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Characterizations of real hypersurfaces of type A in a complex space form in terms of the structure Jacobi operator

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Abstract. Let M be a real hypersurface of a complex space form with almost contact metric structure (ϕ, ξ, η, g). In this paper, we study real hypersurfaces in a complex space form whose structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is ξ-parallel. In particular, we prove that the condition $\nabla_\xi R_\xi = 0$ characterize the homogeneous real hypersurfaces of type A in a complex projective space $P_n\mathbb{C}$ or a complex hyperbolic space $H_n\mathbb{C}$ when $R_\xi S = SR_\xi$ holds on M, where S denotes the Ricci tensor of type (1,1) on M.

1. Introduction

Let $(M_n(c), J, \tilde{g})$ be a complex n-dimensional complex space form with Kähler structure $(J, \tilde{g})$ of constant holomorphic sectional curvature $4c$ and let M be an orientable real hypersurface in $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from $(J, \tilde{g})$.

In 1970's, the fourth author [17], [18] classified the homogeneous real hypersurfaces of $P_n\mathbb{C}$ into six types. On the other hand, Cecil and Ryan [3] extensively studied a Hopf hypersurface, which is realized as tubes over certain submanifolds in $P_n\mathbb{C}$, by using its focal map. By making use of those results and the mentioned work of the fourth author, Kimura [11] proved

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the local classification theorem for Hopf hypersurfaces of $P_n\mathbb{C}$ whose all principal curvatures are constant. For the case a complex hyperbolic space $H_n\mathbb{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi's list or Berndt's list, a particular type of tubes over totally geodesic $P_k\mathbb{C}$ or $H_k\mathbb{C}$ $(0 \leq k \leq n-1)$ adding a horosphere in $H_n\mathbb{C}$, which is called type A, has a lot of nice geometric properties. For example, Okumura [13] (resp. Montiel and Romero [12]) showed that a real hypersurface in $P_n\mathbb{C}$ (resp. $H_n\mathbb{C}$) is locally congruent to one of real hypersurfaces of type A if and only if the Reeb flow ξ is isometric or equivalently the structure operator ϕ commutes with the shape operator A.

It is known that there are no real hypersurfaces with parallel Ricci tensors in a nonflat complex space form (see [7], [10]). This result says that there does not exist locally symmetric real hypersurfaces in a nonflat complex space form. The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Cho and the first author start the study on real hypersurfaces in a complex space form by using the operator R_ξ in [5] and [6]. Recently Ortega, Pérez and Santos [15] have proved that there are no real hypersurfaces in a complex projective space $P_n\mathbb{C}, n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. More generally, such a result has been extended by [16]. Moreover some works have studied several conditions on the structure Jacobi operator R_ξ and given some results on the classification of real hypersurfaces of type A in complex space form ([5],[6],[8],[12] and [13]). One of them, Cho and the first author proved the following:

Theorem 1.1 (Cho and Ki [6])**.** *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi A = AR_\xi$. Then M is a Hopf hypersurface in $M_n(c)$. Further, M is locally congruent to one of the following hypersurfaces*:

(1) *In cases that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*

- (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
- (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

(2) *In cases* $M_n(c) = H_n\mathbb{C}$,

- (A_0) *a horosphere*;
- (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane* $H_{n-1}\mathbb{C}$;
- (A_2) *a tube over a totally geodesic* $H_k\mathbb{C}$ $(1 \leq k \leq n-2)$.

In a continuing work [8] they proved the following:

Theorem 1.2 (Ki and Liu [8])**.** *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi S = SR_\xi$. Then M is the same types as those in Theorem 1.1 provided that $\eta(A\xi)^2 + 3c \neq 0$, where S denotes the Ricci tensor of M.*

In this paper we improve Theorem 1.2. Our main result appear in Theorem 5.1.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

Let M be a real hypersurface of a nonflat complex space form $M_n(c)$, $c \neq 0$ and C be a unit normal vector on M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Kähler metric $\tilde{g}$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)C, \quad \tilde{\nabla}_X C = -AX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$ and A is the shape operator of M in $M_n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we may see that M induces an almost contact metric structure (ϕ, ξ, η, g), namely

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M.

Since J is parallel, we verify, using the Gauss and Weingarten formulas, that

$$\nabla_X \xi = \phi AX, \tag{2.1}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.2}$$

Since the ambient space is of constant holomorphic sectional curvature $4c$, we have the following Gauss and Codazzi equations respectively:

$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &- 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.3}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.4}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

In the sequel, to write our formulas in convention forms, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, $\gamma = \eta(A^3\xi)$ and for a function f we denote by ∇f the gradient vector field of f.

If we put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector ξ. From (2.1), we get

$$\phi U = -A\xi + \alpha\xi, \tag{2.5}$$

which enables us to $g(U,U) = \beta - \alpha^2$. If we put

$$A\xi = \alpha\xi + \mu W, \tag{2.6}$$

where W is a unit vector field orthogonal to ξ. Then we get $U = \mu\phi W$, which shows that W is also orthogonal to U. Further we have

$$\mu^2 = \beta - \alpha^2. \tag{2.7}$$

Thus we see that ξ is a principal curvature vector, that is $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.

In this paper, we basically use the technical computations with the orthogonal triplet $\{\xi, U, W\}$ and their associated scalars α, β and μ.

Because of (2.1), (2.5) and (2.6), it is seen that

$$g(\nabla_X \xi, U) = \mu g(AW, X) \tag{2.8}$$

and

$$\mu g(\nabla_X W, \xi) = g(AU, X) \tag{2.9}$$

for any vector field X on M.

Differentiating (2.5) covariantly along M and making use of (2.1) and (2.2), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX \tag{2.10}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha, \tag{2.11}$$

where we have used (2.4). From (2.1) and (2.10), it is verified that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{2.12}$$

The curvature equation (2.3) gives the structure Jacobi operator R_ξ:

$$R_\xi(X) = R(X,\xi)\xi = c\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{2.13}$$

for any vector field X on M.

We shall denote the Ricci tensor of type (1,1) by S. Then it follows from (2.3) that

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X, \tag{2.14}$$

which implies

$$S\xi = 2c(n-1)\xi + hA\xi - A^2\xi, \tag{2.15}$$

where $h = \mathrm{Tr}A$. From (2.13) and (2.14), we have

$$\begin{aligned}(R_\xi S - SR_\xi)(X) = & -\eta(AX)A^3\xi + \eta(A^3X)A\xi - \eta(A^2X)(hA\xi - c\xi) \\ & + \{h\eta(AX) - c\eta(X)\}A^2\xi - ch\{\eta(AX)\xi - \eta(X)A\xi\}.\end{aligned} \tag{2.16}$$

3. Real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$ and $R_\xi S = SR_\xi$

We set $\Omega = \{p \in M; \mu(p) \neq 0\}$ and suppose that Ω is non-empty, that is, ξ is not a principal curvature vector on M. Hereafter, unless otherwise stated, we discuss our arguments on the open subset Ω of M.

Differentiating (2.13) covariantly, we obtain

$$\begin{aligned} g((\nabla_X R_\xi)Y, Z) &= g(\nabla_X(R_\xi Y) - R_\xi(\nabla_X Y), Z) \\ &= -c\{\eta(Z)g(\nabla_X\xi, Y) + \eta(Y)g(\nabla_X\xi, Z)\} \\ &\quad + (X\alpha)g(AY, Z) + \alpha g((\nabla_X A)Y, Z) \\ &\quad - \eta(AZ)\{g((\nabla_X A)\xi, Y) + g(A\phi AX, Y)\} \\ &\quad - \eta(AY)\{g((\nabla_X A)\xi, Z) + g(A\phi AX, Z)\}, \end{aligned}$$

which together with (2.11) yields

$$\begin{aligned} g((\nabla_\xi R_\xi)Y, Z) = &-c\{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha)g(AY, Z) \\ &+ \alpha g((\nabla_\xi A)Y, Z) - \eta(AZ)\{3g(AU, Y) + Y\alpha\} \\ &- \eta(AY)\{3g(AU, Z) + Z\alpha\}, \end{aligned} \tag{3.1}$$

where u is a 1-form dual to U with respect to g, that is $u(X) = g(U, X)$.

At first we assume that $\nabla_\xi R_\xi = 0$. Then we have from (3.1)

$$\begin{aligned} \alpha(\nabla_\xi A)X + (\xi\alpha)AX = c\{u(X)\xi + \eta(X)U\} + \eta(AX)(3AU + \nabla\alpha) \\ + \{3g(AU, X) + X\alpha\}A\xi. \end{aligned} \tag{3.2}$$

If we put $X = \xi$ in this and make use of (2.11), we find

$$\alpha AU + cU = 0, \tag{3.3}$$

which shows that $\alpha \neq 0$ on Ω.

If we differentiate (3.3) covariantly along Ω, and use itself again, then we obtain

$$-c(X\alpha)U + \alpha^2(\nabla_X A)U + \alpha^2 A\nabla_X U + c\alpha\nabla_X U = 0, \tag{3.4}$$

which, together with (2.4) and (2.5), implies that

$$\begin{aligned} c\{(Y\alpha)u(X) - (X\alpha)u(Y)\} + c\alpha^2\mu\{\eta(X)w(Y) - \eta(Y)w(X)\} \\ + \alpha^2\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + c\alpha du(X, Y) = 0, \end{aligned} \tag{3.5}$$

where w is a dual 1-form of W with respect to g, that is $w(X) = g(W, X)$. Here, du is the exterior derivative of a 1-form u given by

$$du(X, Y) = Y(u(X)) - X(u(Y)) - u([X, Y]).$$

If we replace X by U in (3.5), then it follows that

$$c\{\mu^2 \nabla\alpha - (U\alpha)U\} + \alpha^2 A\nabla_U U + c\alpha\nabla_U U = 0, \tag{3.6}$$

because U and W are mutually orthogonal. Combining (2.10) to (3.2) and using (2.4), we obtain

$$\begin{aligned}\alpha^2\phi\nabla_X U = {} & \alpha^2(X\alpha)\xi - c\alpha u(X)\xi + \alpha(\xi\alpha)AX + c\alpha^2\phi X \\ & - \eta(AX)(\alpha\nabla\alpha - 3cU) - \{\alpha(X\alpha) - 3cu(X)\}A\xi \\ & - c\alpha\{u(X)\xi + \eta(X)U\} - \alpha^2 A\phi AX + \alpha^3\phi AX.\end{aligned}$$

Applying ϕ to this and using (2.8), we have

$$\begin{aligned}&\alpha^2\nabla_X U + \alpha^2\mu g(AW, X)\xi - \alpha\eta(AX)\phi\nabla\alpha \\ &\quad = -\alpha(\xi\alpha)\phi AX + c\alpha^2\{X - \eta(X)\xi\} + 3c\mu\eta(AX)W + \alpha(X\alpha)U \\ &\qquad - 3cu(X)U + \alpha^3 AX - c\alpha\mu\eta(X)W - \alpha^3\eta(AX)\xi + \alpha^2\phi A\phi AX.\end{aligned} \tag{3.7}$$

Putting $X = U$ in this and using (2.5), (2.6) and (3.3), we get

$$\alpha^2\nabla_U U = -c\mu(\xi\alpha)W + \left\{\alpha(U\alpha) - 3c\mu^2\right\}U + c\mu\alpha\phi AW. \tag{3.8}$$

If we replace X by ξ in (3.5) and take account of (3.2), then we obtain

$$c\alpha\mu^2\xi + \{\alpha(U\alpha) - 3c\mu^2\}A\xi + \alpha^2 A(\nabla_\xi U) + c\alpha\nabla_\xi U = 0.$$

By the way, using (2.12) and (3.3), we see that

$$\alpha\nabla_\xi U = 3c\mu W + \alpha^2 A\xi - \alpha\beta\xi + \alpha\phi\nabla\alpha.$$

From two equations, it follows that

$$\alpha A\phi\nabla\alpha + c\phi\nabla\alpha + (U\alpha)A\xi + \mu(\alpha^2 + 3c)\{AW - \mu\xi - \frac{1}{\alpha}(\mu^2 - c)W\} = 0, \tag{3.9}$$

where we have used (2.6).

Now, differentiating (2.6) covariantly and using (2.1) and (2.4), we find

$$(\nabla_\xi A)X - c\phi X + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W,$$

which together with (3.2) implies that

$$\begin{aligned}\mu\alpha\nabla_X W = \alpha A\phi AX - \alpha^2\phi AX - c\alpha\phi X - (\xi\alpha)AX \\ + c\{u(X)\xi + \eta(X)U\} + \eta(AX)(3AU + \nabla\alpha) \\ + \{3g(AU, X) + X\alpha\}A\xi - \alpha(X\alpha)\xi - \alpha(X\mu)W.\end{aligned} \tag{3.10}$$

Further, we assume that

$$R_\xi SX = SR_\xi X \tag{3.11}$$

for any vector field X. Then (2.16) becomes

$$\begin{aligned}\eta(AX)A^3\xi - \eta(A^3X)A\xi + \eta(A^2X)(hA\xi - c\xi) \\ - \{h\eta(AX) - c\eta(X)\}A^2\xi + ch\{\eta(AX)\xi - \eta(X)A\xi\} = 0,\end{aligned} \tag{3.12}$$

which shows that

$$\alpha A^3\xi = (\alpha h - c)A^2\xi + (\gamma - \beta h + ch)A\xi + c(\beta - h\alpha)\xi,$$

Combining above equations, we obtain

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi, \tag{3.13}$$

where we have put $\mu^2\rho = \gamma - \beta\alpha$ and $\mu^2(\beta - \rho\alpha) = \beta^2 - \alpha\gamma$ on Ω. Using the last two equations, we can write (3.12) as

$$\mu(\rho - h)(\beta - \rho\alpha - c)(\eta(X)W - w(X)\xi) = 0, \tag{3.14}$$

where we have used (2.6).

Remark 1. *$\beta - \rho\alpha - c \neq 0$ on Ω.*

Indeed, if not, then (3.13) reformed as $A^2\xi = \rho A\xi + c\xi$ on this subset. From this and (2.13) we verify that $R_\xi A = AR_\xi$ on the set. According to Theorem 1.1, it is seen that $\Omega = \emptyset$ because $\nabla_\xi R_\xi = 0$ was assumed. Therefore $\beta - \rho\alpha - c \neq 0$ everywhere on Ω.

From (2.6), (2.7) and (3.13) we see that $AW = \mu\xi + (\rho - \alpha)W$ on Ω. If we put $g(AW, W) =: \lambda$, then we have

$$AW = \mu\xi + \lambda W. \tag{3.15}$$

Further, we have

$$h = \alpha + \lambda \tag{3.16}$$

by virtue of (3.14) and Remark 1.

Using (2.7) and (3.15), the equation (3.9) is deformed as

$$\alpha A\phi\nabla\alpha + c\phi\nabla\alpha + (U\alpha)A\xi + \frac{1}{\alpha}\mu(\alpha^2 + 3c)(\rho\alpha + c - \beta)W = 0.$$

Taking an inner prodct W to this and making use of (3.15), we obtain

$$(-\beta + \rho\alpha + c)\{\alpha(U\alpha) - \mu^2(\alpha^2 + 3c)\} = 0,$$

which shows that

$$\alpha(U\alpha) = \mu^2(\alpha^2 + 3c) \tag{3.17}$$

because of Remark 1.

Because of (3.3), (3.8), (3.15) and (3.17), we see from (3.6)

$$\alpha\mu\nabla\alpha = \alpha\mu(\xi\alpha)\xi + (\lambda\alpha + c)(\xi\alpha)W + (\alpha^2 + 3c)\mu U,$$

which tells us that

$$\mu\alpha(W\alpha) = (\lambda\alpha + c)\xi\alpha. \tag{3.18}$$

Combining above two equations, it is clear that

$$\alpha\nabla\alpha = \alpha(\xi\alpha)\xi + \alpha(W\alpha)W + (\alpha^2 + 3c)U. \tag{3.19}$$

Now, differentiating (3.15) covariantly, and using (2.1), we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\phi AX + (X\lambda)W + \lambda\nabla_X W, \tag{3.20}$$

which implies that

$$g((\nabla_X A)W, W) = \frac{2c}{\alpha}u(X) + X\lambda, \tag{3.21}$$

$$\mu(\nabla_\xi A)W = (\lambda - \alpha)AU - cU + \mu\nabla\mu, \tag{3.22}$$

where we have used (2.4), (2.9) and (3.3).

If we put $X = \mu W$ in (3.2) and take account of (2.7), (3.3) and (3.22), then we obtain

$$\alpha\left(\frac{1}{2}\alpha\nabla\beta - \beta\nabla\alpha\right) + c(3\mu^2 - \lambda\alpha)U = -\mu\alpha(\xi\alpha)AW + \mu\alpha(W\alpha)A\xi,$$

which together with (2.6), (3.15) and (3.18) yields

$$\alpha^2\nabla\beta - \beta\nabla\alpha^2 + 2c(3\mu^2 - \lambda\alpha)U = (\xi\alpha)\{2\alpha(\lambda\alpha - \mu^2)\xi + 2cA\xi\}. \tag{3.23}$$

From (2.7) we have

$$\alpha\mu\nabla\mu = \alpha\left(\frac{1}{2}\nabla\beta - \alpha\nabla\alpha\right).$$

Substituing (3.19) and (3.23) into this, and making use of (2.6), (3.13) and (3.18), we obtain

$$\frac{1}{2}\alpha^2\nabla\mu^2 = \alpha(\alpha\mu^2 + c\lambda)U + \xi\alpha\{(\lambda\alpha + 2c)A\xi - c\alpha\xi\}. \tag{3.24}$$

Now, we prove

Lemma 1. *$\xi\alpha = W\alpha = 0$ on Ω.*

Proof. The equation (3.24) is rewritten as

$$\frac{1}{2}\alpha^2(Y\mu^2) = \alpha(\alpha\mu^2 + c\lambda)u(Y) + (\xi\alpha)\{(\lambda\alpha + 2c)\eta(AY) - c\alpha\eta(Y)\}.$$

Differentiating this with respect to a vector field X, and taking the skew-symmetric parts for X and Y, we eventually have

$$\begin{aligned}
0 = {} & \{\alpha(X\alpha) + \alpha^2u(X)\}(Y\mu^2) - \{\alpha(Y\alpha) + \alpha^2u(X)\}(X\mu^2) \\
& - (X\alpha)\{2\alpha\mu^2u(Y) + c\lambda u(Y) + \lambda\varepsilon\eta(AY) - c\varepsilon\eta(Y)\} \\
& + (Y\alpha)\{2\alpha\mu^2u(X) + c\lambda u(X) + \lambda\varepsilon\eta(AX) - c\varepsilon\eta(X)\} \\
& - (X\lambda)\{c\alpha u(Y) + \alpha\varepsilon\eta(AY)\} + (Y\lambda)\{c\alpha u(X) + \alpha\varepsilon\eta(AX)\} \\
& - (X\varepsilon)\{(\lambda\alpha + 2c)\eta(AY) - c\alpha\eta(Y)\} \\
& + (Y\varepsilon)\{(\lambda\alpha + 2c)\eta(AX) - c\alpha\eta(X)\} \\
& - 2\alpha(\alpha\mu^2 + c\lambda)du(X,Y) - 2(\lambda\alpha + 3c)d\eta(X,Y) \\
& - 2\varepsilon\mu(\lambda\alpha + 2c)dw(X,Y),
\end{aligned} \tag{3.25}$$

where we have put $\varepsilon := \xi\alpha$. Putting $X = U$ and $Y = \alpha\xi$ in this equation and making use of (3.3) , we find

$$\begin{aligned}0 = &\alpha^2(\xi\mu^2)\{(U\alpha) + \alpha\mu^2\} - \alpha^2\varepsilon U\mu^2 - (U\alpha)(2\alpha\mu^4 + c\lambda\mu^2)\\ &- \alpha^2\varepsilon^2(-\alpha\lambda + c) - 4c\alpha\mu^2(U\lambda) + \alpha^4\varepsilon(\xi\lambda) - \alpha^2(U\varepsilon)(\lambda\alpha + c)\\ &- 2\alpha(\alpha\mu^2 + c\lambda)du(U, \alpha\xi) - 2\varepsilon\alpha(\lambda\alpha + 3c)d\eta(U, \alpha\xi)\\ &- 2\varepsilon\mu(\lambda\alpha + 2c)dw(U, \alpha\xi)\end{aligned}$$

Let Ω_0 be the set of points such that $(\xi\alpha)_p \neq 0$ at $p \in \Omega$ and suppose that $\Omega_0 \neq \emptyset$. Then from above equation we have

$$\begin{aligned}&\alpha^3(U\lambda) + \frac{\alpha^2}{\varepsilon}(\lambda\alpha + c)(U\varepsilon) - \frac{c}{\varepsilon}\alpha^2\mu^2(\xi\lambda)\\ =\ &2\mu^2(\lambda\alpha + c)(2\alpha^2 + 3c) - 2\alpha\mu^2(\alpha\mu^2 + c\lambda)\\ &+ \mu^2(\alpha^2 + 3c)(-\lambda\alpha + c) + \alpha\mu^2(2\alpha\mu^2 + c\lambda)\\ &+ \alpha(\lambda\alpha + c)(\alpha\mu^2 + c\lambda) + \alpha^2\mu^2(\lambda\alpha + 3c)\\ &+ 3c\mu^2(\lambda\alpha + 2c),\end{aligned} \tag{3.26}$$

on Ω_0, where we have used (3.17) and (3.24).

On the other hand, from (3.23) we get

$$\alpha^2(X\beta) - \beta(X\alpha^2) + 2c(3\mu^2 - \lambda\alpha)u(X) = 2\varepsilon\{\alpha(\lambda\alpha - \mu^2)\eta(X) + c\eta(AX)\}.$$

Using the same method as that used to derive (3.26), we can deduce from this equation the following

$$\begin{aligned}&2\alpha^3(U\lambda) + \frac{2\alpha^2}{\varepsilon}(\lambda\alpha - \mu^2 + c)(U\varepsilon) - \frac{2c\alpha^2\mu^2}{\varepsilon}(\xi\lambda)\\ =\ &- 12c\mu^2(\lambda\alpha + c) + 4\alpha\mu^2(\alpha\mu^2 + c\lambda) + 2\mu^2(4\alpha^2 + 4c + \mu^2)(\alpha^2 + 3c)\\ &- 2\alpha\mu^2(4\alpha\beta + 12\alpha c + 3c\lambda) - 2c(3\mu^2 - \lambda\alpha)(\lambda\alpha + c)\\ &+ 2\alpha^2\mu^2(\lambda\alpha - \mu^2 + c) + 6c^2\mu^2,\end{aligned} \tag{3.27}$$

on Ω_0. From (3.21), (3.22) and (3.24), we get

$$\xi\lambda = W\mu = \frac{\varepsilon}{\alpha^2}(\lambda\alpha + 2c), \tag{3.28}$$

which together with (3.26) implies that

$$\begin{aligned}2\alpha^3(U\lambda) &= 4\mu^2(\lambda\alpha+c)(2\alpha^2+3c) - 4\alpha\mu^2(\alpha\mu^2+c\lambda)\\ &\quad + 2\mu^2(\alpha^2+3c)(-\lambda\alpha+c) + 2\alpha\mu^2(2\alpha\mu^2+c\lambda)\\ &\quad + 2\alpha(\lambda\alpha+c)(\alpha\mu^2+c\lambda) + 2\alpha^2\mu^2(\lambda\alpha+3c) + 3c\mu^2(\lambda\alpha+2c)\end{aligned}$$

From (3.27), (3.28) and the above equation, it follows that

$$\begin{aligned}\frac{\alpha^2}{\varepsilon}(U\varepsilon) &= (2\alpha^2-3c)\mu^2 + (\lambda\alpha+c)(4\alpha^2+15c)\\ &\quad - (4\alpha^2+\lambda\alpha+3c)(\alpha^2+3c)\\ &\quad + \alpha^2(\lambda\alpha+15c+4\alpha^2) + 3c^2 - 3c\alpha\lambda,\end{aligned} \tag{3.29}$$

on Ω_0.

Now, we know from (3.19)

$$Y\alpha = \varepsilon\eta(Y) + (W\alpha)w(Y) + \frac{1}{\alpha}\left(\alpha^2+3c\right)u(Y). \tag{3.30}$$

In the same way as above, it is, using (3.30), verified that

$$\begin{aligned}0 &= \varepsilon\{(X\alpha)\eta(Y) - (Y\alpha)\eta(X)\}\\ &\quad + \alpha\{(X\varepsilon)\eta(Y) - (Y\varepsilon)\eta(X)\}\\ &\quad + (W\alpha)\{(X\alpha)w(Y) - (Y\alpha)w(X)\}\\ &\quad + \alpha\{X(W\alpha)w(Y) - Y(W\alpha)w(X)\}\\ &\quad + 2\alpha\{(X\alpha)u(Y) - (Y\alpha)u(X)\}\\ &\quad + 2\alpha\varepsilon d\eta(X,Y) + 2\alpha(W\alpha)dw(X,Y) + 2(\alpha^2+3c)du(X,Y).\end{aligned}$$

Putting $X = U$ and $Y = \xi$ in this and using (2.9) and (3.3), we find

$$\begin{aligned}0 &= \varepsilon(U\alpha) + \alpha(U\varepsilon) - 2\alpha(\xi\alpha)\mu^2 + 2\alpha\varepsilon d\eta(U,\xi)\\ &\quad + 2\alpha(W\alpha)dw(U,\xi) + 2(\alpha^2+3c)du(U,\xi),\end{aligned}$$

which together with (3.17) and (3.18) implies that

$$\frac{\alpha^2}{\varepsilon}(U\varepsilon) = (\alpha^2+6c)(\lambda\alpha+c) + \mu^2(2\alpha^2-3c),$$

on Ω_0. Substituting this into (3.29), we find on Ω_0

$$(\alpha\lambda+c)(\alpha^2+c) = 0.$$

Since $\xi\alpha \neq 0$ on Ω_0, we get $\alpha^2 + c \neq 0$ which shows that

$$\alpha\lambda + c = 0. \tag{3.31}$$

So we have $W\alpha = 0$ by virtue of (3.18). Thus (3.19) is reduced to

$$\alpha\nabla\alpha = \alpha\varepsilon\xi + (\alpha^2 + 3c)U.$$

Using the same method as that used to derive (3.25) from (3.24), we can derive from this the following

$$X(\alpha\varepsilon)\eta(Y) - Y(\alpha\varepsilon)\eta(X) + 2\alpha(X\alpha)u(Y) - 2\alpha(Y\alpha)u(X) \\ + \alpha\varepsilon g((\phi A + A\phi)X, Y) + (\alpha^2 + 3c)(g(\nabla_X U, Y) - g(\nabla_Y U, X)) = 0. \tag{3.32}$$

Now, we can take a orthonormal basis $\{e_0 = \xi,\ e_1 = (1/\mu)U,\ e_2, \ldots, e_n,\ \phi e_1 = (1/\mu)\phi U,\ \phi e_2, \ldots, \phi e_n\}$. Putting $X = \phi e_i$ and $Y = e_i$ and summing up for $i = 0, \ldots, n$, we have $\alpha = h$ on Ω_0, which together with (3.16), implies that $\lambda = 0$. This contradicts (3.31). □

4. Lemmas

In the following, we will continue our discussions on Ω in M which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi S = SR_\xi$. Then (3.19) and (3.24) are reduced respectively to

$$\alpha\nabla\alpha = (\alpha^2 + 3c)U, \tag{4.1}$$

$$\alpha\mu\nabla\mu = (\alpha\mu^2 + c\lambda)U \tag{4.2}$$

by virtue of Lemma 1. Using these, we can write (3.7) and (3.10) as the followings respectively.

$$\nabla_X U = \alpha AX + cX - (\mu^2 + c)\eta(X)\xi - \mu\lambda w(X)\xi \\ -\frac{c}{\alpha}\mu\eta(X)W + u(X)U + \phi A\phi AX - \eta(AX)A\xi, \tag{4.3}$$

$$\mu\alpha\nabla_X W = -2cu(X)\xi + \{\alpha\eta(AX) + c\eta(X)\}U - \frac{c}{\mu}\lambda u(X)W \\ +\alpha A\phi AX - \alpha^2\phi AX - c\alpha\phi X. \tag{4.4}$$

By taking the skew-symmetric part of $g(A\nabla_X U, Y)$, we see, using (4.3), that

$$g(A\nabla_X U, Y) - g(A\nabla_Y U, X) = \mu c\left(1 + \frac{\lambda}{\alpha}\right)(\eta(Y)w(X) - \eta(X)w(Y)).$$

Substituting (4.1) and the last equation into (3.5), we find

$$du(X, Y) = \mu\lambda(\eta(Y)w(X) - \eta(X)w(Y)). \tag{4.5}$$

Putting $X = W$ in (4.4) and making use of (3.3) and (3.15), we get

$$\alpha\mu\nabla_W W = \left\{\mu^2 - c - \lambda\left(\alpha + \frac{c}{\alpha}\right)\right\}U. \tag{4.6}$$

Lemma 2. $\alpha^2 + 3c = 0$ *on* Ω.

Proof. Since we have $\varepsilon = 0$, (3.32) becomes

$$(\alpha^2 + 3c)du(X, Y) = 0,$$

which connected to (4.5) yields $\lambda(\alpha^2 + 3c) = 0$.

Now, we suppose that $\alpha^2 + 3c \neq 0$ on Ω, and then we restrict the arguments on such place. Then we have $\lambda = 0$. Thus, by putting $X = W$ in (3.20) and using (3.15) and (4.2), we have

$$(\nabla_W A)W + A\nabla_W W = 0.$$

We also have from (3.21) $(\nabla_W A)W = (2c/\alpha)U$ because of (2.4). So we have $2cU + \alpha A\nabla_W W = 0$. This, connected with (4.6) implies that $\mu^2 + c = 0$ by virtue of (3.3) and $\lambda = 0$. Therefore μ is constant on this subset, a contradiction because of (4.2). Thus we arrive at the conclusion. $\square$

By the same method as in the proof of Lemma 2, we verify from (4.2) that

$$c\{(X\lambda)u(Y) - (Y\lambda)u(X)\} + (\alpha\mu^2 + c\lambda)du(X, Y) = 0,$$

where we have used Lemma 2. Replacing Y by U in this and making use of (4.5), we find $\mu^2(X\lambda) = (U\lambda)u(X)$. Hence above equation becomes $(\alpha\mu^2 + c\lambda)du(X, Y) = 0$, which together with (4.5) yields

$$\alpha\mu^2 + c\lambda = 0. \tag{4.7}$$

Thus μ is constant because of (4.2). So we see that λ so dose by virtue of Lemma 2. Using (4.7) and Lemma 2, we can write (4.6) as

$$\lambda\nabla_W W = (\alpha - \lambda)U. \tag{4.8}$$

λ being constant, we verify, using (2.4) and (3.21), that $(\nabla_W A)W = (2c/\alpha)U$. If we put $X = W$ in (3.20) and take account of this, then we obtain

$$A\nabla_W W - \lambda\nabla_W W = \left(\lambda - \frac{2c}{\alpha}\right)U,$$

where we have used λ and μ are constant. From this and (4.8) it is seen that

$$6\lambda - \alpha = 0. \tag{4.9}$$

Combining (4.7) to (4.9) we have

Lemma 3. $6\mu^2 + c = 0$ *on* Ω.

Using (4.9), Lemma 2 and Lemma 3, we can write (4.4) as

$$\begin{aligned}\mu\nabla_X W = \mu\{u(X)W + w(X)U\} - \frac{2c}{\alpha}\{u(X)\xi + \eta(X)U\}\\ + A\phi AX - \alpha\phi AX - c\phi X,\end{aligned} \tag{4.10}$$

which implies that

$$\mu dw(X,Y) = 2g(A\phi AX, Y) - \alpha g((\phi A - A\phi)X, Y) - 2cg(\phi X, Y). \tag{4.11}$$

If we replace X by ξ or U, then we have respectively

$$\nabla_\xi W = 0, \quad \nabla_U W = -\frac{c}{\alpha}\mu\xi \tag{4.12}$$

by virture of (3.3), (4.7) and Lemma 2.

From (4.5) and Lemma 3, we see that $\nabla_U U = 0$. Putting $X = U$ in (3.4), we verify, using this and Lemma 2, that

$$(\nabla_U A)U = 0. \tag{4.13}$$

On the other hand, (3.2) turns out to be

$$(\nabla_\xi A)X = \frac{c}{\alpha}\{u(X)\xi + \eta(X)U\} + \eta(AX)U + u(X)A\xi, \tag{4.14}$$

by virtue of (3.3) and Lemma 2, which implies

$$(\nabla_\xi A)W = \mu U. \tag{4.15}$$

5. The proof of Main theorem

We continue our arguments under the same hypotheses of the section 4. Now we prove

Theorem 5.1. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$ whose Ricci tensor S commutes with R_ξ, namely $R_\xi S = SR_\xi$. Then M satisfies $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of the following*:

(I) *in case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*

- (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
- (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*

(II) *in case that $M_n(c) = H_n\mathbb{C}$,*

- (A_0) *a horosphere,*
- (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,*
- (A_2) *a tube over a totally geodesic $H_k\mathbb{C}(1 \leq k \leq n-2)$.*

Proof. Differentiating (4.10) covariantly and using (2.1) and (2.2), we find

$$\begin{aligned}\mu\nabla_Y\nabla_X W = {} & \mu\{Y(u(X))W + u(X)\nabla_Y W + Y(w(X))U + \eta(X)\nabla_Y U\} \\ & -\frac{2c}{\alpha}\{Y(u(X))\xi + u(X)\nabla_Y\xi + Y(\eta(X))U + \eta(X)\nabla_Y U\} \\ & +\nabla_Y(A\phi AX) - \alpha\nabla_Y(\phi AX) - c\nabla_Y(\phi X).\end{aligned}$$

If we take the skew-symmetric part of X and Y, and put $X = \xi$ and $Y = U$, we have

$$\alpha^2\nabla_W W = 6cU,$$

where we have used (2.3), (3.3) and $\nabla_U U = 0$. From (4.8) we have $\lambda = -\alpha$, which contradicts (4.9).

Therefore we conclude that $\Omega = \emptyset$, that is, $A\xi = \alpha\xi$ on M. So we see in addition that α is constant on M (see [9]). Thus, from (3.2) we verify that $\alpha\nabla_\xi A = 0$. Accordingly, we have $\alpha(A\phi - \phi A) = 0$ by virtue of (2.1) and (2.4). Here, we note the case $\alpha = 0$ corresponds to the case of tube of radius $\pi/4$ in $P_n(\mathbb{C})$ (see [3]). But, in the case of $H_n(\mathbb{C})$ it is known that α never vanishes for Hopf hypersurfaces (cf. [1]). Due to Okumura's work or Montiel and Romero's work stated in the Introduction, we complete the proof.

□

Finally we prove

Corollary 1. *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $S\xi = g(S\xi, \xi)\xi$. Then M is the same type as those stated in Theorem 1.1.*

Proof. By (2.15) we have $g(S\xi, \xi) = h\alpha - \beta + 2c(n-1)$. From this and our assumption $S\xi = g(S\xi, \xi)\xi$ we see that $A^2\xi = hA\xi + (\beta - h\alpha)\xi$ and hence $A^3\xi = (h^2 + \beta - h\alpha)A\xi + h(\beta - h\alpha)\xi$. Substituting these into (2.16), we obtain $R_\xi S = SR_\xi$. This completes the proof. □

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