Toyama Math. J.
Vol. 30(2007), 45-56

Weakly selfadjoint operators

Saichi IZUMINO and Noboru NAKAMURA

Dedicated to Emeritus Professor Masahiro Nakamura
in honour of his eighty-eighth birthday

Abstract. Extending the notion of a weakly positive operator introduced by Wigner, we define a weakly selfadjoint operator as the product of two selfadjoint operators, one of whose factors is positive, and show some facts on the operator.

1. Throughout this note we consider bounded linear operators acting on a Hilbert space H, calling simply operators. An operator A is positive if $(Ax, x) \geq 0$ for all $x \in H$, and is denoted by $A \geq 0$. By Wigner [7], a weakly positive operator was introduced as the product of two positive invertible operators. In [4], M. Nakamura and one of the authors studied such an operator in the light of the recent knowledge. In succession of the consideration we define a weakly selfadjoint operator as the product of two selfadjoint operators, one of whose factors is positive, and show some results on the operator.

2. In the similar fashion as in [7] for a weakly positive operator, we can define a weakly selfadjoint invertible operator T as an operator such that T is similar to a selfadjoint operator S by some positive invertible operator X, that is,

2000 *Mathematics Subject Classification.* 47A10, 47A12, 47A30, 47A63.
Key words and phrases. Kantorovich constant, weakly positive operator, spectral radius, weakly selfadjoint operator.

$$T = XSX^{-1}. \tag{2.1}$$

It is easy to see that this definition is equivalent to that given as before, for, if we put $A = X^2$ and $B = X^{-1}SX^{-1}$, then $T = AB$ with the first factor A being positive and the second factor B being selfadjoint.

Now first we want to state a basic fact on invertibility for a weakly selfadjoint operator:

Proposition 1. *Let $T = AB$ be a weakly selfadjoint operator. Then T is invertible if and only if both A and B are invertible. For the spectrum $\sigma(T)$ of T we have*

$$\sigma(T) = \sigma(A^{\frac{1}{2}}BA^{\frac{1}{2}}) \quad if \quad A \geq 0, \tag{2.2}$$

and

$$\sigma(T) = \sigma(B^{\frac{1}{2}}AB^{\frac{1}{2}}) \quad if \quad B \geq 0. \tag{2.2$'$}$$

Proof. If T is invertible, then $A(BT^{-1}) = (BT^{-1})^*A = I$ (the identity operator on H), so that A is invertible. Hence $B = A^{-1}T$ is also invertible. Conversely, if both A and B are invertible, then T is clearly invertible. For (2.2), first if A is invertible, then $T = A^{\frac{1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})A^{-\frac{1}{2}}$, so that $\sigma(T) = \sigma(A^{\frac{1}{2}}BA^{\frac{1}{2}})$. Next if A is not invertible, then T is not invertible from the above discussion, and further easily we see that $A^{\frac{1}{2}}BA^{\frac{1}{2}}$ is not invertible. Hence $0 \in \sigma(T), \sigma(A^{\frac{1}{2}}BA^{\frac{1}{2}})$. Now from the fact $\sigma(XY)\backslash\{0\} = \sigma(YX)\backslash\{0\}$ for any operators X and Y (say, [3, p. 87]), we have

$$\sigma(T) = [\sigma(A^{\frac{1}{2}}(A^{\frac{1}{2}}B))\backslash\{0\}]\cup\{0\} = [\sigma((A^{\frac{1}{2}}B)A^{\frac{1}{2}})\backslash\{0\}]\cup\{0\} = \sigma(A^{\frac{1}{2}}BA^{\frac{1}{2}}).$$

Similarly we can obtain (2.2$'$).

Corollary 2. *If T is weakly selfadjoint, then*

$$\sigma(T) \subset \mathbb{R} = (-\infty, \infty). \tag{2.3}$$

Related to Corollary 2 the spectrum $\sigma(T)$ of $T = AB$ is not always in $\mathbb{R}$ if we only assume that both A and B are selfadjoint, without positivity of one factor A or B. In fact, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix},$$

then $T = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix}$ and $\sigma(T) = \left\{ \dfrac{-3 \pm \sqrt{3}i}{2} \right\} \not\subset \mathbb{R}.$

We denote by $W(T)$ and $\overline{W(T)}$ the numerical range and its closure of an operator T, respectively. Then as a fact between the spectrum and numerical range of the product of two operators, we know the following general result due to Nakamoto [5] related to (2.3): If A or B is positive then

$$\sigma(AB) \subset \overline{W(A)W(B)}.$$

3. In [4], the following result due to Wang was shown by using the fact $\sigma(T) \subset \mathbb{R}^+ = [0, \infty)$ for a weakly positive operator T : If $A \geq 0,\ AB \geq 0$ and $\| B \| \leq 1$, then $A \geq AB$. As an extension of this fact, we have the following:

Theorem 3. *Let A be selfadjoint, $AB \geq 0$ and $\| B \| \leq 1$. Then $|A| \geq AB$.*

Proof. Since $T_\varepsilon := (|A| + \varepsilon I)^{-1}(AB)$ for $\varepsilon > 0$ is weakly selfadjoint, we have $\sigma(T_\varepsilon) \subset \mathbb{R}$. Further, we have

$$\| T_\varepsilon \| \leq \| (|A| + \varepsilon I)^{-1} A \| \| B \| \leq 1,$$

so that $\sigma(T_\varepsilon) \subset [-1, 1]$. Hence

$$(|A| + \varepsilon I)^{-\frac{1}{2}} AB (|A| + \varepsilon I)^{-\frac{1}{2}} \leq r(T_\varepsilon) I \leq I$$

or

$$AB \leq |A| + \varepsilon I.$$

Taking the limit as $\varepsilon \to 0$, we have the desired inequality.

With the similar method as in [4, Added in Proof. 2], applying monotonicity of the square roots of positive operators, we can give a very short alternative proof of the above theorem: Since $AB \geq 0$ and $\| B \| \leq 1$, we see

$$(AB)^2 = ABB^*A \leq \| B \|^2 A^2 \leq A^2,$$

so that by taking the square roots of both sides, we have the desired inequality.

4. Generalizing Carlin-Noble definition of the square root of a weakly positive matrix [2], M. Nakamura and one of the authors [4] defined the function of a weakly positive invertible operator for every continuous function defined on an interval containing its spectrum. With the similar way we can extend the definition to any weakly selfadjoint invertible operator as follows:

Theorem 4. *Let $T = AB$ be a weakly selfadjoint invertible operator with selfajoint factors A and B. For any continuous function f defined on an interval containing $\sigma(T)$, put*

$$f(T) = \begin{cases} A^{\frac{1}{2}} f(A^{\frac{1}{2}} B A^{\frac{1}{2}}) A^{-\frac{1}{2}} & \text{if } A \geq 0, \\ B^{-\frac{1}{2}} f(B^{\frac{1}{2}} A B^{\frac{1}{2}}) B^{\frac{1}{2}} & \text{if } B \geq 0. \end{cases} \tag{4.1}$$

Then $f(T)$ is well-defined, that is, the definition does not depend on the selfadjoint factors representing T as the product. Further, if both factors A and B are positive, then the right two definitions of $f(T)$ in (4.1) are identical.

As a result, if f is defined on $\mathbb{R}$, then we can define $f(T)$ for any weakly selfadjoint operator T by (4.1).

For the spectral mapping theorem with respect to a weakly selfadjoint operator, using the definition (4.1), we can show the following:

Theorem 5. *With the same assumption as in Theorem 4, the spectrum $\sigma(f(T))$ is well-defined, and $\sigma(f(T)) = f(\sigma(T))$.*

5. For a weakly positive operator the following theorem due to Bourin is well-known:

Bourin's theorem. (cf. [1], [4, Theorem 3].) *Let $T = AB$ be a weakly positive operator with positive operators A and B. If B satisfies*

$$m_B I \leq B \leq M_B I \tag{5.1}$$

for some scalars $0 < m_B < M_B$, then

$$\| T \|^2 \leq K_B r(T)^2, \tag{5.2}$$

where $r(T)$ *is the spectral radius of* T *and* $K_B = K(m_B, M_B) = \dfrac{(M_B + m_B)^2}{4M_B m_B}$, *called the Kantorovich constant of* B *with respect to* (5.1).

Since $\| T^* \| = \| T \|$ and $r(T^*) = r(T)$, we see that if

$$m_A I \leq A \leq M_A I \tag{5.3}$$

for some scalars $0 < m_A < M_A$, then we have

$$\| T \|^2 \leq K_A r(T)^2, \tag{5.4}$$

instead of (5.2). The inequality (5.2) is related to the Kantorovich constant K_B with respect to the right factor B and (5.4) is related to the Kantorovich constant K_A with respect to the left factor A, so we shall often call the former the *right* Bourin's inequality, and the latter the *left* Bourin's one.

Now as a slight extension of Bourin's theorem, we state the following fact for a weakly selfadjoint operator:

Theorem 6. *Let* $T = AB$ *be a weakly selfadjoint invertible operator. Suppose that* A *is positive (and invertible since* T *is invertible). Define* $\tilde{T} = A|B|$. *If* $mI \leq |B| = (B^2)^{\frac{1}{2}} \leq MI$ *for some scalars* $0 < m < M$, *then* $\| \tilde{T} \| = \| T \|$ *and*

$$\| \tilde{T} \|^2 \leq K_{|B|} r(\tilde{T})^2. \tag{5.5}$$

Proof. First note that $\| T \|^2 = \| AB^2A \| = \| A|B|^2A \| = \| \tilde{T} \|^2$. For the inequality (5.5), it is immediate from the Bourin's inequality applied to $\tilde{T}$.

6. In [4], a parametrized decomposition of a weakly positive operator was considered. We here want to discuss the similar decomposition of a weakly selfadjoint operator. Let $T = AB$ be a weakly selfadjoint invertible operator. Consider the equation

$$ABX = XBA \tag{6.1}$$

for a selfadjoint X. Then we can put $X = A$ or B^{-1} as a solution of (6.1), so that if we define

(6.2) $$C(s,t) = sA + tB^{-1}$$

with real parameters s and t, this operator is a little wider selfadjoint solution of (6.1). Choosing s and t such that $C(s,t)$ is invertible, we define

(6.3) $$A(s,t) = ABC(s,t) \quad \text{and} \quad B(s,t) = C(s,t)^{-1}.$$

Then we have a decomposition

(6.4) $$T = A(s,t)B(s,t)$$

of T with selfadjoint operators $A(s,t)$ and $B(s,t)$. If we consider the equation

(6.5) $$XAB = BAX$$

instead of (6.1), then since $X = A^{-1}$ and B satisfy (6.5), defining $C'(s,t)$ by

(6.6) $$C'(s,t) = sA^{-1} + tB,$$

we have another decomposition $T = A'(s,t)B'(s,t)$ of T such that

(6.7) $$A'(s,t) = C'(s,t)^{-1} \quad \text{and} \quad B'(s,t) = C'(s,t)AB.$$

Now, say, corresponding to the decomposition (6.4) of T by the parametrized factors $A(s,t)$ and $B(s,t)$ in (6.3), we have the following Bourin's (left and right) inequalities: If we assume $A(s,t) \geq 0$, then for $\tilde{T} = A(s,t)|B(s,t)|$

(6.8) $$\| \tilde{T} \|^2 = \| T \|^2 \leq K_{A(s,t)} r(\tilde{T})^2 \quad \text{(the left Bourin's inequality)}$$

and

(6.9) $$\| \tilde{T} \|^2 = \| T \|^2 \leq K_{|B(s,t)|} r(\tilde{T})^2 \quad \text{(the right Bourin's inequality).}$$

7. We here consider the left and right Bourin's inequalities ((6.8) and (6.9)) for the parametrized decompositions of a weakly selfadjoint operator with a concrete example. Since their left hand sides are constant, we only observe their right hand sides. Now let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}.$$

Then $\quad A \geq 0,\ T = AB = \begin{bmatrix} 2 & 2 \\ 1 & -4 \end{bmatrix}, \quad |B| = \frac{1}{\sqrt{29}} \begin{bmatrix} 7 & -3 \\ -3 & 22 \end{bmatrix},$

$$I \leq A \leq 2I, \quad \frac{-3+\sqrt{29}}{2} I \leq |B| \leq \frac{3+\sqrt{29}}{2} I,$$

$$\tilde{T} = A|B| = \frac{1}{\sqrt{29}} \begin{bmatrix} 14 & -6 \\ -3 & 22 \end{bmatrix}, \| \tilde{T} \| = \| T \| = \sqrt{20} \ \text{ and}$$

$$r(\tilde{T}) = \frac{18+\sqrt{34}}{\sqrt{29}}.$$

For the Kantorovich constant, we then have $K_A = 9/8$ and $K_{|B|} = 29/20$. Consequently, for the left and right Bourin's inequalities ((5.4) and (5.2)) with respect to $\tilde{T}$, both of the left hand sides are the same value, $\| \tilde{T} \|^2 = \| T \|^2 = 20$, and the right hand sides are

$$K_A r(\tilde{T})^2 = \frac{9(18+\sqrt{34})^2}{232} = 22.03\cdots$$

and

$$K_{|B|} r(\tilde{T})^2 = \frac{(18+\sqrt{34})^2}{20} = 28.39\cdots,$$

respectively.

Now consider the parametrized operators $A(s,t)$ and $B(s,t)$, say, in the case (6.3). Then from (6.2), we see

$$C(s,t) = sA + tB^{-1} = \frac{1}{5} \begin{bmatrix} 10s+4t & t \\ t & 5s-t \end{bmatrix},$$

$$A(s,t) = ABC(s,t) = \begin{bmatrix} 4s+2t & 2s \\ 2s & -4s+t \end{bmatrix}$$

and

$$B(s,t) = C(s,t)^{-1} = \frac{1}{t^2-2st-10s^2} \begin{bmatrix} -5s+t & t \\ t & -10s-4t \end{bmatrix}.$$

To apply Theorem 6, we have to assume that $A(s,t)$ or $B(s,t)$ is positive (and invertible). Say, assume that $A(s,t) > 0$. So we have to consider the left and right Bourin's inequalities (6.8) and (6.9) for our example. As shown before, both the left hand sides of those (Bourin's) inequalities have the common value $\| T \|^2 = 20$. However, the right hand sides of them depend on the parameters s and t, and change the values as we shall show.

First note that $A(s,t) > 0$ is equivalent to

$$\text{(7.1)} \qquad \begin{cases} 4s + 2t \geq 0, & -4s + t \geq 0 \\ \det A(s,t) = 2(t^2 - 2st - 10s^2) > 0. \end{cases}$$

If we denote by D_A, the set of all points (s,t) satisfying (7.1) in the coordinate plane, then we have

$$\text{(7.2)} \qquad D_A = D_+ \cup D_-,$$

where

$$D_+ = \{(s,t); s > 0,\ t > (1 + \sqrt{11})s\}$$

and

$$D_- = \{(s,t); s < 0,\ t > (1 - \sqrt{11})s\}.$$

In this case we see that $B(s,t) \not\geq 0$ or $\det B(s,t) < 0$, in fact, we see, from (6.4), (7.1) and $\det T = -10$,

$$\det B(s,t) = \det A(s,t)^{-1} T = \frac{\det T}{\det A(s,t)} = -\frac{10}{\det A(s,t)} < 0.$$

By an elementary computation, we then have

$$|B(s,t)| = \frac{1}{(t^2 - 2st - 10s^2)\sqrt{29t^2 + 50st + 25s^2}}$$

$$\times \begin{bmatrix} 7t^2 - 20st - 25s^2 & -3t(t+5s) \\ -3t(t+5s) & 2(11t^2 + 35st + 25s^2) \end{bmatrix},$$

$$\tilde{T}(s,t) = A(s,t)|B(s,t)| = \frac{1}{\sqrt{29t^2 + 50st + 25s^2}} \begin{bmatrix} 14t + 10s & -6t - 10s \\ -3t + 5s & 22t + 20s \end{bmatrix}$$

and

$$r(\tilde{T}(s,t)) = \frac{18t+15s+\sqrt{34t^2+40st-25s^2}}{\sqrt{29t^2+50st+25s^2}} \tag{7.3}$$

Further we can obtain the following inequalities:

$$m_{A(s,t)-}I \leq A(s,t) \leq m_{A(s,t)+}I, \; m_{|B(s,t)|-}I \leq |B(s,t)| \leq m_{|B(s,t)|+}I, \tag{7.4}$$

where

$$m_{A(s,t)\pm} = \frac{3t \pm \sqrt{t^2+16st+80s^2}}{2} \; (>0),$$

and

$$m_{|B(s,t)|\pm} = \frac{\pm 3(t+5s)+\sqrt{29t^2+50st+25s^2}}{2(t^2-2st-10s^2)} \; (>0).$$

Hence the Kantorovich constants $K_{A(s,t)}$ and $K_{|B(s,t)|}$ are given as

$$K_{A(s,t)} = \frac{9t^2}{8(t^2-2st-10s^2)} \text{ and } K_{|B(s,t)|} = \frac{29t^2+50st+25s^2}{20(t^2-2st-10s^2)}. \tag{7.5}$$

Now observe the right hand side of the left Bourin's inequality. Write it $R_l(s,t)$. Then we have, from (7.3) and (7.5),

$$R_l(s,t) = K_{A(s,t)} r(\tilde{T})^2 = \frac{9t^2(18t+15s+\sqrt{34t^2+40st-25s^2})^2}{8(t^2-2st-10s^2)(29t^2+50st+25s^2)} \tag{7.6}$$

$$\left(= \frac{9t^2(|18t+15s|+\sqrt{34t^2+40st-25s^2})^2}{8(t^2-2st-10s^2)(29t^2+50st+25s^2)}, \quad \text{because } 18t+15s>0 \right)$$

for $(s,t) \in D_A$. Dividing by s^4 both the numerator and denominator of the last term of (7.6), and putting $u = t/s$ and $R_l(u) = R_l(s,t)$, we can rewrite (7.6) as follows:

$$R_l(u) = \frac{9u^2(|18u+15|+\sqrt{34u^2+40u-25})^2}{8(u^2-2u-10)(29u^2+50u+25)} \tag{7.7}$$

for $u > 1+\sqrt{11}$ or $u < 1-\sqrt{11}$.

Now, for the behavior of $R_l(u)$ we have:

(i) $\lim_{u\to\pm\infty} R_l(u) = \frac{9(18+\sqrt{34})^2}{232} = K_A r(T)^2$.

(ii) $\lim_{u\to 1\pm\sqrt{11}} R_l(u) = \infty$.

(iii) $R_l(u)$ is increasing for $u < 1 - \sqrt{11}$, and decreasing for $u > 1 + \sqrt{11}$.

Similarly, for the right Bourin's inequality, its right hand side, denoted by $R_r(s,t)$, is given as

$$R_r(s,t) = K_{|B(s,t)|} r(\tilde{T})^2 = \frac{(18t + 15s + \sqrt{34t^2 + 40st - 25s^2})^2}{20(t^2 - 2st - 10s^2)} \tag{7.8}$$

or, putting $u = t/s$ and $R_r(u) = R_r(s,t)$ in (7.8), we have

$$R_r(u) = \frac{(|18u + 15| + \sqrt{34u^2 + 40u - 25})^2}{20(u^2 - 2u - 10)} \tag{7.9}$$

for $u > 1 + \sqrt{11}$ or $u < 1 - \sqrt{11}$.

For the behavior of $R_r(u)$ we have:

(iv) $\lim_{u \to \pm\infty} R_r(u) = \frac{(18 + \sqrt{34})^2}{20} = K_{|B|} r(T)^2$.

(v) $\lim_{u \to 1 \pm \sqrt{11}} R_r(u) = \infty$.

(vi) $R_r(u)$ is decreasing for $u < -5$ or $u > 1 + \sqrt{11}$, and increasing for $-5 < u < 1 - \sqrt{11}$.

(vii) The minimum of $R_r(u)$ is 20, which is attained at $u = -5$, so that, say, for $(s,t) = (-1,5)$ the minimum is attained. The corresponding factors of T are

$$A(-1,5) = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}, \quad B(-1,5) = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix},$$

and then $\tilde{T} = \frac{1}{\sqrt{5}} \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$.

Finally, we want to state the outline for the case $B(s,t) > 0$. Let

$$D_B = \{(s,t); s > 0,\ (1 - \sqrt{11})s < t < (1 + \sqrt{11})s\}.$$

Then $B(s,t) > 0$ is equivalent to $(s,t) \in D_B$. Let us define $\hat{T} = |A(s,t)|B(s,t)$ (similarly as $\tilde{T} = A(s,t)|B(s,t)|$ before). Then since

$$|A(s,t)| = \frac{1}{\sqrt{t^2 + 16st + 80s^2}} \begin{bmatrix} 2(t^2 + 10st + 20s^2) & 6st \\ 6st & -(t^2 + 4st - 40s^2) \end{bmatrix},$$

we have

$$\hat{T} = \frac{1}{\sqrt{t^2 + 16st + 80s^2}} \begin{bmatrix} 2t + 20s & 2t \\ -t & 4t + 40s \end{bmatrix}.$$

The corresponding left and right Bourin's inequalities are

$$\| \hat{T} \|^2 = 20 \leq K_{|A(s,t)|} r(\hat{T})^2$$

$$= \frac{(3t + 30s + \sqrt{-t^2 + 20st + 100s^2})^2}{-8(t^2 - 2st - 10s^2)}$$

$$\left(= \frac{(3u + 30 + \sqrt{-u^2 + 20u + 100})^2}{-8(u^2 - 2u - 10)} \right)$$

and

$$\| \hat{T} \|^2 = 20 \leq K_{B(s,t)} r(\hat{T})^2$$

$$= \frac{9(t + 5s)^2 (3t + 30s + \sqrt{-t^2 + 20st + 100s^2})^2}{-20(t^2 - 2st - 10s^2)(t^2 + 16st + 80s^2)}$$

$$\left(= \frac{9(u + 5)^2 (3u + 30 + \sqrt{-u^2 + 20u + 100})^2}{-20(u^2 - 2u - 10)(u^2 + 16u + 80)} \right),$$

respectively, for $1 - \sqrt{11} < u = t/s < 1 + \sqrt{11}$.

References

[1] J. Bourin, *Compression, Dilation, and Matrix Inequalities*, Monograph in RGMIA, 2004.

[2] H. J. Carlin and G. A. Noble, *Circuit properties of coupled dispersive lines with applications to wave guide modeling*, Proc. Network and Signal Theory, edited by J. K. Skwirzynski and J. O. Scanlan, Peter Pergrinus, Inc., London, 1973, 258–269.

[3] T. Furuta, *Invitation to Linear Operators, From Matrices to Bounded Linear Operators on a Hilbert Space*, Taylor and Francis Inc., New York, (2001).

[4] S. Izumino and M. Nakamura, *Wigner's weakly positive operators*, Scientiae Mathematicae Japonicae, 65 (2007), 1095–1102.

[5] R. Nakamoto, *On a theorem of Bouldin*, Proc. Japan Acad., 47 (1971), 979–981.

[6] D. Wang, *Löwner-Heinz and Reid inequalities*, Math. Japon., 49 (1999), 247–250.

[7] E. P. Wigner, *On weakly positive operators*, Canadian J. Math., 15 (1965), 313–317.

Saichi Izumino
University of Toyama,
Gofuku, Toyama 930-8555, Japan
email: s-izumino@h5.dion.ne.jp

Noboru Nakamura
Fujikoshi-kogyo Senior Highschool,
Higashi-ishigane 7-5, Toyama, 930-0964, Japan
email: nakamura@fujikoshiko.ac.jp

(Received December 22, 2006)