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CR-warped product submanifolds of locally conformal Kaehler manifolds

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Abstract. In the present paper characterizations in terms of the canonical structures P and F on a CR-submanifold of a locally conformal Kaehler manifold are worked out under which the submanifold reduces to a CR-warped product submanifold.

1. Introduction

A submanifold of an almost Hermitian manifold $\bar{M}$ is called a CR-product if it is locally a Riemannian product of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$. These submanifolds are CR-submanifolds in the sense of A.Bejancu [1]. On the other hand, if g_T and $g_\perp$ are Riemannian metrics on N_T and $N_\perp$ and f a positive differentiable function on N_T, then a more general class of product manifolds is constructed by homothetically warping the product metric onto the fibers. More explicitly, the warped product metric g on the product manifold $N_T \times N_\perp$ is defined as $g = g_T + f^2 g_\perp$. The manifold $M = (N_T \times N_\perp, g)$ isometrically immersed in $\bar{M}$, denoted by $N_T \times_f N_\perp$ is known as a CR-warped product submanifold. The warped product obtained by interchanging the two factors of CR-warped product namely, $N_\perp \times_f N_T$ is known as a warped product CR-submanifold. The function f defined on the first factor of a

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warped product manifold is known as a warping function. If the warping function is constant, the above warped products are simply CR-products in $\bar{M}$. In general a CR-warped product (as well as a warped product CR) submanifold is a CR-submanifold and an actual product (while CR-products are product manifolds just locally) yet the induced metric is not a product metric unless f is constant. Warped product manifolds are not only interesting objects with differential geometric point of view but they also play an important role in Physics. For these findings and other applications, the study of warped product spaces assumes significance and thus many research articles have recently appeared exploring existence of warped products as submanifolds in known spaces. To this end, it is proved that warped product CR-submanifolds of a Kaehler manifold are trivial whereas there are examples of non-trivial CR-warped products of a Kaehler manifold (cf. [7]). Extending the study to the setting of l.c.K. manifolds, it is shown that non-trivial warped products of the above forms do exist in l.c.K. manifolds. Many extrinsic geometric properties of these warped products are obtained in [5],[15],[16] and [17] etc. As warped product manifolds provide an excellent setting to model space-time near black holes or bodies with high gravitational fields moreover, for their occurance in geometric studies, it is natural to seek characterizations under which a CR-submanifold is a CR-warped product. In the present article, some conditions in terms of the canonical structures on a CR-submanifold are worked out which yield a CR-warped product.

2. Preliminaries

An almost Hermitian manifold $(\bar{M}, J, g)$ of dimension $2m$ is called a *locally conformal Kaehler* (l.c.K.) manifold if there is a closed 1-form ω (known as the *Lee-form*) globally defined on $\bar{M}$ such that $\omega \wedge \Omega = d\Omega$, where Ω is the Kaehler 2-form associated with the almost complex structure J and Hermitian metric g, defined as $\Omega(U, V) = g(U, JV)$ for all vector fields U, V on $\bar{M}$. $\bar{M}$ is globally conformal Kaehler (resp. Kaehler) if the Lee-form ω is exact (resp. $\omega = 0$). A simply connected l.c.K. manifold is g.c.K. For a l.c.K. manifold we define the *Lee-vector field* λ as $g(U, \lambda) = \omega(U)$. Let

$T(\bar{M})$ be the tangent bundle on $\bar{M}$ and $\bar{\nabla}$, the Levi-Civita connection on $\bar{M}$. Then we have

$$(\bar{\nabla}_U J)V = \theta(V)U - \omega(V)JU - g(U,V)A - \Omega(U,V)\lambda, \qquad (2.1)$$

for any $U, V \in T(\bar{M})$, where $\theta = \omega \circ J$ and $A = -J\lambda$ are the *anti-Lee form* and the *anti-Lee vector field* respectively (cf. [16]). In terms of the Lee vector field, the above relation takes the form

$$(\bar{\nabla}_U J)V = g(\lambda, JV)U - g(\lambda, V)JU + g(JU, V)\lambda + g(U,V)J\lambda. \qquad (2.2)$$

Let M be a submanifold of $\bar{M}$ and $T(M)$, $T^{\perp}(M)$ denote respectively the tangent and the normal bundles on M. If ∇ and $\nabla^{\perp}$ are the induced Riemannian connections on TM and $T^{\perp}(M)$ respectively, then we have the following fundamental equations for the submanifold

$$\bar{\nabla}_U V = \nabla_U V + h(U,V), \quad \bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N \qquad (2.3)$$

for each $U, V \in T(M)$ and $N \in T^{\perp}(M)$ where h and A_N are respectively the second fundamental form and the shape operator (corresponding to the normal vector field N) for the immersion of M into $\bar{M}$. They are related as $g(A_N U, V) = g(h(U,V), N)$, where g denotes the Riemannian metric on $\bar{M}$ as well the induced metric on M.

For any $U \in T(M)$, we put

$$PU = tan(JU) \text{ and } FU = nor(JU).$$

where tan_x and nor_x are the natural projections associated to the direct sum decomposition

$$T_x\bar{M} = T_x M \oplus T^{\perp}_x M, \quad x \in M$$

Similarly, we put

$$tN = tan(JN) \text{ and } fN = nor(JN).$$

That is, P (resp. f) is a $(1,1)$ tensor field on TM (resp. $T^{\perp}(M)$ whereas t (resp. F) is a tangential (resp. normal) valued 1-form on $T^{\perp}(M)$ (resp.

TM). The covariant derivatives of the tensor fields P and F are defined as

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{2.4}$$

$$(\bar{\nabla}_U F)V = \nabla_U^{\perp} FV - F\nabla_U V, \tag{2.5}$$

A *CR-submanifold* M of an almost Hermitian manifold $\bar{M}$ is a submanifold endowed with two orthogonal complementary distributions D and $D^{\perp}$ such that D is holomorphic i.e., $JD_x = D_x$ and $D^{\perp}$ is totally real i.e., $JD_x^{\perp} \subseteq T_x^{\perp}(M)$ for each $x \in M$.That is, we have

$$T(M) = D \oplus D^{\perp}.$$

A CR-submanifold is *proper* if both the distributions D and $D^{\perp}$ are non-trivial. In view of the above decomposition we define projection operators B and C onto D and $D^{\perp}$ respectively, that is, for any $U \in T(M)$, we write

$$U = BU + CU, \quad BU \in D \quad \text{and} \quad CU \in D^{\perp}.$$

Similarly, we write

$$T^{\perp}(M) = JD^{\perp} \oplus \nu$$

where ν is the orthogonal complementary distribution of $JD^{\perp}$ in $T^{\perp}(M)$ and an invariant subbundle of $T^{\perp}(M)$. It is straight forward to see that

$$t(T^{\perp}M) = D^{\perp}, \text{ and } f(T^{\perp}M) \subset \nu. \tag{2.6}$$

A submanifold M of an almost Hermitian manifold $\bar{M}$ is called a *CR-product* if it is locally a Riemannian product of a holomorphic submanifold N_T and a totally real submanifold $N_{\perp}$ of $\bar{M}$. Obviously, a CR-product is a CR-submanifold. B.Y.Chen [6] obtained the following characterization for CR-submanifolds of a Kaehler manifold to be CR-products.

Theorem 2.1 ([6]). *A CR-submanifold of a Kaehler manifold $\bar{M}$ is a CR-product if and only if P is parallel i.e., $\bar{\nabla}P = 0$.*

As a consequence of the above Theorem, it was noted that

Theorem 2.2 ([6]). *A CR-submanifold of a Kaehler manifold $\bar{M}$ is a CR-product if and only if $A_{JD^\perp}D = 0$.*

If $\bar{M}$ is a locally conformal Kaehler manifold and M is a submanifold of $\bar{M}$ then by making use of equations (2.2), (2.3) and (2.4), it is easy to derive that

$$\begin{aligned}(\bar{\nabla}_U P)V =& A_{FV}U + th(U,V) + g(U,V)(J\lambda)_T + g(JU,V)(\lambda)_T \\ & - g(\lambda,V)PU - g(J\lambda,V)U, \qquad (2.7)\\ (\bar{\nabla}_U F)V =& fh(U,V) - h(U,PV) + g(U,V)(J\lambda)_N \\ & + g(JU,V)(\lambda)_N - g(\lambda,V)FU. \qquad (2.8)\end{aligned}$$

for each $U, V \in TM$, where the suffix T and N denote the tangential and normal parts respectively.

3. CR-submanifolds of l.c.K. manifolds

The study of CR-submanifolds was initiated in Kaehler manifolds by A. Bejancu [1] and was later taken up by B.Y. Chen [6]. Subsequently, it was extended to more general settings of locally conformal Kaehler and nearly Kaehler manifolds (cf. [4], [11], [12], [14] and [19] etc.). We recall here some of the results relevant to the present article. For instance, Some useful formulas for CR-submanifold of an l.c.K. manifold are stated in the following Lemma.

Lemma 3.1 ([16]). *Let M be a CR-submanifold of an l.c.K. manifold then we have*

(i) $g(\nabla_U Z, X) = g(JA_{JZ}U, X) - \theta(Z)g(U, JX) + \omega(Z)g(U,X) - \omega(X)g(U,Z)$,

(ii) $A_{JW}Z - A_{JZ}W = \theta(Z)W - \theta(W)Z$,

(iii) $A_{J\mu}JX - A_\mu X = \omega(\mu)X - \theta(\mu)JX$

for all $X, Y \in D$, $Z, W \in D^\perp$, $\mu \in \nu$ and U tangent to M.

With regard to the integrability conditions for the distributions in this setting, we have.

Lemma 3.2 ([14]). *On a CR-submanifold of a locally conformal Kaehler manifold,*

(i) $D^{\perp}$ *is involutive,*

(ii) D *is involutive if and only if*

$$g(h(X,JY) - h(JX,Y)) + 2g(X,JY)\lambda, JZ) = 0$$

for all X, Y *in* D *and* Z *in* $D^{\perp}$.

Further, as an extension of the necessary and sufficient condition for CR-products, D.Blair and S. Dragomir proved

Theorem 3.1 ([4]). *Let* M^m *be a CR-submanifold of an l.c.K. manifold* $\bar{M}^{2n}$. *Then the following statements are equivalent*

(i) P *is parallel*

(ii) M *is locally a Riemannian product* $M^{2p} \times M^q$ *where* M^{2p} *(resp.* M^q*) is a holomorphic (resp. totally real) submanifold of* $\bar{M}^{2n}$, *and either* $\lambda_T = 0$ *i.e.,* M^{2n} *is normal to the Lee-field of* $\bar{M}^{2n}$, *or* $\lambda_T \neq 0$ *and then* $\lambda_T \in D$ *and* $p = 1$ *i.e.,* M^2 *is a complex curve in* $\bar{M}^{2n}$.

The equivalent version of the above characterization in terms of the shape operator is obtained by K. Matsumoto as:

Theorem 3.2 ([14]). *A CR-submanifold of a l.c.K. manifold* $\bar{M}$ *is a CR-product if and only if the endomorphism* P *is parallel or equivalently the following condition is satisfied*

$$A_{JZ}X = g(J\lambda, Z)X - g(\lambda, Z)JX - g(J\lambda, X)Z \tag{3.1}$$

for all X *in* D *and* Z *in* $D^{\perp}$.

Now, by (2.7), it is easy to derive that

$$g((\bar{\nabla}_X P)Y - (\bar{\nabla}_Y P)X, Z) = 2g(JX, Y)g(\lambda, Z)$$

Making use of (2.4) while taking account of the fact that $PU \in D$, the above equation yields

$$g(\nabla_X PY - \nabla_Y PX, Z) = 2g(JX, Y)g(\lambda, Z)$$

Hence, we conclude,

Theorem 3.3. *On a proper CR-submanifold of an l.c.K. manifold, the Lee-vector field λ is orthogonal to $D^\perp$ if and only if $\nabla_X PY - \nabla_Y PX$ lies in D for each $X, Y \in D$.*

As an immediate consequence of the above Theorem it follows that

Corollary 3.1. *On a CR-product submanifold of an l.c.K. manifold, λ is orthogonal to $D^\perp$.*

The above observation was proved in [14].

Theorem 3.4. *Let M be a CR-submanifold of an l.c.K. manifold $\bar{M}$ with λ orthogonal to $D^\perp$. Then the holomorphic distribution on M is involutive if and only if*

$$g((\bar{\nabla}_X P)JY - (\bar{\nabla}_{JX} P)Y, Z) = 0,$$

for each $X, Y \in D$ and $Z \in D^\perp$.

Proof. As λ is assumed to be orthogonal to $D^\perp$, by making use of (2.7), we obtain

$$g((\bar{\nabla}_X P)JY - (\bar{\nabla}_{JX} P)Y, Z) = g(h(JX, Y) - h(X, JY) - 2\Omega(X, Y)\lambda, JZ).$$

The assertion follows on using Lemma 3.2.

4. Warped product submanifolds of l.c.K. manifolds

R.L. Bishop and B. O'Neill [3] introduced the notion of warped product manifolds while constructing examples of manifolds of negative curvatures. Such manifolds emerge in various geometric studies e.g, Kenmotsu manifolds and a surface of revolution are warped product manifolds. Moreover, a sphere and even $R^n - \{0\}$ are locally isometric to warped product manifolds. Apart from being differential geometrically important spaces, warped products have got applications in Physics too. For instance, the best relativistic Schwarzschild model of space-time near black holes or bodies with high gravitational field is a warped product manifold (cf. [10]). The definition of warped product manifolds was formulated as:

Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively, and f a positive differentiable function on B, known as *warping function*. The *warped product* $B \times_f F$ is the product manifold $B \times F$ with the Riemannian metric $g = g_B + f^2 g_F$. A warped product is said to be *non-trivial* if its warping function is non-constant. In fact, a trivial warped product manifold $B \times_f F$ is a Riemannian product $B \times F^f$ where F^f is the Riemannian manifold with Riemannian metric $f^2 g_F$ which is homothetic to the original metric g_F of F. On a warped product manifold $M = B \times_f F$ the following facts are observed in [3]

$$\nabla_X Y \in TB \tag{4.1}$$

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z \tag{4.2}$$

$$\nabla_Z W = -g(Z, W) \nabla f, \tag{4.3}$$

for each $X, Y \in TB$, $Z, W \in TF$, where ∇f denotes the gradient of f defined as

$$g(\nabla f, X) = Xf. \tag{4.4}$$

It follows from formulae (4.1) and (4.3) that B is totally geodesic and F is totally umbilical in M.

An n-dimensional submanifold M of a Riemannian manifold $\bar{M}$ is called a *warped product submanifold* of $\bar{M}$ if

(i) M is a warped product manifold of two submanifolds N_1 and N_2 of $\bar{M}$,

(ii) The two submanifolds are orthogonal i.e., $g(U_1, U_2) = 0$ for any $U_1 \in TN_1$ and $U_2 \in TN_2$.

Let N_T and $N_\perp$ be holomorphic and totally real submanifolds of an almost Hermitian manifold $\bar{M}$ such that $M = N_T \times_f N_\perp$ (resp. $N_\perp \times_f N_T$) is a warped product submanifold of $\bar{M}$. Then M is called a *CR-warped product submanifold* (resp. *warped product CR-submanifold*) of $\bar{M}$ with the induced metric on $M = N_T \times_f N_\perp$ (resp. $N_\perp \times_f N_T$) as $g = g_T + f^2 g_\perp$ (resp.$g_\perp + f^2 g_T$) where g_T and $g_\perp$ are Riemannian metrics on N_T and $N_\perp$ respectively. B.Y.Chen showed that each CR-submanifold of a Kaehler manifold which is a warped product CR-submanifold is in fact a CR-product, however, non-trivial CR-warped products do exist in Kaehler manifolds (cf. [7]). As a step forward, warped product manifolds are investigated in the setting of l.c.K. manifolds and many interesting extrinsic geometric properties are observed. For instance, V. Bonanzinga and K.Matsumoto proved the following.

Lemma 4.1 ([5]). *Let $M = N_\perp \times_f N_T$ be a warped product CR-submanifold of an l.c.K. manifold. Then $Z \ln f = g(\lambda, Z)$, for each $Z \in TN_\perp$. Therefore M is a CR-product if the Lee-vector field λ is normal to $N_\perp$.*

N. Jamal et al. [17] considered warped products $N_1 \times_f N_2$ in an l.c.K. manifold with one of the factors a holomorphic submanifold and extended the above formula by proving the following

Theorem 4.1. **([17])** *Let $M = N_1 \times_f N_2$ be a warped product submanifold of an l.c.K. manifold $\bar{M}$ with one of the factors a holomorphic submanifold. Then*

$$U_1 \ln f = g(\lambda, U_1)$$

for each $U_1 \in TN_1$, or else M is a CR-warped product submanifold.

5. CR-warped product submanifolds of an l.c.K. manifold

Throughout this section it is assumed that M is a CR-submanifold of an l.c.K. manifold $\bar{M}$ isometrically immersed as a warped product submanifold $N_T \times_f N_\perp$. For each (x, u) in M and for each $X \in T_x(N_T)$, there is a unique vector in D at (x, u) whose projection under $\pi_T : N_T \times_f N_\perp \to N_T$ is the vector X. In this way, a vector field X on N_T is identified with a vector field lying in the holomorphic distribution D. Similarly a vector field Z on $N_\perp$ is identified with a vector field in the totally real distribution $D^\perp$.

Lemma 5.1. *On a CR-warped product submanifold of an l.c.K. manifold, the Lee-vector field λ is orthogonal to $D^\perp$.*

Proof. For any $X, Y \in D$ and $Z \in D^\perp$,

$$\begin{aligned} g(h(X,Y), JZ) &= g(\bar{\nabla}_X Y, JZ) \\ &= g((\bar{\nabla}_X J)Y, Z) - g(\nabla_X JY, Z) \\ &= g((\bar{\nabla}_X J)Y, Z) + g(JY, \nabla_X Z) \end{aligned}$$

The second term in the right hand side of the above equation vanishes due to formula (4.2) whereas the first term on using (2.2) reduces to $g(JX, Y)g(\lambda, Z) + g(X, Y)g(J\lambda, Z)$. Comparing symmetric and skew symmetric terms, it can be deduced that

$$g(h(X,Y), JZ) = g(X,Y)g(J\lambda, Z) \tag{5.1}$$

and,

$$g(JX, Y)g(\lambda, Z) = 0. \tag{5.2}$$

The assertion follows from (5.2).

Lemma 5.2. *Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold of an l.c.K. manifold $\bar{M}$ then we have*

$$(\bar{\nabla}_U P)Z = g(CU, Z)P(\nabla \ln f).$$

for each $U \in TM$ *and* $Z \in D^{\perp}$.

Proof. As $PZ = 0$ for each $Z \in D^{\perp}$, by formula (2.4), we write

$$g((\bar{\nabla}_U\ P)Z, X) = g(\nabla_U Z, PX)$$

for each $X \in D$. Thus, we have:

$$g((\bar{\nabla}_U P)Z, X) = g(\nabla_{BU} Z, PX) + g(\nabla_{CU} Z, PX).$$

Taking account of (4.2) and (4.4) in the right hand side, the equation takes the form

$$g((\bar{\nabla}_U P)Z, X) = g(CU, Z)g(P\nabla \ln f, X)$$

As $(\bar{\nabla}_U P)Z$ lies in D, it follows from the above equation that

$$(\nabla_U P)Z = g(CU, Z)P(\nabla \ln f).$$

This proves the Lemma.

If λ_T and λ_N denote the tangential and normal parts of the Lee-vector field λ, then we have

Lemma 5.3. *On a CR-warped product submanifold of an l.c.K. manifold,*

(i) $th(X, Y) = -g(X, Y)t\lambda_N$

(ii) $th(X, Z) = (JX \ln f + g(J\lambda, X))Z$

for each $X, Y \in D$ *and* $Z \in D^{\perp}$.

Proof. Equation (5.1) can be written as:

$$g(th(X, Y), Z) = -g(X, Y)g(J\lambda, Z). \tag{5.3}$$

The assertion (i) follows on taking account of the fact that $t\xi \in D^{\perp}$ for all $\xi \in T^{\perp}M$.

To prove statement (ii), consider $g(h(JX,Z),JW)$, for $X \in D$ and $Z,W \in D^{\perp}$.

$$\begin{aligned} g(h(JX,Z),JW) &= g(\bar{\nabla}_Z JX, JW) \\ &= g((\bar{\nabla}_Z J)X + J\bar{\nabla}_Z X, W) \end{aligned}$$

Using formulae (2.2) and (4.2), the above equation reduces to

$$g(h(JX,Z),JW) = (X \ln f - g(\lambda, X))g(Z,W).$$

Replacing X by JX, the equation takes the form

$$g(th(X,Z),W) = (JX \ln f + g(J\lambda, X))g(Z,W).$$

Again taking account of the fact that $t\xi \in D^{\perp}$ for each $\xi \in T^{\perp}(M)$, the statement (ii) follows from the last equation.

Theorem 5.1. *A proper CR-submanifold M of an l.c.K. manifold $\bar{M}$ with the Lee-vector field λ orthogonal to $D^{\perp}$ is a CR-warped product submanifold if and only if there exist a C^{∞}-function μ on M such that $Z\mu = 0$ for all $Z \in D^{\perp}$ and*

$$\begin{aligned} (\bar{\nabla}_U P)V = &(PBV\mu)CU + g(CU,CV)P\nabla\mu + g(BU,BV)P\lambda_T \\ &+ g(JBU,BV)\lambda_T - g(\lambda,BV)JBU - g(J\lambda,BV)BU \qquad (5.4) \end{aligned}$$

for all $U,V \in TM$.

Proof. Let $M = N_T \times_f N_{\perp}$ be a CR-warped product submanifold of an l.c.K. manifold $\bar{M}$. Then by Lemma 5.1, λ is orthogonal to the submanifold $N_{\perp}$. Now, by formula (2.7),

$$\begin{aligned} (\bar{\nabla}_X P)Y = &th(X,Y) + g(X,Y)(J\lambda)_T + g(JX,Y)\lambda_T \\ &- g(\lambda,Y)JX - g(J\lambda,Y)X \qquad (5.5) \end{aligned}$$

for each $X,Y \in TN_T$. On the other hand, by formula (2.4) and the fact that $PZ = 0$ for each $Z \in TN_{\perp}$, we have

$$(\bar{\nabla}_Z P)Y = (PY \ln f)Z. \qquad (5.6)$$

and by Lemma 5.2

$$(\bar{\nabla}_U P)Z = g(CU, Z)P\nabla \ln f. \tag{5.7}$$

Combining (5.5), (5.6), (5.7) and using Lemma 5.3(i), we obtain

$$\begin{aligned}(\bar{\nabla}_U P)V &= -g(BU, BV)t\lambda_N + g(BU, BV)(J\lambda)_T - (PBV \ln f)CU \\ &\quad +g(CU, CV)P\nabla \ln f + g(JBU, BV)\lambda_T \\ &\quad -g(\lambda, BV)JBU - g(J\lambda, BV)BU.\end{aligned}$$

As $N_\perp$ is orthogonal to λ, $(J\lambda)_T = P\lambda_T + t\lambda_N$. Using this fact while combining the first two terms in the right hand side of the above equation we obtain (5.4).

Conversely, suppose that M is a CR-submanifold of an l.c.K. manifold with Lee-vector field λ orthogonal to $D^\perp$ and (5.4) holds for all $U, V \in TM$ and μ, a C^∞-function on M with $Z\mu = 0$ for all $Z \in D^\perp$. Then, the holomorphic distribution D on M is involutive by virtue of Theorem 3.4. Moreover, it follows from (5.4) that $(\bar{\nabla}_X P)Y$ lies in D for each $X, Y \in D$. Hence, by formula (2.4) $\nabla_X PY \in D$ showing that the leaves of D are totally geodesic in M.

Let $N_\perp$ be a leaf of $D^\perp$, ∇' the Levi-Civita connection on $N_\perp$ and h' the second fundamental form of $N_\perp$ into M. Then by Gauss formula,

$$0 = g(\nabla'_Z W, X) = g(\nabla_Z W - h'(Z, W), X)$$

for any $X \in D$ and $Z, W \in D^\perp$. That means

$$\begin{aligned}g(h'(Z, W), X) &= g(\nabla_Z W, X) \\ &= g(\bar{\nabla}_Z W, X) \\ &= g(\bar{\nabla}_Z JW - (\bar{\nabla}_Z J)W, JX) \\ &= -g(A_{JW}Z, JX) - g(\bar{\nabla}_Z J)W, JX).\end{aligned}$$

It is easy to see that, the second term in the right hand side of the above equation takes the form $g((\nabla_Z P)W - A_{JW}Z, JX)$. Thus the above equation on making use of (5.4) yields

$$g(h'(Z, W), X) = -g(Z, W)g(\nabla\mu, X).$$

Hence,

$$h'(Z,W) = -g(Z,W)\nabla\mu.$$

This shows that leaves of $D^\perp$ are totally umbilical in M with mean curvature vector $\nabla\mu$. Moreover the condition $Z\mu = 0$ for all $Z \in D^\perp$ implies that mean curvature is parallel. That is, the leaves of $D^\perp$ are extrinsic spheres in M. Hence, by virtue of a result in [9] we get that M is locally a CR-warped product submanifold $N_T \times_f N_\perp$ of $\bar{M}$. This proves the Theorem.

Theorem 5.2. *Let M be a CR-submanifold of an l.c.K. manifold $\bar{M}$ with λ orthogonal to $D^\perp$. Then M is a CR-warped product submanifold if and only if*

$$A_{JZ}X = g(J\lambda, Z)X - ((JX\mu) + g(J\lambda, X))Z \tag{5.8}$$

for each $X \in D$, $Z \in D^\perp$ and μ a C^∞-function on M such that $Z\mu = 0$, for all $Z \in D^\perp$.

Proof. By formula (2.7),

$$A_{JZ}X = (\bar{\nabla}_X P)Z - th(X,Z) + g(J\lambda, Z)X$$

If M is a CR-warped product submanifold of $\bar{M}$, then by (5.4), $(\bar{\nabla}_X P)Z = 0$. Thus on applying Lemma 5.3(ii), we obtain (5.8).

Conversely, let M be a CR-submanifold such that (5.8) holds for a C^∞-function μ on M with $Z\mu = 0$ for all $Z \in D^\perp$. Then by (5.8)

$$g(h(X,JY), JZ) = g(A_{JZ}X, JY) = g(J\lambda, Z)g(X, JY)$$

and,

$$g(h(JX,Y), JZ) = g(J\lambda, Z)g(JX, Y)$$

On subtracting we get

$$g(h(X,JY) - h(JX,Y)) + 2g(X,JY)\lambda, FZ) = 0.$$

Thus, D is involutive on M. Moreover, since the totally real distribution is involutive (cf. Lemma 3.2), M is foliated by the leaves N_T and $N_\perp$.

Now, for any $X, Y \in D$, $Z \in D^{\perp}$, by (5.8) we may write

$$(A_{JZ}X, Y) = g(J\lambda, Z)g(X, Y)$$

On applying Weingarten formula, the above relation is written as

$$g(\bar{\nabla}_X JZ, Y) + g(J\lambda, Z)g(X, Y) = 0.$$

That can further be simplified to yield

$$g(Z, \nabla_X JY) - g(Z, (\bar{\nabla}_X J)Y) + g(J\lambda, Z)g(X, Y) = 0,$$

which on using (2.2) takes the form

$$g(\nabla_X JY, Z) = g(X, JY)g(\lambda, Z)$$

As λ is orthogonal to $D^{\perp}$, it follows that the leaves of D are totally geodesic in M.

Let $N_{\perp}$ be a leaf of $D^{\perp}$, ∇' be the Levi-Civita connection on $N_{\perp}$ and h' be the second fundamental form of $N_{\perp}$ into M. Then by the argument of the one given in the proof of Theorem 5.1, we obtain that

$$h'(Z, W) = -g(Z, W)\nabla\mu$$

where $\nabla\mu$ is the gradient of μ. The above relation shows that the leaves of $D^{\perp}$ are totally umbilical in M with mean curvature vector $\nabla\mu$. Moreover, the condition $Z\mu = 0$ for all $Z \in D^{\perp}$, imply that the mean curvature vector is parallel. That is, the leaves of $D^{\perp}$ are extrinsic spheres in M. Hence, by the argument given in the Theorem 5.1 M is locally a CR-warped product submanifold $N_T \times_{\mu} N_{\perp}$ of $\bar{M}$. This proves the Theorem completely.

Now, in terms of the normal valued 1-form F, the characterization for CR-warped product submanifolds is given as:

Theorem 5.3. *A proper CR-submanifold M of a l.c.K. manifold $\bar{M}$ with λ orthogonal to $D^{\perp}$ is a CR-warped product submanifold if and only if*

$$(\bar{\nabla}_U F)V = -BV(\mu)JCU + fh(CU, CV) + g(CU, CV)J\lambda)_N \tag{5.9}$$

for any $U, V \in T(M)$ and $W \in D^{\perp}$, where μ is a C^{∞}-function on M such that $W\mu = 0$, for all $W \in D^{\perp}$.

Proof. Let $M = N_T \times_f N_{\perp}$ be a CR-warped product of a l.c.K.manifold $\bar{M}$. As N_T is totally geodesic in M, on using (2.5) and (4.2), we obtain

$$g((\bar{\nabla}_U F)X, \xi) = -X \ln f g(JCU, \xi)$$

for any $U \in TM$ $X \in D$ and $\xi \in T^{\perp}M$. Therefore, we conclude that

$$(\bar{\nabla}_U F)X = -(X \ln f)JCU \tag{5.10}$$

On the other hand, using the fact that JU is orthogonal to $D^{\perp}$ for any $U \in TM$ and λ is orthogonal to the submanifold $N_{\perp}$, it follows from (2.8) that

$$(\bar{\nabla}_U F)Z = fh(U, Z) + g(CU, Z)(J\lambda)_N \tag{5.11}$$

for any $U \in TM$ and $Z \in D^{\perp}$. Formula (5.9) is proved by virtue of equations (5.10) and (5.11).

Conversely, suppose that on a CR-submanifold M of a l.c.K. manifold with λ orthogonal to $D^{\perp}$, we have

$$(\bar{\nabla}_U F)V = -BV(\mu)JCU + fh(U, CV) + g(CU, CV)J\lambda_N$$

for any $U, V \in T(M)$ and $W \in D^{\perp}$, where μ is a C^{∞}-function on M such that $W\mu = 0$, for all $W \in D^{\perp}$.Then it is easy to see that

$$(\bar{\nabla}_X F)Y = 0$$

for $X, Y \in D$. Thus, by (2.5) $F\nabla_X Y = 0$. That means $\nabla_X Y \in D$ i.e., D is parallel. Therefore, D is integrable and its leaves are totally geodesic in M.

Let $N_{\perp}$ be a leaf of $D^{\perp}$, ∇' the Levi-Civita connection on $N_{\perp}$ and h' the second fundamental form of $N_{\perp}$ into M. Then, by similar argument as given in the proof of Theorem 5.1, $h'(Z, W) = -g(Z, W)\nabla\mu$, which shows that the leaves of $D^\perp$ are totally umbilical in M with mean curvature vector $\nabla\mu$. Moreover the condition $W\mu = 0$ for all $W \in D^\perp$ implies that mean curvature is parallel. That is, the leaves of $D^\perp$ are extrinsic spheres in M. Hence, we conclude that M is locally a CR-warped product submanifold of $\bar{M}$ with warping function e^μ. This proves the Theorem.

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