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Weakly selfadjoint operators II

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Abstract. In this note, we show some basic results related to the spectrum of a weakly selfadjoint operator.

1. Throughout this note, an operator means a bounded linear operator on a Hilbert space H. An operator A is positive (denoted by $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. For an operator A, the numerical range $W(A)$ means the following set in the complex plane:

$$W(A) = \{(Ax, x) \ ; \ x \in H, \| x \| = 1\}.$$

For the set $W(A)$ the following general facts are well-known (cf. [3, § 2.5]).

(i) $W(A)$ is convex.

(ii) $\overline{W(A)}$, the closure of $W(A)$, contains the spectrum $\sigma(A)$ of A, so that

$$\mathrm{co}\sigma(A) \subset \overline{W(A)},$$

where $\mathrm{co}\sigma(A)$ is the convex hull of $\sigma(A)$.

We define the spectral radius $r(A)$ of an operator A by

$$r(A) = \sup\{|\lambda|; \ \lambda \in \sigma(A)\}.$$

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Now if $\sigma(A)$ lies in the real line $\mathbb{R}$ and if we put

$$m_A = \inf\{\lambda\ ; \lambda \in \sigma(A)\} \quad \text{and} \quad M_A = \sup\{\lambda\ ; \lambda \in \sigma(A)\},$$

then from convexity of $W(A)$ and the fact $m_A, M_A \in \sigma(A)$, we can see that

$$\text{co}\sigma(A) = [m_A, M_A] = W(A) = \overline{W(A)}. \tag{1.1}$$

In [5], S. Izumino and the author defined a weakly selfadjoint operator, analogously as a weakly positive operator defined in [4], and gave some fundamental results on the operator in the light of the recent knowledge. In succession of the consideration, in this note, we shall show some further facts related to the spectrum of the operator.

2. R. Bouldin [1] (cf. [6, Theorem 2]) showed the following:

Theorem B. If one of operators A and B is positive, then $\text{co}\sigma(AB) \subset \overline{W(A)} \cdot \overline{W(B)}$.

Applying this result, we show the following fact.

Proposition 1. *If $T = AB$ is a weakly selfadjoint operator as the product of two selfadjoint operators A and B, one of whose factors is positive, then*

$$\text{co}\sigma(T) \subset \text{co}\sigma(A)\ \text{co}\sigma(B). \tag{2.1}$$

Proof. From the above (1.1) we see

$$\text{co}\sigma(A) = W(A) = \overline{W(A)} \quad \text{and} \quad \text{co}\sigma(B) = W(B) = \overline{W(B)}.$$

Hence we have

$$\overline{W(A)} \cdot \overline{W(B)} = \text{co}\sigma(A)\ \text{co}\sigma(B),$$

so that by Theorem B we have the desired (2.1).

From Proposition 1, we have the following:

Corollary 2 (cf. [5, Corollary 2]). *If T is a weakly selfadjoint operator, then*

$$\sigma(T) \subset \mathbb{R} = (-\infty, \infty). \tag{2.2}$$

Related to Corollary 2 the following interesting fact is known between the spectral radius $r(T)$ and the norm of Re $T = (T + T^*)/2$.

Proposition 3 (cf. [2, Lemma 2]). *If* $\sigma(T)$ *is contained in* $\mathbb{R}$, *then*

$$r(T) \leq \| \text{ Re } T \|. \tag{2.3}$$

Hence, for a weakly selfadjoint T, (2.3) holds.

For completeness we give a proof of (2.3): Recall that

$$\mathrm{co}\sigma(T) \subset \overline{W(T)}.$$

Using the relation Re $W(T) = W(\text{Re } T)$, we have

$$\text{Re } \overline{W(T)} = \overline{W(\text{Re } T)} \subset [- \| \text{ Re } T \|, \| \text{ Re } T \|].$$

Hence

$$\sigma(T) = \text{Re } \sigma(T) \subset [- \| \text{ Re } T \|, \| \text{ Re } T \|],$$

so that we obtain the desired inequality (2.3).

Example. Let $T = AB$ with $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then by easy calculation we obtain $r(T) = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ and $\| \text{ Re } T \| = 2$, which shows the above inequality (2.3).

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