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Real hypersurfaces in complex space forms whose shape operator commutes with the structure Jacobi operator

U-Hang Ki, Chunji Li and Setsuo Nagai

Abstract. It is known that there are no real hypersurfaces with parallel Ricci tensor S in a nonflat complex space form ([6]). In this paper we investigate real hypersurfaces in a nonflat complex space form under condition that the structure Jacobi operator R_ξ commutes with the shape operator A.

Introduction

A Kähler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$.

As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}_n$ or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0, c = 0$ or $c < 0$.

Let M be a real hypersurface of $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J and the Kaehlerian metric of $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. A real hypersurface is said to be a *Hopf hypersurface* if the structure vector field ξ of M is principal.

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Typical examples of real hypersurfaces in a complex projective space $P_n\mathbb{C}$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroups of the projective unitary group $PU(n+1)$. The complete classification of them was obtained by ([16]) as follows:

Theorem T ([16]) *Let M be a homogeneous real hypersurface of $P_n\mathbb{C}$. Then M is a tube of radius r over one of the following Kähler submanifolds*:
(A_1) *A hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,*
(A_2) *a totally geodesic $P_k\mathbb{C}$ $(1 \leqq k \leqq n-2)$, where $0 < r < \frac{\pi}{2}$,*
(B) *a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,*
(C) *$P_1\mathbb{C} \times P_{(n-1)/2}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n(\geqq 5)$ is odd,*
(D) *a complex Grassmann $G_{2,5}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*
(E) *a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

Due to Takagi's theorem we can see that every homogeneous real hypersurface in $P_n\mathbb{C}$ is a Hopf hypersurface. However, in $H_n\mathbb{C}$ there exists a homogeneous real hypersurface which is not a Hopf hypersurface (see [12]). Also Berndt([1]) classified all Hopf real hypersurfaces with constant principal curvatures in a complex hyperbolic space $H_n\mathbb{C}$ as follows:

Theorem B ([1]) *Let M be a real hypersurface of $H_n\mathbb{C}$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following*:

(A_0) *a self-tube, that is, a horosphere,*
(A_1) *a geodesic hypersphere or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*
(A_2) *a tube over a totally geodesic $H_k\mathbb{C}(1 \le k \le n-2)$,*
(B) *a tube over a totally real hyperbolic space $H_n\mathbb{R}$.*

We denote by ∇, S and R_ξ be the Levi-Civita connection, the Ricci tensor and the structure Jacobi operator with respect to the structure vector field ξ of M respectively.

We know that there are no real hypersurfaces with parallel Ricci tensor in $M_n(c), n \ge 3, c \ne 0$ ([6]).

If we pay a particular attention to the fact that for each Hopf hypersurface M in $M_n(c), c \neq 0$, then $R_\xi A = AR_\xi$ or $S\xi = g(S\xi, \xi)\xi$ is satisfied. Therefore, it is natural to consider a problem that if a real hypersurface M in $M_n(c), c \neq 0$ satisfies $R_\xi A = AR_\xi$ or $S\xi = g(S\xi, \xi)\xi$, is M a Hopf hypersurface ? Recently, there are many studies on partial answers to this problem ([3] $\sim$ [10] etc.). The following facts are used in this paper without proof.

THEOREM HKK ([3]) *Let M be a real hypersurface of a nonflat complex space form which satisfies $\nabla_\xi S = 0$ and $S\xi = g(S\xi, \xi)\xi$. If $g(\nabla_\xi \xi, \nabla_\xi \xi) = \mu^2$ is constant, then M is a Hopf hypersurface.*

THEOREM KN ([9]) *Let M be a real hypersurface in a complex projective space $P_n\mathbb{C}$. Then the following are equivalent*:

(1) *M is a Hopf hypersurface in the ambient space $P_n\mathbb{C}$.*

(2) *The structure vector ξ is an eigenvector with constant eigenvalue of the Ricci tensor S of M and $\nabla_{\phi\nabla_\xi \xi} S = 0$ holds.*

THEOREM KSN ([10]) *Let M be a real hypersurface in $P_n\mathbb{C}$ which satisfies $R_\xi S = SR_\xi$ and $\nabla_{\phi\nabla_\xi \xi} S = 0$. If $g(S\xi, \xi)$ is constant on M, then M is a Hopf hypersurface.*

The main purpose of the present paper is to establish the following:

THEOREM. *Let M be a real hypersurface in $M_n(c)$, $c \neq 0$. Then the followings are equivalent provided that $6(\mathrm{Tr}\, A)^2 + c \neq 0$:*

(1) *M is a Hopf hypersurface in the ambient space in $M_n(c)$.*

(2) *$R_\xi A = AR_\xi$ and $\nabla_{\phi\nabla_\xi \xi} S = 0$ hold on M.*

COROLLARY. *Let M be a real hypersurface in $P_n\mathbb{C}$. Then the followings are equivalent*:

(1) *M is a Hopf hypersurface in the ambient space $P_n\mathbb{C}$.*

(2) *$R_\xi A = AR_\xi$ and $\nabla_{\phi\nabla_\xi \xi} S = 0$ hold on M.*

1. Preliminaries

Let M be a real hypersurface of a complex space form $M_n(c)$ with parallel almost complex structure J and N be a unit normal vector field on M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_Y X = \nabla_Y X + g(AY, X)N, \ \tilde{\nabla}_X N = -AX,$$

for any vector fields X and Y on M, where ∇ and g denote the Riemannian connection and the Riemannian metric induced from $\tilde{g}$ respectively, and A denotes the shape operator in the direction of N. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \ JN = -\xi.$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$\eta(\xi) = 1, \ \phi\xi = 0, \ \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M.

From the fact $\tilde{\nabla} J = 0$ and by using of the Gauss and Weingarten formulas, we obtain

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{1.1}$$

$$\nabla_X \xi = \phi AX. \tag{1.2}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations respectively:

$$\begin{aligned} R(X,Y)Z = \ & \tfrac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X \\ & -g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX \\ & -g(AX, Z)AY, \end{aligned} \tag{1.3}$$

$$(1.4)\qquad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}$$

for any vector fields X, Y and Z on M, where R denotes Riemannian curvature tensor of M

In the following, to write our formulas in convention forms, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, $\gamma = \eta(A^3\xi)$ and $h = \mathrm{Tr}\, A$, and for a function f we denote by ∇f the gradient vector field of f.

We denote the Ricci tensor of type $(1,1)$ by S. Then we have from (1.3)

$$(1.5)\qquad SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X,$$

which together with (1.2) implies that

$$(1.6)\qquad \begin{aligned}(\nabla_X S)Y = \; & -\tfrac{3}{4}c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} + (Xh)AY \\ & +(hI - A)(\nabla_X A)Y - (\nabla_X A)AY,\end{aligned}$$

where I is the identity tensor.

We put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector fields ξ. Then, using (1.2), we see that

$$(1.7)\qquad \phi U = -A\xi + \alpha\xi,$$

which shows that $g(U,U) = \beta - \alpha^2$. We easyly see that ξ is a principal curvature vector, that is, $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.

If $A\xi - g(A\xi,\xi)\xi \neq 0$, then we can put

$$(1.8)\qquad A\xi = \alpha\xi + \mu W,$$

where W is a unit vector field orthogonal to ξ. Then by (1.2) we see that $U = \mu\phi W$ and hence $g(U,U) = \mu^2$. So we have

$$(1.9)\qquad \mu^2 = \beta - \alpha^2.$$

Further, W is also orthogonal to U.

Using (1.2) and (1.8), we see that

$$(1.10)\qquad \mu g(\nabla_X W, \xi) = g(AU, X),$$

$$(1.11)\qquad g(\nabla_X \xi, U) = \mu g(AW, X).$$

Now, differentiating (1.7) covariantly along M and making use of (1.1) and (1.2), we find

$$\begin{aligned} & g(\phi X, \nabla_Y U) + \eta(X) g(AU + \nabla\alpha, Y) \\ = \; & g((\nabla_Y A)X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y), \end{aligned} \tag{1.12}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha \tag{1.13}$$

because of (1.4).

Because of properties of the almost contact metric structure, we also have from (1.12)

$$\nabla_X U + g(A^2\xi, X)\xi = \phi(\nabla_X A)\xi + \phi A\phi AX + \alpha AX. \tag{1.14}$$

By the definition of U, (1.2) and (1.12), it is verified that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha, \tag{1.15}$$

which shows that

$$\mu g(\nabla_\xi U, W) = \alpha\mu^2 - 3g(AU, U) - U\alpha. \tag{1.16}$$

From the Gauss equation (1.3) the structure Jacobi operator R_ξ is given by

$$R_\xi X = R(X, \xi)\xi = \frac{c}{4}(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi$$

for any vector field X on M.

From this and (1.5), we have

$$\begin{aligned} & g(R_\xi Y, AX) - g(R_\xi X, AY) \\ = \; & g(A^2\xi, Y)g(A\xi, X) - g(A^2\xi, X)g(A\xi, Y) \\ & + \tfrac{c}{4}\left\{g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\right\}. \end{aligned} \tag{1.17}$$

2. Structure Jacobi operator of real hypersurfaces

Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. If it satisfies $R_\xi A = AR_\xi$, then we have from (1.17)

$$A^2\xi = \rho A\xi + \frac{c}{4}\xi, \tag{2.1}$$

which shows that

$$\beta = \rho\alpha + \frac{c}{4}. \tag{2.2}$$

We set $\Omega = \{p \in M \mid \mu(p) \neq 0\}$, and suppose that $\Omega \neq \emptyset$, that is, ξ is not a principal curvature vector on M. From now on we discuss our arguments on the open set Ω of M unless otherwise stated.

Combining (1.8) to (2.1), we verify that

$$AW = \mu\xi + (\rho - \alpha)W \tag{2.3}$$

and hence

$$A^2W = \rho AW + \frac{c}{4}W \tag{2.4}$$

by virtue of $\mu \neq 0$.

Differentiating (2.1) covariantly along Ω and making use of (1.2), we find

(2.5)
$$\begin{aligned} &g((\nabla_X A)\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) - \rho g(A\phi AX, Y) \\ &= (X\rho)g(A\xi, Y) + \rho g((\nabla_X A)\xi, Y) + \tfrac{c}{4}g(\phi AX, Y), \end{aligned}$$

which together with (1.4) and (1.13) yields

$$(\nabla_\xi A)A\xi = \rho AU - \frac{c}{4}U + \frac{1}{2}\nabla\beta.$$

If we put $X = \xi$ in (2.5) and use (1.13) and the last equation, we get

$$3A^2U - 2\rho AU - \frac{c}{2}U = (\xi\rho)A\xi - A\nabla\alpha + \rho\nabla\alpha - \frac{1}{2}\nabla\beta, \tag{2.6}$$

where we have used (1.4).

Differentiating (2.3) covariantly, we find

$$\begin{aligned} &(\nabla_X A)W + A\nabla_X W \\ = \; &(X\mu)\xi + \mu\nabla_X\xi + X(\rho - \alpha)W + (\rho - \alpha)\nabla_X W. \end{aligned} \tag{2.7}$$

By taking the inner product (2.7) with W and taking account of (1.8) and (1.10), we obtain

$$g((\nabla_X A)W, W) = -2g(AU, X) + X\rho - X\alpha \tag{2.8}$$

since W is a unit vector field orthogonal to ξ. We also have by applying ξ to (2.7)

$$\mu g((\nabla_X A)W, \xi) = (\rho - 2\alpha)g(AU, X) + \mu(X\mu), \tag{2.9}$$

where we have used (1.10), which connected to (1.4) gives

$$\mu(\nabla_W A)\xi = (\rho - 2\alpha)AU - \frac{c}{2}U + \mu\nabla\mu, \tag{2.10}$$

$$\mu(\nabla_\xi A)W = (\rho - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu. \tag{2.11}$$

Putting $X = \xi$ in (2.8) and making use of (2.9), we find

$$W\mu = \xi\rho - \xi\alpha. \tag{2.12}$$

Now, define a 1-form u by $u(X) = g(U, X)$ for any vector field X. Using (1.4) and (2.5), we verrfy that

$$\begin{aligned} & \tfrac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y)\} + \tfrac{c}{2}(\rho - \alpha)g(\phi Y, X) \\ & -g(A^2\phi AX, Y) + g(A^2\phi AY, X) + 2\rho g(\phi AX, AY) \\ & -\tfrac{c}{2}\{g(\phi AY, X) - g(\phi AX, Y)\} \\ = \; & g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + (Y\rho)g(A\xi, X) \\ & -(X\rho)g(A\xi, Y). \end{aligned} \tag{2.13}$$

If we replace X by μW to both sides of (2.12) and use (1.13), (2.3), (2.4), (2.9) and (2.10), then we obtain

$$\begin{aligned} & (3\alpha - 2\rho)A^2U + (2\rho^2 - 2\rho\alpha + c)AU + \tfrac{c}{4}(\alpha - \rho)U \\ & = \mu A\nabla\mu + (\alpha - \rho)\mu\nabla\mu + \mu^2(\nabla\rho - \nabla\alpha) - \mu(W\rho)A\xi. \end{aligned} \tag{2.14}$$

Putting $X = \mu W$ in (1.14) and using (2.3) and (2.10), we find

$$\begin{aligned} \mu\nabla_W U = \; & (2\rho - 3\alpha)\phi AU + \mu\phi\nabla\mu \\ & +\mu(\rho\alpha - \alpha^2 + \tfrac{c}{2})W - \mu^2(\rho - \alpha)\xi. \end{aligned} \tag{2.15}$$

3. Real hypersurface satisfying $\nabla_{\phi U}S = 0$ and $R_\xi A = AR_\xi$

In this section, we will continue our arguments under the same hypothesis $R_\xi A = AR_\xi$ as in Section 2. Further, assume that $\nabla_{\phi U}S = 0$ and hence $\nabla_W S = 0$ on Ω because of $\mu \neq 0$. By replacing X by μW in (1.6), we find

$$\begin{aligned} &-\tfrac{3}{4}c(\rho-\alpha)\{u(Y)\xi+\eta(Y)U\}+\mu(Wh)AY+\mu h(\nabla_W A)Y \\ &=\mu A(\nabla_W A)Y+\mu(\nabla_W A)AY, \end{aligned} \tag{3.1}$$

where we have used (1.2) and (2.3). Putting $Y = W$ in this and making use of (2.3), (2.8) and (2.10), we find

$$\begin{aligned} &(Wh)AW \\ =\ &(2h-\rho)AU-2A^2U-\tfrac{c}{2}U+A\nabla\rho-A\nabla\alpha \\ &+\tfrac{1}{2}\nabla\beta+(h-\rho)\nabla\alpha+(\rho-h-\alpha)\nabla\rho. \end{aligned} \tag{3.2}$$

If we replace Y by ξ and take account of (2.3), (2.8) and (2.10), then we obtain

$$\begin{aligned} &\mu A\nabla\mu+(\alpha-h)\mu\nabla\mu+\mu^2(\nabla\rho-\nabla\alpha) \\ =\ &\mu(Wh)A\xi+(2\alpha-\rho)A^2U+(h\rho-2\alpha h+\rho\alpha+c)AU \\ &+\tfrac{c}{4}(5\alpha-3\rho-2h)U. \end{aligned} \tag{3.3}$$

On the other hand, we have from (1.5) and (2.1)

$$S\xi=\frac{c}{4}(2n-3)\xi+(h-\rho)A\xi. \tag{3.4}$$

Differentiating this covariantly, we find

$$\begin{aligned} &(\nabla_X S)\xi+S\nabla_X\xi \\ =\ &\tfrac{c}{4}(2n-3)\nabla_X\xi+X(h-\rho)A\xi+(h-\rho)(\nabla_X A)\xi \\ &+(h-\rho)A\nabla_X\xi. \end{aligned}$$

Replacing X by μW in this and using $\nabla_X S = 0$ and (2.10), we obtain

$$\begin{aligned} (\rho-\alpha)SU=\ &\tfrac{c}{4}(2n-3)(\rho-\alpha)U+\mu(Wh-W\rho)A\xi \\ &+(h-\rho)\left\{(\rho-2\alpha)AU-\tfrac{c}{2}U+\mu\nabla\mu\right\} \\ &+(\rho-\alpha)(h-\rho)AU, \end{aligned}$$

where we have used the fact that $\mu\nabla_W\xi = (\rho-\alpha)U$, which together with (1.5) implies that

$$\begin{aligned} &\mu W(\rho-h)A\xi + (\rho-h)\mu\nabla\mu \\ = \;& (\rho-\alpha)A^2U - c(\rho-\alpha)U - \tfrac{c}{2}(h-\rho)U \\ &+ \{(h-\rho)(\rho-2\alpha) - \rho(\rho-\alpha)\}\, AU. \end{aligned} \tag{3.5}$$

Remark 3.1. $\rho - \alpha \neq 0$ on Ω. In fact, if not, then we have $\rho-\alpha = 0$ and hence $\mu^2 = \dfrac{c}{4}$ by virtue of (2.2). Then (2.14) becomes $\alpha A^2U + cAU = -\mu(W\alpha)A\xi$ and therefore $W\alpha = 0$. So we have

$$\alpha A^2U + cAU = 0. \tag{3.6}$$

Further, (2.6), (3.2) and (3.5) are reduced respectively to

$$2A^2U - 2\alpha AU - \frac{c}{2}U = (\xi\alpha)A\xi - A\nabla\alpha, \tag{3.7}$$

$$2A^2U = (2h-\alpha)AU - \frac{c}{2}U, \tag{3.8}$$

$$(h-\rho)\left\{\alpha AU + \frac{c}{2}U\right\} = 0 \tag{3.9}$$

because of $\nabla\mu = 0$. Combining (3.6) to (3.9), we see that $(h-\rho)AU = 0$, which connected to (3.8) gives $h - \rho = 0$. Thus (3.8) is led to

$$2A^2U = \alpha AU - \frac{c}{2}U. \tag{3.10}$$

Comparing (3.6) with (3.10), we have $3\alpha^2 + 4c = 0$ and consequently α is constant. Therefore (3.7) turns out to be $3A^2U - 2\alpha AU - \dfrac{c}{2}U = 0$, which together with (3.6) and (3.10) will produce a contradiction. Accordingly $\rho - \alpha \neq 0$ on Ω is proved. In what follows $\rho - \alpha \neq 0$ is satisfied everywhere.

In the previous paper [9], two of the present authors proved the following fact:

Remark 3.2. (Lemma 3.2 of [9]) Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If it satisfies $\nabla_{\phi U}S = 0$ and $S\xi = \sigma\xi$ for some constant σ, then we have $\xi\alpha = 0$, $W\alpha = 0$, $\xi h = 0$ and $Wh = 0$ on Ω.

Lemma 3.3. *$\xi\alpha = 0$, $\xi\rho = 0$, $W\alpha = 0$ and $Wh = 0$ on Ω.*

Proof. Since U is orthogonal to the structure vector ξ, if we take the inner product (2.6) with ξ, then we obtain

$$\xi\mu = W\alpha, \tag{3.11}$$

where we have used (1.8) and (2.2).

Taking the inner product (3.5) with ξ or W, and using (2.12) and (3.11), we also have respectively

$$\begin{aligned} \alpha(W\rho - Wh) &= (h-\rho)W\alpha, \\ \mu(W\rho - Wh) &= (h-\rho)(\xi\rho - \xi\alpha), \end{aligned} \tag{3.12}$$

which enables us to obtain $(h-\rho)\{\mu(W\alpha) - \alpha(\xi\rho - \xi\alpha)\} = 0$ and hence

$$\mu(W\alpha) = \alpha(\xi\rho - \xi\alpha). \tag{3.13}$$

In fact, if not, then we have $h = \rho$. So (3.4) implies $S\xi = \frac{c}{4}(2n-3)\xi$ on this subset. Thus, Remark 3.2 tells us that $W\alpha = 0$, $\xi\rho = 0$ and $\xi\alpha = 0$, a contradiction. Therefore (3.13) is established.

On the other hand, applying (2.14) by ξ and making use of (2.12) and (3.11), we also have

$$\alpha(W\rho) = (2\alpha - \rho)W\alpha + 2\mu(\xi\rho - \xi\alpha),$$

which together with (2.2) and (3.13) implies that

$$\mu\alpha(W\rho) = (\rho\alpha + \frac{c}{2})(\xi\rho - \xi\alpha). \tag{3.14}$$

By the way, if we take the inner product (3.2) with W and take account of (2.12), we obtain

$$\begin{aligned} &(\rho-\alpha)(Wh - W\rho) \\ = \; &2\mu(\xi\rho - \xi\alpha) + (h - 2\rho + 2\alpha)W\alpha + (\rho - h - \alpha)W\rho. \end{aligned} \tag{3.15}$$

So we verify, using (3.12) $\sim$ (3.14), that $(\rho - h)W\alpha = 0$ and hence $W\alpha = 0$ by virtue of Remark 3.2. Thus, (3.11) tells us that $\xi\mu = 0$, that is, $\xi\beta = 2\alpha(\xi\alpha)$. From this, (2.2) and (3.13) we see that $(\rho-\alpha)\xi\alpha = 0$. Therefore it is seen that $\xi\alpha = 0$ because of Remark 3.1. From this, $W\alpha = 0$ and (3.13) we verify that $\alpha(\xi\rho) = 0$, which together with (3.14) yields $\xi\rho = 0$. Thus (3.12) and (3.15) imply that $(\rho - h)W\rho = 0$. Therefore $W\rho = Wh = 0$ are satisfied. This completes the proof of the lemma. □

Because of Lemma 3.3 and (2.2), equations (2.6), (2.14), (3.2), (3.3) and (3.5) are led to respectively to as follows:

$$A\nabla\alpha = -3A^2U + 2\rho AU + \frac{c}{2}U + \frac{1}{2}(\rho\nabla\alpha - \alpha\nabla\rho), \tag{3.16}$$

$$\begin{aligned} &\mu A\nabla\mu + (\alpha - \rho)\mu\nabla\mu + \mu^2(\nabla\rho - \nabla\alpha) \\ = \ &(3\alpha - 2\rho)A^2U + (2\rho^2 - 2\rho\alpha + c)AU + \tfrac{c}{4}(\alpha - \rho)U, \end{aligned} \tag{3.17}$$

$$\begin{aligned} A\nabla\rho - A\nabla\alpha = \ &2A^2U + (\rho - 2h)AU + \tfrac{c}{2}U \\ &+(\rho - h)(\nabla\alpha - \nabla\rho) + \alpha\nabla\rho - \tfrac{1}{2}\nabla\beta, \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\mu A\nabla\mu + (\alpha - h)\mu\nabla\mu + \mu^2(\nabla\rho - \nabla\alpha) \\ = \ &(2\alpha - \rho)A^2U + (h\rho - 2\alpha h + \rho\alpha + c)AU \\ &+\tfrac{c}{4}(5\alpha - 3\rho - 2h)U, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &(h - \rho)\mu\nabla\mu \\ = \ &(\alpha - \rho)A^2U + (2\rho^2 + 2\alpha h - h\rho - 3\rho\alpha)AU \\ &+\tfrac{c}{2}(\rho + h - 2\alpha)U. \end{aligned} \tag{3.20}$$

From (3.16) and (3.18), we have

$$(h - \rho)(\nabla\rho - \nabla\alpha) = A\nabla\rho + A^2U + (2h - 3\rho)AU - cU. \tag{3.21}$$

Since we have

$$2\mu A\nabla\mu = \alpha A\nabla\rho + (\rho - 2\alpha)A\nabla\alpha$$

by virtue of (2.2) and (3.19), it follows that

$$\begin{aligned} &\alpha A\nabla\rho + (\rho - 2\alpha)A\nabla\alpha + 2\mu^2(\nabla\rho - \nabla\alpha) - 2(h - \alpha)\mu\nabla\mu \\ &= 2(2\alpha - \rho)A^2U + 2(h\rho - 2\alpha h + \rho\alpha + c)AU + \tfrac{c}{2}(5\alpha - 3\rho - 2h)U. \end{aligned}$$

Further, using (3.16) and (3.21), we have

$$\begin{aligned}&(\alpha-\rho)A^2U+(2\rho^2+2\alpha h-3\rho\alpha-2h\rho-2c)AU\\&+\tfrac{c}{2}(2h+4\rho-5\alpha)U\\=\;&2(h-\alpha)\mu\nabla\mu+\tfrac{1}{2}(\rho-2\alpha)\nabla\beta\\&+(2\alpha^2-\rho\alpha-\alpha h-\tfrac{c}{2})(\nabla\rho-\nabla\alpha)+(2\rho\alpha-\rho^2)\nabla\alpha,\end{aligned}$$

which together with (2.2) and (3.20) implies that

$$\begin{aligned}&(2h\rho+4c)AU\\=\;&c(3\rho-3\alpha+h)U+(h\alpha+c)\nabla\rho-(h\rho+c)\nabla\alpha.\end{aligned}\tag{3.22}$$

If we apply this by A, then we obtain

$$(2h\rho+4c)A^2U=c(3\rho-3\alpha+h)AU+(h\alpha+c)A\nabla\rho-(h\rho+c)A\nabla\alpha,$$

which connected to (3.16), (3.20), (3.21) and (3.22) yields

$$\begin{aligned}&4A^2U+2(3\alpha-4\rho-h)AU+(3h\rho-3h\alpha-c)U\\&=(\alpha+h-2\rho)\nabla\rho-(h-\rho)\nabla\alpha.\end{aligned}\tag{3.23}$$

From this and (3.20), it follows that

$$\begin{aligned}&2(3\alpha^2-\rho\alpha-5\alpha h-6c)AU\\&+\left\{c(8\rho-6\alpha+h)-3h(\rho-\alpha)^2\right\}U\\=\;&(2\rho^2-\rho\alpha-4\alpha h+\alpha^2-h\rho-3c)\nabla\rho\\&+(3\alpha h-3\alpha\rho+\rho^2+2h\rho+3c)\nabla\alpha.\end{aligned}\tag{3.24}$$

Lemma 3.4. $\alpha\neq 0$ *on* Ω.

Proof. If not, we have by (2.2) $\beta=\dfrac{c}{4}$ and hence $\mu^2=\dfrac{c}{4}$. Thus, (3.16), (3.17), (3.20) and (3.22) turn out respectively to

$$3A^2U=2\rho AU+\frac{c}{2}U,\tag{3.25}$$

$$\frac{c}{4}\nabla\rho=-2\rho A^2U+(2\rho^2+c)AU-\frac{c}{4}\rho U,\tag{3.26}$$

$$\rho A^2U=(2\rho-h)\rho AU+\frac{c}{2}(\rho+h)U,\tag{3.27}$$

$$2(h\rho + 2c)AU = c(h + 3\rho)U + c\nabla\rho \tag{3.28}$$

because of $\alpha = 0$. However we notice here that $\rho \neq 0$ on this set by virtue of (3.25) and (3.26).

From (3.25) and (3.27) we obtain

$$\rho(3h - 4\rho)AU = \frac{c}{2}(2\rho + 3h)U. \tag{3.29}$$

On the other hand, we also have by using (3.26) and (3.28)

$$4\rho A^2U + \rho(h - 4\rho)AU = \frac{c}{2}(2\rho + h)U,$$

or, using (3.27)

$$3h\rho AU = \frac{c}{2}(2\rho + 3h)U.$$

Thus, using (3.29), we have $\rho AU = 0$ and hence $U = 0$ because of (3.25) and $\rho \neq 0$, a contradiction. Consequently we have $\alpha \neq 0$ on Ω. □

Lemma 3.5. *$h - \rho \neq 0$ on Ω.*

Proof. If not, we have $h - \rho = 0$. So (3.20) becomes

$$(\rho - \alpha)\left\{A^2U - \rho AU - cU\right\} = 0$$

on this set. Since $\rho - \alpha \neq 0$ by Remark 3.1, it follows that

$$A^2U = \rho AU + cU.$$

Since $g(S\xi, \xi) = \frac{c}{4}(2n - 3)$ is constant because of (3.4), owing to Lemma 3.1 of [9], we see that $AU = \lambda U$, where $\mu^2\lambda = g(AU, U)$. Consequently we obtain

$$\lambda^2 = \rho\lambda + c \tag{3.30}$$

on this subset, which together with (3.21) yields

$$A\nabla\rho = 0. \tag{3.31}$$

By the way, we also have from (3.23)

$$(\alpha - \rho)\nabla\rho = \left\{4\lambda^2 + (6\alpha - 10\rho)\lambda + 3\rho^2 - 3\rho\alpha - c\right\}U,$$

or, using (3.30)

$$(\alpha-\rho)\nabla\rho = 3\left\{(\rho-\alpha)(\rho-2\lambda)+c\right\}U.$$

From this and (3.31), we see that

$$(\rho-\alpha)(\rho-2\lambda)+c=0, \tag{3.32}$$

and hence $\nabla\rho = 0$ on this set because $\rho-\alpha \neq 0$ on Ω. So, using (3.30), we see that λ is constant. Making use of (3.32), we have $\nabla\alpha = 0$. Thus, (3.16) tells us that

$$3\lambda^2 = 2\rho\lambda + \frac{c}{2},$$

which together with (3.30) implies that

$$2\lambda^2 + 3c = 0 \tag{3.33}$$

and $5\lambda = 3\rho$ because of $\lambda \neq 0$ on this set. So, using (3.30) and (3.32), we have $11\rho = 3\alpha$.

From these and (2.2) we verify that

$$\beta - \alpha^2 + \frac{6}{11}\alpha^2 - \frac{c}{4} = 0.$$

Therefore, it is contradictory because of (3.33). Thus, $h-\rho \neq 0$ on Ω is proved. □

From (3.22) and (3.24) we have

$$fU = \sigma\nabla\rho + \rho\nabla\alpha, \tag{3.34}$$

where we have put

$$\begin{aligned} f = \ & (5c\alpha - c\rho + 3\rho^3 - 6\alpha\rho^2 + 3\alpha^2\rho)h^2 \\ & -2c(6\alpha^2 - 5\alpha\rho - 2c + \rho^2)h \\ & +c(\rho - 3\alpha)(2c + 3\alpha\rho - 3\alpha^2), \end{aligned} \tag{3.35}$$

$$\begin{aligned} \sigma = \ & (h\alpha + c)(3\alpha^2 - \alpha\rho - 5\alpha h - 6c) \\ & -(h\rho + 2c)(2\rho^2 - \alpha\rho - 4\alpha h + \alpha^2 - h\rho - 3c), \end{aligned} \tag{3.36}$$

$$\tau = (\alpha\rho + 5\alpha h - 3\alpha^2 + 6c)(h\rho + c) \\ -(h\rho + 2c)(3\alpha h - 3\alpha\rho + \rho^2 + 2h\rho + 3c). \tag{3.37}$$

From (3.34) we obtain $fu(Y) = \sigma(Y\rho) + \tau(Y\alpha)$ for any vector field Y. Differentiating this covariantly and taking the skew-symmetric parts obtained, we find

$$(Xf)u(Y) - (Yf)u(X) + fdu(X,Y) \\ = (X\sigma)Y\rho - (Y\sigma)X\rho + (X\tau)Y\alpha - (Y\tau)X\alpha, \tag{3.38}$$

where the exterior derivative du of 1-form u is given by

$$du(X,Y) = Yu(X) - Xu(Y) - u([X,Y]).$$

Now we prove

Lemma 3.6. *If $\xi h = 0$, then we have $f = 0$ on Ω.*

Proof. Since $\xi h = 0$ is assumed, by putting $X = \xi$ in (3.38) and using Lemma 3.3, we obtain $fdu(\xi, Y) = 0$ because f, σ and τ are polynomials with respect to h, ρ and α. Hence $f = 0$ on Ω. In fact, if not, we have $du(\xi, X) = 0$ for any vector X, that is, $g(\nabla_\xi U, X) + g(\nabla_X \xi, U) = 0$, which together with (1.11), (1.15) and (2.3) implies that $\phi(3AU + \nabla\alpha) + \mu\rho W = 0$. Thus, it follows that

$$\nabla\alpha = \rho U - 3AU \tag{3.39}$$

on this subset. Here we have used $\xi\alpha = 0$. From this and (3.16), we deduce that

$$\alpha\nabla\rho = -\rho AU + (\rho^2 + c)U \tag{3.40}$$

on this set. From the last two equations, we verify that

$$\mu\nabla\mu = (3\alpha - 2\rho)AU + (\rho^2 - \rho\alpha + \frac{c}{2})U,$$

where we have used (1.9) and (2.2). Thus, (3.20) tells us that

$$A^2U = hAU + (\rho^2 - \rho h + c)U \tag{3.41}$$

on this set because of Remark 3.1. Substituting (3.39), (3.40) and (3.41) into (3.21), we find $AU = hU$ and hence $\rho^2 - \rho h + c = 0$ on this subset. Therefore (3.40) implies that $\nabla\rho = 0$ by virtue of Lemma 3.4. So we see that $\nabla h = 0$ on this set. Since we have $AU = hU$ and hence $\nabla\alpha = (\rho - 3h)U$ because of (3.39), we verify that $h^2\rho - h\rho^2 + 2c\rho - 3c\alpha = 0$ and thus $\nabla\alpha = 0$ because ρ and h are constant. Here we have used (3.22). Therefore (3.34) implies $f = 0$ on this subset, a contradiction. This completes the proof of Lemma 3.6. □

Lemma 3.7. $f = 0$ *on* Ω.

Proof. If we replace Y by W in (3.38) and make use of Lemma 3.3, then we obtain $f\,du(X, W) = 0$ because f, σ and τ are polynomials with respect to h, ρ and α.

Let Ω_0 be a set of points in Ω such that $f(p) \neq 0$ at $p \in \Omega$ and suppose that $\Omega_0 \neq \emptyset$. Then we have $du(W, X) = 0$, that is, $g(\nabla_W U, X) + g(\nabla_X W, U) = 0$ on Ω_0. Thus, using (1.11), (1.16) and (2.3), we are led to

$$U\alpha = \rho\mu^2 - 3g(AU, U) \tag{3.42}$$

on Ω_0.

If we take the inner product (2.15) with μW and make use of (1.9) and (2.2), then we obtain the following on Ω_0:

$$\begin{aligned} &\mu^2 g(\nabla_W U, W) \\ = \; &(3\alpha - 2\rho)g(AU, U) + (\alpha - \tfrac{1}{2}\rho)U\alpha \\ &-\tfrac{1}{2}\alpha U\rho + \mu^2(\alpha\rho - \alpha^2 + \tfrac{c}{2}), \end{aligned}$$

which together with (3.42) implies that

$$\begin{aligned} \mu^2 g(\nabla_W U, W) = \; &-\tfrac{1}{2}\rho g(AU, U) - \tfrac{1}{2}\alpha U\rho \\ &+\mu^2(2\alpha\rho - \alpha^2 - \tfrac{1}{2}\rho^2 + \tfrac{c}{2}). \end{aligned} \tag{3.43}$$

On the other hand, differentiating (3.22) covariantly, we find

$$\begin{aligned} &2X(h\rho)AU + (2h\rho + 4c)\{(\nabla_X A)U + A\nabla_X U\} \\ = \; &cX(3\rho - 3\alpha + h)U + c(3\rho - 3\alpha + h)\nabla_X U + X(h\rho)\nabla\rho \\ &+(h\alpha + c)\nabla_X^2\rho - X(h\rho)\nabla\alpha - (h\alpha + c)\nabla_X^2\alpha, \end{aligned}$$

from which, taking the skew-symmetric part and using (1.4) and (1.7), we have

$$\begin{aligned}
& 2X(h\rho)g(AU,Y) - 2Y(h\rho)g(AU,X) \\
& +\tfrac{c}{2}(h\rho+2c)\mu(\eta(X)w(Y) - \eta(Y)w(X)) \\
& +(2h\rho+4c)\{g(A\nabla_X U,Y) - g(A\nabla_Y U,X)\} \\
= \; & cX(3\rho-3\alpha+h)u(Y) - cY(3\rho-3\alpha+h)u(X) \\
& +c(3\rho-3\alpha+h)du(X,Y) + X(h\alpha)Y\rho \\
& -Y(h\alpha)X\rho - X(h\rho)Y\alpha + Y(h\rho)X\alpha,
\end{aligned}$$

on Ω_0, where we have defined a 1-form w by $w(X)=g(W,X)$ for any vector field X. Since $du(W,X)=0$, by putting $X=W$ and $Y=\xi$ in the last equation, we obtain

$$\begin{aligned}
& (h\rho+2c)\left\{g(\nabla_W U,\alpha\xi+\mu W) - g(\nabla_\xi U,\mu\xi+(\rho-\alpha)W) - \tfrac{c}{4}\mu\right\} \\
= \; & 0
\end{aligned}$$

on Ω_0, where we have used (1.8), (2.3) and Lemma 3.3. We notice here that $h\rho+2c\neq 0$. In fact, if not, then $\xi h=0$ and hence $f=0$ because of Lemma 3.6. Therefore, the last equation is led to

$$g(\nabla_W U,W) + (\rho-\alpha)^2 = 0$$

since we have (1.9), (1.11), (2.2) and (2.3), which connected to (3.43) gives

$$\alpha U\rho = (\rho^2+c)\mu^2 - \rho g(AU,U). \tag{3.44}$$

If we take the inner product (3.22) with U and make use of (3.42) and (3.44), then we get

$$(\rho+\alpha)g(AU,U) = (2\rho\alpha - 3\alpha^2 + 2\alpha h + \rho^2 + c)\mu^2,$$

which shows that

$$(\rho+\alpha)\xi g(AU,U) = 2\alpha\mu^2(\xi h) \tag{3.45}$$

on Ω_0 by virtue of Lemma 3.3.

We also have from (3.24), (3.42) and (3.44)

$$\begin{aligned}
&(6\alpha^3-10\alpha^2\rho-\alpha^2h-3c\alpha+2\rho^3+2\alpha\rho^2+2\rho\alpha h\\
&-\rho^2h-3c\rho)g(AU,U)\\
=\;&\mu^2\left\{3\alpha(\rho-\alpha)^2h+c(6\alpha^2-8\rho\alpha-\alpha h)\right.\\
&+(\rho^2+c)(2\rho^2-\alpha\rho-4\alpha h+\alpha^2-\rho h-3c)\\
&\left.+3\alpha^2\rho h-3\alpha^2\rho^2+\alpha\rho^3+2\alpha\rho^2h+3c\alpha\rho\right\},
\end{aligned}$$

which enables us to obtain

$$\begin{aligned}
&(6\alpha^3-10\alpha^2\rho-3c\alpha+2\rho^3+2\alpha\rho^2-3c\rho-(\rho-\alpha)^2h)\xi g(AU,U)\\
=\;&(\rho-\alpha)^2g(AU,U)\xi h+\mu^2\left\{3\alpha(\rho-\alpha)^2-c\alpha-(\rho^2+c)(\rho+4\alpha)\right.\\
&\left.+3\alpha^2\rho+2\alpha\rho^2\right\}\xi h,
\end{aligned}$$

on Ω_0, where we have used Lemma 3.3. From this and (3.45) we have on Ω_0 the following:

$$\begin{aligned}
&2\alpha\mu^2\left\{6\alpha^3-10\alpha^2\rho-3c\alpha+2\rho^3+2\alpha\rho^2-3c\rho-(\rho-\alpha)^2h\right\}\\
=\;&(\rho+\alpha)(\rho-\alpha)^2g(AU,U)+\mu^2(\rho+\alpha)\left\{3\alpha(\rho-\alpha)^2-c\alpha\right.\\
&\left.-(\rho^2+c)(\rho+4\alpha)+3\alpha^2\rho+2\alpha\rho^2\right\}.
\end{aligned}$$

Owing to Lemma 3.3 and Remark 3.1, we have

$$2\alpha\mu^2(\xi h)+(\rho+\alpha)\xi g(AU,U)=0$$

on Ω_0, which together with (3.45) yields $\xi h=0$ and hence $f=0$ on Ω because of Lemma 3.4 and Lemma 3.6. Thus, Lemma 3.7 is proved. □

4. Principal curvatures corresponding to $\nabla_\xi\xi$

We continue our arguments under the same hypotheses $R_\xi A=AR_\xi$ and at the same time $\nabla_W S=0$ as in Section 3. Then by Lemma 3.7 we see that

$$\begin{aligned}
&(5c\alpha-c\rho+3\rho^3-6\alpha\rho^2+3\alpha^2\rho)h^2-2c(6\alpha^2-5\alpha\rho-2c+\rho^2)h\\
&+c(\rho-3\alpha)(2c+3\alpha\rho-3\alpha^2)=0,
\end{aligned}\tag{4.1}$$

where we have used (3.34) and (3.35).

Applying (3.24) by A, we find

$$\begin{aligned} & 2(3\alpha^2-\rho\alpha-5\alpha h-6c)A^2U \\ = & \left\{3h(\rho-\alpha)^2-c(8\rho-6\alpha+h)\right\}AU \\ & +(2\rho^2-\rho\alpha-4\alpha h+\alpha^2-h\rho-3c)A\nabla\rho \\ & +(3\alpha h-3\alpha\rho+\rho^2+2h\rho+3c)A\nabla\alpha. \end{aligned}$$

Substituting (3.16) and (3.21) into this and making use of (3.23), we find

$$\begin{aligned} & \lambda_1 AU+\lambda_2 U \\ = & \left\{5h\alpha\rho-6ch-7\alpha^3+2\rho^3-4h\alpha^2-11h^2\alpha+17h\rho^2-9h^2\rho\right. \\ & \left.-27\alpha\rho^2+28\alpha^2\rho\right\}\nabla\rho+\left\{6ch-13h\alpha\rho+5\rho^3+3h\alpha^2+11h^2\alpha\right. \\ & \left.-8h\rho^2+9h^2\rho+2\alpha\rho^2-3\alpha^2\rho\right\}\nabla\alpha, \end{aligned} \tag{4.2}$$

where we have put

$$\begin{aligned} \lambda_1 = \ & 12c\alpha-32ch-4c\rho-54h\alpha\rho-42\alpha^3+8\rho^3+40h\alpha^2 \\ & -42h^2\alpha+50h\rho^2+2h^2\rho-90\alpha\rho^2+116\alpha^2\rho, \end{aligned}$$

$$\begin{aligned} \lambda_2 = \ & 23ch\rho-13ch\alpha-2c\alpha\rho+3c\alpha^2+21h\alpha^3-5c\rho^2-15h\rho^3 \\ & +51h\alpha\rho^2-57h\alpha^2\rho+30h^2\alpha\rho-15h^2\alpha^2-15h^2\rho^2. \end{aligned}$$

By Lemma 3.7, we can deduce from (3.22) and (4.2) the following:

$$\begin{aligned} & (21c\alpha-c\rho+15\rho^3-30\alpha\rho^2+15\alpha^2\rho)h^3 \\ & +(46c\alpha\rho+16c^2+15\rho^4-53c\alpha^2-21c\rho^2-51\alpha\rho^3-21\alpha^3\rho \\ & +57\alpha^2\rho^2)h^2+(39c\alpha^3-28c^2\alpha+4c^2\rho-44c\rho^3+101c\alpha\rho^2 \\ & -88c\alpha^2\rho)h-c(20c\alpha\rho+63\alpha^4+12\rho^4-12c\alpha^2-16c\rho^2 \\ & -147\alpha\rho^3-237\alpha^3\rho+309\alpha^2\rho^2)=0. \end{aligned} \tag{4.3}$$

Similarly, from (3.24) and (4.2) we obtain

$$\begin{aligned} & (21c\alpha-c\rho+12\alpha^3+3\rho^3+6\alpha\rho^2-21\alpha^2\rho)hs \\ & +(2c\alpha\rho+16c^2-90\alpha^4+75\rho^4-39c\alpha^2+9c\rho^2 \\ & -141\alpha\rho^3+189\alpha^3\rho-33\alpha^2\rho^2)h^2 \\ & +(12\rho^5-24c^2\alpha-21c\alpha^3-8c^2\rho-120c\rho^3-144\alpha\rho^4 \\ & +108\alpha^4\rho+137c\alpha\rho^2+12c\alpha^2\rho+360\alpha^2\rho^3-336\alpha^3\rho^2)h \\ & -c(48c\alpha\rho+117\alpha^4+32\rho^4-18c\alpha^2-46c\rho^2 \\ & -389\alpha\rho^3-507\alpha^3\rho+747\alpha^2\rho^2)=0. \end{aligned} \tag{4.4}$$

(In the above arguments we use a computer for calculations).

Let Ψ be the resultant of (4.1) and (4.2) with respect to h, and Θ be that of (4.1), (4.2) and (4.3), that

$$\Psi = -12c^2(\alpha-\rho)(2c+3\alpha\rho-3\rho^2)\Delta,$$

$$\Theta = -36(\alpha-\rho)(3\alpha^4+3\alpha^2c+2c^2-7\alpha^3\rho-c\alpha\rho+5\alpha^2\rho^2-6c\rho^2-\alpha\rho^3)\Delta,$$

where we have put

$$\begin{aligned}\Delta = \; & -21627c^2\alpha^{10}+6129c^3\alpha^8+195c^4\alpha^6-225c^5\alpha^4+16c^6\alpha^2-18225c\alpha^{11}\rho \\ & +138996c^2\alpha^9\rho-29799c^3\alpha^7\rho+3282c^4\alpha^5\rho-242c^5\alpha^3\rho+151632c\alpha^{10} \\ & -378783c^2\alpha^8\rho^2+59958c^3\alpha^6\rho^2-11723c^4\alpha^4\rho^2+2120c^5\alpha^2\rho^2-144c^6\rho^2 \\ & -564003c\alpha^9\rho^3+569628c^2\alpha^7\rho^3-62631c^3\alpha^5\rho^3+12220c^4\alpha^3\rho^3-1662c^5\alpha\rho^3 \\ & -8748\alpha^{10}\rho^4+1222884c\alpha^8\rho^4-516246c^2\alpha^6\rho^4+36528c^3\alpha^4\rho^4-3475c^4\alpha^2\rho^4 \\ & -87c^5\rho^4+71928\alpha^9\rho^5-1686798c\alpha^7\rho^5+290484c^2\alpha^5\rho^5-14869c^3\alpha^3\rho^5 \\ & +114c^4\alpha\rho^5-260604\alpha^8\rho^6+1512720c\alpha^6\rho^6-100390c^2\alpha^4\rho^6+6066c^3\alpha^2\rho^6 \\ & -613c^4\rho^6+545184\alpha^7\rho^7-860682c\alpha^5\rho^7+17684c^2\alpha^3\rho^7-669c^3\alpha\rho^7 \\ & -724248\alpha^6\rho^8+284568c\alpha^4\rho^8+1233c^2\alpha^2\rho^8-713c^3\rho^8+632016\alpha^5\rho^9 \\ & -41889c\alpha^3\rho^9-728c^2\alpha\rho^9-361368\alpha^4\rho^{10}-672c\alpha^2\rho^{10}-251c^2\rho^{10} \\ & +130464\alpha^3\rho^{11}+525c\alpha\rho^{11}-27324\alpha^2\rho^{12}-60c\rho^{12}+2808\alpha\rho^{13}-108\rho^{14}.\end{aligned}$$

From above two equations, we have

$$(3\rho^2-3\alpha\rho-2c)(3\alpha^4+3c\alpha^2+2c^2-7\alpha^3\rho-c\alpha\rho+5\alpha^2\rho^2-6c\rho^2-\alpha\rho^3)\Delta=0,$$

because of Remark 3.1. Further, from this we can deduce that both α and ρ are constants. Thus (3.22) becomes

$$(2h\rho+4c)AU = c(3\rho-3\alpha+h)U. \tag{4.5}$$

Now we demonstrate the following lemma:

Lemma 4.1. *$AU = \lambda U$ on Ω, where the scalar λ is given by $\mu^2\lambda = g(AU,U)$.*

Proof. If not, we have from (4.5)

$$h\rho = -2c, \qquad h = 3(\alpha-\rho) \tag{4.6}$$

on this subset. Since ρ and α are constant, (3.21) and (3.23) are reduced respectively to

$$A^2U + (2h - 3\rho)AU - cU = 0, \tag{4.7}$$

$$4A^2U - 2\rho AU - (h^2 + c)U = 0. \tag{4.8}$$

From the last two equations, we obtain

$$2(4h - 5\rho)AU + (h^2 - 3c)U = 0.$$

Because of our assumption, we have $h^2 = 3c$ and $4h = 5\rho$. From (2.2), (4.7), (4.8) and the last two equations produce a contradiction. □

Because of (1.9) and (2.2) we see that μ is constant by virtue of $\nabla\alpha = \nabla\rho = 0$. Thus, (3.17) implies that

$$(3\alpha - 2\rho)\lambda^2 + (2\rho^2 - 2\rho\alpha + c)\lambda + \frac{c}{4}(\alpha - \rho) = 0, \tag{4.9}$$

where we have used Lemma 4.1.

By using $\nabla\alpha = \nabla\rho = 0$ and Lemma 4.1, we verify that (3.16), (3.18) and (4.5) turn out respectively to

$$3\lambda^2 = 2\rho\lambda + \frac{c}{2},$$

$$2\lambda^2 + (\rho - 2h)\lambda + \frac{c}{2} = 0,$$
$$2h\rho\lambda = c(3\rho - 3\alpha + h - 4\lambda).$$

Combining these and (4.9), we have $h = \lambda$, $3\alpha = 7\lambda$ and $\rho = 3\lambda$. So, we are led to $6h^2 + c = 0$. Thus, we have

Theorem 4.2. *Let M be a real hypersurface in $P_n\mathbb{C}$ which satisfies $R_\xi A = AR_\xi$ and $\nabla_{\phi\nabla_\xi\xi}S = 0$. Then M is a Hopf hypersurface in $P_n\mathbb{C}$.*

Finally, we consider real hypersurfaces in a complex hyperbolic space satisfying $R_\xi A = AR_\xi$ and $\nabla_{\phi\nabla_\xi\xi}S = 0$. Then we have

$$h^2 = -\frac{c}{6}.$$

Let $\lambda_1, \dots, \lambda_{2n-2}$ be principal curvatures corresponding to arbitrary principal curvature vectors orthogonal to U. Then, using $AU = \lambda U$ and $h = \lambda$, we have $\lambda_1 + \cdots + \lambda_{2n-2} = 0$. Hence we have

$$\sum_{i<j} \lambda_i \lambda_j \leq 0, \quad h_{(2)} = h^2 - 2\sum_{i<j} \lambda_i \lambda_j, \tag{4.10}$$

where $h_{(2)} = \mathrm{Tr}\,{}^t AA$.

On the other hand, the scalar curvature r of M is given by

$$r = c(n^2 - 1) + h^2 - h_{(2)}$$

by virtue of (1.5), which together with (4.10) implies $r \leq 0$.

Thus, we have

Theorem 4.3. *Let M be a real hypersurface in $H_n\mathbb{C}$ which satisfies $R_\xi A = AR_\xi$ and $\nabla_{\phi\nabla_\xi\xi} S = 0$. If the scalar curvature of M is nonnegative, then M is a Hopf hypersurface in $H_n\mathbb{C}$.*

References

[1] J. Berndt, Real hypersurfaces with constant principal curvatures in a complex hyperbolic space, *J. Reine Angew. Math.*, **395** (1989), 132–141.

[2] J. T. Cho and U-H. Ki, Real hypersurfaces of a complex projective space in terms of the Jacobi operators, *Acta Math. Hungar*, **80** (1998), 155–167.

[3] T. Y. Hwang, U-H. Ki and N.-G. Kim, Ricci tensors of real hypersurfaces in a nonflat complex space form, *Math. J. Toyama Univ.*, **27** (2004), 1–22.

[4] E.-H. Kang and U-H. Ki, On real hypersurfaces of a complex hyperbolic space, *Bull. Korean Math. Soc.*, **34** (1997), 173–184.

[5] E.-H. Kang and U-H. Ki, Real hypersurfaces satisfying $\nabla_\xi S = 0$ of a complex space form, *Bull. Korean Math. Soc.*, **35** (1998), 819–835.

[6] U-H. Ki, Real hypersurfaces with parallel Ricci tensor of a complex space form, *Tsukuba J. Math.*, **13** (1989), 73–81.

[7] U-H. Ki, N.-G. Kim and S.-B. Lee, On certain real hypersurfaces of a complex space form, *J. Korean Math. Soc.*, **29** (1992), 63–77.

[8] U-H. Ki and C. Li, Real hypersurfaces concerned with the structure Jacobi operator in a nonflat complex space form, *preprint*.

[9] U-H. Ki and S. Nagai, Real hypersurfaces of a nonflat complex space form in terms of the Ricci tensor, *Tsukuba J. Math.*, **29** (2005), 511–532.

[10] U-H. Ki and S. Nagai, The Ricci tensor and structure Jacobi operator of real hypersurfaces in a complex projective space, *preprint*.

[11] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space II, *Tsukuba J. Math.*, **15** (1991), 547–561.

[12] M. Lohnherr and H. Reckziegel, On ruled real hypersurfaces in complex space forms, *Geom. Dedicata*, **74** (1999), 267–286.

[13] S. Maeda, Ricci tensors of real hypersurfaces in a complex projective space, *Proc. Amer. Math. Soc.*, **122** (1994), 1229–1235.

[14] R. Niebergall and P.J. Ryan, *Real Hypersurfaces in Complex Space Forms*, Tight and Taut Submanifolds (Eds. T. E. Cecil and S. S. Chern), 233–305, Cambridge University Press, 1998.

[15] Y. J. Suh, On real hypersurfaces of a complex space form with η-parallel Ricci tensor, *Tsukuba J. Math.*, **14** (1990), 27–37.

[16] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, *Osaka J. Math.*, **10** (1973), 495–506.

[17] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific Publ. Singapore, 1984.

U-Hang Ki
The National Academy of Sciences
Korea
Department of Mathematics
Kyungpook National University
Daegu, 702-701
Korea
E-mail address: uhangki2005@yahoo.co.kr

Chunji Li
Institute of System Science
College of Sciences
Northeastern University
Shenyang, 110-004
P. R. China
E-mail address: chunjili@hanmail.net

Setsuo Nagai
Department of Education
University of Toyama
Toyama, 930-8555
Japan
E-mail address: EZW00314@nifty.com

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