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Forced oscillation criteria for quasilinear elliptic inequalities with $p(x)$-Laplacian via Riccati method*

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Abstract. Forced oscillation criteria for quasilinear elliptic inequalities with $p(x)$-Laplacian are derived by using the Riccati inequality. The approach used is to reduce forced oscillation problems for quasilinear elliptic inequalities with $p(x)$-Laplacian to one-dimensional oscillation problems for Riccati inequalities with variable exponents. More general quasilinear elliptic inequalities with mixed nonlinearities are also investigated.

1. Introduction

There is much current interest in studying oscillations of half-linear elliptic equations with p-Laplacian($p = \alpha + 1$) of the form

$$\nabla \cdot \left(a(x)|\nabla u|^{\alpha-1}\nabla u\right) + c(x)|u|^{\alpha-1}u = 0,$$

where $\alpha > 0$ is a constant, $\nabla = (\partial/\partial x_1, ..., \partial/\partial x_n)$ and the dot $\cdot$ denotes the scalar product (cf. [1, 3–6, 11, 16]). The operator $\nabla \cdot \left(|\nabla u|^{p(x)-2}\nabla u\right)$ is said to be $p(x)$-Laplacian, and becomes p-Laplacian $\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right)$ if

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$p(x) = p$ (constant). The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years (see [7]). These problems arise from nonlinear elasticity theory, electrorheological fluids (cf. [14,20]). In 2007 Zhang [19] studied oscillation problems for the $p(t)$-Laplacian equation

$$\left(|u'|^{p(t)-2}u'\right)' + t^{-\theta(t)}g(t,u) = 0, \quad t > 0.$$

It is noted that the elliptic equation with $p(x)$-Laplacian ($p(x) = \alpha(x) + 1$)

$$\nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) + C(x)|v|^{\alpha(x)-1}v = 0$$

is not half-linear if $\alpha(x)$ is not a constant. However, it is shown that the elliptic inequality with $p(x)$-Laplacian ($p(x) = \alpha(x) + 1$)

$$vQ_0[v] \le 0 \tag{1}$$

is *half-linear* in the sense that a constant multiple of a solution v of (1) is also a solution of (1), where

$$\begin{aligned} Q_0[v] \;&:=\; \nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ &\quad + |\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v + C(x)|v|^{\alpha(x)-1}v. \end{aligned} \tag{2}$$

In fact, it can be shown that

$$(kv)Q_0[kv] = |k|^{\alpha(x)+1}vQ[v] \ (k \in \mathbb{R})$$

(cf. Yoshida [17, Proposition 2.1]). We refer to Allegretto [2] and Yoshida [18] for Picone identity arguments for elliptic operators with $p(x)$-Laplacian.

The objective of this paper is to investigate oscillatory behavior of solutions of the quasilinear elliptic inequality $vQ[v] \le 0$ with $p(x)$-Laplacian ($p(x) = \alpha(x) + 1$) and a forcing term, where

$$\begin{aligned} Q[v] \;&:=\; \nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ &\quad + |\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v \\ &\quad + C(x)|v|^{\alpha(x)-1}v + D(x)|v|^{\beta(x)-1}v + E(x)|v|^{\gamma(x)-1}v - f(x). \end{aligned} \tag{3}$$

We note that $\log|v|$ in (2), (3) has singularities at zeros of v, but $v\log|v|$ becomes continuous at the zeros of v if we define $v\log|v| = 0$ at the zeros, in light of the fact that $\lim_{\varepsilon\to+0}\varepsilon\log\varepsilon=0$. Therefore, we conclude that $vQ[v]$ has no singularities and is continuous in Ω. We remark that $vQ[v]\le 0$ is not half-linear.

The approach used is to reduce the multi-dimensional oscillation problems to one-dimensional oscillation problems for Riccati differential inequalities, and to utilize the Riccati techniques.

In Section 2 we establish Riccati inequality for $vQ[v]\le 0$, and in Section 3 we give oscillation results for $vQ[v]\le 0$ on the basis of the Riccati inequality obtained in Section 2.

2. Riccati inequality

Let Ω be an exterior domain in $\mathbb{R}^n$, that is, Ω includes the domain $\{x\in\mathbb{R}^n; |x|\ge r_0\}$ for some $r_0>0$. We assume that $A(x)\in C(\Omega;(0,\infty))$, $B(x)\in C(\Omega;\mathbb{R}^n)$, $C(x)\in C(\Omega;\mathbb{R})$, $D(x)\in C(\Omega;[0,\infty))$, $E(x)\in C(\Omega;[0,\infty))$, $f(x)\in C(\Omega;\mathbb{R})$, and that $\alpha(x)\in C^1(\Omega;(0,\infty))$, $\beta(x)\in C(\Omega;(0,\infty))$, $\gamma(x)\in C(\Omega;(0,\infty))$, and that $\beta(x)>\alpha(x)>\gamma(x)>0$.

The domain $\mathcal{D}_Q(\Omega)$ of Q is defined to be the set of all functions v of class $C^1(\Omega;\mathbb{R})$ such that $A(x)|\nabla v|^{\alpha(x)-1}\nabla v\in C^1(\Omega;\mathbb{R}^n)$.

A solution $v\in\mathcal{D}_Q(\Omega)$ of $vQ[v]\le 0$ is said to be *oscillatory* in Ω if it has a zero in Ω_r for any $r>0$, where

$$\Omega_r=\Omega\cap\{x\in\mathbb{R}^n;\ |x|>r\}.$$

We use the notation:

$$A(r,\infty)=\{x\in\mathbb{R}^n;\ |x|>r\},$$
$$A[r,\infty)=\{x\in\mathbb{R}^n;\ |x|\ge r\}.$$

Since Ω is an exterior domain in $\mathbb{R}^n$, we see that

$$\Omega_{r_1}=A(r_1,\infty)$$

for some large $r_1\ge r_0$.

Lemma 1. *If $v \in \mathcal{D}_Q(\Omega)$ and $|v(x)| \geq \lambda$ in $A[r_2, \infty)$ for some $\lambda > 0$ and some $r_2 > r_1$, then we obtain the following*:

$$\begin{aligned} & -\nabla \cdot \left(\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{|v|^{\alpha(x)-1}v} \right) \\ \geq\ & C(x) + F(\beta(x), \alpha(x), \gamma(x); D(x), E(x)) - \lambda^{-\alpha(x)}|f(x)| \\ & + \alpha(x)A(x)\left|\frac{\nabla v}{v}\right|^{\alpha(x)+1} + B(x) \cdot \left(\frac{|\nabla v|^{\alpha(x)-1}\nabla v}{|v|^{\alpha(x)-1}v} \right) - \frac{vQ[v]}{|v|^{\alpha(x)+1}} \end{aligned} \tag{4}$$

in $A[r_2, \infty)$, where

$$\begin{aligned} & F(\beta(x), \alpha(x), \gamma(x); D(x), E(x)) \\ =\ & \left(\frac{\beta(x) - \gamma(x)}{\alpha(x) - \gamma(x)} \right) \left(\frac{\beta(x) - \alpha(x)}{\alpha(x) - \gamma(x)} \right)^{\frac{\alpha(x)-\beta(x)}{\beta(x)-\gamma(x)}} D(x)^{\frac{\alpha(x)-\gamma(x)}{\beta(x)-\gamma(x)}} E(x)^{\frac{\beta(x)-\alpha(x)}{\beta(x)-\gamma(x)}}. \end{aligned}$$

Proof. Letting

$$\varphi(v) := |v|^{\alpha(x)-1}v = |v(x)|^{\alpha(x)-1}v(x),$$

we derive the following:

$$\begin{aligned} -\nabla \cdot \left(\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \right) &= -A(x)|\nabla v|^{\alpha(x)-1}\nabla \left(\frac{1}{\varphi(v)} \right) \cdot \nabla v \\ & \quad -\frac{1}{\varphi(v)} \nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v \right) \end{aligned} \tag{5}$$

and

$$\begin{aligned} & A(x)|\nabla v|^{\alpha(x)-1}\nabla \left(\frac{1}{\varphi(v)} \right) \cdot \nabla v \\ & = -\alpha(x)A(x)\left|\frac{\nabla v}{v}\right|^{\alpha(x)+1} - \frac{1}{\varphi(v)} A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x) \cdot \nabla v \end{aligned} \tag{6}$$

(see Yoshida [17, (2.5) and (2.7) in the proof of Lemma 2.1]). Combining (5) with (6) yields

$$\begin{aligned} & -\nabla \cdot \left(\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \right) \\ =\ & \alpha(x)A(x)\left|\frac{\nabla v}{v}\right|^{\alpha(x)+1} - \frac{1}{\varphi(v)}\Big[\nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v \right) \\ & \qquad - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x) \cdot \nabla v\Big]. \end{aligned} \tag{7}$$

Using (3) we see that

$$\begin{aligned}
& \frac{1}{\varphi(v)}\Big[\nabla \cdot \big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla \alpha(x)\cdot \nabla v\Big] \\
= \; & \frac{1}{\varphi(v)}\Big[Q[v] - |\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v - C(x)|v|^{\alpha(x)-1}v \\
& \qquad -D(x)|v|^{\beta(x)-1}v - E(x)|v|^{\gamma(x)-1}v + f(x)\Big] \\
= \; & \frac{vQ[v]}{|v|^{\alpha(x)+1}} - B(x)\cdot\left(\frac{|\nabla v|^{\alpha(x)-1}\nabla v}{|v|^{\alpha(x)-1}v}\right) - C(x) \\
& -\frac{1}{\varphi(v)}\Big[D(x)|v|^{\beta(x)-1}v + E(x)|v|^{\gamma(x)-1}v\Big] + \frac{1}{\varphi(v)}f(x). \qquad (8)
\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}
& \frac{1}{\varphi(v)}\Big[D(x)|v|^{\beta(x)-1}v + E(x)|v|^{\gamma(x)-1}v\Big] \\
= \; & D(x)|v|^{\beta(x)-\alpha(x)} + E(x)|v|^{\gamma(x)-\alpha(x)} \\
\geq \; & F(\beta(x),\alpha(x),\gamma(x);D(x),E(x)) \qquad (9)
\end{aligned}$$

(cf. Jaroš, Kusano and Yoshida [8, p.717]). It can be shown that

$$\frac{1}{\varphi(v)}f(x) \leq \frac{|f(x)|}{|v|^{\alpha(x)}} \leq \lambda^{-\alpha(x)}|f(x)|. \qquad (10)$$

Combining (7)–(10), we arrive at the desired inequality (4). □

Lemma 2. *If $v \in \mathcal{D}_Q(\Omega)$, $vQ[v] \leq 0$ in Ω and $|v(x)| \geq \lambda$ in $A[r_2,\infty)$ for some $\lambda > 0$ and some $r_2 > r_1$, then we obtain the Riccati inequality:*

$$\begin{aligned}
& \nabla \cdot W(x) + F_\lambda(x) + \alpha(x)A(x)^{-1/\alpha(x)}|W(x)|^{1+(1/\alpha(x))} \\
& \qquad + \langle W(x), A(x)^{-1}B(x)\rangle \leq 0 \quad \text{in } A[r_2,\infty), \qquad (11)
\end{aligned}$$

where $\langle U, V\rangle$ denotes the scalar product of U, $V \in \mathbb{R}^n$,

$$W(x) = \frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{|v|^{\alpha(x)-1}v}$$

and

$$F_\lambda(x) = C(x) + F(\beta(x),\alpha(x),\gamma(x);D(x),E(x)) - \lambda^{-\alpha(x)}|f(x)|.$$

Proof. Since

$$|W(x)| = A(x)\left|\frac{\nabla v}{v}\right|^{\alpha(x)},$$

we easily see that

$$\left|\frac{\nabla v}{v}\right|^{\alpha(x)+1} = \left(\frac{|W(x)|}{A(x)}\right)^{\frac{\alpha(x)+1}{\alpha(x)}} = \frac{|W(x)|^{1+(1/\alpha(x))}}{A(x)^{1+(1/\alpha(x))}}$$

and hence

$$\alpha(x)A(x)\left|\frac{\nabla v}{v}\right|^{\alpha(x)+1} = \alpha(x)A(x)^{-1/\alpha(x)}|W(x)|^{1+(1/\alpha(x))}. \tag{12}$$

It is clear that

$$B(x)\cdot\left(\frac{|\nabla v|^{\alpha(x)-1}\nabla v}{|v|^{\alpha(x)-1}v}\right) = \langle W(x), A(x)^{-1}B(x)\rangle. \tag{13}$$

Combining (4), (12), (13), and taking account of $vQ[v] \le 0$, we arrive at the desired inequality (11). □

Lemma 3. *If $v \in \mathcal{D}_Q(\Omega)$, $vQ[v] \le 0$ in Ω and $|v(x)| \ge \lambda$ in $A[r_2,\infty)$ for some $\lambda > 0$ and some $r_2 > r_1$, then we obtain*

$$\begin{aligned}\nabla\cdot\big(\psi(x)W(x)\big) + \psi(x)F_\lambda(x) + \alpha(x)\psi(x)A(x)^{-1/\alpha(x)}|W(x)|^{1+(1/\alpha(x))}\\ +\langle W(x), \psi(x)A(x)^{-1}B(x) - \nabla\psi(x)\rangle \le 0\end{aligned} \tag{14}$$

in $A[r_2,\infty)$ for any $\psi(x) \in C^1(A[r_2,\infty);\mathbb{R})$.

Proof. It is easy to see that

$$\nabla\cdot\big(\psi(x)W(x)\big) = \psi(x)\nabla\cdot W(x) + \langle W(x), \nabla\psi(x)\rangle. \tag{15}$$

Combining (11) with (15) yields the desired inequality (14). □

The following two lemmas follow by the same arguments as were used in Yoshida [17, Lemmas 2.4 and 2.5], and will be omitted.

Lemma 4. *If $v \in \mathcal{D}_Q(\Omega)$, $vQ[v] \le 0$ in Ω and $|v(x)| \ge \lambda$ in $A[r_2, \infty)$ for some $\lambda > 0$ and some $r_2 > r_1$, then we derive the Riccati inequality:*

$$\nabla \cdot \big(\psi(x)W(x)\big) + \hat{F}_\lambda(x) + \frac{\alpha(x)}{\alpha(x)+1} g(x)|W(x)|^{1+(1/\alpha(x))} \le 0 \qquad (16)$$

in $A[r_2, \infty)$ for any $\psi(x) \in C^1(A[r_2,\infty); (0,\infty))$, where

$$g(x) = \frac{\alpha(x)+1}{2}\psi(x)A(x)^{-1/\alpha(x)},$$
$$\hat{F}_\lambda(x) = \psi(x)F_\lambda(x) - \frac{1}{\alpha(x)+1} g(x)^{-\alpha(x)}\psi(x)^{\alpha(x)+1}\left|\frac{B(x)}{A(x)} - \frac{\nabla\psi(x)}{\psi(x)}\right|^{\alpha(x)+1}.$$

Lemma 5. *Assume that the following hypothesis holds:*

(H) $\alpha(x) \equiv \alpha(|x|)$ *in $A[r_1, \infty)$.*

If $v \in \mathcal{D}_Q(\Omega)$, $vQ[v] \le 0$ in Ω and $|v(x)| \ge \lambda$ in $A[r_2, \infty)$ for some $\lambda > 0$ and some $r_2 > r_1$, then we have the Riccati inequality:

$$Y'(r) + \int_{S_r} \hat{F}_\lambda(x)\, dS + \frac{\alpha(r)}{\alpha(r)+1}\Psi(r)^{-1/\alpha(r)}|Y(r)|^{1+(1/\alpha(r))} \le 0 \qquad (17)$$

for $r \ge r_2$, where

$$S_r = \{x \in \mathbb{R}^n;\ |x| = r\},$$
$$\Psi(r) = \int_{S_r} g(x)^{-\alpha(r)}\psi(x)^{\alpha(r)+1}\, dS,$$
$$Y(r) = \int_{S_r} \psi(x)\langle W(x), \nu(x)\rangle\, dS,$$

$\nu(x)$ being the unit exterior normal vector x/r on S_r.

3. Oscillation results

In this section we present oscillation results for $vQ[v] \le 0$ by using Riccati inequality in Section 2.

Theorem 1. *Assume that the hypothesis (H) is satisfied, and that there exists a function $\psi(x) \in C^1(A[r_1,\infty); (0,\infty))$ such that the Riccati inequality (17) has no solution on $[r, \infty)$ for all large r and any $\lambda > 0$. Then,* *for every solution $v \in \mathcal{D}_Q(\Omega)$ of $vQ[v] \le 0$, either v is oscillatory in Ω or satisfies the condition*

$$\liminf_{|x|\to\infty} |v(x)| = 0. \tag{18}$$

Proof. Suppose to the contrary that there is a nonoscillatory solution $v \in \mathcal{D}_Q(\Omega)$ of $vQ[v] \le 0$ such that $\liminf_{|x|\to\infty} |v(x)| > 0$. Then, there is a number $r_2 > r_1$ such that $|v(x)| \ge \lambda$ in $A[r_2, \infty)$ for some $\lambda > 0$. Lemma 5 implies that (17) has a solution $Y(r)$ on $[r_2, \infty)$ for some $r_2 > r_1$ and some $\lambda > 0$. This contradicts the hypothesis and completes the proof. □

Now we need to investigate sufficient conditions for Riccati inequality (17) to have no solution on $[r, \infty)$ for all large r and any $\mu > 0$.

Let

$$D = \{(r, s) \in \mathbb{R}^2;\ r \ge s \ge r_1\},$$
$$D_0 = \{(r, s) \in \mathbb{R}^2;\ r > s \ge r_1\}$$

and we consider the kernel function $H(r, s)$, which is defined, continuous and sufficiently smooth on D, so that the following conditions are satisfied:

(K_1) $H(r, s) \ge 0$ and $H(r, r) = 0$ for $r \ge s \ge r_1$;

(K_2) there exists a constant $k_0 > 0$ such that

$$\lim_{r\to\infty} \frac{H(r, s)}{H(r, r_1)} = k_0 \quad \text{for all } s \ge r_1;$$

(K_3) $\dfrac{\partial H}{\partial s}(r, s) \le 0$, $-\dfrac{\partial H}{\partial s}(r, s) = h(r, s)H(r, s)$ for $(r, s) \in D_0$, where $h(r, s) \in C(D_0; \mathbb{R})$

(cf. Kong [10], Philos [13]).

We let $\rho(s) \in C^1([r_1, \infty); (0, \infty))$ and define an integral operator A_τ^ρ by

$$A_\tau^\rho(y; r) = \int_\tau^r H(r, s)y(s)\rho(s)\, ds, \quad r \ge \tau \ge r_1,$$

where $y \in C([\tau, \infty); \mathbb{R})$. It is easily seen that A_τ^ρ is linear and positive, and in fact satisfies the following:

(A_1) $A_\tau^\rho(k_1 y_1 + k_2 y_2; r) = k_1 A_\tau^\rho(y_1; r) + k_2 A_\tau^\rho(y_2; r) \quad$ for $k_1, k_2 \in \mathbb{R}$;

(A$_2$) $A^\rho_\tau(y;r) \geq 0 \quad$ for $y \geq 0$;

(A$_3$) $A^\rho_\tau(y';r) = -H(r,\tau)y(\tau)\rho(\tau) + A^\rho_\tau([h-\rho^{-1}\rho']y;r)$

(see Wong [15]).

Theorem 2. *Assume that the hypothesis* (H) *of Lemma* 5 *holds. If there exist functions* $\psi(x) \in C^1(A[r_1,\infty);(0,\infty))$ *and* $\rho(s) \in C^1([r_1,\infty);(0,\infty))$ *such that for any* $\lambda > 0$

$$\limsup_{r\to\infty} \frac{1}{H(r,r_1)} A^\rho_{r_1}\left(\int_{S_r} \hat{F}_\lambda(x)\,dS - \frac{1}{\alpha(r)+1}\left|h - \frac{\rho'}{\rho}\right|^{\alpha(r)+1}\Psi;\, r\right) = \infty,$$

then, for every solution $v \in \mathcal{D}_Q(\Omega)$ *of* $vQ[v] \leq 0$*, either* v *is oscillatory in* Ω *or satisfies the condition* (18)*.*

Proof. The proof follows by using exactly the same arguments as in Yoshida [17, Theorem 3.2], and will be omitted. □

Following the classical idea of Kamenev [9], we define $H(r,s)$ and $\rho(s)$ by

$$H(r,s) = (r-s)^\mu,\ \ \mu > 1,\ \ (r,s) \in D,$$
$$\rho(s) = s^\nu,\ \ \nu \in \mathbb{R}.$$

Then we obtain the following corollary.

Corollary. *Assume that the hypothesis* (H) *of Lemma* 5 *is satisfied, and, moreover, that* $\mu > 1$ *and* ν *is a real number. If there exists a function* $\psi(x) \in C^1(A[r_1,\infty);(0,\infty))$ *such that for any* $\lambda > 0$

$$\limsup_{r\to\infty} \frac{1}{r^\mu}\int_{r_1}^r \Big[\omega_n s^{\nu+n-1}(r-s)^\mu M[\hat{F}_\lambda](s) - \frac{1}{\alpha(s)+1} s^{\nu-\alpha(s)+1}|\nu r - (\mu+\nu)s|^{\alpha(s)+1}(r-s)^{\mu-\alpha(s)-1}\Psi(s)\Big]\,ds = \infty,$$

then, for every solution $v \in \mathcal{D}_Q(\Omega)$ *of* $vQ[v] \leq 0$*, either* v *is oscillatory in* Ω *or satisfies the condition* (18)*, where* ω_n *denotes the surface area of the unit sphere* S_1 *and* $M[\hat{F}_\lambda](r)$ *denotes the spherical mean of* $\hat{F}_\lambda(x)$ *over the sphere* S_r*.*

In addition to the hypotheses (K$_1$)–(K$_3$) we suppose the following:

(K$_4$) $\dfrac{\partial H}{\partial r}(r,s) = \tilde{h}(r,s)H(r,s)$ for $(r,s) \in D_0$, where $\tilde{h}(r,s) \in C(D_0;\mathbb{R})$.

Theorem 3. *Assume that the hypothesis* (H) *of Lemma* 5 *holds. If there are functions* $\psi(x) \in C^1(A[r_1,\infty);(0,\infty))$ *and* $\rho(s) \in C^1([r_1,\infty);(0,\infty))$ *such that for each* $\xi \geq r_1$ *and for any* $\lambda > 0$

$$\limsup_{r\to\infty} A_\xi^\rho\left(\int_{S_r} \hat{F}_\lambda(x)\,dS - \frac{1}{\alpha(r)+1}\left|h - \frac{\rho'}{\rho}\right|^{\alpha(r)+1}\Psi;\, r\right) > 0,$$
$$\limsup_{r\to\infty} \tilde{A}_\xi^\rho\left(\int_{S_r} \hat{F}_\lambda(x)\,dS - \frac{1}{\alpha(r)+1}\left|\tilde{h} + \frac{\rho'}{\rho}\right|^{\alpha(r)+1}\Psi;\, r\right) > 0,$$

then, for every solution $v \in \mathcal{D}_Q(\Omega)$ *of* $vQ[v] \leq 0$*, either* v *is oscillatory in* Ω *or satisfies the condition* (18)*.*

Proof. The proof is quite similar to that of Yoshida [17, Theorem 3.4], and hence is omitted. □

Remark. In the case where $n = 1$, Li and Li [12] established oscillation results for the second-order nonlinear differential equation

$$\left[a(t)|y'(t)|^{\sigma-1}y'(t)\right]' + q(t)f(y(t)) = r(t)$$

which are related to Theorem 1.

Example. We consider the quasi-linear elliptic inequality

$$v\Big(\nabla\cdot\big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v$$
$$+C(x)|v|^{\alpha(x)-1}v + D(x)|v|^{\beta(x)-1}v + E(x)|v|^{\gamma(x)-1}v - f(x)\Big) \leq 0 \quad (19)$$

in an exterior domain Ω, where

$$\alpha(x) = \alpha(|x|) = 1 + e^{-|x|},\ \beta(x) = 2 + e^{-|x|},\ \gamma(x) = e^{-|x|},$$
$$A(x) = \frac{(\alpha(x)+1)^{\alpha(x)+1}}{3\cdot 6^{\alpha(x)}|x|^{n+1}},$$
$$C(x) = |x|^{1-n},\ D(x) = |x|^{3-3n},\ E(x) = |x|^{n-1},$$
$$f(x) = \frac{\tilde{f}(x)}{|x|^{n-1+\delta}}\ (\tilde{f}(x) \text{ is a bounded function in } \Omega,\ 0 < \delta < 1).$$

A simple computation shows that

$$\begin{aligned} F(\beta(x),\alpha(x),\gamma(x);D(x),E(x)) &= 2D(x)^{1/2}E(x)^{1/2} \\ &= 2|x|^{1-n} \end{aligned}$$

and that

$$\begin{aligned} F_\lambda(x) &= C(x)+F(\beta(x),\alpha(x),\gamma(x);D(x),E(x))-\lambda^{-\alpha(x)}|f(x)| \\ &= 3|x|^{1-n}-e^{-(1+e^{-|x|})\log\lambda}|f(x)|. \end{aligned}$$

Choosing $\psi(x)=1$, we observe that $\hat{F}_\lambda(x)=F_\lambda(x)$ and

$$M[\hat{F}_\lambda](r)=3r^{1-n}-e^{-(1+e^{-r})\log\lambda}M[|f(x)|](r).$$

Since

$$g(x)=\frac{\alpha(x)+1}{2}A(x)^{-1/\alpha(x)},$$

we easily obtain

$$\Psi(r)=\omega_n\frac{2^{\alpha(r)}r^{n-1}}{(\alpha(r)+1)^{\alpha(r)}}M[A](r).$$

Letting $\mu=3$ and $\nu=0$, we see that

$$\begin{aligned} &\frac{1}{r^\mu}\int_{r_1}^{r}\Bigg[\omega_n s^{\nu+n-1}(r-s)^\mu M[\hat{F}_\lambda](s) \\ &\qquad -\frac{1}{\alpha(s)+1}s^{\nu-\alpha(s)+1}|\nu r-(\mu+\nu)s|^{\alpha(s)+1}(r-s)^{\mu-\alpha(s)-1}\Psi(s)\Bigg]ds \\ =\ &\frac{1}{r^3}\int_{r_1}^{r}\Bigg[\omega_n s^{n-1}(r-s)^3 M[\hat{F}_\lambda](s) \\ &\qquad -\frac{1}{\alpha(s)+1}s^{-\alpha(s)+1}|3s|^{\alpha(s)+1}(r-s)^{2-\alpha(s)}\Psi(s)\Bigg]ds \\ =\ &\frac{\omega_n}{r^3}\int_{r_1}^{r}\Bigg[(r-s)^3 M[\hat{F}_\lambda](s)s^{n-1} \\ &\qquad -(r-s)^{2-\alpha(s)}M[A](s)\frac{3\cdot 6^{\alpha(s)}s^{n+1}}{(\alpha(s)+1)^{\alpha(s)+1}}\Bigg]ds \\ =\ &\frac{\omega_n}{r^3}\int_{r_1}^{r}\left[3(r-s)^3-(r-s)^{1-e^{-s}}\right]ds \\ &\qquad -\frac{\omega_n}{r^3}\int_{r_1}^{r}(r-s)^3 s^{n-1}e^{-(1+e^{-s})\log\lambda}M[|f(x)|](s)\,ds. \qquad (20) \end{aligned}$$

It can be shown that

$$\int_{r_1}^{r} (r-s)^3 ds = \frac{(r-r_1)^4}{4} = \mathrm{O}\big(r^4\big) \ (r \to \infty), \tag{21}$$

$$\int_{r_1}^{r} (r-s)^{1-e^{-s}} ds = \mathrm{O}\big(r^2\big) \ (r \to \infty) \tag{22}$$

(see Yoshida [17, Example 4.1]). It is easy to check that $e^{-(1+e^{-s})\log\lambda} \le K_\lambda$ and

$$M[|f(x)|](r) \le \frac{K}{r^{n-1+\delta}},$$

where $K_\lambda = \max\{1, e^{-2\log\lambda}\}$ and K is a positive constant. Hence, it follows that for any $\lambda > 0$

$$\int_{r_1}^{r} (r-s)^3 s^{n-1} e^{-(1+e^{-s})\log\lambda} M[|f(x)|](s)\, ds = K_\lambda \mathrm{O}\big(r^{4-\delta}\big) \ (r \to \infty). \tag{23}$$

Combining (20)–(23) yields

$$\lim_{r\to\infty} \Bigg\{ \frac{\omega_n}{r^3} \int_{r_1}^{r} \Big[3(r-s)^3 - (r-s)^{1-e^{-s}}\Big]\, ds - \frac{\omega_n}{r^3} \int_{r_1}^{r} (r-s)^3 s^{n-1} e^{-(1+e^{-s})\log\lambda} M[|f(x)|](s)\, ds \Bigg\} = \infty \tag{24}$$

for any $\lambda > 0$. Since all the hypotheses of Corollary are satisfied in view of (20) and (24), it follows from Corollary that for every solution $v \in \mathcal{D}_Q(\Omega)$ of (19), either v is oscillatory in Ω or satisfies the condition (18)

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