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Interpolation sequence for the spaces $H^q_+(\phi)(q \geq 1)$

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Abstract. Let ϕ be a subadditive increasing real valued function defined on $[0, \infty)$ and which satisfies $\phi(x) = 0$ if and only if $x = 0$. For $q \geq 1$ we define $H^q(\phi)$ to be the set of all functions f which are analytic in the open unit disc and satisfy

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \Big[\phi\big(|f(re^{i\theta})|\big)\Big]^q \, d\theta < \infty.$$

And $H^q_+(\phi)$ to be the subspace of $H^q(\phi)$ of functions which satisfy

$$\lim_{r \to 1} \int_0^{2\pi} \Big[\phi\big(|f(re^{i\theta})|\big)\Big]^q \, d\theta = \int_0^{2\pi} \Big[\phi\big(|f(e^{i\theta})|\big)\Big]^q \, d\theta.$$

In this paper we prove some interpolation theorems for $H^q_+(\phi)$

1. Introduction

Let us recall some definitions. We call a real-valued function ϕ defined on $[0, \infty)$ a modulus function, if ϕ is an increasing continuous subadditive function that satisfies the condition that $\phi(x) = 0$ if and only if $x = 0$.

Let $q \geq 1$. By the class $H^q(\phi)$ we mean the collection of all analytic functions f defined in the open unit disc Δ which satisfy

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \Big[\phi\big(|f(re^{i\theta})|\big)\Big]^q \, d\theta < \infty.$$

For $q = 1$ the spaces were studied in details in [2, 3]. If $\phi(x) = x$, then $H^q(\phi)$ becomes the usual Banach space H^q. For $\phi(x) = x^p$, $0 < p \leq 1$

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and $q = 1$, then $H^q(\phi)$ becomes the usual F-spaces H^q. If q = 1 and $\phi(x) = \log(1+x^p)$, then $H^q(\phi)$ becomes N_p. If $q = 1$ and $\phi(x) = \log(1+x)$, then $H^q(\phi)$ becomes the class N of functions of bounded characteristic. For $N_p = N_+$ for $0 < p \leq 1$ [8], where

$$N_p = \{f \in H^+(D) \cap H^p(\phi) : \lim_{r\to 1} \int_0^{2\pi} \log(1 + |f(r\mathrm{e}^{\mathrm{i}\theta})|^p)\,\mathrm{d}\theta = \int_0^{2\pi} \log(1 + |f(\mathrm{e}^{\mathrm{i}\theta})|^p)\,\mathrm{d}\theta\},$$

and

$$N^+ = \{f \in N : \lim_{r\to 1} \int_0^{2\pi} \log^+(|f(r\mathrm{e}^{it})|)\,\mathrm{d}t = \int_0^{2\pi} \log^+(|f(\mathrm{e}^{it})|)\,\mathrm{d}t\}.$$

In general the spaces $H^q(\phi)$ are not F-Spaces; see [13, p. 453] for an example. We shall assume that ϕ satisfies the additional condition that $\phi(\mathrm{e}^t)$ is a convex function of t, and consequently $H^q(\phi) \subseteq N$.

A function $f \in H^q(\phi)$ is said to belong to the class $H^q_+(\phi)$ if

$$\lim_{r\to 1} \int_0^{2\pi} \Big[\varphi\big(|f(r\mathrm{e}^{\mathrm{i}\theta})|\big)\Big]^q\,\mathrm{d}\theta = \int_0^{2\pi} [\varphi(|f(\mathrm{e}^{\mathrm{i}\theta})|)]^q\,\mathrm{d}\theta.$$

For the class $H^q_+(\phi)$, which is a vector space , we define a distance function by

$$\rho(f, 0) = \left[\frac{1}{2\pi} \int_0^{2\pi} [\varphi(|f(\mathrm{e}^{\mathrm{i}\theta})|)]^q\,\mathrm{d}\theta\right]^{\frac{1}{q}} \tag{1.1}$$

For the spaces $H^q(p \geq 1)$, $H^q(0 < p \leq 1)$. [5], N_+ [13] $N^+_p(p > 0)$ and $(\log^+ H)^\alpha(a > 1)$, [12] become special cases of $H^q_+(\phi)$.

Let $\alpha = (\alpha_n) = (\alpha_1, \ldots, \alpha_n, \ldots)$, be a sequence of real numbers such that $\alpha_n \to 0$.

Let

$$l^q(\phi, \alpha) = \{(c_n) : c_n \in C \text{ and } d((c_n), 0) = \Big[\sum_{n=1}^{\infty} \alpha_n[\varphi(|c_n)|)]^q\Big]^{\frac{1}{q}} < \infty\}$$

Let X be the class of analytic functions in Δ and $\{z_n\}$ is a given sequence in Δ. When a complex sequence $\{c_n\}$ is given, the interpolation problem asks if a function $f \in X$ exists such that $f\{z_n\} = c_n$ for all n.

Let Y be a class of complex sequences. If for every sequence $(c_n) \in Y$ there is an $f \in X$ such that $f\{z_n\} = c_n$, then the sequence $\{z_n\}$ is called a universal interpolation sequence for the pair (X, Y), simply written z_n is u. i. s. for (X, Y). We are interested in the pair $H_+^q(\phi), l^q(\phi, \alpha)$ where $\alpha = (1 - |z_n|^2)$.

A sequence $\{z_n\}$ in Δ is called uniformly separated sequence (u. s. s.) if

$$\sum_{n=1}^{\infty} \big(1 - |z_n|\big) < \infty, \text{ and } \prod_{\substack{n=1 \\ m \neq n}}^{\infty} \left| \frac{z_n - z_n}{1 - \bar{z}_m z_n} \right| \geq \delta > 0 \quad (m = 1, 2, \ldots).$$

L. Carleson [1] showed that $\{z_n\}$ is u. i. s. for (H^∞, l^∞) if and only if $\{z_n\}$ is u. s. s. For the pair $\big(H_+^q(\phi), l^q(\phi, \alpha)\big)$ we have the following results: for $q \geq 1, \phi(x) = x$, [11] proved that $\{z_n\}$ is u. i. s. for $\big(H_+^q(\phi), l^q(\phi, \alpha)\big)$ if and only if $\{z_n\}$ is u. s. s. The same result was proved for $q = 1$ and $\phi(x) = x^p, 0 < p \leq 1$ by [9]. For $q = 1, \phi(x) = \log(1 + x)$,[14] showed that $\{z_n\}$ is u. s. s., then $\{z_n\}$ is u. i. s. and if $\{z_n\}$ is u. i. s., then

$$(1 - |z_n|^2) \log \frac{1}{|\mathrm{B}_n(z_n)|} \to 0 \text{ as } n \to \infty,$$

where

$$\mathrm{B}_n(z) = \prod_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{|z_m|}{z_m} \frac{z_m - z}{1 - \bar{z}_m z} \, .$$

In this paper we obtain results which generalize the above mentioned results.

In section 2 we will prove that $H_+^q(\phi)$ and $l^q(\phi, \alpha)$ are F-spaces in the sense of Banach, [4].

In section 3 we prove that if $\{z_n\}$ is u. s. s. then $\{z_n\}$ is u. i. s. for $\big(H_+^q(\phi), l^q(\phi, \alpha)\big)$, we also proved that if ϕ satisfies $\lim\limits_{x \to \infty} \varphi^{-1}(ax)\varphi^{-1}(\frac{1}{x}) < \infty$ for all real a and $T_{\phi,q}\big(H_+^q(\phi)\big) = l^q(\phi, \alpha)$ then $\{z_n\}$ is u. s. s. where

$$T_{\phi,q}(f(z)) = \Big(\big(\phi^{-1}(1 - |z_n|^2)\big)^{\frac{1}{q} - 1} f(z_n) \Big).$$

2. The spaces $H_+^q(\phi), l^q(\phi, \alpha)$.

In this section we will show that the spaces $H_+^q(\phi)$ and $l^q(\phi, \alpha)$ are F-spaces.

Theorem 2.1. *$H^q_+(\phi)$ is an F-space in the sense of Banach,[4]. That is*

i. Let f_n be functions in $H^q_+(\phi)$ such that $\rho(f_n, 0) \to 0$ as $n \to \infty$. Then for any $\alpha \in C, \rho(\alpha f_n, 0) \to 0$ as $n \to \infty$.

ii. Let $\alpha_n \in C$ be such that $\alpha_n \to 0$. Then for each $f \in H^q_+(\phi)$, $\rho(\alpha_n f, 0) \to 0$ as $n \to \infty$.

iii. $H^q_+(\phi)$ is complete with respect to the metric (1.1).

Proof. i. Suppose $\{f_n\}$ is a sequence in $H^q_+(\phi)$, $\rho(f_n, 0) \to 0$ and $\beta \in C$. Now

$$\rho(\beta f_n, 0) = \left(\frac{1}{2\pi}\int_0^{2\pi}[\varphi(|\beta f_n(\theta)|^q \, d\theta\right)^{\frac{1}{q}} \le \big[|\beta| + 1\big]\rho(f_n, 0) \to 0,$$

where $\big[|\beta|\big]$ is the greatest integer in $|\beta|$.

ii. Suppose $\beta_n \in C, \beta_n \to 0$ and $f \in H^q_+(\phi)$, without loss of generality we may assume $|\beta_n| \le 1$, so

$$[\varphi(|\beta_n f|)]^q \le [\phi(|f|)]^q \text{ and } \varphi(|\beta_n f(\theta)|) \to 0 \text{ a. e.}$$

Hence by Lebesgue convergence theorem we get

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi}\big[\phi(|\beta_n f(\theta)|\big]^q \, d\theta = 0.$$

Thus $\rho(\beta_n f, 0) \to 0$ as $n \to \infty$.

iii. Suppose $\{f_n\}$ is a Cauchy sequence in $H^q_+(\phi)$. By Lemma 3 in [2] applied to $\big[\phi(|f|)\big]^q$ which is subharmonic for $q \ge 1$, from [6, Lemma 5.1], we get

$$|f(z)| \le \varphi^{-1}\left(\frac{c\rho(f, 0)}{\Big((1 - |z|)^{\frac{1}{q}}\Big)}\right) \quad \text{for } z \in \Delta.$$

Therefore

$$|f_n(z) - f_m(z)| \le \varphi^{-1}\left(\frac{c\rho(f_n, f_m)}{\Big((1 - r)^{\frac{1}{q}}\Big)}\right) < \varepsilon$$

for $n, m > N(\varepsilon)$ and for all $z \in \{w : |w| \leq r < 1\}$. Hence $\{f_n(z)\}$ is a Cauchy sequence in C. Since $\{f_n(z)\}$ converges uniformly on compact subsets of Δ and $f_n(z)$ is analytic, then $\{f_n\}$ converges to an analytic function f. Clearly, $\{\phi(|f_n|)\}$ converges uniformly on compact subsets to $\phi(|f|)$. Therefore

$$\begin{aligned}\int_0^{2\pi}\Big[\varphi\big(|f(re^{i\theta})|\big]^q\,d\theta &= \lim_{n\to\infty}\int_0^{2\pi}[\varphi(|f_n(re^{i\theta})|)]^q\,d\theta\\ &\leq \lim_{n\to\infty}\int_0^{2\pi}[\varphi(|f_n(\theta)|)]^q\,d\theta \leq \mathrm{M},\end{aligned}$$

hence, $f \in H^q(\phi)$. But $H^q(\phi) \subseteq N$, so $\lim_{r\to 1} f(re^{i\theta}) = f(\theta)$ a. e. and $f_n(\theta) \to f(\theta)$ in measure. Now choose a subsequence f_{n_j} such that $f_{n_j}(\theta) \to f(\theta)$ a. e., then

$$\begin{aligned}\rho(f, f_n) &= \left(\frac{1}{2\pi}\int_0^{2\pi}[\varphi(|f(\theta) - f_n(\theta|)]^q\,d\theta\right)^{\frac{1}{q}}\\ &\leq \underline{\lim}_{j\to\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}[\varphi(|f_{n_j}(\theta) - f_n(\theta)|]^q\,d\theta\right)^{\frac{1}{q}}\\ &\leq \rho(f_{n_j}, f_n) + \varepsilon \quad \text{for large } j.\end{aligned}$$

Thus if n_j and n are sufficiently large, we have

$$\rho(f, f_n) \to 0, \text{ as } n \to \infty.$$

It remains to show that $f \in H_+^q(\phi)$. Since $f_{n_j}(\theta) \to f(\theta)$ a. e., then there exists a compact set $E \subset [0, 2\pi]$ such that $m(E) > 2\pi - \varepsilon$ and $f_{n_j} \to f$ uniformly on E, hence

$$\left(\frac{1}{2\pi}\int_0^{2\pi}[\varphi(|f_{n_j}(\theta)|)]^q\,d\theta\right)^{\frac{1}{q}} \leq \left(\frac{1}{2\pi}\int_E[\phi(|f(\theta)|)]^q\,d\theta\right)^{\frac{1}{q}} + \varepsilon \quad \text{, for large } j.$$

Also,

$$\begin{aligned}\left(\int_{E^c}[\varphi(|f_{n_j}(\theta)|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}} &\le \left(\int_{E^c}[\varphi(|f_{n_j}(\theta)-f(\theta)|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}}\\ &\quad+\left(\int_{E^c}[\varphi(|f(\theta)|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}}\\ &\le \rho(f_{n_j},f)+\left(\int_{E^c}[\varphi(|f(\theta)|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}}\\ &\le \varepsilon+\left(\int_{E^c}[\varphi(|f(\theta)|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}},\end{aligned}$$

but $f_{n_j}\in H^q_+(\varphi)$, so

$$\left(\int_0^{2\pi}[\varphi(|f_{n_j}(r\mathrm{e}^{\mathrm{i}\theta})|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}} \le \left(\int_0^{2\pi}[\varphi(|f(\theta)|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}}+\varepsilon,$$

letting $j\to\infty$,

$$\left(\int_0^{2\pi}[\varphi(|f(r\mathrm{e}^{\mathrm{i}\theta})|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}} \le \left(\int_0^{2\pi}[\varphi(|f(\theta)|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}}+\varepsilon,$$

for any $\varepsilon>0$. Hence

$$\lim_{r\to1}\left(\int_0^{2\pi}[\varphi(|f(r\mathrm{e}^{\mathrm{i}\theta})|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}} = \left(\int_0^{2\pi}[\varphi(|f(\theta)|)]^q\,\mathrm{d}\theta\right)^{\frac{1}{q}},$$

this proves that $f\in H^q_+(\phi)$.

□

Theorem 2.2. *$l^q(\varphi,\alpha)$ is an F-space.*

Proof. Parts (i), (ii) are exactly the same as in Theorem 2.1. For completeness, suppose $\{x_n\}$ is a Cauchy sequence, let $\varepsilon>0$ be given, then there exists N such that

$$d(x_k,x_m)<\varepsilon \quad \text{for all } k,m>N(\varepsilon).$$

Hence

$$\sum_{n=1}^{\infty}\alpha_n\left(\varphi(|x_n^k-x_n^m|)\right)^q<\varepsilon^q, \tag{2.1}$$

where $x_k = (x_n^k), x_m = (x_n^m)$. Therefore

$$|x_n^k - x_n^m| < \varphi^{-1}\left[\frac{\varepsilon}{\alpha_n^{\frac{1}{q}}}\right].$$

Thus $\{x_n^k\}$ is a Cauchy sequence in C for each n. So it must converge to some $x_n \in C$.

Let $x = \{x_n\}$, then Minkowski's inequality gives

$$\left(\sum_{n=1}^{\infty} \alpha_n \left(\varphi(|x_n|)\right)^q\right)^{\frac{1}{q}} \leq \left(\sum_{n=1}^{\infty} \alpha_n \left(\varphi(|x_n - x_n^k|)\right)^q\right)^{\frac{1}{q}} + \left(\sum_{n=1}^{\infty} \alpha_n \left(\varphi(|x_n^k|)\right)^q\right)^{\frac{1}{q}},$$

but $\{x_n\}$ is a Cauchy sequence, hence the second term of the right side is bounded by M which does not depend on k, also

$$\sum \alpha_n \left(\varphi(|x_n - x_n^k|)\right)^q = \lim_{m\to\infty} \sum \alpha_n \left(\varphi(|x_n^m - x_n^k|)\right)^q < \varepsilon^q$$

by (2.1). Thus $x_n \to x$ and $x \in l^q(\phi, \alpha)$. □

3. Interpolation Theorems

We will assume an additional condition on f, for each $g \in H^1$ the function $\log\left[\varphi^{-1}\left(|g(z)|\right)^{\frac{1}{q}} + 1\right]$ is integrable on the unit circle.

Now we prove a theorem which generalizes theorem 1 in [14].

Theorem 3.1. *If $\{z_n\}$ is u. s. s. then $\{z_n\}$ is u. i. s. for $\left(H_+^q(\phi), l^q(\phi\alpha)\right)$, where $\alpha = (1 - |z_n|^2)$.*

Proof. Let $c = (c_n) \in l^q(\phi, \alpha)$ and let

$$g(z) = \sum_{n=1}^{\infty} \left(1 - |z_n|^2\right)^2 [\varphi|c_n|)]^q \frac{\mathrm{B}_n(z)}{\mathrm{B}_n(z_n)} \cdot \frac{1}{(1 - \bar{z}_n z)^2}.$$

clearly g is analytic in Δ since $\sum_{n=1}^{\infty} (1 - |z_n|^2)[\varphi(|c_n|)]^q < \infty$. Also $g \in H^1$, because

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |g(re^{\mathrm{i}\theta})|\,\mathrm{d}\theta &\leq \sum_{n=1}^{\infty}\left(1 - |z_n|^2\right)^2 \left[(\varphi|c_n|)\right]^q \frac{1}{|\mathrm{B}_n(z_n)|}\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{|1 - \bar{z}_n z|} \\ &\leq \frac{1}{\delta}\sum_{n=1}^{\infty}\left(1 - |z_n|^2\right)\left[\varphi(|c_n|)\right]^q < \infty.\end{aligned}$$

Since $\log\bigl[\varphi^{-1}(|g(z)|^{\frac{1}{q}})+1\bigr]$ is integrable, hence by [7, p. 53] there exists $f_1 \in H^1$ such that $|f_1(z)| = \phi^{-1}(|g(z)|^{\frac{1}{q}})+1$. Since $\mathrm{B}_k(z_n) = 0$ for all $k \neq n$, $g(z_n) = [\phi(|c_n|)]^q$. Hence, $|f_1(z_n)| = |c_n| + 1$ and $f_1(z_n) = (|c_n|+1)\mathrm{e}^{\mathrm{i}\theta_n}$, where $c_n = |c_n|\mathrm{e}^{\mathrm{i}\alpha_n}$ and $f_3(z_n) = \mathrm{e}^{\mathrm{i}\alpha_n}$, then $(f_1f_2 - f_3)(z_n) = c_n$. Since $\varphi^q(|f_1(z)|) \leq |g(z)| + \varphi^q\,(1)$, we have $f_1 \in H^q(\phi)$ and so is $f_1f_2 - f_3$.

By Carleson's Theorem there exists functions f_2, f_3 in H^∞ such that $f_2(z_n) = \mathrm{e}^{\mathrm{i}(\alpha_n-\theta_n)}$. □

The following theorem generalizes the one given by [11].

Theorem 3.2. *Suppose ϕ satisfies*

$$\lim_{x\to\infty} \varphi^{-1}(ax)\varphi^{-1}\Bigl(\frac{1}{x}\Bigr) < \infty,$$

for all real a and $T_{\phi,q}\bigl(H^q_+(\phi)\bigr) = l^q(\phi,\alpha)$, then $\{z_n\}$ is u. s. s.

We need the following lemma:

Lemma. *The sequence $E = \{e_n\}$ is bounded in $l^q(\phi,\alpha)$ in the sense of topological vector space W. Rudin [10], where $e_n = (0,0,\ldots,0,1,0,\ldots)$ and 1 appears in the nth place.*

Proof. Let $s > 0$ be given and let $\mathrm{B}_s = \{x \in l^q(\phi,\alpha) \|x\| < s\}$. We need to show that there exists r_0 such that $E \subset r\mathrm{B}_s$ for all $r > r_0$. Let r_0 be such that $(\phi^q)^{-1}(s) = \frac{1}{r_0}$, then for $r > r_0$

$$\Bigl\|\frac{e_n}{r}\Bigr\| = \varphi^q\Bigl(\frac{1}{r}\Bigr) < \varphi^q\Bigl(\frac{1}{r_0}\Bigr) = s,$$

this implies $\frac{e_n}{r} \in \mathrm{B}_s$. Therefore $E \subset r\mathrm{B}_s$. □

Proof of Theorem 3.2. Let

$$N^q_\varphi = \{f \in H^q_+(\varphi) \,:\, f(z_n) = 0 \quad \text{for all } n\}.$$

The quotient space $H^q_+(\varphi)/N^q_\varphi$ is an F-space and $T_{\phi,q}$ induces a one-to-one bounded linear functional $\Delta_{\varphi,q}$ (since $T_{\varphi,q}$ is bounded) from $H^q_+(\varphi)/N^q_\varphi$ onto $l^q(\phi,\alpha)$. Hence, the inverse is bounded which implies that $\Delta^{-1}_{\phi,q}(E)$

is bounded on $H_+^q(\varphi)$, i. e. there exists $\mathrm{M} > 0$ such that $\Delta_{\phi,q}^{-1}(E) \subset \mathrm{B_M}$, which means that for all n, there exists $f_n \in H_+^q(\varphi)$ such that

$$\|f_n\| \leq \mathrm{M}, \tag{3.1}$$

and

$$f_n(z_k) = \begin{cases} \varphi^{-1}(1-|z_k|^2)^{\left(\frac{1}{q}\right)-1} & n = k, \\ 0 & n \neq k. \end{cases}$$

For $n > k$, let

$$F_{n_k}(z) = f_k(z) \prod_{\substack{j=1 \\ j \neq k}} \frac{1-\bar{z}_j z}{z - z_j},$$

then

$$\|F_{n_k}\| \leq \|f_k\| \text{ and } F_{n_k} \in H_+^q(\varphi),$$

so by [2]

$$|F_{n_k}(z)| \leq \varphi^{-1}\left(\frac{c\|F_{n_k}\|}{(1-|z|^2)^{\frac{1}{q}}}\right), \tag{3.2}$$

but,

$$|F_{n_k}(z)| = \left|\prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1-\bar{z}_j z_k}{z_k - z_j}\right| \left|f_k(z_k)\right|,$$

thus

$$|F_{n_k}(z_k)| = \left[\varphi^{-1}\left(1-|z_k|^2\right)^{\left(\frac{1}{q}\right)-1}\right] \prod_{\substack{j=1 \\ j \neq k}}^{n} \left|\frac{1-\bar{z}_j z_k}{z_k - z_j}\right|,$$

and by using (3.1) and (3.2) we get

$$\left|\prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1-\bar{z}_j z_k}{z_k - z_j}\right| \leq \varphi^{-1}(\alpha_k)\varphi^{-1}\left(\frac{c\mathrm{M}}{\alpha_k}\right) = \varphi^{-1}(x_k)\varphi^{-1}\left(\frac{1}{x_k}\right) < \Delta < \infty,$$

for x_k large, where $x_k = \frac{\alpha_k}{c\mathrm{M}}$ and Δ is a constant. Thus $\{z_k\}$ is u. s. s. □

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