K_p(a,b) \le K_q(a,b)$. The same inequality is also obtained for $p = 0$ or $q = 0$. □

From the inequalities $K_{-1}(a,b) \le K_0(a,b) \le K_{1/2}(a,b) \le K_1(a,b) \le K_2(a,b)$, we have:

Corollary 2.3. For $a, b > 0, a \neq b$,

$$\frac{2ab}{a+b} < ab\frac{\ln b - \ln a}{b-a} < \sqrt{ab} < L(a,b) \left(= \frac{b-a}{\ln b - \ln a}\right) < \frac{a+b}{2}.$$

As a property of a convex (concave) function we know:

Proposition 2.4. *If $\Phi(x)$ is a convex (concave) function, then, for a continuous function $f(t) \geq 0$ on $[0,1]$,*

$$\int_0^1 \Phi(f(t))dt \geq (\leq)\ \Phi\left(\int_0^1 f(t)dt\right). \tag{2.1}$$

In the above (2.1), put $\Phi(x) = x^p$ for $p \geq 1$. Then $\Phi(x)$ is convex, so that we obtain the following inequality:

$$\int_0^1 f(t)^p dt \geq \left(\int_0^1 f(t)dt\right)^p. \tag{2.2}$$

Further, if $0 < p \leq q$, then put $q/p = p_1 \geq 1$. By (2.2),

$$\int_0^1 f(t)^{p_1} dt \geq \left(\int_0^1 f(t)dt\right)^{p_1}.$$

Replacing $f(t)$ by $f(t)^p$, we obtain

$$\int_0^1 f(t)^q dt \geq \left(\int_0^1 f(t)^p dt\right)^{p_1}.$$

Therefore

$$\left(\int_0^1 f(t)^q dt\right)^{1/q} \geq \left(\int_0^1 f(t)^p dt\right)^{1/p}. \tag{2.3}$$

If $f(t)$ is (strictly) positive and $p \leq q < 0$, then replace, in the above (2.3), $f(t)$ by $\frac{1}{f(t)}$ and p, q by $-q, -p$, respectively. We again obtain the same inequality. Now define $\Phi_p(f)$ by

$$\Phi_p(f) := \left(\int_0^1 f(t)^p dt\right)^{1/p} \quad for \quad p \neq 0.$$

Moreover, define $\Phi_0(f)$ by

$$\Phi_0(f) = \lim_{p\to 0} \Phi_p(f) = \exp\left(\int_0^1 \ln f(t)dt\right).$$

Then we obtain the inequality (2.3) for $p = 0$ or $q = 0$. Now we have:

Proposition 2.5. *If $f(t)$ is a positive continuous function on $[0,1]$, then $\Phi_p(f)$ is an increasing function of p $(-\infty < p < \infty)$.*

Putting $f(t) = (1-t)a + tb$, we have:

Corollary 2.6. *$D_p(a,b)$ is an increasing function of p $(-\infty < p < \infty)$ with $a, b > 0$.*

3. Order relations among the four interpolating families of means

Now we prepare useful lemmas related to the order between $N_p(a,b)$ and $D_p(a,b)$, and that between $M_p(a,b)$ and $K_p(a,b)$.

First the following fact can be obtained from [9], Proposition 4.2:

Lemma 3.1. *Let $a, b > 0$ and $a \neq b$. Then*

(i) $N_p(a,b) < D_p(a,b)$ *for* $0 \leq p < 1$,

(ii) $N_1(a,b) = D_1(a,b)\left(=\frac{a+b}{2}\right)$,

(iii) $N_p(a,b) > D_p(a,b)$ *for* $p > 1$.

Lemma 3.2 ([3], Theorem 3.1). *Let $a, b > 0$ and $a \neq b$. Then*

(i) $M_p(a,b) > K_p(a,b)$ *for* $0 \leq p < 1$,

(ii) $M_1(a,b) = K_1(a,b)\left(= L(a,b)\right)$,

(iii) $M_p(a,b) < K_p(a,b)$ *for* $p > 1$.

Based on the above facts, we first have order relations among the four interpolating families of means with respect to the parameter p with $0 \le p \le 1$.

Proposition 3.3. *Let $a, b > 0$ and $a \neq b$. Then*

(i) $K_0(a,b) < M_0(a,b) = N_0(a,b) < D_0(a,b)$,

(ii) $K_p(a,b) < M_p(a,b) < N_p(a,b) < D_p(a,b)$ *for* $0 < p < 1$,

(iii) $K_1(a,b) = M_1(a,b) < N_1(a,b) = D_1(a,b)$.

Proof. For (i), we can easily see that

$$K_0(a,b) = ab\frac{\ln b - \ln a}{b-a} < K_{1/2}(a,b) = \sqrt{ab} = M_0(a,b) = N_0(a,b)$$
$$= D_{-2}(a,b) < D_0(a,b) = \frac{1}{e}\left(a^a b^{-b}\right)^{\frac{1}{a-b}}.$$

For (ii), the second inequality $M_p(a,b) < N_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}$ has been given by Theorem 2.1 in [3], $N_p(a,b) < D_p(a,b)$ by Lemma 3.1 (i), and that $K_p(a,b) < M_p(a,b)$ by Lemma 3.2.

For (iii), we can easily see

$$K_1(a,b) = M_1(a,b) = \frac{b-a}{\ln b - \ln a} < N_1(a,b) = D_1(a,b) = \frac{a+b}{2}. \qquad \square$$

As for the order relations among $M_p(a,b), K_p(a,b), D_p(a,b)$ and $N_p(a,b)$ for $p > 1, p \neq 3$, though the result is restricted to a neighbourhood of $b/a = 1$, we have:

Proposition 3.4. *The following inequalities (i) and (ii) hold if* $0 < |b/a - 1| < \varepsilon$ *with a sufficiently small number* $\varepsilon > 0$.

(i) $M_p(a,b) < K_p(a,b) < D_p(a,b) < N_p(a,b)$ *for* $1 < p < 3$,

(ii) $M_p(a,b) < D_p(a,b) < K_p(a,b) < N_p(a,b)$ *for* $p > 3$.

To prove this proposition, we prepare the next lemma for $M_p(a,b)$ $(p > 0)$ with $a = 1$ and $b = x$.

Lemma 3.5. *Applying Taylor's expansion to the mean $M_p(1,x)$, we have*

$$M_p(1,x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4(p+2)}(x-1)^2 + o[x-1]^3, \tag{3.1}$$

where $0 \le o[x-1]^3 \le c[x-1]^3$ *for some constant* $c > 0$.

Before the proof of this lemma we state well-known formulae related to the Beta and the Gamma functions:

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}, \ \Gamma(z) = (z-1)\Gamma(z-1) \quad (z > 1).$$

Proof of Lemma 3.5. By Theorem 2.1 in [3], we have

$$\frac{1}{M_p(1,x)} = \frac{\displaystyle\int_0^1 \frac{t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1}}{(1-t+x^p t)^{\frac{1}{p}}} dt}{\displaystyle\int_0^1 t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1} dt}.$$

From the definition of the Beta function, we have

$$\int_0^1 t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1} dt = B\left(\frac{1}{p}, \frac{1}{p}\right).$$

Hence

$$\begin{aligned}
\frac{1}{M_p(1,x)} &= \frac{1}{B\left(\frac{1}{p}, \frac{1}{p}\right)} \int_0^1 \frac{t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1}}{(1-(1-(1+(x-1))^p)t)^{\frac{1}{p}}} dt \\
&= \frac{1}{B\left(\frac{1}{p}, \frac{1}{p}\right)} \int_0^1 \frac{t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1}}{\left(1-(1-(1+p(x-1)+\frac{p(p-1)(x-1)^2}{2}+\cdots))t\right)^{\frac{1}{p}}} dt \\
&= \frac{1}{B\left(\frac{1}{p}, \frac{1}{p}\right)} \int_0^1 \frac{t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1}}{\left(1+\left(p(x-1)+\frac{p(p-1)(x-1)^2}{2}+\cdots\right)t\right)^{\frac{1}{p}}} dt \\
&= \frac{1}{B\left(\frac{1}{p}, \frac{1}{p}\right)} \int_0^1 t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1} \\
&\qquad \times \left(1+\left(p(x-1)+\frac{p(p-1)(x-1)^2}{2}+\cdots\right)t\right)^{-\frac{1}{p}} dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{B\left(\frac{1}{p},\frac{1}{p}\right)}\int_0^1 t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1} \\
&\quad\times\left(1-\frac{1}{p}\cdot\left(p(x-1)+\frac{p(p-1)(x-1)^2}{2}+\cdots\right)t\right. \\
&\quad\left.+\frac{1}{2}\cdot\frac{1}{p}\cdot\left(\frac{1}{p}+1\right)\left(p(x-1)+\frac{p(p-1)(x-1)^2}{2}+\cdots\right)^2 t^2+\cdots\right)dt \\
&= \frac{1}{B\left(\frac{1}{p},\frac{1}{p}\right)}\left(B\left(\frac{1}{p},\frac{1}{p}\right)-\frac{1}{p}\left(p(x-1)+\frac{p(p-1)(x-1)^2}{2}+\cdots\right)\right. \\
&\quad\times\int_0^1 t^{\left(\frac{1}{p}+1\right)-1}(1-t)^{\frac{1}{p}-1}dt+\frac{1}{2}\cdot\frac{1}{p}\cdot\left(\frac{1}{p}+1\right) \\
&\times\left.\left(p(x-1)+\frac{p(p-1)(x-1)^2}{2}+\cdots\right)^2\int_0^1 t^{\left(\frac{1}{p}+2\right)-1}(1-t)^{\frac{1}{p}-1}dt-\cdots\right) \\
&= 1-\frac{1}{p}\frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\left(p(x-1)+\frac{p(p-1)(x-1)^2}{2}+\cdots\right) \\
&+\frac{1}{2}\cdot\frac{1}{p}\cdot\frac{1+p}{p}\cdot\frac{B\left(\frac{1}{p}+2,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\left(p(x-1)+\frac{p(p-1)(x-1)^2}{2}+\cdots\right)^2-\cdots \\
&= 1-\frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}(x-1) \\
&\quad-\left(\frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\frac{p-1}{2}-\frac{1+p}{2}\frac{B\left(\frac{1}{p}+2,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\right)(x-1)^2-\cdots.
\end{aligned}
$$

Now if we put

$$
\begin{aligned}
X(p) = &-\frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}(x-1) \\
&-\left(\frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\frac{p-1}{2}-\frac{1+p}{2}\frac{B\left(\frac{1}{p}+2,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\right)(x-1)^2-\cdots,
\end{aligned}
$$

then since $\frac{1}{M_p(1,x)}=1+X(p)$, we have

$$
M_p(1,x) = (1+X(p))^{-1}=1-X(p)+X(p)^2-\cdots
$$

$$
\begin{aligned}
&= 1 + \frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}(x-1) \\
&\quad + \left(\frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\frac{p-1}{2} - \frac{p+1}{2}\frac{B\left(\frac{1}{p}+2,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\right) \\
&\quad \times (x-1)^2 + \cdots + \left(\frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\right)^2 (x-1)^2 + o[x-1]^3.
\end{aligned}
$$

Put $\phi(p) = \dfrac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}$ and

$$
\psi(p) = \left(\frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\frac{p-1}{2} - \frac{p+1}{2}\frac{B\left(\frac{1}{p}+2,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\right) + \left(\frac{B\left(\frac{1}{p}+1,\frac{1}{p}\right)}{B\left(\frac{1}{p},\frac{1}{p}\right)}\right)^2.
$$

Then

$$
M_p(1,x) = 1 + \phi(p)(x-1) + \psi(p)(x-1)^2 + o[x-1]^3.
$$

Now using the formulae between the Beta and the Gamma functions, we can show

$$
\begin{aligned}
\phi(p) &= \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}+1\right)} = \frac{1}{2} \quad \text{and} \\
\psi(p) &= \frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}+1\right)}\frac{p-1}{2} - \frac{p+1}{2}\frac{\Gamma\left(\frac{1}{p}+2\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}+2\right)} \\
&\quad + \left(\frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}+1\right)}\right)^2 \\
&= \frac{1}{2}\cdot\frac{p-1}{2} - \frac{p}{2}\cdot\frac{p+1}{p}\cdot\frac{p+1}{2p+4} + \left(\frac{1}{2}\right)^2 = -\frac{1}{4(p+2)}.
\end{aligned}
$$

Hence we have the desired Taylor's expansion of $M_p(1,x)$. □

Proof of Proposition 3.4. For $p \geq 1$, to compare the four means $M_p(a,b), K_p(a,b), D_p(a,b)$ and $N_p(a,b)$, we can take $a = 1$ and $b = x$

without loss of generality. First from (3.1) in Lemma 3.5, we have

$$M_p(1,x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4(p+2)}(x-1)^2 + o[x-1]^3.$$

For the other means $K_p(1,x), D_p(1,x)$ and $N_p(1,x)$, we can obtain

$$K_p(1,x) = 1 + \frac{1}{2}(x-1) + \frac{p-2}{12}(x-1)^2 + o[x-1]^3,$$

$$D_p(1,x) = 1 + \frac{1}{2}(x-1) + \frac{p-1}{24}(x-1)^2 + o[x-1]^3,$$

and

$$N_p(1,x) = 1 + \frac{1}{2}(x-1) + \frac{p-1}{8}(x-1)^2 + o[x-1]^3.$$

Now for (i), if $\frac{p-2}{12} < \frac{p-1}{24}$, that is, $p < 3$ then the inequality $K_p(1,x) < D_p(1,x)$ holds for x such that $0 < |x-1| < \varepsilon$, with a sufficiently small $\varepsilon > 0$. The other inequalities are obvious from Lemmas 3.1 and 3.2. And also, for the first inequality of (ii), it is clear that $-\frac{1}{4(p+2)} < \frac{p-1}{24}$ for $p > -2$. Therefore the first inequality of (ii) holds for $p > 3$. Furthermore from Lemmas 3.1 and 3.2, we obtain the other inequalities of (ii). □

Remark 3.6. The inequality $K_3(a,b) > D_3(a,b)$ holds for $a, b > 0, a \neq b$. Hence *Proposition 3.4 (ii) is also valid for* $p = 3$. For this, it suffices to show that $K_3(1,x) > D_3(1,x)$ holds for any $x > 0, x \neq 1$: Since

$$K_3(1,x) = \frac{2}{3} \cdot \frac{(x^2+x+1)}{(x+1)} \text{ and } D_3(1,x) = \left\{ \frac{(x+1)(x^2+1)}{4} \right\}^{1/3},$$

we see

$$K_3(1,x)^3 - D_3(1,x)^3 = \frac{1}{108(x+1)^3}\{32(x^2+x+1)^3 - 27(x+1)^4(x^2+1)\}.$$

Now we have to prove $f(x) := 32(x^2+x+1)^3 - 27(x+1)^4(x^2+1) > 0$. If we put $u = (x+1)^2$ and $v = x^2+1$, then from $x^2+x+1 = \frac{u+v}{2}$ and $1 < \frac{u}{v} < 2$, we obtain, putting $t = \frac{u}{v}$, for $1 < t < 2$,

$$f(x) = 32\left(\frac{u+v}{2}\right)^3 - 27u^2v = v^3(4t^3 - 15t^2 + 12t + 4) = v^3(4t+1)(t-2)^2 > 0.$$

Remark 3.7. Related to Proposition 3.4, we can prove that if $0 \leq p \leq 2$, then $D_p(a,b) \geq K_p(a,b)$ for $a,b>0$. The proof is as follows:

First for $0 \leq p \leq 1$, $D_p(a,b) \geq K_p(a,b)$ by Proposition 3.3. Next for $1 \leq p \leq 2$, since $0 \leq p-1 \leq 1$, we have, by concavity of $x \longmapsto x^{p-1}$,

$$(1-t)a^{p-1} + tb^{p-1} \leq \{(1-t)a+tb\}^{p-1}.$$

Based on the above inequality, we obtain $K_p(a,b) \leq D_p(a,b)$ for $1 \leq p \leq 2$, by the following inequalities,

$$\int_0^1 \{(1-t)a^{p-1} + tb^{p-1}\}^{\frac{1}{p-1}} dt \leq \int_0^1 \{(1-t)a+tb\} dt \leq \left[\int_0^1 \{(1-t)a+tb\}^p dt\right]^{\frac{1}{p}}.$$

□

Remark 3.8. In [9], by using convexity of $X \longmapsto X^{-1}$ for positive matrices X, the following inequality

$$K_p(A,B) \leq K_{p+1}(A,B)$$

was shown for all positive matrices A,B and all real numbers p. Moreover the functions $K_p(1,x)$ was shown to be operator monotone for $-1 \leq p \leq 2$ ([6], Proposition 4.2). For the function $D_{p-1}(1,x) = \left(\frac{x^p-1}{p(x-1)}\right)^{\frac{1}{p-1}}$, it is known to be matrix monotone if $-2 \leq p \leq 2$ ([1], Theorem 3). For the function $N_p(1,x) = \left(\frac{1+x^p}{2}\right)^{1/p}$, it is known to be matrix monotone if (and only if) $-1 \leq p \leq 1$ ([1], Theorem 4).

Remark 3.9. A hypergeometric function ${}_2F_1(\alpha,\beta,\gamma;z)$ with parameters α,β,γ is defined by

$${}_2F_1(\alpha,\beta,\gamma;z) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k(1)_k} z^k,$$

where $(\lambda)_k = \prod_{i=1}^k (\lambda+i-1)$.

In [3], R. Bhatia and R.-C. Li showed the following expression with respect to $M_p(a,b)$ (for $a,b,p>0$):

$$\frac{1}{M_p(a,b)} = (\max\{a,b\})^{-1}\left[1+\sum_{k=1}^{\infty}\frac{\prod_{i=0}^{k-1}(\frac{1}{p}+i)^2}{\prod_{i=0}^{k-1}(\frac{2}{p}+i)\cdot k!}\left\{1-\left(\frac{\min\{a,b\}}{\max\{a,b\}}\right)^p\right\}^k\right].$$

From this fact we obtain:

$$\begin{aligned}\frac{1}{M_p(a,b)} &= (\max\{a,b\})^{-1}\left[1+\sum_{k=1}^{\infty}\frac{\left(\frac{1}{p}\right)_k\left(\frac{1}{p}\right)_k}{\left(\frac{2}{p}\right)_k(1)_k}\left\{1-\left(\frac{\min\{a,b\}}{\max\{a,b\}}\right)^p\right\}^k\right]\\ &= (\max\{a,b\})^{-1}{}_2F_1\left(\frac{1}{p},\frac{1}{p},\frac{2}{p};\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right).\end{aligned}$$

In particular, if $p=2$, then we have

$$\frac{1}{AG(a,b)} = (\max\{a,b\})^{-1}{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right).$$

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