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A note on convergence in Bergman spaces over bounded symmetric domains in $C^N (N > 1)$

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Abstract. In this paper we prove a result that generalizes the result of [7] on the unit disk to bounded star-shaped circular domains.

1. Introduction

Let D be a bounded symmetric domain in C^N $(N > 1)$, and $O \in D$. D is circular and star-shaped with respect to the origin, i. e. $tz \in D$ when $z \in D$ and $t \in C$ with $|t| < 1$, [4]. We denote by $H(D)$ the space of holomorphic functions on D.

For $p > 0$ the Bergman space A^p is defined on D by

$$A^p = A^p(D) = \left\{ f \ : \ f \in H(D) \text{ and } \left|f\right|_{A^p} = \left(\frac{1}{V} \int_D \left|f(z)\right|^p dv_z \right)^{\frac{1}{p}} < \infty \right\},$$

or equivalently [5],

$$A'^p = \left\{ f \ : \ f \in H(D) \text{ and } \left|f\right|_{A'^p} = \sup_{0 \le r < 1} \left(\frac{1}{V} \int_D \left|f(rz)\right|^p dv_z \right)^{\frac{1}{p}} < \infty \right\}. \quad (1)$$

Where V is the Euclidean volume of D and dv_z is the Euclidean element of volume at $z \in D$.

It is well known that a complete orthonormal system (CONS) of homogenous polynomials $\{\Psi_{k\nu}\}\nu = 1, \dots ; m_k = \binom{N+k-1}{k}$; $k = 0, 1, \dots$ exist on a bounded star-shaped domain [2].

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We will have the following lemmas which will be used in the proof of Theorem 2.1.

Lemma 1.1. *Let D be a bounded star-shaped circular domain. Then any holomorphic function on D has a Fourier series expansion*

$$f(z) = \sum_{k=0}^{\infty} \sum_{\nu=1}^{m_k} c_{k\nu}(f)\psi_{k\nu}(z);\ c_{k\nu}(f) = \lim_{r\to 1} \int_D f(rz)\bar{\psi}_{k\nu}\, dv_z\,, \tag{2}$$

where the series converges absolutely and uniformly on compact subsets of D.

Proof. The proof of Lemma 1.1 follows the method of the proof of the Lemma in [3] and the proof of Theorem [1]. □

Lemma 1.2.

$$A^p(D) = A'^p(D) \text{ and } \|f\|_{A^p} = \|f\|_{A'^p}\,,$$

[5] where D is bounded star-shaped circular domain in C^N $(N > 1)$.

2. The following Theorem will extend a special case of a result of [7].

Theorem 2.1. *Let D be a bounded star-shaped circular domain in C^N $(N > 1)$ and $f \in A^p(D)$ $(0 < p < \infty)$, then $\|f_r - f\|_{A^p} \to 0$ as $r \to 1$, where $f_r(z) = f(rz)$.*

Furthermore, the set of polynomials in z is dense in $A^p(D)$; also $A^p(D)$ is separable.

Proof. Since $rz \in \bar{D}$ for fixed $r, 0 \le r < 1$, $f_r(z) = f(rz)$ is holomorphic on $\bar{D}$ and hence bounded on $\bar{D}$. Thus $f_r \in A^p$ and hence to L^p. Also for $z \in D$, $\lim_{r\to 1} f_r(z) = f(z)$, by continuity of f.

Finally by (1) and Lemma 1.2, we have $\|f_r\|_{A^p} \to \|f\|_{A^p}$ as $r \to 1$.

Thus the hypothesis of [6] are satisfied, so that

$$\|f_r - f\|_{A^p} \to 0 \text{ as } r \to 1\,.$$

Let $f \in A^p$ $(0 < p < \infty)$. Given $\epsilon > 0$, there exists r_0 $(r_0 < 1)$, such that

$$\|f - f_{r_0}\|_{A^p} < \frac{\epsilon}{2}\,. \tag{3}$$

Now let $S_{n,r_0}(z)$ denote the n$^{\text{th}}$ partial sum of the Fourier series (2) of $f_{r_0}(z)$. Since $S_{n,r_0} \to f_{r_0}$ uniformly on $\bar{D}$ by Lemma 1.1,

$$\|S_{n,r_0} - f_{r_0}\|_{A^p} < \frac{\epsilon}{2}\,, \tag{4}$$

for n sufficiently large, thus by (3) and (4)

$$\|S_{n,r_0} - f\|_{A^p} < \|S_{n,r_0} - f_{r_0}\|_{A^p} + \|f - f_{r_0}\|_{A^p} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

Hence, the linear combination of $\{\Psi_{k\nu}\}$ is dense in A^p, but as in [3], $\sum\limits_{\nu=1}^{m_k} c_{k\nu}\psi_{k\nu}(z) = \sum\limits_{\nu=1}^{m_k} A_{k\nu}Z_{k\nu}$, where $Z_{k\nu}$ denote the monomial $z_1^{\nu_1} \dots\ z_N^{\nu_n}$ $\big(k = \nu_1 + \dots + \nu_N;\ k = 0, 1, \dots;\ \nu = 1, \dots;\ m_k = \binom{N+k-1}{k}\big)$.

Thus the polynomials are dense in A^p. □

3. We have the following Corollaries:

Corollary 1. *Let $f, g \in A^2(D)$, then*

$$(f, g) = \lim_{r\to 1}\left(\frac{1}{V}\int\limits_D f(rz)\overline{g(rz)}dv_z\right).$$

Proof.

$$\big|(f, g) - (f_r, g_r)\big| \le \frac{1}{V}\int\limits_D \big|f(z)\overline{g(z)} - f_r(z)\overline{g_r(z)}\big|\, dv_z$$

$$= \frac{1}{V}\int\limits_D \big|f(z)\big(\overline{g(z)} - \overline{g_r(z)}\big) + \big(f(z) - f_r(z)\big)\overline{g_r(z)}\big|\, dv_z$$

$$\le \frac{1}{V}\int\limits_D \big|f(z)\big|\,\big|\overline{g(z) - g_r(z)}\big|\, dv_z + \frac{1}{V}\int\limits_D \big|f(z) - f_r(z)\big|\,\big|g_r(z)\big|\, dv_z\,.$$

By Schwarz inequality,

$$|(f, g) - (f_r, g_r)| < \frac{1}{V}\left(\int\limits_D |f(z)|^2\, dv_z\right)^{\frac{1}{2}}\left(\int\limits_D |\bar{g(z)} - \overline{g(rz)}|^2\, dv_z\right)^{\frac{1}{2}}$$

$$+ \frac{1}{V}\left(\int\limits_D |f(z) - f(rz)|^2\, dv_z\right)^{\frac{1}{2}}\left(\int\limits_D |\overline{g(rz)}|^2\, dv_z\right)^{\frac{1}{2}},$$

or

$$|(f,g)-(f_r,g_r)| \le \|f\|_{A^2}\,\|g-g_r\|_{A^2} + \|f-f_r\|_{A^2}\,\|g_r\|_{A^2}. \tag{5}$$

Now $\|g_r\|_{A^2} \to \|g\|_{A^2}$ as $r \to 1$ by Lemma 1.2, so the right side of (5) tends to zero by Theorem 2.1.

Therefore

$$(f,g) = \lim_{r\to 1}(f_r,g_r) = \lim_{r\to 1}\left(\frac{1}{V}\int\limits_D f(rz)\overline{g(rz)}\,dv_z\right).$$

□

Corollary 2. *For $f \in A^p$ $(1 \le p < \infty)$,*

$$c_{k\nu}(f) = \int\limits_D f(z)\bar{\psi}_{k\nu}\,dv_z\,, \tag{6}$$

where $c_{k\nu}$ is given by (2).

Proof. By Holder's inequality for $1 < p < \infty$,

$$\begin{aligned}&\left|\int\limits_D \left(f_r(z)\bar{\psi}_{k\nu}(z) - f(z)\bar{\psi}_{k\nu}(z)\right)dv_z\right| \\ &\le \left(\int\limits_D |f(rz)-f(z)|^p\,dv_z\right)^{\frac{1}{p}}\left(\int\limits_D |\overline{\psi_{k\nu}(z)}|^q\,dv_z\right)^{\frac{1}{q}},\end{aligned} \tag{7}$$

where $\frac{1}{p}+\frac{1}{q}=1$. The right side of (7) equals $\|f_r-f\|_{A^p}\|\psi_{1\nu}\|_{A^q}$. But $\psi_{k\nu}$ is homogeneous polynomial on C^N $(N>1)$, so it is bounded on compact set $\bar{D}$ and hence is in A^q. By Theorem 2.1 $\|f_r-f\|_{A^p} \to 0$ as $r \to 1$. Thus formula (6) follows. If p=1

$$\left|\int\limits_D \left(f_r(z)\bar{\psi}_{k\nu} - f(z)\bar{\psi}_{k\nu}\right)dv_z\right| \le \left(\int\limits_D |f(rz)-f(z)|\,dv_z\right)\sup_{z\in D}\left|\psi_{k\nu}(z)\right| \tag{8}$$

$$= \left\|f_r - f\right\|_{A^1}\sup_{z\in D}\left|\psi_{k\nu}(z)\right|.$$

But $\sup\limits_{z\in D}|\psi_{k\nu}(z)|$ is finite since $\psi_{k\nu}$ is bounded on D, so that the right side of (8) tend to Zero as $r \to 1$ by Theorem 2.1. Thus (6) follows for $p=1$. □

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